

# Aggregate Welfare of the Anas-Arnott Housing Model

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Anas and Arnott (1991) formulate an elegant housing market model. Anas and Arnott (1996) derive the utilitarian social welfare function corresponding to the Anas-Arnott model (1991). They make use of this function to analyze the Anas-Arnott model. This paper shows some properties of this social welfare function.

## 1. Introduction

Housing has two important characteristics that make it distinct from other goods. First, housing is a durable good. Secondly, housing has large quality differences. Sweeney (1974a, b and c) provides some early theoretical models incorporating these characteristics. After his seminal papers, many researchers have formulated other theoretical housing market models. Among those theoretical models, the model formulated by Anas and Arnott (1991) is the only one which has been applied empirically (See Anas and Arnott (1993b, 1994)). We call their model the Anas-Arnott model in this paper. Anas and Arnott (1993a) prove that there exists a unique stationary equilibrium in a simplified version of the Anas-Arnott model, where only vacant land and one type of housing exist. Anas and Arnott (1996) solve a sequence maximization problem where a social planner maximizes an aggregate welfare of agents in the model under some economic constraints to show that the Anas-Arnott dynamic equilibrium can be characterized as the social optimum. After assuming that all exogenous variables in the Anas-Arnott model are stationary, Yamazaki (2000) analyzes the stationary version of the dynamic programming problem corresponding to the sequence maximization problem in Anas and Arnott (1996) to prove that there exists a unique stationary equilibrium in the Anas-Arnott model. Anas, Arnott and Yamazaki (2000) assume that all financial exogenous variables such as conversion costs of housing stocks asymptotically grow at the rate of interest and prove that there exists a unique balanced growth path in the model. Three papers just mentioned, Anas and Arnott (1996), Yamazaki (2000) and Anas, Arnott and Yamazaki (2000), analyze a dynamic optimization problem of maximizing the utilitarian social welfare corresponding to the Anas-Arnott model. However, they do not carefully examine the social welfare function.

The purpose of this paper is to show some properties of the social welfare function.

## 2. The Anas-Arnott Model

In the Anas and Arnott model, land and housing investors face uncertainty in construction and conversion costs. The idiosyncratic costs affecting an investor's construction or demolition options are assumed to be i.i.d. across investors of the same type and serially uncorrelated. Each period's asset markets open at the beginning of the period, before that period's uncertainty is realized.

The supply side of the general AA model is described as follows. A land investor pays  $V_{0t}$ , the price of land, to buy one unit of land at the beginning of the year  $t$ . During the year, the land is rented for a non-housing use for an exogenous annual rental,  $R_{0t}$ , received at the beginning of the year. At the end of the year, the cost of constructing a house with quality level  $k \in \{1, 2, \dots, K\}$  on the land is revealed to each investor as a random draw from a distribution of costs. Those land investors who experience a sufficiently low differential cost of construction will build a house, while others will not. A house investor pays  $V_{kt}$ , the price of one house with quality level  $k \in \{1, 2, \dots, K\}$ , at the beginning of the year  $t$ . At the end of the year, the cost of converting a house with quality level  $k$  to a house with quality level  $k' \in \{1, 2, \dots, K\}$  is revealed to each investor. Converting to the house with quality level 0 means demolishing the house. Those investors who experience a sufficiently low differential cost of converting to a house with quality level  $k'$  will convert their house to the house with quality level  $k'$ , while the others will not. Conversion profits  $\hat{\Pi}$  are measured at the beginning of each year and are the sum of two parts: (1) the deterministic profit  $\Pi$  received from renting and conversion, and (2) the random idiosyncratic non-financial cost  $c$  associated with the activity. Hence,

$$(1) \quad \hat{\Pi}_{kk't} = \Pi_{kk't} - c_{kk't}$$

where  $k \in \{1, 2, \dots, K\}$  and  $k' \in A(k) \subset \{1, 2, \dots, K\}$ . The deterministic part of the conversion profits is specified as

$$(2) \quad \Pi_{kk't} = \delta(V_{k't+1} - C_{kk't}) - V_{kt}$$

where  $\delta = 1/(1+r)$  with  $r$  the interest rate and  $C_{kk't}$  is the expected cost of converting the

house with quality  $k$  to the house with quality  $k'$  at year  $t$ . The c.d.f. of each of the idiosyncratic non-financial costs is assumed to be the double exponential distribution. Under this distributional assumption, the expected discounted conversion profit of converting the house with quality  $k$  to the house with quality  $k'$  at year  $t$  can be written as

$$(3) \quad \Omega_{kt}(\mathbf{V}_{t+1}) = E \left[ \max_{k' \in A(k)} \{ \hat{\Pi}_{kk't} \} \right] = \frac{1}{\Phi_k} \ln \left( \sum_{k' \in A(k)} \exp \Phi_k \Pi_{kk't} \right)$$

where  $\Phi_k$  is the cost dispersion parameter for the house investor which is inversely related to the variance of the idiosyncratic cost.  $\mathbf{V}_{t+1}$  is the vector of housing prices at year  $t+1$ .  $\omega_{kt}(R_{kt})$ , the expected profit at the beginning of year  $t$  from the occupancy/vacancy decision, is defined as

$$(4) \quad \omega_{kt}(R_{kt}) = E \left[ \max \{ \pi_{kot}, \pi_{kvt} \} \right]$$

where

$$(5) \quad \pi_{kot} = R_{kt} - D_{kot} - d_{kot} \quad \text{and} \quad \pi_{kvt} = -D_{kvt} - d_{kvt}.$$

$[D_{kot}, D_{kvt}]$  are the expected maintenance costs for type- $k$  occupied and vacant units at the beginning of year  $t$  respectively and  $[d_{kot}, d_{kvt}]$  are the idiosyncratic maintenance costs for occupied and vacant units.  $[d_{kot}, d_{kvt}]$  are double exponentially distributed with dispersion parameter  $\phi_k$ . Then (4) can be written as

$$(4') \quad \omega_{kt}(R_{kt}) = \frac{1}{\phi_k} \ln \left[ \exp \phi_k (R_{kt} - D_{kot}) + \exp \phi_k (-D_{kvt}) \right]$$

For vacant land, we assume that it can always be rented to an alternative use for some exogenous rent  $R_{0t}$ . Hence,  $\omega_{0t}(R_{0t}) = R_{0t}$ . Under the assumption of double exponential distribution, the investor's conversion choice probabilities are calculated as follows.

$$Q_{kk't}(\mathbf{V}_{t+1}) = \text{prob} \left[ \hat{\Pi}_{kk't} > \hat{\Pi}_{kst} : s, k' \in A(k), s \neq k' \right]$$

$$\begin{aligned}
 (6) \quad &= \frac{\exp \Phi_k \Pi_{kk't}}{\sum_{s \in A(k)} \exp \Phi_k \Pi_{kst}} \\
 &= \frac{\exp \Phi_k \delta(V_{k't+1} - C_{kk't})}{\sum_{s \in A(k)} \exp \Phi_k \delta(V_{st+1} - C_{kst})}
 \end{aligned}$$

Similarly,

$$\begin{aligned}
 (7) \quad q_{kot}(R_{kt}) &= \text{prob}[\pi_{kot} > \pi_{kvt}] \\
 &= \frac{\exp \phi_k(R_{kt} - D_{kot})}{\exp \phi_k(R_{kt} - D_{kot}) + \exp \phi_k(-D_{kvt})}
 \end{aligned}$$

Now we turn to explaining the demand side of the general Anas-Arnott model. Households are divided into  $h=1,2,\dots,H$  demand groups.  $\mathbf{N}_t = (N_{1t}, N_{2t}, \dots, N_{Ht})$  is the exogenous vector of the number of households by year in each demand group.  $\mathbf{y}_t = (y_{1t}, y_{2t}, \dots, y_{Ht})$  is the exogenous vector of the household income in demand group  $h$  in year  $t$ .  $\mathbf{Y}_t = (Y_{0t}, Y_{2t}, \dots, Y_{Kt})$  is the exogenous vector of variables denoting submarket characteristics other than rent.  $\hat{U}_{hkt} = U_{hkt} + u_{hkt}$  is the utility that a household in group  $h$  enjoys from renting a housing unit in submarket  $k$  in year  $t$ . The systematic utility is specified as

$$(8) \quad U_{hkt} = \alpha_h(y_{ht} - R_{kt}) + \beta_h Y_{kt}.$$

The idiosyncratic utility  $u_{hkt}$  is assumed to be independently and identically double exponentially distributed for each  $k$ . Then the probability that a household in demand group  $h$  chooses submarket  $k$  is given by stochastic utility maximization:

$$P_{hkt}(R_{0t}, \mathbf{R}_t) = \text{prob}[\hat{U}_{hkt} > \hat{U}_{hst}; \forall s \in \{0, 1, \dots, K\}, s \neq k]$$

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$$\begin{aligned}
 (9) \quad &= \frac{\exp \delta_h U_{hkt}}{\sum_{s=0}^K \exp \delta_h U_{hst}} \\
 &= \frac{\exp(\delta_h(\alpha_h(y_{ht} - R_{kt}) + \beta_h Y_{kt}))}{\sum_{s=0}^K \exp(\delta_h(\alpha_h(y_{ht} - R_{st}) + \beta_h Y_{st}))},
 \end{aligned}$$

where  $\mathbf{R}_t = (R_{1t}, \dots, R_{Kt})$  is the vector of rental prices at year  $t$ . The expected maximized utility can be written

$$(10) \quad \Psi_{ht}(\mathbf{R}_t) = E \left[ \max \left\{ \hat{U}_{hkt}; k \in \{0, 1, \dots, K\} \right\} \right] = \frac{1}{\delta_h} \ln \left( \sum_{s=0}^K \exp \delta_h U_{hst} \right)$$

We have described the demand and supply sides of the general AA model. The market clearing condition can be written as

$$(11) \quad \sum_{h=1}^H P_{hkt}(\mathbf{R}_t) N_{ht} = q_{kot}(\mathbf{R}_t) S_{kt},$$

where  $S_{kt}$  is the stock of housing with quality level  $k$  at year  $t$ . We denote the vector of  $S_{kt}$  as  $\mathbf{S}_t$ . The stock of housing (including the stock of land) adjusts forward in time according to a Markovian process driven by the conversion probabilities given by (6), that is,

$$(12) \quad S_{kt+1} = \sum_{z=0}^K Q_{zkt}(\mathbf{V}_{t+1}) S_{zt}$$

for all  $k$  and each  $t$ . For each  $t$  there are  $K+1$  such equations, but one is redundant by the fact that the total stock of built-up plus vacant land is a given and time-invariant constant,  $L$ :

$$(13) \quad \sum_{z=0}^K S_{zt} = L.$$

In a competitive asset market, the price of each asset is bid up in such a way that the expected economic profit from each asset type is zero at the beginning of each year. Hence,

$$\begin{aligned}
 (14) \quad V_{kt} &= E \left[ \max_{k' \in A(k)} \{ \hat{\Pi}_{kk't} \} \right] + E \left[ \max \{ \pi_{kot}, \pi_{kvt} \} \right] \\
 &= \frac{1}{\Phi_k} \ln \left( \sum_{k' \in A(k)} \exp \Phi_k \Pi_{kk't} \right) + \omega_{kt}(R_{kt})
 \end{aligned}$$

This equation can be called the asset bid-price equation. The Anas-Arnott model has several exogenous variables. To avoid notational complexity, we define a vector of exogenous financial variables:  $\mathbf{M}_t \equiv (\mathbf{y}_t, \mathbf{Y}_t, -R_{0t}, -\mathbf{C}_t, -\mathbf{D}_{ot}, -\mathbf{D}_{vt})$ , where  $\mathbf{C}_t \equiv (C_{00t}, \dots, C_{0Kt}, C_{10t}, \dots, C_{KKt})$ ,  $\mathbf{D}_{ot} \equiv (D_{00t}, D_{10t}, \dots, D_{K0t})$  and  $\mathbf{D}_{vt} \equiv (D_{0vt}, D_{1vt}, \dots, D_{Kvt})$ . We are ready to define the equilibrium concepts.

**Definition 1:** Given  $\mathbf{N}_t$ ,  $\mathbf{S}_0$  and  $\mathbf{M}_t$ , a (infinite-time-horizon) dynamic equilibrium given by  $\{\mathbf{S}_t, \mathbf{Q}_t, \mathbf{P}_t, \mathbf{q}_t, \mathbf{V}_t, \mathbf{R}_t\}_{t=1}^{\infty}$  satisfies (6), (7), (9), (11), (12), (13) and (14).

We are also interested in a special class of such dynamic equilibria in which each asset price, stock and rent remain invariant from period to period. We will call such an equilibrium a stationary equilibrium.

**Stationary Assumption (for Exogenous Variables):**  $\mathbf{N}_t = \mathbf{N}$  and  $\mathbf{M}_t = \mathbf{M}$  for each  $t$ , where  $\mathbf{N}$  and  $\mathbf{M}$  are some constant vectors.

Together with this assumption, if there exists a stationary equilibrium,

$$(6') \quad Q_{kk'}(\mathbf{V}) = \frac{\exp \Phi_k \delta(V_{k'} - C_{kk'})}{\sum_{s \in A(k)} \exp \Phi_k \delta(V_s - C_{ks})},$$

$$(7') \quad q_{ko}(R_k) = \frac{\exp \phi_k (R_k - D_{ko})}{\exp \phi_k (R_k - D_{ko}) + \exp \phi_k (-D_{kv})},$$

$$(9') \quad P_{hk}(\mathbf{R}) = \frac{\exp(\delta_h(\alpha_h(y_h - R_k) + \beta_h Y_k))}{\sum_{s=0}^K \exp(\delta_h(\alpha_h(y_h - R_s) + \beta_h Y_s))},$$

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$$(11') \quad \sum_{h=1}^H P_{hk}(\mathbf{R})N_h = q_{ko}(\mathbf{R})S_k,$$

$$(12') \quad S_k = \sum_{z=0}^K Q_{zk}(\mathbf{V})S_z,$$

$$(13') \quad \sum_{z=0}^K S_z = L$$

and

$$(14') \quad V_k = \frac{1}{\Phi_k} \ln \left( \sum_{k' \in A(k)} \exp \Phi_k \Pi_{kk'} \right) + \omega_{kt}(R_k)$$

must hold at the stationary equilibrium. The following is the formal definition of the stationary equilibrium.

**Definition 2:** Given  $\mathbf{N}$ ,  $\mathbf{S}_0$  and  $\mathbf{M}$ , a stationary dynamic equilibrium given by  $\{\mathbf{S}, \mathbf{Q}, \mathbf{P}, \mathbf{q}, \mathbf{V}, \mathbf{R}\}$  satisfies (6'), (7'), (9'), (11'), (12'), (13') and (14').

We have described the general Anas-Arnott model. In the next section, we derive social welfare of the model.

### 3. Social Welfare

Anas and Arnott (1996) consider an aggregate expected welfare of consumers and investors in the general Anas-Arnott model:

$$(15) \quad Z_t = Z(\mathbf{S}_t, \mathbf{R}_t, \mathbf{V}_{t+1}; \mathbf{V}_t, \mathbf{N}_t, \mathbf{M}_t) \\ = \sum_{h=1}^H N_{ht} \Psi_{ht}(\mathbf{R}_t, \mathbf{M}_t) + \sum_{k=0}^K S_{kt} (\Omega_{kt}(\mathbf{V}_{t+1}, \mathbf{M}_t) + \omega_{kt}(R_{kt}, \mathbf{M}_t) - V_{kt}) - \Theta_t L$$

where  $\Theta_t$  is the rent at year  $t$ , claimed by the original owners of the  $L$  units of land and  $\tilde{\mathbf{M}}_t$  is defined as  $\tilde{\mathbf{M}}_t = (\mathbf{M}_t, \Theta_t)$ . Anas and Arnott (1996) show that a simple calculation eliminates  $(\mathbf{R}_t, \mathbf{V}_t, \mathbf{V}_{t+1})$  from the utilitarian welfare function  $Z_t$  in (15):

$$(15') \quad Z_t = Z(\mathbf{P}_t, \mathbf{q}_t, \mathbf{Q}_t, \mathbf{S}_t; \mathbf{N}_t, \mathbf{M}_t).$$

Using this  $Z_t$  in (15'), we can write the present value of the aggregate welfare  $Z_t$  over time as

$$(16) \quad W = \sum_{t=0}^{\infty} \delta^t Z(\mathbf{P}_t, \mathbf{q}_t, \mathbf{Q}_t, \mathbf{S}_t; \mathbf{N}_t, \mathbf{M}_t).$$

Anas, Arnott and Yamazaki (2000) maximize (16) with respect to  $(\mathbf{P}_t, \mathbf{q}_t, \mathbf{Q}_t, \mathbf{S}_t)$  subject to (11), (12) and (13) to show that the first-order conditions of this maximization problem is equivalent to the conditions of the dynamic equilibrium in Definition 1. This result is just an example of welfare theorem. This also means that there exists a representative agent whose preferences represent those of underlying consumers and profits of housing investors in the model. Anderson et al (1992) derive the utilitarian social welfare function corresponding to a basic logit model, which has only consumers, and show that this function generates the logit demand function. They also examine the social welfare function to show that it constitutes an indirect utility function. One of major objectives of this paper is to show that the social welfare function (15) corresponding to the Anas-Arnott model is also an indirect utility function.  $Z_t$  in (15) consists of  $\Omega_{kt}(\mathbf{V}_{t+1}, \mathbf{M}_t)$ ,  $\omega_{kt}(R_{kt}, \mathbf{M}_t)$  and  $\Psi_{ht}(\mathbf{R}_t, \mathbf{M}_t)$ . First, we want to show that these functions satisfy four properties of a profit function or an indirect utility function (See e.g. Varian (1992)).

Anas and Arnott (1991) assume that each idiosyncratic utility or cost is i.i.d. according to the double exponential distribution:

$$(17) \quad G(x < z) = \exp - [\exp - \gamma(z - \eta)], \quad \gamma > 0,$$

where  $x$  stands for a random realization of idiosyncratic utility or cost. The mode  $\eta$  of the distribution is assumed to be  $-g/\gamma$ , where  $g$  is Euler's constant. This assumption implies that  $E(x) = \eta + g/\gamma = 0$ . The variance is  $Var(x) = \pi^2/6\gamma^2$ .  $\gamma$  is called the dispersion parameter, which is inversely proportional to the standard deviation of the distribution. First, we want to show

**LEMMA 1:** *Suppose that  $x$  follows (17) with mean zero and dispersion parameter  $\gamma$ . Let  $\alpha$  be any positive real number. Denote the dispersion parameter of  $\alpha x$  as  $\gamma(\alpha)$ . Then*



$$\gamma(\alpha) = \frac{1}{\alpha} \gamma.$$

**Proof:** Clearly, the mean of  $\alpha x$  is zero. Hence,  $\frac{\pi^2}{6\gamma^2(\alpha)} = E((\alpha x)^2) = \alpha^2 E(x^2) = \frac{\alpha^2 \pi^2}{6\gamma^2}$ .

This implies the conclusion. ■

Next, we show

**LEMMA 2:** Let  $\mathbf{x} = (x_1, \dots, x_n) \in \mathbf{R}^n$  and  $\mathbf{m} = (m_1, \dots, m_n) \in \mathbf{R}^n$ . Suppose that each element of  $\mathbf{m}$  follows (17) with mean zero and dispersion parameter  $\gamma$ . Define a function of  $\mathbf{x}$  and  $\mathbf{m}$  as

$$(18) \quad F(\mathbf{x}; \mathbf{m}) \equiv \frac{1}{\gamma} \ln \left( \sum_{i=1}^n \exp(\gamma x_i) \right).$$

Then this function is

- (1) continuous in  $\mathbf{x}$ ,
- (2) increasing in  $\mathbf{x}$ ,
- (3) convex in  $\mathbf{x}$ , and
- (4) homogeneous of degree one with respect to  $\mathbf{x}$  and  $1/\gamma$ .

**Proof:** Since all partial derivatives of  $F$  with respect to  $\mathbf{x}$  are positive:

$$(19) \quad F_j \equiv \frac{\partial F}{\partial x_j} = \frac{\exp(\gamma x_j)}{\sum_{i=1}^n \exp(\gamma x_i)} > 0, j=1,2,\dots,n,$$

the statement (2) is proved. Since  $F$  is differentiable with respect to  $\mathbf{x}$ ,  $F$  is continuous in  $\mathbf{x}$ . Hence, the statement (1) is proved.

After differentiating (19) with respect to  $\mathbf{x}$ , we get

$$(20) \quad F_{jj} \equiv \frac{\partial^2 F}{\partial x_j^2} = \gamma \frac{\exp(\gamma x_j) \left( \sum_{i=1}^n \exp(\gamma x_i) \right) - (\exp(\gamma x_j))^2}{\left( \sum_{i=1}^n \exp(\gamma x_i) \right)^2} > 0$$

and

$$(21) \quad F_{jk} \equiv \frac{\partial^2 F}{\partial x_k \partial x_j} = \gamma \frac{-(\exp(\gamma x_j))(\exp(\gamma x_k))}{\left(\sum_{i=1}^n \exp(\gamma x_i)\right)^2} < 0.$$

Clearly,

$$(22) \quad \sum_{k=0}^n F_{jk} = 0 \quad \text{for } j=1,2,\dots,n.$$

Because of the signs of (20) and (21), (22) is equivalent to

$$(22') \quad |F_{jj}| = \sum_{k \neq j} |F_{jk}| \quad \text{for } j=1,2,\dots,n.$$

Denote the Hessian (determinant) of the function  $F$  as  $H$  and the successive (or leading)  $k \times k$  principal minors of the Hessian matrix as  $H_k$ . Because of (22'),

$$(23) \quad |F_{jj}| > \sum_{k \neq j} |F_{jk}| \quad \text{for } j=1,2,\dots,n-1.$$

Hence, together with (20), the matrix of  $H_{n-1}$  is positive dominant diagonal. Therefore,

$$(24) \quad H_k > 0 \quad \text{for } k=1,2,\dots,n-1,$$

while (22) implies

$$(25) \quad H_n = H = 0.$$

Hence, (24) and (25) prove the statement (3), that is, the function  $F$  is convex.

Let  $\alpha$  be any positive real number. Then, by Lemma 1,

$$(26) \quad \begin{aligned} F(\alpha \mathbf{x}; \alpha \mathbf{m}) &= \frac{1}{\gamma(\alpha)} \ln \left( \sum_{i=1}^n \exp(\gamma(\alpha) \alpha x_i) \right) \\ &= \frac{\alpha}{\gamma} \ln \left( \sum_{i=1}^n \exp \left( \frac{\gamma}{\alpha} \alpha x_i \right) \right) \\ &= \alpha \frac{1}{\gamma} \ln \left( \sum_{i=1}^n \exp(\gamma x_i) \right) = \alpha F(\mathbf{x}; \mathbf{m}). \end{aligned}$$

Hence, the statement (4) is proved. ■

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Now we are ready to prove

**PROPOSITION 1:** *The function  $\Psi_{ht}$  given by (3) is*

- (1) *continuous in  $\mathbf{R}_t$ ,*
- (2) *decreasing in  $\mathbf{R}_t$ , increasing in  $y_{ht}$  and increasing in  $\mathbf{Y}_t$ ,*
- (3) *convex in  $\mathbf{R}_t$ , and*
- (4) *homogeneous of degree one with respect to  $\mathbf{R}_t$ ,  $y_{ht}$ ,  $\mathbf{Y}_t$  and  $1/\delta_{ht}$ .*

The property (4) means that the indirect utility function is homogeneous of degree zero in real term. Roughly speaking, it is homogeneous of degree one with respect to  $\mathbf{R}_t$ ,  $y_{ht}$ ,  $\mathbf{Y}_t$  and  $\mathbf{u}_{ht}$ , the vector of idiosyncratic utilities of a housing demand group  $h$ .

**Proof:** Consider  $U_{hkt} = y_{ht} - R_{kt} + \beta_h Y_{kt}$  and  $\mathbf{u}_{ht}$  as  $x_k$  and  $\mathbf{m}$  in Lemma 2, respectively. Of course, we need to set  $n$  in the lemma to be  $K$ . Since  $\frac{\partial \Psi_{ht}}{\partial U_{hkt}} = -\frac{\partial \Psi_{ht}}{\partial R_{kt}}$

$= \frac{\partial \Psi_{ht}}{\partial y_{ht}} = \frac{1}{\beta_h} \frac{\partial \Psi_{ht}}{\partial Y_{kt}} > 0$  and  $\frac{\partial^2 \Psi_{ht}}{\partial U_{hjt} \partial U_{hkt}} = \frac{\partial^2 \Psi_{ht}}{\partial R_{jt} \partial R_{kt}}$ , it is clear that the properties (1)-(3) in

Lemma 2 proves (1)-(3) in Proposition 1. Since  $\alpha y_{ht} - \alpha R_{kt} + \alpha \beta_h Y_{kt} = \alpha U_{hkt}$  for any  $\alpha > 0$  and since the dispersion parameter  $\delta_{ht}$  of  $\mathbf{u}_{ht}$  corresponds to the dispersion parameter  $\gamma$  in Lemma 2, the property (4) in Proposition 1 follows from the property (4) in Lemma 2. ■

Next, we want to prove

**PROPOSITION 2:** *The function  $\Omega_{kt}$  given by (7) is*

- (1) *continuous in  $\mathbf{V}_{t+1}$  and  $\mathbf{C}_{kt} \equiv (C_{k0t}, C_{k1t}, \dots, C_{kKt})$ ,*
- (2) *increasing in  $\mathbf{V}_{t+1}$  and decreasing in  $\mathbf{C}_{kt}$ ,*
- (3) *convex in  $\mathbf{V}_{t+1}$  and  $-\mathbf{C}_{kt}$ , and*
- (4) *homogeneous of degree one with respect to  $\mathbf{V}_{t+1}$ ,  $-\mathbf{C}_{kt}$  and  $1/\Phi_{kt}$ .*

**Proof:** Interpret  $\frac{V_{st+1} - C_{kst}}{(1+r)} \equiv W_{kst}$  and  $-\mathbf{c}_{kt} \equiv -(c_{k0t}, \dots, c_{kKt})$  as  $x_s$  and  $\mathbf{m}$  in Lemma 2, respectively. Of course, we need to set  $n$  in the lemma to be  $K$  and to start the subscript  $s$  in the above variables from zero, instead of one in the lemma. Since  $\frac{\partial \Omega_{kt}}{\partial W_{kst}} = \frac{1}{(1+r)} \frac{\partial \Omega_{kt}}{\partial V_{st+1}} = -\frac{1}{(1+r)m_{ks}} \frac{\partial \Omega_{kt}}{\partial C_{kst}}$ , the property (2) of Lemma 2 proves that  $\Omega_{kt}$  is increasing in  $\mathbf{V}_{t+1}$  and decreasing in  $\mathbf{C}_{kt}$ . Hence, the statement (2) in Proposition 2 has been proved. The statement (1) follows from the statement (2). By Lemma 2,  $\Omega_{kt}$  is convex in  $\mathbf{W}_{kt} \equiv (W_{k0t}, \dots, W_{kKt})$ . Hence, given  $\mathbf{W}_{kt}$  and  $\mathbf{W}'_{kt} \equiv (W'_{k0t}, \dots, W'_{kKt}) \neq \mathbf{W}_{kt}$ , for any  $\lambda \in (0,1)$

$$(27) \quad \frac{1}{\Phi_{kt}} \ln \left( \sum_{s=0}^K b(s,k) \exp \Phi_{kt} (\lambda W_{kst} + (1-\lambda)W'_{kst}) \right) \\ \leq \lambda \frac{1}{\Phi_{kt}} \ln \left( \sum_{s=0}^K b(s,k) \exp \Phi_{kt} (W_{kst}) \right) + (1-\lambda) \frac{1}{\Phi_{kt}} \ln \left( \sum_{s=0}^K b(s,k) \exp \Phi_{kt} (W'_{kst}) \right)$$

Since (27) holds for any combination of  $\mathbf{W}_{kt}$  and  $\mathbf{W}'_{kt} \neq \mathbf{W}_{kt}$ , for any combination of  $\mathbf{V}_{t+1}$ ,  $-\mathbf{C}_{kt}$ ,  $\mathbf{V}'_{t+1}$  and  $-\mathbf{C}'_{kt}$ ,

$$(28) \quad \frac{1}{\Phi_{kt}} \ln \left( \sum_{s=0}^K b(s,k) \exp \Phi_{kt} \left( \frac{(\lambda V_{st+1} + (1-\lambda)V'_{st+1}) + (\lambda(-C_{kst}) + (1-\lambda)(-C'_{kst}))}{(1+r)m_{ks}} \right) \right) \\ \leq \lambda \frac{1}{\Phi_{kt}} \ln \left( \sum_{s=0}^K b(s,k) \exp \Phi_{kt} \left( \frac{V_{st+1} + (-C_{kst})}{(1+r)m_{ks}} \right) \right) \\ + (1-\lambda) \frac{1}{\Phi_{kt}} \ln \left( \sum_{s=0}^K b(s,k) \exp \Phi_{kt} \left( \frac{V'_{st+1} + (-C'_{kst})}{(1+r)m_{ks}} \right) \right).$$

This means that  $\Omega_{kt}$  is convex in  $\mathbf{V}_{t+1}$  and  $-\mathbf{C}_{kt}$ . Hence, the property (3) has been proved. Finally, as the property (4) in Proposition 1, the property (4) in Proposition 2 follows from the property (4) in Lemma 2. ■

## Aggregate Welfare of the Anas-Arnott Housing Model

**PROPOSITION 3:** The function  $\omega_{kt}$  given by (5) is

- (1) continuous in  $R_{kt}$  and  $\mathbf{D}_{kt} \equiv (D_{kot}, D_{kvt})$ ,
- (2) increasing in  $R_{kt}$  and decreasing in  $\mathbf{D}_{kt}$ ,
- (3) convex in  $R_{kt}$  and  $-\mathbf{D}_{kt}$ , and
- (4) homogeneous of degree one with respect to  $R_{kt}$ ,  $-\mathbf{D}_{kt}$  and  $1/\phi_{kt}$ .

**Proof:** Since the proof of this proposition is analogous to the one for Proposition 2, we omit the proof of this proposition. ■

Before proceeding, we define  $1/\delta_t \equiv (1/\delta_{1t}, \dots, 1/\delta_{Ht})$ ,  $1/\Psi_t \equiv (1/\Psi_{1t}, \dots, 1/\Psi_{Ht})$  and  $1/\Phi_t \equiv (1/\Phi_{0t}, \dots, 1/\Phi_{Kt})$ . Now we are ready to prove

**PROPOSITION 4:** The utilitarian welfare function  $Z_t$  given by (15) is

- (1) continuous in  $\mathbf{R}_t$ ,  $\mathbf{V}_t$  and  $\mathbf{V}_{t+1}$ ,
- (2) convex in  $\mathbf{R}_t$ ,  $\mathbf{V}_t$ ,  $\mathbf{V}_{t+1}$ ,  $\mathbf{M}_t$  and  $\Theta_t$ , and
- (3) homogeneous of degree one with respect to  $\mathbf{R}_t$ ,  $\mathbf{V}_t$ ,  $\mathbf{V}_{t+1}$ ,  $\mathbf{M}_t$ ,  $\Theta_t$ ,  $1/\delta_t$ ,  $1/\Psi_t$  and  $1/\Phi_t$ .

**Proof:** Continuity of  $Z_t$  is clearly satisfied. Proposition 1, 2 and 3 imply that  $\Psi_{ht}$ ,  $\Phi_{kt}$  and  $\omega_{kht}$  are convex in  $\mathbf{R}_t$ ,  $\mathbf{V}_t$ ,  $\mathbf{V}_{t+1}$ ,  $\mathbf{M}_t$  and  $\Theta_t$ . Since a weighted sum of convex functions is convex,  $Z_t$  is convex in  $\mathbf{R}_t$ ,  $\mathbf{V}_t$ ,  $\mathbf{V}_{t+1}$ ,  $\mathbf{M}_t$  and  $\Theta_t$ . Proposition 1, 2 and 3 also prove that  $Z_t$  is homogeneous of degree one with respect to  $\mathbf{R}_t$ ,  $\mathbf{V}_t$ ,  $\mathbf{V}_{t+1}$ ,  $\mathbf{M}_t$ ,  $\Theta_t$ ,  $1/\delta_t$ ,  $1/\Psi_t$  and  $1/\Phi_t$ . ■

Since  $[\Omega_{kt} + \omega_{kt} - V_{kt}] = 0$  along the equilibrium path, we define

$$(29) Z^*(\mathbf{R}_t, \Theta_t) \equiv \sum N_{ht} \Psi_{ht} - \Theta_t A.$$

This function  $Z^*$  is the reduced form of the utilitarian social welfare function  $Z_t$  corresponding to the Anas-Arnott model. Lastly, we state

**PROPOSITION 5:** *The function  $Z^*$  given by (29) is*

- (1) *continuous in  $\mathbf{R}_t$ ,*
- (2) *decreasing in  $\mathbf{R}_t$ , increasing in  $\mathbf{y}_t$ , increasing in  $\mathbf{Y}_t$  and decreasing in  $\Theta_t$ ,*
- (3) *convex in  $\mathbf{R}_t$  and  $\Theta_t$ , and*
- (4) *homogeneous of degree one with respect to  $\mathbf{R}_t$ ,  $\mathbf{y}_t$ ,  $\mathbf{Y}_t$ ,  $\Theta_t$  and  $1/\Psi_t$*

**Proof:** Since  $G^*$  is linear in  $\Psi_{ht}$  and  $\Theta_t$ , Proposition 1 proves the statements (1) and (2) except that  $G^*$  is decreasing in  $\Theta_t$ . Since the partial derivative of  $G^*$  with respect to  $\Theta_t$  is  $-A$ , it is decreasing in  $\Theta_t$ . Clearly, the statements (2) and (3) of Proposition 4 proves the statements (3) and (4), respectively. ■

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