Study on Scalarization Methods in Set-valued Optimization

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Chapter 1

Introduction

Optimization is a methodology widely used in economics, engineering, and other areas. One of mathematical fundamentals in optimization is convex analysis. It is an area in nonlinear analysis based on the convexity of functions. A basic aim of optimization is to obtain the minimum (maximum) value of an objective function. Some existence and boundedness of the minimum value are ensured by some convexities and continuities of the objective function. Then, we usually treat a real-valued function defined on a convex set in a vector space, and study several characterization related to the convexity of the function. We often receive much benefit from useful results on several types of inequality, e.g., minimax inequality, variational inequality, alternative theorem, and so on (see [2, 3, 4, 15, 16, 23]). With a request from some actuality problems and a development of mathematical research, the above theorems replacing real-valued functions with vector-valued (multi objective) ones is studied by many researchers (see [1, 8, 14, 19]). For considering inequality on vector-valued functions, they define a preorder with respect to an ordering cone for pairs of vectors (see [18, 6]).

In more generalized case, we assume an objective function having images given in the form of a set. We call the function a set-valued map. Then, how do we define the optimality of the function? We used to define it by an ordering cone for pairs of sets learned by the case of vectors. However, there is no natural standard of the ordering. In this thesis, we use the six kinds of set-relations proposed by Kuroiwa, Tanaka, and Ha in [11]. The set-relations are naturally generalized ones of preorders for vectors.

In set-valued analysis, we often use scalarizing functions from sets to reals. An important role of the scalarizing functions is characterization of set-valued maps related to several properties used in convex analysis. We consider the composition of a set-valued map and a scalarizing function, and we obtain an inequality of the map by some inherited convexities and continuities of the composition from the map. In this way, called a scalarizing method, we obtain inequality theorems above for set-valued maps. For example, Fan-Takahashi minimax inequality theorem for set-valued maps ([13]) and Ekeland's variational principle for set-valued maps ([10]).

The topics of the thesis are, consideration of inherited properties of scalarizing functions (Chap.3) and to propose Ricceri type inequality theorem for set-valued maps (Chap.4). We obtain several inherited properties of scalarizing functions with monotonicity and convexity. A kind of scalarizing functions with the properties is the unified type of scalarizing functions in [13]. The scalarizing functions have inherited properties of continuities. We study this scalarizing function and propose necessary conditions of the composition having convexity. As one of applications, we prove set-valued version of Ricceri type inequality theorem ([17]) related to Fan-Takahashi minimax inequality theorem ([4, 23]) by using the method with respect to the scalarizing functions. In the case of Fan-Takahashi minimax inequality theorems for set-valued maps ([13]), they prove four types of them with each other set-relations. We prove only two types of Ricceri type inequality theorem because an assumption of the theorem has just needed convexity (not enough to quasiconvexity.)

The organization of the thesis is as follows. In Chapter 2, we recall several definitions about cone-topologies and properties of set-valued maps with respect to the set-relations. In Chapter 3, we introduce some properties of scalarizing functions. In section 3.1, we show several inherited properties of scalarizing functions related to convexities. In section 3.2, we propose several necessary conditions for convexity of compositions. In Chapter 4, we prove that set-valued Ricceri type minimax theorem.

Chapter 2

Mathematical settings

In this chapter, we recall some basic definitions which will use in the paper. We take symbols same in the after chapters.

2.1 Definitions in convex analysis

Let E be a topological vector space, $A, B \subset E$ and $\alpha \in \mathbb{R}$ where $\mathbb{R}$ is the set of all reals. The algebraic sum $A + B$ and the scalar multiplication αA are defined as follows:

$$A + B := \{a + b \mid a \in A,\ b \in B\},$$

$$\alpha B := \{\alpha b \mid b \in B\}.$$

In particular, in the case that B is the empty set $\emptyset$, we use

$$A + \emptyset = \emptyset,$$

$$\alpha\emptyset = \begin{cases} \emptyset & (\alpha \neq 0), \\ \theta_E & (\alpha = 0) \end{cases}$$

where θ_E is the zero vector of E.

Moreover, for a subset $A \in E$, we denote the topological interior of A by $\operatorname{int} A$.

Throughout the paper, let $\mathbb{R}$ be the set of all real numbers, $\mathbb{N}$ the set of all natural numbers, $[a, b]$ the line-segment bounded by real numbers a and b including end points, $]a, b[$ the line-segment bounded by real numbers a and b without end points. We assume that the field of each vector space is the real field.

Definition 2.1 (convex set)**.** A set $A \subset E$ is called *convex set* (shortly, *convex*) if for any $x, y \in A$ and $\lambda \in]0, 1[$,

$$\lambda x + (1 - \lambda)y \in A.$$

The empty set $\emptyset$ is convex.

Definition 2.2 (cone)**.** A set $A \subset E$ is called *cone* if for any $x \in A$ and $\lambda \geq 0$,

$$\lambda x \in A.$$

Proposition 2.3. *Let $A \subset E$ be cone. Then, A is convex if and only if*

$$A + A = A.$$

Proof. Assume that cone A is a convex set. Then, for every $x, y \in A$, we have

$$\frac{x}{2} + \frac{y}{2} = \frac{(x + y)}{2} \in A,$$

and hence $x + y \in A$. thus, we obtain $A + A \subset A$. Since cone contains θ_E, we obtain $A + A \supset A$. Conversely, we assume that the set A satisfies $A + A = A$.

For any $x, y \in A$ and $\lambda \in]0, 1[$, we obtain $\lambda x \in A$ and $(1 - \lambda)y \in A$ because A is cone. Since $A + A = A$, we obtain $\lambda x + (1 - \lambda)y \in A$, that is, A is convex. ∎

Definition 2.4 (pointed cone)**.** A cone set $A \subset E$ is called *pointed cone* if

$$A \cap (-A) = \{\theta_E\}.$$

Let Y be a topological vector space attached an ordering cone $C \subset Y$ (that is, $C \neq \{\theta_Y\}$, pointed, $C + C = C$, $\lambda C \subset C$ for any $\lambda \geq 0$ and $\operatorname{int} C$.) This cone C give the partial order $\leq_C$ on Y as

$$x \leq_C y \text{ if } y - x \in C \text{ for all } x, y \in C.$$

The ordering cone C has important role in the sense of following definitions. We recall several definitions of C-notion proposed in [14].

Definition 2.5 (C-notions, [14])**.** Let $A \subset Y$. The set A is called

(i) C-*convex* if $A + C$ is convex,

(ii) C-*closed* if $A + C$ is closed,

(iii) C-*proper* if $A + C$ is proper, that is, $A + C \neq Y$,

(iv) C-*bounded* if for each neighborhood U of θ_Y, there exists $t \geq 0$ such that $A \subset tU + C$,

(v) C-compact if for any open cover formed $\{V_\lambda + C\}_\Lambda$ of A, there exists $n \in \mathbb{N}$ such that $A \subset \cap_{i=1}^{n}(V_{\lambda_i} + C)$.

Next, let us recall several definitions of some functions. Let f be a real-valued function on E, $\mathcal{V}(x)$ the family of all open neighborhoods of a point $x \in E$.

Definition 2.6 (convex (concave) function)**.** A function f is called a *convex function* on E (shortly, *convex*) if for any $x, y \in E$ and $\lambda \in]0,1[$,

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y).$$

If $-f$ is convex, then f is called a *concave function*.

Definition 2.7 (quasiconvex (quasiconcave) function)**.** A function f is called a *quasiconvex function* on E (shortly, *quasiconvex*) if for any $x, y \in E$ and $\lambda \in]0,1[$,

$$f(\lambda x + (1-\lambda)y) \leq \max\{f(x), f(y)\}.$$

If $-f$ is quasiconvex, then f is called a *qusiconcave function*.

Definition 2.8 (lower (upper) semicontinuous)**.** (i) A function f is called a *lower semicontinuous function* (*l.s.c.*) on E if for each $\bar{x} \in E$ and $W \in \mathcal{V}(f(\bar{x}))$, there exists $U \in \mathcal{V}(\bar{x})$ such that for each $x \in U$ there exists $y \in W$ such that $y < f(x)$.

(ii) A function f is called an *upper semicontinuous function* (*u.s.c.*) on E if for each $\bar{x} \in E$ and $W \in \mathcal{V}(f(\bar{x}))$ there exists $U \in \mathcal{V}(\bar{x})$ such that for each $x \in U$ there exists $y \in W$ such that $f(x) < y$.

Definition 2.9 (finitely notions)**.** (i) A set $A \in E$ is called *finitely open* (*closed*, *compact*) if $A \cap S$ is open (resp. closed, compact) for each finite dimensional subspace $S \subset E$.

(ii) A function f is called *finitely lower* (*upper*) *semicontinuous* if f is lower (resp. upper) semicontinuous on S for each finite dimensional subspace $S \subset E$.

2.2 Properties of set-valued maps by set-relations

Let E be a topological vector space, Y an ordered topological vector space with an ordering cone C and F a set-valued map from E to $2^Y \setminus \{\emptyset\}$. We call F is (a set notion)-valued if every image $F(\cdot)$ is (the set notion).

When we consider optimality of set-valued maps, we need a relation between two sets. We introduce the following set-relations using ordering cone.

Definition 2.10 (set-relations, [11])**.** Let $A, B \in 2^Y \setminus \{\emptyset\}$. Then, we denote

(i) $A \subset \bigcap_{b\in B}(b - C)$, equivalently $B \subset \cap_{a\in A}(a + C)$ by $A \leq_C^{(1)} B$,

(ii) $A \cap (\bigcap_{b\in B}(b - C)) \neq \emptyset$ by $A \leq_C^{(2)} B$,

(iii) $B \subset A + C$ by $A \leq_C^{(3)} B$,

(iv) $(\bigcap_{a\in A}(a + C)) \cap B \neq \emptyset$ by $A \leq_C^{(4)} B$,

(v) $A \subset (B - C)$ by $A \leq_C^{(5)} B$,

(vi) $A \cap (B - C) \neq \emptyset$, equivalently $(A + C) \cap B \neq \emptyset$ by $A \leq_C^{(6)} B$.

Proposition 2.11 ([12])**.** *For any nonempty sets $A, B \subset Y$, the following statements hold:*

(i) *for each $j = 1, \ldots, 6$,*

$A \leq_C^{(j)} B$ *implies* $(A + y) \leq_C^{(j)} (B + y)$ *for any* $y \in Y$ *and*

$A \leq_C^{(j)} B$ *implies* $\alpha A \leq_C^{(j)} \alpha B$ *for any* $\alpha > 0$,

(ii) *for each* $j = 1, \ldots, 5$, $\leq_C^{(j)}$ *is transitive,*

(iii) *for each* $j = 3, 5, 6$, $\leq_C^{(j)}$ *is reflexive,*

(iv) *for each* $j = 1, \ldots, 6$, $A \leq_C^{(j)} B$ *and* $y_1 \leq_C y_2$ *for some* $y_1, y_2 \in Y$ *imply* $A + y_1 \leq_C^{(j)} B + y_2$.

Proposition 2.12 ([12])**.** *For any nonempty sets* $A, V \subset Y$, *a direction* $k \in \operatorname{int} C$ *and* $j = 1, \ldots, 6$, *the following statements hold*:

$A \leq_C^{(j)} (tk + V)$ *implies* $A \leq_C^{(j)} (sk + V)$ *for any* $s \geq t$,

$(tk + V) \leq_C^{(j)} A$ *implies* $(sk + V) \leq_C^{(j)} A$ *for any* $s \leq t$.

By using the set-relations, we can define convexities and concavities for set-valued maps as the followings.

Definition 2.13 ([11])**.** For each $j = 1, \ldots, 6$,

(i) a map F is called *type* (j) *C-convex* if for each $x_1, x_2 \in E$ and $\lambda \in]0, 1[$,

$$F(\lambda x_1 + (1 - \lambda)x_2) \leq_C^{(j)} \lambda F(x_1) + (1 - \lambda)F(x_2),$$

(ii) a map F is called *type* (j) *properly quasi C-convex* if for each $x_1, x_2 \in E$ and $\lambda \in]0, 1[$,

$$F(\lambda x_1 + (1 - \lambda)x_2) \leq_C^{(j)} F(x_1) \text{ or } F(\lambda x_1 + (1 - \lambda)x_2) \leq_C^{(j)} F(x_2),$$

(iii) a map F is called *type (j) naturally quasi C-convex* if for each $x_1, x_2 \in E$ and $\lambda \in]0,1[$, there exists $\mu \in [0,1]$ such that

$$F(\lambda x_1 + (1-\lambda)x_2) \leq_C^{(j)} \mu F(x_1) + (1-\mu)F(x_2).$$

Definition 2.14 ([11])**.** For each $j = 1, \ldots, 3$, a map F is called *type (j)-lower quasiconvex* if for each $x_1, x_2 \in X$ and $\lambda \in]0,1[$,

$$F(\lambda x_1 + (1-\lambda)x_2) \leq_C^{(j)} (F(x_1) + C) \cap (F(x_2) + C).$$

Definition 2.15 ([5])**.** A map F is called *Ferro type (-1) quasiconvex* if for each $y \in Y$, the set

$$F^{-1}(y - C) := \{x \in X \mid F(x) \cap (y - C) \neq \emptyset\}$$

is convex.

Definition 2.16 ([12])**.** For each $j = 1, \ldots, 6$,

(i) a map F is called *type (j) C-concave* if for each $x_1, x_2 \in X$ and $\lambda \in]0,1[$,

$$\lambda F(x_1) + (1-\lambda)F(x_2) \leq_C^{(j)} F(\lambda x_1 + (1-\lambda)x_2),$$

(ii) a map F is called *type (j) properly quasi C-concave* if for each $x_1, x_2 \in X$ and $\lambda \in]0,1[$,

$$F(x_1) \leq_C^{(j)} F(\lambda x_1 + (1-\lambda)x_2) \text{ or } F(x_2) \leq_C^{(j)} F(\lambda x_1 + (1-\lambda)x_2),$$

(iii) a map F is called *type (j) naturally quasi C-concave* if for each $x_1, x_2 \in X$ and $\lambda \in]0,1[$, there exists $\mu \in [0,1]$ such that

$$\mu F(x_1) + (1-\mu)F(x_2) \leq_C^{(j)} F(\lambda x_1 + (1-\lambda)x_2).$$

Definition 2.17 ([11])**.** For each $j = 1, 4, 5$, a map F is called *type* (j)*-lower quasiconcave* if for each $x_1, x_2 \in X$ and $\lambda \in]0, 1[$,

$$(F(x_1) - C) \cap (F(x_2) - C) \leq_C^{(j)} F(\lambda x_1 + (1 - \lambda)x_2).$$

Definition 2.18 ([5])**.** A map F is called *Ferro type* (-1) *quasiconcave* if for each $y \in Y$, the set

$$F^{-1}(y + C) := \{x \in X \mid F(x) \cap (y + C) \neq \emptyset\}$$

is convex.

Proposition 2.19. *Let F be a set-valued map, the following statements hold*:

(i) *F is type* (3)*-lower quasiconvex if and only if F is Ferro type* (-1) *quasiconvex,*

(i) *F is type* (5)*-lower quasicocave if and only if F is Ferro type* (-1) *quasiconcave.*

Proof. We proof only convex equality (i). For any $y \in Y$, $x_1, x_2 \in F^{-1}(y - C)$ and $\lambda \in]0.1[$, there exists $y_1 \in F(x_1) \cap (y - C)$ such that $y_1 \in y - C$. we have $y \in y_1 + C$, that is, $y \in F(x_1) + C$. By the same way, $y \in F(x_2) + C$ then

$$y \in (F(x_1) + C) \cap (F(x_1) + C) \subset F(\lambda x_1 + (1 - \lambda)x_2) + C.$$

There exists $y_3 \in F(\lambda x_1 + (1 - \lambda)x_2)$ such that $y \in y_3 + C$, that is, $y_3 \in y - C$. Therefore, $\lambda x_1 + (1 - \lambda)x_2 \in F^{-1}(y - C)$.

The converse can be proved in a similar way for any $x_1, x_2 \in E$, $\lambda \in]0, 1[$ and $y \in (F(x_1) + C) \cap (F(x_2) + C)$. ∎

Above definitions include the C-convexity and C-concavity for vector-valued functions with respect to $\leq_C$.

Moreover, we introduce two kinds of continuities for set-valued maps related to l.s.c. It is related to u.s.c. when C is replaced by $(-C)$.

Definition 2.20 (C-continuity, [14])**.** (i) F is called a *C-lower continuous* if for each $\bar{x} \in E$ and open set W with $F(\bar{x}) \cap W \neq \emptyset$, there exists $U \in \mathcal{V}(\bar{x})$ such that $F(y) \cap (W + C) \neq \emptyset$ for all $y \in U$.

(ii) F is called a *C-upper continuous* if for each $\bar{x} \in E$ and open set W with $F(\bar{x}) \subset W$, there exists $U \in \mathcal{V}(\bar{x})$ such that $F(y) \subset W + C$ for all $y \in U$.

Chapter 3

Scalarizing functions for set-valued maps

In this chapter, we introduce scalarizing function for set-valued maps and properties of them. When we analyze set-valued maps, we often characterize set-valued maps by scalarizing functions. Thorough out this paper, we define the composite function $(\psi \circ F)$ of a scalarizing function ψ and a set-valued map F as $\psi \circ F(\cdot) := \psi(F(\cdot))$.

3.1 Inherited properties of monotone and/or convex scalarizing functions

Let X be a topological vector space, Y an ordered topological vector space attached an ordering cone C, F a set-valued map from X to $2^Y \setminus \{\emptyset\}$ and ψ a scalarizing function from $2^Y \setminus \{\emptyset\}$ to $\mathbb{R}$. We consider scalarizing function with j-monotonicity or convexity defined as the followings.

Definition 3.1. (i) A scalarizing function ψ is said to be *convex* if for each $A_1, A_2 \in 2^Y \setminus \{\emptyset\}$ and $\lambda \in]0,1[$,

$$\psi(\lambda A_1 + (1-\lambda)A_2) \leq \lambda\psi(A_1) + (1-\lambda)\psi(A_2).$$

(ii) A scalarizing function ψ is said to be *concave* if for each $A_1, A_2 \in 2^Y \setminus \{\emptyset\}$ and $\lambda \in]0,1[$,

$$\psi(\lambda A_1 + (1-\lambda)A_2) \geq \lambda\psi(A_1) + (1-\lambda)\psi(A_2).$$

Definition 3.2. For each $j = 1, \ldots, 6$, a function ψ is said to be *j-monotone with respect to $\leq_C^{(j)}$* if

$$A \leq_C^{(j)} B \textit{ implies } \psi(A) \leq \psi(B).$$

Definition 3.3. (i) A function ψ is said to be *quasiconvex* if for any $\alpha \in \mathbb{R}$, $\mathrm{lev}(\psi, \leq, \alpha) := \{A \in 2^Y \setminus \{\emptyset\} \mid \psi(A) \leq \alpha\}$ is convex.

(ii) A function ψ is said to be *quasiconcave* if for any $\alpha \in \mathbb{R}$, $\mathrm{lev}(\psi, \geq, \alpha) := \{A \in 2^Y \setminus \{\emptyset\} \mid \psi(A) \geq \alpha\}$ is convex.

We show properties of convexity and concavity of the composition of ψ and F. The following statements are proved by similar ways, hence we prove only Proposition 3.5 and 3.8.

Proposition 3.4 ([9])**.** *Let ψ be j-monotone with respect to $\leq_C^{(j)}$ for each $j = 1, \ldots, 6$ and convex. Then the following statements hold*:

(i) *if F is type (j) C-convex, then $\psi \circ F$ is convex,*

(ii) *if F is type (j) naturally quasi C-convex, then $\psi \circ F$ is quasiconvex.*

Proposition 3.5 ([9])**.** *Let ψ be j-monotone with respect to $\leq_C^{(j)}$ for each $j = 1, \ldots, 6$ and quasiconvex. Then the following statements hold:*

(i) *if F is type (j) C-convex, then $\psi \circ F$ is quasiconvex,*

(ii) *if F is type (j) naturally quasi C-convex, then $\psi \circ F$ is quasiconvex.*

Proof. We prove only (ii) because (i) is proved in a similar way. For each $j = 1, \ldots, 6$, we suppose that ψ is quasiconvex, j-monotone with respect to $\leq_C^{(j)}$ and F is type (j) naturally quasi C-convex. Let $\alpha \in \mathbb{R}$, $x_1, x_2 \in \operatorname{lev}(\psi \circ F, \leq, \alpha) := \{x \in X \mid \psi \circ F(x) \leq \alpha\}$, and $\lambda \in]0, 1[$. Since $\psi \circ F(x_1), \psi \circ F(x_2) \leq \alpha$, $F(x_1), F(x_2) \in \operatorname{lev}(\psi, \leq, \alpha)$. As F is type (j) naturally quasi C-convex, there exists $\mu \in [0, 1]$ such that

$$F(\lambda x_1 + (1 - \lambda)x_2) \leq_C^{(j)} \mu F(x_1) + (1 - \mu)F(x_2).$$

Since ψ is quasicovex, for this μ,

$$\mu F(x_1) + (1 - \mu)F(x_2) \in \operatorname{lev}(\psi, \leq, \alpha).$$

As ψ is j-monotone with respect to $\leq_C^{(j)}$,

$$\begin{aligned} \psi \circ F(\lambda x_1 + (1 - \lambda)x_2) &\leq \psi(\mu F(x_1) + (1 - \mu)F(x_2)) \\ &\leq \alpha. \end{aligned}$$

Therefore, $\lambda x_1 + (1 - \lambda)x_2 \in \operatorname{lev}(\psi \circ F, \leq, \alpha)$, that is, $\psi \circ F$ is quasiconvex. ∎

Proposition 3.6 ([9])**.** *Let ψ be j-monotone with respect to $\leq_C^{(j)}$ for each $j = 1, \ldots, 6$ and concave. Then the following statements hold:*

(i) *if F is type (j) C-concave, then $\psi \circ F$ is concave,*

(ii) *if F is type (j) naturally quasi C-concave, then $\psi \circ F$ is quasiconcave.*

Proposition 3.7 ([9])**.** *Let ψ be j-monotone with respect to $\leq_C^{(j)}$ for each $j = 1, \ldots, 6$ and quasiconcave. Then the following statements hold*:

(i) *if F is type (j) C-concave, then $\psi \circ F$ is quasiconcave,*

(ii) *if F is type (j) naturally quasi C-concave, then $\psi \circ F$ is quasiconcave.*

Proposition 3.8 ([9])**.** *Let ψ be j-monotone with respect to $\leq_C^{(j)}$for each $j = 1, \ldots, 6$. Then the following statements hold*:

(i) *if F is type (j) properly quasi C-convex, then $\psi \circ F$ is quasiconvex,*

(ii) *if F is type (j) properly quasi C-concave, then $\psi \circ F$ is quasiconcave.*

Proof. We prove only (i) because (ii) is proved in a similar way. Suppose that ψ is j-monotone with respect to $\leq_C^{(j)}$ and F is type (j) properly quasi C-convex. Let $\alpha \in \mathbb{R}$, $x_1, x_2 \in X$, and $\lambda \in]0, 1[$. Since F is type (j) properly quasi C-convex,

$$F(\lambda x_1 + (1 - \lambda)x_2) \leq_C^{(j)} F(x_1) \text{ or } F(\lambda x_1 + (1 - \lambda)x_2) \leq_C^{(j)} F(x_2),$$

and that ψ is j-monotone with respect to $\leq_C^{(j)}$,

$$\psi \circ F(\lambda x_1 + (1 - \lambda)x_2) \leq \max\{\psi \circ F(x_1), \psi \circ F(x_2)\},$$

that is, $\psi \circ F$ is quasiconvex. ∎

The results as mentioned above are collected as below. We denote “convex”, “quasiconvex”, “concave”, and “quasiconcave” by “cv”, “qcv”, “cc”, and “qcc” for short in table, respectively.

Table 3.1: Summary on several types of convexity and concavity of composite functions

Assumptions			Conclusions
ψ		F	$\psi \circ F$
j-monotone	cv	type (j) C-convex	cv
	cv	type (j) naturally quasi C-convex	qcv
	qcv	type (j) C-convex	
		type (j) naturally quasi C-convex	
	cc	type (j) C-concave	cc
	cc	type (j) naturally quasi C-concave	qcc
	qcc	type (j) C-concave	
		type (j) naturally quasi C-concave	
j-monotone		type (j) properly quasi C-convex	qcv
		type (j) properly quasi C-concave	qcc

3.2 The unified types of scalarizing functions

For an example of j-monotone convex scalarizing function, we introduce the unified types of scalarizing functions proposed in [12].

Definition 3.9 ([12])**.** Let $A, V \in 2^Y \setminus \{\emptyset\}$ and $k \in \operatorname{int} C$. For each $j = 1, \ldots, 6$, we define scalarizing functions $I_{k,V}^{(j)}$ and $S_{k,V}^{(j)}$ from $2^Y \setminus \{\emptyset\}$ to $\overline{\mathbb{R}}$ by

$$I_{k,V}^{(j)} := \inf\{t \in \mathbb{R} \mid A \leq_C^{(j)} (tk + V)\}, S_{k,V}^{(j)} := \sup\{t \in \mathbb{R} \mid (tk + V) \leq_C^{(j)} A\}.$$

These functions are called *unified types of scalarizing functions* for sets.

We assume that $+\infty - \infty = +\infty$ and $\alpha(+\infty) = +\infty$, $\alpha(-\infty) = -\infty$

for $\alpha > 0$. These scalarizing functions have convexity or concavity, j-monotonicity and C-continuity.

Proposition 3.10. *For $V \in 2^Y \setminus \{\emptyset\}$ and a $k \in \operatorname{int} C$, the following statements hold:*

(i) *for each $j = 1, 2, 3$, $I^{(j)}_{k,V}$ is convex,*

(ii) *for each $j = 4, 5, 6$, if V is $(-C)$-convex , then $I^{(j)}_{k,V}$ is convex.*

Proof. We prove the case of $j = 5$ in (ii). Others in this Proposition are proved in a similar way. Suppose that V is $(-C)$-convex. Let $A_1, A_2 \in 2^Y \setminus \{\emptyset\}$, $\lambda \in]0, 1[$, $\alpha_1 := I^{(5)}_{k,V}(A_1)$ and $\alpha_2 := I^{(5)}_{k,V}(A_2)$. We need to consider three cases: (a) $\alpha_1 = +\infty$ or $\alpha_2 = +\infty$, (b) $\alpha_1, \alpha_2 \in \mathbb{R}$, (c) otherwise. We consider only (b) because (a) and (c) can be proved easily. For any $s > 0$, by Proposition 2.12 and definition of $I^{(5)}_{k,V}$,

$$A_1 \subset (\alpha_1 + s)k + V - C,\ A_2 \subset (\alpha_2 + s)k + V - C.$$

Since V is $(-C)$-convex,

$$\begin{aligned} \lambda A_1 + (1-\lambda)A_2 &\subset \lambda\{(\alpha_1 + s)k + V - C\} + (1-\lambda)\{(\alpha_2 + s)k + V - C\} \\ &\subset \{\lambda\alpha_1 + (1-\lambda)\alpha_2 + s\}k + V - C. \end{aligned}$$

Therefore,

$$I^{(5)}_{k,V}(\lambda A_1 + (1-\lambda)A_2) \le \lambda\alpha_1 + (1-\lambda)\alpha_2 + s.$$

As $s > 0$ is arbitrary,

$$I^{(5)}_{k,V}(\lambda A_1 + (1-\lambda)A_2) \le \lambda\alpha_1 + (1-\lambda)\alpha_2.$$

∎

Proposition 3.11. *For $V \in 2^Y \setminus \{\emptyset\}$ and $k \in \operatorname{int} C$, the following statements hold:*

(i) *for each $j = 1, 4, 5$, $S^{(j)}_{k,V}$ is concave,*

(ii) *for each $j = 2, 3, 6$, if V is C-convex, then $S^{(j)}_{k,V}$ is concave.*

Proof. In similar ways in Proposition 3.10, the statements are proved. ∎

Proposition 3.12 ([12])**.** *For $V \in 2^Y \setminus \{\emptyset\}$, $k \in \operatorname{int} C$ and $j = 1, \ldots, 5$, $I^{(j)}_{k,V}$ and $S^{(j)}_{k,V}$ are j-monotone with respect to $\leq^{(j)}_C$.*

Proposition 3.13 ([22])**.** *For $V \in 2^Y \setminus \{\emptyset\}$ and $k \in \operatorname{int} C$, the following statements hold:*

(i) (a) *If F is $(-C)$-lower continuous on E then $S^{(3)}_{k,V} \circ F$ is upper semicontinuous in E,*

(b) *if F is C-upper continuous on E then $S^{(3)}_{k,V} \circ F$ is lower semicontinuous in E,*

(ii) (a) *If F is C-lower continuous on E then $S^{(5)}_{k,V} \circ F$ is lower semicontinuous in E,*

(b) *if F is $(-C)$-upper continuous on E then $S^{(5)}_{k,V} \circ F$ is upper semicontinuous in E,*

(iii) (a) *If F is $(-C)$-lower continuous on E then $I^{(3)}_{k,V} \circ F$ is upper semicontinuous in E,*

(b) *if F is C-upper continuous on E then $I^{(3)}_{k,V} \circ F$ is lower semicontinuous in E,*

(iv) (a) *If F is C-lower continuous on E then $I^{(5)}_{k,V} \circ F$ is lower semicontinuous in E,*

(b) *if F is $(-C)$-upper continuous on E then $I^{(5)}_{k,V} \circ F$ is upper semicontinuous in E.*

Proposition 3.14 ([13]). *Let $A, V \in 2^Y \setminus \{\emptyset\}$ and $k \in \operatorname{int} C$. Then, the following statements hold:*

(i) *if A is C-bounded and V C-proper then $S^{(3)}_{k,V}(A) \in \mathbb{R}$,*

(i) *if A is $(-C)$-proper and V $(-C)$-bounded then $S^{(5)}_{k,V}(A) \in \mathbb{R}$,*

(iii) *if A is C-proper and V C-bounded then $I^{(3)}_{k,V}(A) \in \mathbb{R}$,*

(iv) *if A is $(-C)$-bounded and V $(-C)$-proper then $I^{(5)}_{k,V}(A) \in \mathbb{R}$.*

Proposition 3.15 ([10]). *For $V \in 2^Y \setminus \{\emptyset\}$ and $k \in \operatorname{int} C$, the following statements hold:*

(i) *if V is C-proper, then $I^{(3)}_{k,V}(V) = 0$,*

(ii) *if V is $(-C)$-proper, then $I^{(5)}_{k,V}(V) = 0$.*

Proposition 3.16 ([10]). *For $V \in 2^Y \setminus \{\emptyset\}$, $k \in \operatorname{int} C$ and $r \in \mathbb{R}$, the following statements hold:*

(i) *if A is C-closed, then $I^{(3)}_{k,V}(A) \leq r$ implies $A \leq^{(3)}_C rk + V$,*

(ii) *if V is $(-C)$-closed, then $I^{(5)}_{k,V}(A) \leq r$ implies $A \leq^{(5)}_C rk + V$.*

Proposition 3.17. *For $V \in 2^Y \setminus \{\emptyset\}$ and $k \in \operatorname{int} C$, the following statements hold:*

(i) *if V is C-proper, then* $S^{(3)}_{k,V}(V) = 0$,

(ii) *if V is $(-C)$-proper, then* $S^{(5)}_{k,V}(V) = 0$.

Proposition 3.18. *For $V \in 2^Y \setminus \{\emptyset\}$, $k \in \operatorname{int} C$ and $r \in \mathbb{R}$, the following statements hold:*

(i) *if V is C-closed, then $S^{(3)}_{k,V}(A) \geq r$ implies $rk + V \leq^{(3)}_C A$,*

(ii) *if A is $(-C)$-closed, then $S^{(5)}_{k,V}(A) \geq r$ implies $rk + V \leq^{(5)}_C A$.*

In Proposition 3.16, $I^{(3)}_{k,V}(A) \leq r$ and $I^{(5)}_{k,V}(A) \leq r$ are characterized by $\leq^{(3)}_C$ and $\leq^{(5)}_C$, respectively. The inverse of the relation of these are clear by the definition of $I^{(3)}_{k,V}$ and $I^{(5)}_{k,V}$. However, it is unclear for the set-relation between A and $rk + V$ in the case of $I^{(j)}_{k,V}(A) < r$ (without equality.) Such problems are discussed as the following.

Theorem 3.19 ([9])**.** *For $A, V \in 2^Y \setminus \{\emptyset\}$, $k \in \operatorname{int} C$ and $r \in \mathbb{R}$, $I^{(3)}_{k,V}(A) < r$ implies $A \leq^{(3)}_{\operatorname{int} C} rk + V$. The converse is true if V is C-compact.*

Proof. First, we prove $\Rightarrow$. Since $I^{(3)}_{k,V}(A) < r$, there exists $\epsilon > 0$ such that $I^{(3)}_{k,V}(A) < r - \epsilon$. Then there exists $\overline{r} \in \mathbb{R}$ such that

$$I^{(3)}_{k,V}(A) < \overline{r} < r - \epsilon \text{ and } A \leq^{(3)}_C \overline{r}k + V \quad (\Leftrightarrow \overline{r}k + V \subset A + C).$$

By Proposition 2.12, we obtain $(\overline{r}+s)k+V \subset A+C$ for any $s > 0$. Therefore,

$$\begin{aligned}(r - \epsilon + s)k + V &= (\overline{r} + s)k + V + (r - \epsilon - \overline{r})k \\ &\subset A + C + \operatorname{int} C \\ &= A + \operatorname{int} C.\end{aligned}$$

We choose $s = \epsilon$, then $rk + V \subset A + \operatorname{int} C$, that is, $A \leq^{(3)}_{\operatorname{int} C} rk + V$.

In the case that V is C-compact. We show that $A \leq^{(3)}_{\operatorname{int} C} rk + V$ implies $I^{(3)}_{k,V}(A) < r$. Since $A \leq^{(3)}_{\operatorname{int} C} rk + V \Leftrightarrow rk + V \subset A + \operatorname{int} C$, for any $v \in V$, there exists $a_v \in A$ such that

$$rk + v \in a_v + \operatorname{int} C.$$

As each $a_v + \operatorname{int} C$ is an open set, there exists $\epsilon_v > 0$ such that

$$v \in a_v - (r - \epsilon_v)k + \operatorname{int} C \subset a_v - (r - \epsilon_v)k + C.$$

Let $\alpha_v \in]0, \epsilon_v[$. Then

$$v \in a_v - (r - \epsilon_v + \alpha_v)k + \alpha_v k + C \subset A - (r - \epsilon_v + \alpha_v)k + \operatorname{int} C + C.$$

Therefore,

$$V \subset \bigcup_{v \in V} \{A - (r - \epsilon_v + \alpha_v)k + \operatorname{int} C + C\},$$

that is, the family $\{A - (r - \epsilon_v + \alpha_v)k + \operatorname{int} C + C \mid v \in V\}$ is a cover of V and each $A - (r - \epsilon_v + \alpha_v)k + \operatorname{int} C + C$ is an open set. Since V is C-compact, there exists $\{v_1, \ldots, v_n\} \subset V$ such that

$$V \subset \bigcup_{i=1}^{n} \{A - (r - \epsilon_{v_i} + \alpha_{v_i})k + \operatorname{int} C + C\}.$$

Let $j \in \{1, \ldots, n\}$ be the one such that

$$r - \epsilon_{v_j} + \alpha_{v_j} = \max\{r - \epsilon_{v_i} + \alpha_{v_i} \mid i = 1, \ldots, n\}.$$

Then, for each $i \in \{1, \ldots, n\}$,

$$A - (r - \epsilon_{v_i} + \alpha_{v_i})k + \operatorname{int} C + C \subset A - (r - \epsilon_{v_j} + \alpha_{v_j})k + C.$$

Therefore, $V + (r - \epsilon_{v_j} + \alpha_{v_j})k \subset A + C$. As a result,

$$I^{(3)}_{k,V}(A) \leq (r - \epsilon_{v_j} + \alpha_{v_j}) < r.$$

∎

We have similar statements, which are Theorem 3.20 and 3.21, as well as Theorem 3.19.

Theorem 3.20 ([9]). *For $A, V \in 2^Y \setminus \{\emptyset\}$, $k \in \operatorname{int} C$ and $r \in \mathbb{R}$, the following statements hold*:

(i) $I^{(5)}_{k,V}(A) < r$ *implies* $A \leq^{(5)}_{\operatorname{int} C} rk + V$. *The converse is true if A is $(-C)$-compact,*

(ii) $S^{(3)}_{k,V}(A) > r$ *implies* $rk + V \leq^{(3)}_{\operatorname{int} C} A$. *The converse is true if A is C-compact,*

(iii) $S^{(5)}_{k,V}(A) > r$ *implies* $rk + V \leq^{(5)}_{\operatorname{int} C} A$. *The converse is true if V is $(-C)$-compact.*

Theorem 3.21 ([9]). *For $A, V \in 2^Y \setminus \{\emptyset\}$, $k \in \operatorname{int} C$ and $r \in \mathbb{R}$, the following statements hold*: *For $j = 1, 2, 4, 6$,*

(i) *if* $I^{(j)}_{k,V}(A) < r$, *then* $A \leq^{(j)}_{\operatorname{int} C} rk + V$,

(ii) *if* $S^{(j)}_{k,V}(A) > r$, *then* $rk + V \leq^{(j)}_{\operatorname{int} C} A$.

Proof. In a similar way of the proof of in Theorem 3.19, the statements are proved. ∎

In [12, 13, 22], the authors study the inherited properties related to convexity or continuity of $I^{(j)}_{k,V} \circ F$ and $S^{(j)}_{k,V} \circ F$ on some assumptions of F. In the below, we show several inverse results, that is, we drive convexities and concavities of F from those of $I^{(j)}_{k,V} \circ F$ and $S^{(j)}_{k,V} \circ F$.

Theorem 3.22 ([9])**.** *For $k \in \operatorname{int} C$, the following statements hold:*

(i) *if F is type (3)-lower quasiconvex, then $I^{(3)}_{k,V} \circ F$ is quasiconvex for any $V \in 2^Y \setminus \{\emptyset\}$. The converse is true if F is C-closed-valued,*

(ii) *let F be C-closed-valued, C-convex-valued and C-proper-valued. If $S^{(3)}_{k,V} \circ F$ is convex for any C-closed, C-convex and C-proper set $V \in 2^Y \setminus \{\emptyset\}$, then F is type (3) C-convex,*

(iii) *let F be $(-C)$-closed-valued and $(-C)$-proper-valued. If $S^{(5)}_{k,V} \circ F$ is convex for any $(-C)$-closed and $(-C)$-proper set $V \in 2^Y \setminus \{\emptyset\}$, then F is type (5) C-convex,*

(iv) *let F be C-closed-valued and C-proper-valued. If $S^{(3)}_{k,V} \circ F$ is quasiconvex for any C-closed and C-proper set $V \in 2^Y \setminus \{\emptyset\}$, then F is type (3) properly quasi C-convex,*

(v) *let F be $(-C)$-closed-valued and $(-C)$-proper-valued. If $S^{(5)}_{k,V} \circ F$ is quasiconvex for any $(-C)$-closed and $(-C)$-proper set $V \in 2^Y \setminus \{\emptyset\}$, then F is type (5) properly quasi C-convex,*

(vi) *let F be $(-C)$-closed-valued and cone-valued. If $I^{(5)}_{k,V} \circ F$ is quasiconvex for any $(-C)$-closed cone $V \in 2^Y \setminus \{\emptyset\}$, then F is type (5) C-convex.*

Proof. (i) Assume that F is type (3)-lower quasiconvex. We prove that $\operatorname{lev}(I^{(3)}_{k,V} \circ F, \leq, \alpha) := \{x \in X \mid I^{(3)}_{k,V} \circ F(x) \leq \alpha\}$ is convex for any $\alpha \in \mathbb{R}$. Let $x_1, x_2 \in \operatorname{lev}(I^{(3)}_{k,V} \circ F, \leq, \alpha)$ and $\lambda \in]0,1[$. We need to consider two cases:

(a) $I^{(3)}_{k,V} \circ F(x_1) \in \mathbb{R}$ or $I^{(3)}_{k,V} \circ F(x_2) \in \mathbb{R}$, (b) $I^{(3)}_{k,V} \circ F(x_1) = -\infty$ and $I^{(3)}_{k,V} \circ F(x_2) = -\infty$. We only consider (a) because (b) can be proved in a similar way of (a). In the case of (a), for $i = 1, 2$, $I^{(3)}_{k,V} \circ F(x_i) < \alpha + s$ for any $s > 0$. By Theorem 3.19, $(\alpha + s)k + V \subset F(x_i) + \operatorname{int} C \subset F(x_i) + C$. Therefore,

$$(\alpha + s)k + V \subset (F(x_1) + C) \cap (F(x_2) + C).$$

Since F is type (3)-lower quasiconvex, we have

$$(\alpha + s)k + V \subset F(\lambda x_1 + (1 - \lambda)x_2) + C.$$

That is, $I^{(3)}_{k,V}(F(\lambda x_1 + (1 - \lambda)x_2)) \leq \alpha + s$. As $s > 0$ is arbitrary,

$$I^{(3)}_{k,V} \circ F(\lambda x_1 + (1 - \lambda)x_2) \leq \alpha.$$

Conversely, We show that if $I^{(3)}_{k,V} \circ F$ is quasiconvex for any $V \in 2^Y \setminus \{\emptyset\}$ and F is C-closed-valued, then F is type (3)-lower quasiconvex. Let $x_1, x_2 \in X$, $\lambda \in]0, 1[$ and $y \in (F(x_1) + C) \cap (F(x_2) + C)$. Since $F(x_1) \leq^{(3)}_C \{y\}$ and the monotonicity of $I^{(3)}_{k,V}$, $I^{(3)}_{k,\{y\}}(F(x_1)) \leq I^{(3)}_{k,\{y\}}(\{y\})$. By $\{y\}$ is C-proper and Proposition 3.15, $I^{(3)}_{k,\{y\}}(\{y\}) = 0$. As a result, $I^{(3)}_{k,\{y\}}(F(x_i)) \leq 0$ for $i = 1, 2$. Since $I^{(3)}_{k,\{y\}} \circ F$ is quasiconvex,

$$I^{(3)}_{k,\{y\}} \circ F(\lambda x_1 + (1 - \lambda)x_2) \leq 0.$$

Since F is C-closed-valued and Proposition 3.16,

$$F(\lambda x_1 + (1 - \lambda)x_2) \leq^{(3)}_C \{y\},$$

that is, $y \in F(\lambda x_1 + (1 - \lambda)x_2) + C$.

(ii) Let $x_1, x_2 \in X, \lambda \in]0,1[$, and $V := F(\lambda x_1 + (1-\lambda)x_2)$. Then V is C-closed, C-convex and C-proper set. Therefore,

$$\begin{aligned} 0 &= S^{(3)}_{k,V}(V) \\ &= S^{(3)}_{k,V}(F(\lambda x_1 + (1-\lambda)x_2)) \\ &\leq \lambda S^{(3)}_{k,V} \circ F(x_1) + (1-\lambda) S^{(3)}_{k,V} \circ F(x_2) \\ &\leq S^{(3)}_{k,V}(\lambda F(x_1) + (1-\lambda)F(x_2)). \end{aligned}$$

By Proposition 3.18, $F(\lambda x_1 + (1-\lambda)x_2) \leq^{(3)}_C \lambda F(x_1) + (1-\lambda)F(x_2)$.

(iii) In a similar way of (ii), this statement is proved.

(iv) We give the proof by the method of contradiction. We assume that F is not type (3) properly quasi C-convex. Then, there exist $x_1, x_2 \in X$ and $\lambda \in]0,1[$ such that

$$F(\lambda x_1 + (1-\lambda)x_2) \not\leq^{(3)}_C F(x_1) \text{ and } F(\lambda x_1 + (1-\lambda)x_2) \not\leq^{(3)}_C F(x_2).$$

Let $V := F(\lambda x_1 + (1-\lambda)x_2)$. By the contraposition of Proposition 3.18, $S^{(3)}_{k,V}(F(x_i)) < 0$ for $i = 1, 2$. Since $S^{(3)}_{k,V} \circ F$ is quasiconvex,

$$0 = S^{(3)}_{k,V}(V) = S^{(3)}_{k,V} \circ F(\lambda x_1 + (1-\lambda)x_2) < 0.$$

This is contradiction.

(v) In a similar way of (iv), this statement is proved.

(vi) Let $x_1, x_2 \in X$, $\lambda \in]0,1[$, and $V := F(x_1) \cup F(x_2)$. For $i = 1, 2$, $F(x_i) \subset 0k + V - C$. By the definition of $I^{(5)}_{k,V}$, $I^{(5)}_{k,V}(F(x_i)) \leq 0$. Since $I^{(5)}_{k,V} \circ F$ is quasiconvex,

$$I^{(5)}_{k,V} \circ F(\lambda x_1 + (1-\lambda)x_2) \leq 0.$$

By Proposition 3.16, $F(\lambda x_1 + (1-\lambda)x_2) \leq^{(5)}_C V$. As F is cone-valued,

$$V \subset \lambda F(x_1) + (1-\lambda)F(x_2) - C.$$

As a result, $F(\lambda x_1 + (1-\lambda)x_2) \leq_C^{(5)} \lambda F(x_1) + (1-\lambda)F(x_2)$. ∎

Theorem 3.23. *For $k \in \operatorname{int} C$, the following statements hold:*

(i) *if F is type (5)-lower quasiconcave, then $S_{k,V}^{(5)} \circ F$ is quasiconvex for any $V \in 2^Y \setminus \{\emptyset\}$. The converse is true if F is $(-C)$-closed-valued,*

(ii) *let F be $(-C)$-closed-valued, $(-C)$-convex-valued and $(-C)$-proper-valued. If $I_{k,V}^{(5)} \circ F$ is concave for any $(-C)$-closed, $(-C)$-convex and $(-C)$-proper set $V \in 2^Y \setminus \{\emptyset\}$, then F is type (5) C-concave,*

(iii) *let F be C-closed-valued and $(-C)$-proper-valued. If $I_{k,V}^{(3)} \circ F$ is concave for any C-closed and C-proper set $V \in 2^Y \setminus \{\emptyset\}$, then F is type (3) C-concave,*

(iv) *let F be $(-C)$-closed-valued and $(-C)$-proper-valued. If $I_{k,V}^{(5)} \circ F$ is quasiconcave for any $(-C)$-closed and $(-C)$-proper set $V \in 2^Y \setminus \{\emptyset\}$, then F is type (5) properly quasi C-concave,*

(v) *let F be C-closed-valued and C-proper-valued. If $I_{k,V}^{(3)} \circ F$ is quasiconcave for any C-closed and C-proper set $V \in 2^Y \setminus \{\emptyset\}$, then F is type (3) properly quasi C-concave,*

(vi) *let F be C-closed-valued and cone-valued. If $S_{k,V}^{(3)} \circ F$ is quasiconcave for any C-closed cone $V \in 2^Y \setminus \{\emptyset\}$, then F is type (3) C-concave.*

Proof. By a similar ways to the proof of theorem 3.22, the statements are proved. ∎

Chapter 4

Ricceri type inequality theorem for set-valued maps

In this chapter, we present two results by using the properties of two scalarizing functions $S^{(3)}_{k,V}$ and $S^{(5)}_{k,V}$ introduced in Chapter 3.

We know following theorem proposed by B. Ricceri.

Theorem 4.1 ([17]). *Let E be a real topological vector space, S a linear subspace in E, D a nonempty subset of S, X a non-empty finitely closed and convex subset of E, K a finitely compact subset of X with $\theta_E \in K$, $\tilde{\tau}$ a topology on K with respect to which K is compact and f a real-valued function on $X \times S$. We assume that f satisfies the following conditions:*

(1) *for every $x \in X$, the function $f(x, \cdot)$ is concave in S,*

(2) *the function $f(\cdot, y)$ is finitely lower semicontinuous in X for every $y \in (X-X) \cap S$, is $\tilde{\tau}$-lower semicontinuous in K for every $y \in D$, is finitely continuous in X and $\tilde{\tau}$-continuous in K.*

Then, for any convex real-valued function ψ *on* S *with* $\psi(\theta_E) = 0$ *and*

$$f(x, x) > f(x, \theta_E) + \psi(x) \qquad \text{for all} \quad x \in (X \cap S) \setminus I_{K,D,X},$$

there exists $\hat{x} \in K$ *such that*

$$f(\hat{x}, y) \leq f(\hat{x}, \theta_E) + \psi(y) \qquad \text{for all} \quad y \in D.$$

We notice that when we generalize inequality theorem as above for set-valued maps, we consider two cases from usually order $\leq$ on reals. Those are $\leq_C^{(j)}$ and $\not\leq_{\mathrm{int}\,C}^{(j)}$ because set-relations are not total order as $\leq$. Now, we consider only $\not\leq_{\mathrm{int}\,C}^{(3)}$ or $\not\leq_{\mathrm{int}\,C}^{(5)}$. Then we obtain the following two theorems, which are generalized ones of Theorem 4.1 for set-valued maps.

Theorem 4.2 ($\not\leq_{\mathrm{int}\,C}^{(5)}$ type, [21])**.** *Let* E *be a real topological vector space,* S *a linear subspace in* E*,* D *a nonempty subset of* S*,* X *a non-empty finitely closed and convex subset of* E*,* K *a finitely compact subset of* X *with* $\theta_E \in K$*,* $\tilde{\tau}$ *a topology on* K *with respect to which* K *is compact,* Y *an ordered topological vector space with an ordering cone* C *and* F *a set-valued map from* $X \times S$ *to* $2^Y \setminus \{\emptyset\}$*. We assume that* F *satisfies the following conditions:*

(1) F *is* $(-C)$*-proper,*

(2) *for every* $x \in X$*, the map* $F(x, \cdot)$ *is type (5)* C*-concave in* S*,*

(3) *the map* $F(\cdot, y)$ *is finitely* C*-lower continuous in* X *for every* $y \in (X - X) \cap S$ *and* $\tilde{\tau}$*-*C*-lower continuous in* K *for every* $y \in D$*, the map* $F(\cdot, \theta_E)$ *is singleton-valued, finitely continuous in* X *and* $\tilde{\tau}$*-continuous in* K*.*

Then, for any C-convex vector-valued map ψ from S to Y with $\psi(\theta_E) = \theta_Y$ and

$$F(x, \theta_E) + \psi(x) \leq^{(5)}_{\operatorname{int} C} F(x, x) \qquad \textit{for all} \quad x \in (X \cap S) \setminus I_{K,D,X},$$

there exists $\hat{x} \in K$ such that

$$F(\hat{x}, \theta_E) + \psi(y) \not\leq^{(5)}_{\operatorname{int} C} F(\hat{x}, y) \qquad \textit{for all} \quad y \in D.$$

Proof. Let $V := \{\theta_Y\}, k \in \operatorname{int} C$ be fixed. We consider the set-valued map B from $X \times S$ to $2^Y \setminus \{\emptyset\}$ defined by

$$B(x, y) := F(x, y) - F(x, \theta_E) - \psi(y).$$

We consider the composite function $S^{(5)}_{k,V} \circ B$, and we denote it by A. By Proposition 3.14, A has real-valued images without $\pm\infty$. Then, there exists $\hat{x} \in K$ such that $A(\hat{x}, y) \leq 0$ for all $y \in D$ because A holds the following conditions:

(a) $A(x, \cdot)$ is concave for any $x \in X$,

(b) $A(x, \theta_E) = 0$,

(c) $A(\cdot, y)$ is finitely lower semicontinuous in X for any $y \in (X - X) \cap S$,

(d) $A(\cdot, y)$ is $\tilde{\tau}$-lower semicontinuous in K for any $y \in D$,

(e) $A(x, x) > 0$ for all $x \in (X \cap S) \setminus I_{K,D,X}$.

We show each proof of the five conditions above.

(a) By assumption (2) and the C-concavity of vector-valued function $-\psi$, it follows from (iv) of Proposition 2.11 that $B(x, \cdot)$ is type (5) C-concave in S. From Proposition 3.6, it follows that $S^{(5)}_{k,V} \circ B(x, \cdot)$ is concave.

(b) Since $F(x, \theta_E)$ is singleton and $\psi(\theta_E) = \theta_Y$, we get $B(x, \theta_E) = \{\theta_Y\} = V$. By Proposition 3.17, $S^{(5)}_{k,V}(V) = 0$.

(c) Let $y \in (X - X) \cap S$ be fixed. For each finite dimensional subspace S' in E, we take $x \in X \cap S'$ and an open subset W of Y with $(F(x, y) - F(x, \theta_E)) \cap W \neq \emptyset$. Hence, there exist $w_1 \in F(x, y)$ and $w_2 \in (-F(x, \theta_E))$ such that $w_1 + w_2 \in W$, and there exists $U_{\theta_Y} \in \mathcal{V}(\theta_Y)$ such that $2U_{\theta_Y} + w_1 + w_2 \subset W$ (see the first lemma in section 9 of [7]). We put $W_1 := U_{\theta_Y} + w_1$ and $W_2 := U_{\theta_Y} + w_2$. Both W_1 and W_2 are open, and they satisfy $w_1 \in (F(x, y) \cap W_1) \neq \emptyset$ and $w_2 \in (-F(x, \theta_E) \cap W_2) \neq \emptyset$, respectively. By the C-lower continuity of $F(\cdot, y)$ and $F(\cdot, \theta_E)$, there exist $U^{(1)}_x, U^{(2)}_x \in \mathcal{V}(x)$ such that $F(z_1, y) \cap (W_1 + C) \neq \emptyset$ and $(-F(z_2, \theta_E)) \cap (W_2 + C) \neq \emptyset$ for any $z_1 \in U^{(1)}_x$ and $z_2 \in U^{(2)}_x$. We put $U_x := U^{(1)}_x \cap U^{(2)}_x$, then $U_x \in \mathcal{V}(x)$ and

$$(F(z, y) - F(z, \theta_E)) \cap (W_1 + W_2 + C) \neq \emptyset \qquad \text{for all } z \in U_x.$$

We know $(W_1 + W_2) \subseteq W$, so we obtain $F(\cdot, y) - F(\cdot, \theta_E)$ is finitely C-lower continuous. Thus, $B(\cdot, y)$ is C-lower continuous. By (ii)-(a) of Proposition 3.13, $S^{(5)}_{k,V} \circ B(\cdot, y)$ is lower semicontinuous on $X \cap S'$, that is, $A(\cdot, y)$ is finitely lower semicontinuous.

(d) It can be proved in a similar way to the proof of (c).

(e) For each $x \in (X \cap S) \setminus I_{K,D,X}$, by assumption and (i) of Proposition 2.11, we have

$$\{\theta_Y\} = V \leq^{(5)}_{\operatorname{int} C} B(x, x).$$

Thus, $V = \{\theta_Y\} \subset B(x, x) - \operatorname{int} C$. Since $B(x, x) - \operatorname{int} C$ is open, there exists $t > 0$ such that $(tk + V) \subset B(x, x) - \operatorname{int} C$, which implies that $0 < t \leq (S^{(5)}_{k,V} \circ B)(x, x)$.

Therefore, by Theorem 4.1, we obtain there exists $\hat{x} \in K$ such that $(S^{(5)}_{k,V} \circ B)(\hat{x}, y) \leq 0$ for all $y \in D$. By the definition of $S^{(5)}_{k,V}$, for each $y \in D$ and $s > 0$,

$$\{\theta_Y\} \quad \not\subseteq \quad B(\hat{x}, y) - sk - C.$$

By $\cup_{s>0}(-sk - C) = -\mathrm{int}\, C$, we obtain

$$\{\theta_Y\} \quad \not\leq^{(5)}_{\mathrm{int}\, C} \quad B(\hat{x}, y).$$

Now, since $F(x, \theta_E)$ is singleton, we obtain

$$F(\hat{x}, \theta_E) + \psi(y) \quad \not\leq^{(5)}_{\mathrm{int}\, C} \quad F(\hat{x}, y).$$

∎

Theorem 4.3 ($\not\leq^{(3)}_{\mathrm{int}\, C}$ type, [20]). *Let E be a real topological vector space, S a linear subspace in E, D a nonempty subset of S, X a non-empty finitely closed and convex subset of E, K a finitely compact subset of X with $\theta_E \in K$, $\tilde{\tau}$ a topology on K with respect to which K is compact, Y an ordered topological vector space with an ordering cone C and F a set-valued map from $X \times S$ to $2^Y \setminus \{\emptyset\}$. We assume that F satisfies the following conditions:*

(1) *F is compact-valued,*

(2) *for every $x \in X$, the map $F(x, \cdot)$ is type (3) C-concave in S,*

(3) *the map $F(\cdot, y)$ is finitely C-upper continuous in X for every $y \in (X - X) \cap S$ and $\tilde{\tau}$-C-upper continuous in K for every $y \in D$, the map $F(\cdot, \theta_Y)$ is singleton-valued map, finitely continuous in X and $\tilde{\tau}$-continuous in K.*

Then, for any C-convex vector-valued map ψ from S to Y with $\psi(\theta_E) = \theta_Y$ and

$$F(x, \theta_E) + \psi(x) \leq^{(3)}_{\operatorname{int} C} F(x, x) \qquad \text{for all} \quad x \in (X \cap S) \setminus I_{K,D,X},$$

there exists $\hat{x} \in K$ such that

$$F(\hat{x}, \theta_E) + \psi(y) \not\leq^{(3)}_{\operatorname{int} C} F(\hat{x}, y) \qquad \text{for all} \quad y \in D.$$

Proof. The proof of Theorem 4.3 can be proved in a similar way of the proof of Theorem 4.2. We consider the set-valued map B from $X \times S$ to $2^Y \setminus \{\emptyset\}$ defined by

$$B(x, y) := F(x, y) - F(x, \theta_E) - \psi(y).$$

We consider the composite function $S^{(3)}_{k,V} \circ B$ for fixed $V := \{\theta_Y\}$ and $k \in \operatorname{int} C$, and we denote it by A. Then, we have following five conditions:

(a) $A(x, \cdot)$ is concave for any $x \in X$,

(b) $A(x, \theta_E) = 0$,

(c) $A(\cdot, y)$ is finitely lower semicontinuous in X for any $y \in (X - X) \cap S$,

(d) $A(\cdot, y)$ is $\tilde{\tau}$-lower semicontinuous in K for any $y \in D$,

(e) $A(x, x) > 0$ for all $x \in (X \cap S) \setminus I_{K,D,X}$.

We show each proof of three statements (c), (d) and (e) above because the proofs of (a) and (b) are the same as (a) and (b) in the proof of Theorem 4.2.

(c) Let $y \in (X - X) \cap S$ be fixed. For each finite dimensional subspace S' in E, we take $x \in X \cap S'$ and an open subset W of Y with $(F(x, y) -$

$F(x,\theta_E)) \subset W$. Since $F(x,y)$ is compact, $F(x,y) - F(x,\theta_E)$ is compact. Hence, there exist W_1, any open neighborhood of $F(x,y)$, and W_2, any open neighborhood of $-F(x,\theta_E)$, such that $W_1 + W_2 \subset W$. By the C-upper continuity of $F(\cdot,y)$ and $F(\cdot,\theta_E)$, there exist $U_x^{(1)}, U_x^{(2)} \in \mathcal{V}(x)$ such that $F(z_1,y) \subset (W_1 + C)$ and $(-F(z_2,\theta_E)) \subset (W_2 + C)$ for any $z_1 \in U_x^{(1)}$ and $z_2 \in U_x^{(2)}$. We put $U_x := U_x^{(1)} \cap U_x^{(2)}$, then $U_x \in \mathcal{V}(x)$ and

$$(F(z,y) - F(z,\theta_E)) \subset (W_1 + W_2 + C) \qquad \text{for all } z \in U_x.$$

We know $(W_1 + W_2) \subset W$, and then $F(\cdot,y) - F(\cdot,\theta_E)$ is finitely C-upper continuous. Thus, $B(\cdot,y)$ is C-upper continuous on $X \cap S'$. By (i)-(b) of Proposition 3.13, $S_{k,V}^{(3)} \circ B(\cdot,y)$ is finitely lower semicontinuous,

(d) It can be proved in a similar way to the proof of (c).

(e) If $(X \cap S) \setminus I_{K,D,X} = \emptyset$, then (e) is true. Otherwise, for each $x \in (X \cap S) \setminus I_{K,D,X}$, by assumption, we have

$$\{\theta_Y\} = V \leq_{\operatorname{int} C}^{(3)} B(x,x).$$

Thus, $B(x,x) \subset V + \operatorname{int} C$. Since $V + \operatorname{int} C$ is open, there exists $t > 0$ such that $B(x,x) - tk \subset V + \operatorname{int} C$, which implies that $0 < t \leq (S_{k,V}^{(3)} \circ B)(x,x) = A(x,x)$.

Therefore, we obtain there exists $\hat{x} \in K$ such that $(S_{k,V}^{(3)} \circ B)(\hat{x},y) \leq 0$ for all $y \in D$. By the definition of $S_{k,V}^{(3)}$ and Theorem 3.20, for each $y \in D$ and $s > 0$,

$$B(\hat{x},y) \quad \not\subseteq \quad \{\theta_Y\} + sk + C.$$

By $\cup_{s>0}(sk + C) = \operatorname{int} C$, we obtain $\{\theta_Y\} \not\leq_{\operatorname{int} C}^{(3)} B(\hat{x},y)$. Thus,

$$\psi(y) \not\leq_{\operatorname{int} C}^{(3)} F(\hat{x},y) - F(\hat{x},\theta_E).$$

∎

Ricceri propose a inequality theorem as corollary of Theorem 4.1 related to Fan-Takahashi minimax inequality theorem, as the followings.

Theorem 4.4 (Fan-Takahashi minimax inequality theorem, [23]). *Let E be a real Hausdorff topological vector space, X a non-empty compact convex subset of E and f a real function on $X \times X$ satisfying the following conditions:*

(1) *for every $x \in X$, the function $f(x, \cdot)$ is concave in X,*

(2) *for every $y \in X$, the function $f(\cdot, y)$ is lower semicontinuous in X,*

(3) *for every $x \in X$ such that one has $f(x, x) \leq 0$.*

Then, there exists $\hat{x} \in X$ such that $f(\hat{x}, y) \leq 0$ for all $y \in X$.

Theorem 4.5 (Ricceri type inequality theorem, [17]). *Let E be a real topological vector space, X a non-empty compact convex subset of E, $\theta_E \in X$ and f a real function on $X \times E$ satisfying the following conditions:*

(1) *for every $x \in X$, the function $f(x, \cdot)$ is concave in E and $f(x, \theta_E) = 0$,*

(2) *for every $y \in E$, the function $f(\cdot, y)$ is lower semicontinuous in X,*

(3) *for every $x \in X$ such that $X \setminus \cup_{\lambda > 0} \lambda(x - X) \neq \emptyset$, one has $f(x, x) > 0$.*

Then, there exists $\hat{x} \in X$ such that $f(\hat{x}, y) \leq 0$ for all $y \in X$.

Theorem 4.4 is equivalent to Ky Fan minimax inequality theorem of Theorem 1 in [4]. Ricceri proposed Theorem 4.5 which is a reasonable substitute of Theorem 4.4. Clearly, both third conditions in the above theorems can not occur at the same time. However the two theorems have the same result, and so they are mutually exclusive.

A set-valued version of Theorem 4.4 is already proposed by the scalarization method ([13]). Finally, I introduce two set-valued minequality theorems related to Fan-Takahashi minimax inequality theorem by the corollaries of Theorem 4.2 and 4.3.

Corollary 4.6 ($\not\leq^{(5)}_{\operatorname{int} C}$ type, [21])**.** *Let E be a real topological vector space, Y an ordered topological vector space with an ordering cone C, X a non-empty compact convex subset of E, $\theta_E \in X$ and F a set-valued map from $X \times E$ to $2^Y \setminus \{\emptyset\}$ satisfying the following conditions:*

(1) *F is $(-C)$-proper,*

(2) *for every $x \in X$, $F(x, \cdot)$ is type (5) C-concave in E and $F(x, \theta_E) = \{\theta_Y\}$,*

(3) *for every $y \in E$, $F(\cdot, y)$ is C-lower continuous in X,*

(4) *for every $x \in X$ such that $X \setminus \cup_{\lambda>0}\lambda(x - X) \neq \emptyset$, one has $\{\theta_Y\} \leq^{(5)}_{\operatorname{int} C} F(x, x)$.*

Then, there exists $\hat{x} \in X$ such that $\{\theta_Y\} \not\leq^{(5)}_{\operatorname{int} C} F(\hat{x}, y)$ for all $y \in X$.

Proof. In Theorem 4.2, take $S = E$, $X = K = D$, $\hat{\tau}$ being the relativization to K of the given Hausdorff vector topology on E, $\psi(\cdot) = \theta_Y$ and then observe that $X \setminus I_{X,X,X}$ coincides with $\{x \in X \mid X \setminus \cup_{\lambda>0}\lambda(x - X) \neq \emptyset\}$. ∎

Corollary 4.7 ($\not\leq^{(3)}_{\operatorname{int} C}$ type, [20])**.** *Let E be a real topological vector space, Y an ordered topological vector space with an ordering cone C, X a non-empty* *compact convex subset of* E, $\theta_E \in X$ *and* F *a set-valued map from* $X \times E$ *to* $2^Y \setminus \{\emptyset\}$ *satisfying the following conditions:*

(1) F *is compact-valued,*

(2) *for every* $x \in X$, $F(x, \cdot)$ *is type (3)* C*-concave in* E *and* $F(x, \theta_E) = \{\theta_Y\}$,

(3) *for every* $y \in E$, $F(\cdot, y)$ *is* C*-upper continuous in* X,

(4) *for every* $x \in X$ *such that* $X \setminus \cup_{\lambda > 0} \lambda(x - X) \neq \emptyset$, *one has* $\{\theta_Y\} \leq^{(3)}_{\text{int}\, C} F(x, x)$.

Then, there exists $\hat{x} \in X$ *such that* $\{\theta_Y\} \not\leq^{(3)}_{\text{int}\, C} F(\hat{x}, y)$ *for all* $y \in X$.

Proof. It can be derived from Theorem 4.3. ∎

Chapter 5

Conclusions

In this thesis, we introduce several convex inherited properties of scalarizing functions and prove Ricceri type inequality theorem for set-valued maps. We can consider some optimization problems more generally and new problems for set optimization. The author believes that the study in the thesis will contribute to a further research in set-valued analysis and optimization theory.

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