Geometric Structure of Banach Spaces and Absolute Norms

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Contents

Introduction

The study on Banach space geometry provides many fundamental notions and interesting aspects, and sometimes has surprising results. The basic geometric properties such as uniform convexity, strict convexity, uniform smoothness, smoothness and uniform non-squareness have made great contributions to various fields of Banach space theory.

Strict convexity of Banach spaces was first introduced in 1936 by Clarkson [17] (and independently by Akhiezer and Krein [3]) as the property that the unit sphere contains no nontrivial line segments, that is, $1 - \|2^{-1}(x+y)\| > 0$ whenever $\|x\| = \|y\| = 1$. Clarkson [17] made use of these values to define the "uniform" version of strict convexity. A Banach space X is said to be uniformly convex if $\delta_X(\varepsilon) > 0$ for each $0 < \varepsilon \leq 2$, where $\delta_X(\varepsilon) = \inf\{1 - \|2^{-1}(x+y)\| : x, y \in X,\ \|x\| = \|y\| = 1,\ \|x-y\| \geq \varepsilon\}$. The value $\delta_X(\varepsilon)$ is called the modulus of convexity of X, and can be viewed as a measure of how "convex" the unit ball is. In other words, the modulus of convexity provides a quantification of the geometric structure of the space from the viewpoint of convexity. A situation similar to this also occurs in smoothness. A Banach space X is said to be smooth if each unit vector has a unique norm one support functional. In fact, this is equivalent to the statement that the norm is Gateaux differentiable, which happens if and only if $\lim_{t\to 0^+}((\|x+ty\| + \|x-ty\|)/2 - 1)/t = 0$ whenever $x, y \in X$ and $\|x\| = 1$. This allows us to quantify the geometric structure of the space from the viewpoint of smoothness, namely, the modulus of smoothness of a Banach space X is defined by $\rho_X(t) = \sup\{(\|x+ty\| + \|x-ty\|)/2 - 1 : x, y \in X,\ \|x\| = \|y\| = 1\}$ for each $t \geq 0$. Then X is said to be uniformly smooth if $\lim_{t\to 0^+} \rho_X(t)/t = 0$. An advantage of these quantifications is that the complete duality between uniform convexity and uniform smoothness can be easily deduced by the well-known Lindenstrauss formulas, that is, a Banach space X is uniformly convex if and only if its dual space X^* is uniformly smooth. The same statement still holds if X is replaced with X^*. Thus quantifying geometric structures might lead better results. Note that the same duality does not hold between strict convexity and smoothness in general, though one of those two properties of X^* implies the other of X.

There are some other ideas to quantify geometric structures of Banach spaces. In 1935, Jordan and von Neumann [36] proved that the norm on a vector space X is induced by an inner product if and only if it satisfies the parallelogram law $\|x+y\|^2+\|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$ for each $x, y \in X$. From this result, Clarkson [18] introduced in 1937 the von Neumann-Jordan constant $C_{NJ}(X)$ of a Banach space X as a measure of how "close" to Hilbert spaces. Namely, the constant $C_{NJ}(X)$ is defined to be the smallest positive number C such that

$$\frac{1}{C} \leq \frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} \leq C$$

for each $x, y \in X$. Another equivalent formulation of $C_{NJ}(X)$ is

$$C_{NJ}(X) = \sup\left\{\frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} : x, y \in X,\ (x,y) \neq (0,0)\right\}.$$

It is known that $1 \leq C_{NJ}(C) \leq 2$, and that $C_{NJ}(X) < 2$ if and only if X is uniformly non-square (cf. [47, 77]). Needless to say, $C_{NJ}(X) = 1$ means that X is a Hilbert space. It was also pointed out in [47] that $C_{NJ}(X^*) = C_{NJ}(X)$. Moreover, Kato, Maligranda and Takahashi [39] showed connection between von Neumann-Jordan constants and the fixed point property of Banach spaces. On this background, the von Neumann-Jordan constants were recognized to be one of the most important geometric constants of Banach spaces. However, until 2000's, there are few examples of Banach spaces that the von Neumann-Jordan constants are precisely determined; see, for example, [47, 77]. In response to this, Saito, Kato and Takahashi made use of absolute normalized norms on $\mathbb{C}^2$ in their 2000 paper [69] to provide various examples of such two-dimensional non-L_p-type spaces. Their method was based on the following useful characterization of absolute norms that can be found in Bonsall and Duncan's book [15]: For each absolute normalized norm $\|\cdot\|$ on $\mathbb{C}^2$, define a convex function by $\psi(t) = \|(1-t,t)\|$ for each $t \in [0,1]$. Then $\max\{1-t,t\} \leq \psi(t) \leq 1$ for each t. Conversely, if ψ is a convex function on $[0,1]$ satisfying $\max\{1-t,t\} \leq \psi(t) \leq 1$ for each t, then the following formula defines an absolute normalized norm on $\mathbb{C}^2$.

$$\|(x,y)\|_\psi = \begin{cases} (|x|+|y|)\psi\left(\dfrac{|y|}{|x|+|y|}\right) & \text{if} \quad (x,y) \neq (0,0), \\ 0 & \text{if} \quad (x,y) = (0,0). \end{cases}$$

Furthermore, this norm satisfies $\psi(t) = \|(1-t,t)\|_\psi$ for each $t \in [0,1]$. Let AN_2 be the set of all absolute normalized norms on $\mathbb{C}^2$, and let Ψ_2 be the collection of all convex functions ψ on $[0,1]$ that satisfy $\max\{1-t,t\} \leq \psi(t) \leq 1$ for each t. Then the above argument shows that the sets AN_2 and Ψ_2 are in a one-to-one correspondence under the equation $\psi(t) = \|(1-t,t)\|$ for each $t \in [0,1]$. Thanks to this characterization, they could give some formulas to calculate the von Neumann-Jordan constants by using the properties of corresponding convex functions. It should be noted that the same results are also true for $\mathbb{R}^2$.

Since Saito, Kato and Takahashi applied absolute normalized norms to studying the von Neumann-Jordan constant, the properties of absolute norms have been widely studied, especially in terms of corresponding convex functions. The higher dimensional version of the above characterization was shown in [70]. In the same paper, it was also proved that the space endowed with an absolute norm is strictly convex if and only if the corresponding function is strictly convex. On the other hand, Mitani, Saito and Suzuki [59] characterized the smoothness of an absolute normed space in terms of the smoothness (differentiablity) of the corresponding convex function. In 2002, the notion of ψ-direct sums was introduced by Takahashi, Kato and Saito [78] as a generalization of ℓ_p-direct sums, that is, for each $\psi \in \Psi_2$ and for Banach space X and Y the ψ-direct sum $X \oplus_\psi Y$ is defined as the space $X \times Y$ endowed with the norm $\|(x,y)\|_\psi = \|(\|x\|, \|y\|)\|_\psi$. The same definition naturally works in the higher dimensional case. They also studied the strict convexity of the ψ-direct sum of two Banach spaces. Strict convexity, uniform convexity and local uniform convexity of ψ-direct sums were studied in Kato, Saito and Tamura [43], while smoothness of ψ-direct sums were investigated by Mitani, Oshiro and Saito [54]; see also [44, 46, 57, 68, 91]. Infinite dimensional absolute norms were considered in [58], and infinite ψ-direct sums were introduced and studied in [90]. There are many other papers concerning absolute norms; see, for example, [5, 40, 65, 67, 87] and so on.

In this thesis, we study geometric structure of Banach spaces by using absolute norms. In particular, we present some recent results concerning the duality of James constant, new geometric properties and Tingley's problem.

Chapter 1 is devoted to study the James constants of two-dimensional spaces. The James constant $J(X)$ of a Banach space X was defined in 1990 by Gao and Lau [26] as a measure of how "non-square" the unit ball is, namely, the James constant is defined by

$$J(X) = \sup\{\min\{\|x+y\|, \|x-y\|\} : x, y \in X,\ \|x\| = \|y\| = 1\}.$$

It is known that $\sqrt{2} \leq J(X) \leq 2$ for any Banach space X, and that X is uniformly non-square if and only if $J(X) < 2$ (cf [26, 39]). However, unlike von Neumann-Jordan constant, the James constant of a Banach space need not coincide with that of its dual space. A counterexample can be found in [39]. It is not easy to calculate James constant even in two-dimensional spaces. In Komuro, Saito and Mitani [41, 42] the James constant of absolute octagonal norms were calculated, but the calculation is far from easy. In contrast, the computation of James constant in two-dimensional spaces endowed with symmetric absolute norms is rather easy since it has a simple calculation formula using corresponding convex functions; see [55]. Using the formula, the James constants of some specific spaces were determined in [55]. Moreover, the James constants of the two-dimensional Lorentz sequence space $d^{(2)}(\omega, q)$ and its dual space $d^{(2)}(\omega, q)^*$ had been completely determined; see [56, 60], and also [38, 76]. However, it is not known that whether $J(d^{(2)}(\omega, q))$ is equal to $J(d^{(2)}(\omega, q)^*)$. The first task in the chapter is to show that $J(d^{(2)}(\omega, q)) = J(d^{(2)}(\omega, q)^*)$ for every ω, q. This encourages us to conjecture that $J(X^*) = J(X)$ for every two-dimensional space X endowed with a symmetric absolute norm. As the main result, we shall show that this conjecture is true.

In Chapter 2, we introduce and study new geometric properties of Banach spaces that generalize p-uniform smoothness and q-uniform convexity, where a Banach space X is said to be p-uniformly smooth if there exists $K > 0$ such that $\rho_X(\tau) \leq K\tau^p$ for all $\tau \geq 0$, and q-uniformly convex if there exists $C > 0$ such that $\delta_X(\varepsilon) \geq C\varepsilon^q$ for each $\varepsilon \in [0, 2]$. As in the case of uniform smoothness and uniform convexity, it is known that p-uniform smoothness and q-uniform convexity are the dual properties of each other provided that $1/p + 1/q = 1$. Moreover, these geometric properties are characterized by simple norm inequalities. The purpose of the chapter is to introduce the concepts of ψ-uniform smoothness and ψ^*-uniform convexity of Banach spaces and to present characterizations analogous to that of p-uniform smoothness and q-uniform convexity. Applying these characterizations yields the duality between ψ-uniform smoothness and ψ^*-uniform convexity under certain conditions. To prove the characterizations of p-uniform smoothness and q-uniform convexity, Beckner's inequality [12] and its norm version [49] play fundamental roles. Hence we first investigate the Beckner type inequality and its generalizations. An elementary proof of Beckner's inequality is given. Then we try to generalize Beckner's inequality by using absolute normalized norms on $\mathbb{R}^2$. Finally, we formulate ψ-uniform smoothness and ψ^*-uniform convexity of Banach spaces, and derive some characterizations with the help of generalized Beckner's inequalities.

We study Tingley's problem in Chapter 3. Let X and Y be real Banach spaces. Then the classical Mazur-Ulam theorem states that if $T : X \to Y$ is a surjective isometry then

T is affine. In 1972, Mankiewicz [52] extended this result by showing that if $U \subset X$ and $V \subset Y$ are open and connected and $T_0 : U \to V$ is a surjective isometry then there exists a surjective affine isometry $T : X \to Y$ such that $T_0 = T|_U$. From this, in particular, it turns out that every isometry from the unit ball of X onto that of Y can be extended to an isometric isomorphism between X and Y. Motivated by this observation, Tingley [84] proposed in 1987 the following problem: Suppose that T_0 is an isometry from the unit sphere of X onto that of Y. Then, does T_0 have a linear isometric extension $T : X \to Y$? This problem is called Tignley's problem, and is also known as the isometric extension problem. Many papers, especially in the last decade, have been devoted to the problem, and it has been solved positively for some classical Banach spaces; see, for example, [9, 21, 32, 50, 85, 89]. Some mathematicians began to attack the problem on more general spaces recently. In 2011, Cheng and Dong [16] studied somewhere-flat spaces. One year later, it was shown by Kadets and Martín [37] that the problem has an affirmative answer for finite dimensional polyhedral Banach spaces. Ding and Li [24] studied using the notion of sharp corner points and Tan and Liu [79] introduced the Tingley property and obtained results on almost-CL-spaces. However, surprisingly, Tingley's problem remains open even if $X = Y$ and X is two-dimensional, the simplest setting for the problem. In this chapter, we present some recent results on Tingley's problem. We first give a further property of spherical isometries by using the frames of the unit balls of Banach spaces. To do this, some geometric properties concerning frames are studied. Moreover, we describe a new geometric approach to two-dimensional Tingely's problem. As applications, we solve the problem for several specific spaces endowed with symmetric absolute normalized norms on $\mathbb{R}^2$.

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Preliminaries

Throughout this thesis, the term "Banach space" always means a real Banach space. Let X be a Banach space. Then the unit ball and unit sphere of X are defined by $B_X = \{x \in X : \|x\| \leq 1\}$ and $S_X = \{x \in X : \|x\| = 1\}$, respectively. The norm dual of X is denoted by X^*. For a subset A of X, the symbols A°, $\overline{A}$, $\langle A \rangle$, $[A]$, $\mathrm{co}(A)$, $\mathrm{aco}(A)$ and $\overline{\mathrm{co}}(A)$ denote the interior, closure, linear span, closed linear span, convex hull, absolute convex hull and closed convex hull of A.

In this thesis, we make frequently use of an essential correspondence between absolute norms on $\mathbb{R}^2$ and certain convex functions on $[0,1]$. A norm $\|\cdot\|$ on $\mathbb{R}^2$ is said to be *absolute* if $\|(x,y)\| = \|(|x|,|y|)\|$ for each (x,y), and *normalized* if $\|(1,0)\| = \|(0,1)\| = 1$. Let AN_2 be the set of all absolute normalized norms on $\mathbb{R}^2$, and let Ψ_2 be the collection of all convex functions ψ on $[0,1]$ satisfying $\max\{1-t,t\} \leq \psi(t) \leq 1$ for each t. Then the sets AN_2 and Ψ_2 are in a one-to-one correspondence under the equation $\psi(t) = \|(1-t,t)\|$ for each $t \in [0,1]$ (cf. [15, 69]). We remark that the norm $\|\cdot\|_\psi$ corresponding to a function $\psi \in \Psi_2$ is given by

$$\|(x,y)\|_\psi = \begin{cases} (|x|+|y|)\psi\left(\dfrac{|y|}{|x|+|y|}\right) & \text{if} \quad (x,y) \neq (0,0), \\ 0 & \text{if} \quad (x,y) = (0,0). \end{cases}$$

The most typical examples of absolute normalized norms are the ℓ_p-norms. The function ψ_p corresponding to the norm $\|\cdot\|_p$ is given by

$$\psi_p(t) = \begin{cases} ((1-t)^p + t^p)^{1/p} & \text{if} \quad 1 \leq p < \infty, \\ \max\{1-t,t\} & \text{if} \quad p = \infty. \end{cases}$$

Symmetry of norms often plays important roles in the thesis. A norm $\|\cdot\|$ on $\mathbb{R}^2$ is said to be *symmetric* if $\|(x,y)\| = \|(y,x)\|$ for each (x,y). It is obvious that an absolute norm $\|\cdot\|_\psi$ is symmetric if and only if the corresponding function ψ is symmetric with respect to $1/2$, that is, $\psi(1-t) = \psi(t)$ for each $t \in [0,1]$. Let Ψ_2^S be the set of all such functions in Ψ_2. Then the set AN_2^S of all symmetric absolute normalized norms on $\mathbb{R}^2$

is naturally identified with the set Ψ_2^S in the above sense. We note that $\|\cdot\|_p \in AN_2^S$ and $\psi_p \in \Psi_2^S$ for all $1 \le p \le \infty$.

In general, it is not easy to find the exact norm of the dual space of a Banach space. This is the case even for two-dimensional spaces. However, the above correspondence between absolute norms and convex functions gives a clear way to find dual norms. This is done by calculating the values of the function defined in the following: For each $\psi \in \Psi_2$, let ψ^* be the function on $[0,1]$ given by

$$\psi^*(s) = \max_{0\le t\le 1} \frac{(1-s)(1-t)+st}{\psi(t)}$$

Then it follows that $\psi^* \in \Psi_2$ and $(\mathbb{R}^2, \|\cdot\|_\psi)^* = (\mathbb{R}^2, \|\cdot\|_{\psi^*})$ (from this we also have $\psi^{**} = \psi$); see [54]. The function ψ^* is called the *dual function* of ψ. If $\psi \in \Psi_2^S$, then $\psi^* \in \Psi_2^S$ and the behavior of ψ^* is given by

$$\psi^*(s) = \max_{0\le t\le 1/2} \frac{(1-s)(1-t)+st}{\psi(t)}$$

for each $s \in [0, 1/2]$; see [56] for details.

The notion of direct sums of Banach spaces is a basic and important way to produce a new Banach space from old ones. For example, the ℓ_p-direct sum $X \oplus_p Y$ of two Banach spaces X and Y is defined as the Cartesian product $X \times Y$ endowed with the norm

$$\|(x,y)\|_p = \begin{cases} (\|x\|^p + \|y\|^p)^{1/p} & \text{if} \quad 1 \le p < \infty, \\ \max\{\|x\|, \|y\|\} & \text{if} \quad p = \infty. \end{cases}$$

In other words, the norm on $X \oplus_p Y$ is defined by $\|(x,y)\|_p = \|(\|x\|, \|y\|)\|_p$. An essential condition that the function $\|(x,y)\|_p$ really becomes a norm on $X \times Y$ is the monotonicity of the ℓ_p-norms, where a norm $\|(x,y)\|$ on $\mathbb{R}^2$ is said to be *monotone* if $\|(x_1,y_1)\| \le \|(x_2,y_2)\|$ whenever $x_1 \le x_2$ and $y_1 \le y_2$. In fact, it is known that the notion of monotonicity is equivalent to that of absoluteness; see [13, 15, 70]. Thus ℓ_p-direct sums can be generalized using absolute norms on $\mathbb{R}^2$ in an obvious way. Let X and Y be Banach spaces, and let $\psi \in \Psi_2$. Then the ψ-*direct sum* of X and Y is the space $X \times Y$ endowed with the norm $\|(x,y)\|_\psi = \|(\|x\|, \|y\|)\|_\psi$, and denoted by $X \oplus_\psi Y$ or $(X \oplus Y)_\psi$. As in the case of ℓ_p-direct sums, it is known that $(X \oplus_\psi Y)^* = X^* \oplus_{\psi^*} Y^*$ (cf. [54, Proposition 3.4]).

Chapter 1

The James constants of two-dimensional spaces

In this chapter, we study the duality of James constants of Banach spaces. It is known that the James constant $J(X)$ of a Banach space X need not coincide with that of its dual space. In fact, there is two-dimensional normed space X such that $J(X^*) \neq J(X)$. Here we shall consider the following problem: When does a two-dimensional normed space X satisfy $J(X^*) = J(X)$?

1.1 Basic results

The James constant of a Banach space was defined in 1990 by Gao and Lau [26] as a measure of how "non-square" the unit ball is. We shall start with the definition.

Definition 1.1.1 (Gao and Lau [26]). Let X be a Banach space. Then the James constant of X is defined by $J(X) = \sup\{\min\{\|x+y\|, \|x-y\|\} : x, y \in S_X\}$.

It is known that $\sqrt{2} \leq J(X) \leq 2$ for any Banach space X, and that X is uniformly non-square if and only if $J(X) < 2$ (cf [26, 39]).

We can reduce the amount of calculation of James constants using a generalized orthogonality in Banach spaces. An element x in a Banach space is said to be *isosceles orthogonal* to another element y, denoted by $x \perp_I y$, if $\|x+y\| = \|x-y\|$. This notion was first introduced by James [33]. We remark that $x \perp_I y$ implies $x \perp_I -y$ and $y \perp_I x$. For more details about isosceles orthogonality can be found in the survey of Alonso, Martini and Wu [8].

An examination of the results of Gao and Lau [26] leads the following proposition.

Proposition 1.1.2. *Let X be a Banach space. Then* $J(X) = \sup\{\|x+y\| : x, y \in S_X,\ x \perp_I y\}$.

This reformulation is very effective, in particular, for two-dimensional spaces endowed with symmetric absolute norms. The advantage comes from the following result of Alonso [4] which shows isosceles orthogonality has the *uniqueness property*.

Lemma 1.1.3 (Alonso [4])**.** *Let X be a two-dimensional normed space. Suppose that $x \in S_X$. Then there exists a unique (up to the sign) element $y \in S_X$ such that $x \perp_I y$.*

By the uniqueness property of isosceles orthogonality, we have a useful characteristic of symmetric absolute norms. Namely, if $\psi \in \Psi_2^S$ and $x, y \in \mathbb{R}^2$ with $\|x\|_\psi = \|y\|_\psi = 1$ then $x \perp_I y$ if and only if $x \perp y$, where $\perp$ denotes the usual orthogonality (x makes a right angle with y). From this and Proposition 1.1.2, Mitani and Saito [55] showed the following practical calculation method for James constants.

Proposition 1.1.4 (Mitani and Saito [55])**.** *Let $\psi \in \Psi_2^S$. Then*

$$J((\mathbb{R}^2, \|\cdot\|_\psi)) = \max_{0 \le t \le 1/2} \frac{2-2t}{\psi(t)} \psi\left(\frac{1-2t}{2-2t}\right).$$

Proof. Suppose that $x, y \in S_{(\mathbb{R}^2, \|\cdot\|_\psi)}$, and that $x \perp y$. Without loss of generality, we may assume that x and y are in the first and second quadrants, respectively. Since $\psi(t) = \|(1-t, t)\|_\psi$, we have $x = \psi(t)^{-1}(1-t, t)$ and $y = \psi(t)^{-1}(t, t-1)$ for some $t \in [0,1]$. Then it follows that

$$\begin{aligned}
\|x+y\|_\psi = \|x-y\|_\psi &= \frac{1+|2t-1|}{\psi(t)} \psi\left(\frac{|2t-1|}{1+|2t-1|}\right) \\
&= \begin{cases} \dfrac{2-2t}{\psi(t)} \psi\left(\dfrac{1-2t}{2-2t}\right) & \text{if} \quad 0 \le t \le 1/2, \\ \dfrac{2t}{\psi(t)} \psi\left(\dfrac{2t-1}{2t}\right) & \text{if} \quad 1/2 \le t \le 1. \end{cases}
\end{aligned}$$

This shows

$$J((\mathbb{R}^2, \|\cdot\|_\psi)) = \max\left\{ \max_{0 \le t \le 1/2} \frac{2-2t}{\psi(t)} \psi\left(\frac{1-2t}{2-2t}\right), \max_{1/2 \le t \le 1} \frac{2t}{\psi(t)} \psi\left(\frac{2t-1}{2t}\right) \right\}.$$

However, $\psi \in \Psi_2^S$ assures that

$$\frac{2t}{\psi(t)} \psi\left(\frac{2t-1}{2t}\right) = \frac{2-2(1-t)}{\psi(1-t)} \psi\left(\frac{1-2(1-t)}{2-2(1-t)}\right)$$

for each $t \in [0, 1/2]$. Thus we finally have

$$J((\mathbb{R}^2, \|\cdot\|_\psi)) = \max_{0\le t\le 1/2} \frac{2-2t}{\psi(t)}\psi\left(\frac{1-2t}{2-2t}\right)$$
$$= \max_{1/2\le t\le 1} \frac{2t}{\psi(t)}\psi\left(\frac{2t-1}{2t}\right),$$

as desired. □

Corollary 1.1.5 (Mitani and Saito [55]). *Let $\psi \in \Psi_2^S$.*

(i) *If $\psi \ge \psi_2$ and the function $t \mapsto \psi(t)/\psi_2(t)$ takes the maximum at $1/2$, then $J((\mathbb{R}^2, \|\cdot\|_\psi)) = 2\psi(1/2)$.*

(ii) *If $\psi \le \psi_2$ and the function $t \mapsto \psi(t)/\psi_2(t)$ takes the minimum at $1/2$, then $J((\mathbb{R}^2, \|\cdot\|_\psi)) = 1/\psi(1/2)$.*

Proof. Let $M = \max_{0\le t\le 1} \psi(t)/\psi_2(t) = \psi(1/2)/\psi_2(1/2)$. Then it follows that

$$\frac{2-2t}{\psi(t)}\psi\left(\frac{1-2t}{2-2t}\right) \le \frac{M(2-2t)}{\psi_2(t)}\psi_2\left(\frac{1-2t}{2-2t}\right) = \sqrt{2}M = 2\psi(1/2).$$

This proves (i). An argument similar to this works in (ii). □

It should be noted that the assumption that ψ is symmetric with respect to $1/2$ is redundant in the preceding corollary; see [55, Remark 1].

These formulas allows us to compute the James constants of the space $\mathbb{R}^2$ endowed with symmetric absolute normalized norms rather easily.

Example 1.1.6 (Gao and Lau [26]). Let $1 \le p, q \le \infty$ with $1/p + 1/q = 1$, and let $r = \min\{p, q\}$. Then $J(\ell_p^2) = 2^{1/r}$. This immediately follows from the preceding corollary. We remark that $(\ell_p^2)^* = \ell_q^2$, and hence $J((\ell_p^2)^*) = J(\ell_q^2) = 2^{1/r} = J(\ell_p^2)$.

Example 1.1.7 (Mitani and Saito [55]). For each $1/2 \le \beta \le 1$, let $\psi_\beta = \max\{1 - t, t, \beta\}$. Then the associated norm is given by $\|(x, y)\|_\beta = \max\{|x|, |y|, \beta(|x| + |y|)\}$. The unit ball of the space $(\mathbb{R}^2, \|\cdot\|_\beta)$ is an octagon. By Theorem 1.1.4, it follows that

$$J((\mathbb{R}^2, \|\cdot\|_\beta)) = \begin{cases} 1/\beta & \text{if} \quad 1/2 \le \beta \le 1/\sqrt{2}, \\ 2\beta & \text{if} \quad 1/\sqrt{2} \le \beta \le 1. \end{cases}$$

Now we shall consider the dual norm $\|\cdot\|_\beta^*$ of $\|\cdot\|_\beta$. By an easy computation, one has that

$$\psi_\beta^*(t) = \begin{cases} \dfrac{1-2\beta}{\beta}t + 1 & \text{if} \quad 0 \le t \le 1/2, \\ \dfrac{2\beta-1}{\beta}t + \dfrac{1+\beta}{\beta} & \text{if} \quad 1/2 \le t \le 1. \end{cases}$$

From this, it follows that

$$\frac{2-2t}{\psi_\beta^*(t)}\psi_\beta^*\left(\frac{1-2t}{2-2t}\right) = \frac{(2\beta-2)t+1}{(1-2\beta)t+\beta},$$

which and Theorem 1.1.4 together imply that

$$J((\mathbb{R}^2, \|\cdot\|_\beta^*)) = \begin{cases} 1/\beta & \text{if} \quad 1/2 \le \beta \le 1/\sqrt{2}, \\ 2\beta & \text{if} \quad 1/\sqrt{2} \le \beta \le 1. \end{cases}$$

In particular, we have $J((\mathbb{R}^2, \|\cdot\|_\beta)^*) = J((\mathbb{R}^2, \|\cdot\|_\beta))$ for each β. See [41, 42] for a complete discussion of the James constants of octagonal norms.

1.2 The James constant of $d^{(2)}(\omega, q)$

In this section, we consider the James constant of the two-dimensional Lorentz sequence space $d^{(2)}(\omega, q)$. For each $0 < \omega \le 1 \le q < \infty$, let $d^{(2)}(\omega, q)$ be the space $\mathbb{R}^2$ endowed with the norm $\|(x, y)\|_{\omega,q} = (\max\{|x|^q, |y|^q\} + \omega \min\{|x|^q, |y|^q\})^{1/q}$. Then we obtain $\|\cdot\|_{\omega,q} \in AN_2^S$. The corresponding convex function $\psi_{\omega,q} \in \Psi_2^S$ is given by

$$\psi_{\omega,q}(t) = \begin{cases} ((1-t)^q + \omega t^q)^{1/q} & \text{if} \quad 0 \le t \le 1/2, \\ (t^q + \omega(1-t)^q)^{1/q} & \text{if} \quad 1/2 \le t \le 1. \end{cases}$$

The James constants of two-dimensional Lorentz sequence spaces was first studied by Kato and Maligranda [38].

Proposition 1.2.1 (Kato and Maligranda [38])**.** *Let $0 < \omega \le 1$. If $2 \le q < \infty$, then*

$$J(d^{(2)}(\omega, q)) = 2\left(\frac{1}{1+\omega}\right)^{1/q}.$$

In the same paper, they asked the values of $J(d^{(2)}(\omega, q))$ for $1 \le q < 2$ and $J(d^{(2)}(\omega, q)^*)$. Using the calculation method described in the preceding section, Mitani and Saito [55] gave a partial answer to this problem.

Proposition 1.2.2 (Mitani and Saito [55]). *Let $0 < \omega \le 1 \le q < 2$.*

(i) *If $0 < \omega \le 1 + \sqrt{2}$, then*

$$J(d^{(2)}(\omega, q)) = 2\left(\frac{1}{1+\omega}\right)^{1/q}.$$

(ii) *If $q = 1$ and $\omega > -1 + \sqrt{2}$, then*

$$J(d^{(2)}(\omega, q)) = 2\left(\frac{1}{1+\omega}\right)^{1/q}.$$

The value of $J(d^{(2)}(\omega, q))$ was entirely determined in 2008. This is the work of Mitani, Saito and Suzuki [60] (cf. [61]).

Theorem 1.2.3 (Mitani, Saito and Suzuki [60]). *Let $1 < q < 2 < p < \infty$ with $1/p + 1/q = 1$. If $(\sqrt{2} - 1)^{2-q} < \omega < 1$, then there exists a unique pair of real numbers s_0, s_1 such that*

$$\left(\frac{1-\omega}{\omega(1+\omega)}\right)^{p-1} < s_0 < \omega^{1/(2-q)} < s_1 < 1$$

and $(1 + s_i)^{q-1}(1 - \omega s_i^{q-1}) = \omega(1 - s_i)^{q-1}(1 + \omega s_i^{q-1})$ for $i = 0, 1$. Moreover, the value of $J(d^{(2)}(\omega, q))$ is as follows:

(i) *If $0 < \omega \le (\sqrt{2} - 1)^{2-q}$, then*

$$J(d^{(2)}(\omega, q)) = 2\left(\frac{1}{1+\omega}\right)^{1/q}.$$

(ii) *If $(\sqrt{2} - 1)^{2-q} < \omega \le \sqrt{2}^q - 1$, then*

$$J(d^{(2)}(\omega, q)) = \max\left\{2\left(\frac{1}{1+\omega}\right)^{1/q}, \left(\frac{2(1+s_0)^{q-1}}{1+\omega s_0^{q-1}}\right)^{1/q}\right\}.$$

(iii) *If $\sqrt{2}^q - 1 < \omega < 1$, then*

$$J(d^{(2)}(\omega, q)) = \left(\frac{2(1+s_0)^{q-1}}{1+\omega s_0^{q-1}}\right)^{1/q}.$$

This answered to one of the two questions of Kato and Maligranda. However, another problem remained open, that is, the value of $J(d^{(2)}(\omega, q)^*)$ had not been known yet. In 2009, Mitani and Saito [56] tried to this problem. As a matter of fact, the norm of $d^{(2)}(\omega, q)^*$ had not been clarified until then. Therefore they first needed to compute the dual norm of $d^{(2)}(\omega, q)$.

Proposition 1.2.4 (Mitani and Saito [56]). *Let $0 < \omega \leq 1 < p, q < \infty$ with $1/p + 1/q = 1$. Then the dual function $\psi^*_{\omega,q}$ of $\psi_{\omega,q}$ is given by*

$$\psi^*_{\omega,q}(t) = \begin{cases} ((1-t)^p + \omega^{1-p} t^p)^{1/p} & \text{if} \quad 0 \leq t < \omega/(1+\omega), \\ (1+\omega)^{1/p-1} & \text{if} \quad \omega/(1+\omega) \leq t < 1/(1+\omega), \\ (t^p + \omega^{1-p}(1-t)^p)^{1/p} & \text{if} \quad 1/(1+\omega) \leq t \leq 1. \end{cases}$$

Consequently, the dual norm $\|\cdot\|^*_{\omega,q}$ of $\|\cdot\|_{\omega,q}$ is given by

$$\|(x,y)\|^*_{\omega,q} = \begin{cases} (|x|^p + \omega^{1-p}|y|^p)^{1/p} & \text{if} \quad |y| \leq \omega|x|, \\ (1+\omega)^{1/p-1}(|x|+|y|) & \text{if} \quad \omega|x| \leq |y| \leq \omega^{-1}|x|, \\ (\omega^{1-p}|x|^p + |y|^p)^{1/p} & \text{if} \quad \omega^{-1}|x| \leq |y|. \end{cases}$$

From this, they could attack the problem. An immediate consequence of the preceding proposition and Corollary 1.1.5, we have the following result.

Proposition 1.2.5 (Mitani and Saito [56]). *Let $0 < \omega \leq 1$. If $2 \leq q < \infty$, then*

$$J(d^{(2)}(\omega,q)^*) = 2\left(\frac{1}{1+\omega}\right)^{1/q}.$$

For the case of $1 < q < 2$, the James constant $J(d^{(2)}(\omega,q)^*)$ is determined by using Proposition 1.1.4 as follows:

Theorem 1.2.6 (Mitani and Saito [56]). *Let $1 < q < 2 < p < \infty$ with $1/p + 1/q = 1$. If $(\sqrt{2}-1)^{2-q} < \omega < 1$, then there exists a unique pair of real numbers s^*_0, s^*_1 such that*

$$\frac{1-\omega}{\omega(1+\omega)} < s^*_0 < \omega^{1/(2-q)} < s^*_1 < \omega$$

and $(1+s^*_i)^{p-1}(1-\omega^{1-p}s^{*p-1}_i) = \omega^{1-p}(1-s^*_i)^{p-1}(1+\omega^{1-p}s^{*p-1}_i)$ for $i = 0, 1$. Moreover, the value of $J(d^{(2)}(\omega,q))$ is as follows:

(i) *If $0 < \omega \leq (\sqrt{2}-1)^{2-q}$, then*

$$J(d^{(2)}(\omega,q)^*) = 2\left(\frac{1}{1+\omega}\right)^{1/q}.$$

(ii) *If $(\sqrt{2}-1)^{2-q} < \omega \leq \sqrt{2}^q - 1$, then*

$$J(d^{(2)}(\omega,q)^*) = \max\left\{2\left(\frac{1}{1+\omega}\right)^{1/q}, \left(\frac{2(1+s^*_1)^{p-1}}{1+\omega^{1-p}s^{*q-1}_1}\right)^{1/p}\right\}.$$

(iii) *If* $\sqrt{2}^q - 1 < \omega < 1$, *then*

$$J(d^{(2)}(\omega,q)^*) = \left(\frac{2(1+s_1^*)^{p-1}}{1+\omega^{1-p}s_1^{*q-1}} \right)^{1/p}.$$

Thus two questions proposed by Kato and Maligranda was completely solved by Theorems 1.2.5 and 1.2.6. The key ingredient of the arguments is obviously the correspondence of AN_2^S and Ψ_2^S that underlies Propositions 1.1.4 and 1.2.4. This shows an advantage of the use of ψ-norms.

Incidentally, taking a glance at the results in this section yields $J(d^{(2)}(\omega,q)^*) = J(d^{(2)}(\omega,q))$ whenever $1 < q < 2$ and $0 < \omega \leq (\sqrt{2}-1)^{2-q}$ or $2 \leq q < \infty$. Therefore our next problem is the following: When does the equality $J(d^{(2)}(\omega,q)^*) = J(d^{(2)}(\omega,q))$ hold. In what follows, we assume that $1 < q < 2$ and $(\sqrt{2}-1)^{2-q} < \omega < 1$. We start with the observation of the pair (s_0, s_1) taken in Theorem 1.2.3.

Suppose that (s_0, s_1) is the pair of positive numbers taken in Theorem 1.2.3. Define the real-valued function f on $[0, 1/2]$ by

$$f(t) = \frac{2-2t}{\psi_{\omega,q}(t)}\psi_{\omega,q}\left(\frac{1}{2-2t}\right) = \left(\frac{\omega(1-2t)^q + 1}{(1-t)^q + \omega t^q}\right)^{1/q}$$

We also put

$$g(s) = f\left(\frac{s}{1+s}\right) = \left(\frac{(1+s)^q + \omega(1-s)^q}{1+\omega s^q}\right)^{1/q}$$

for each $s \in [0,1]$. Since $(1+s_i)^{q-1}(1-\omega s_i^{q-1}) = \omega(1-s_i)^{q-1}(1+\omega s_i^{q-1})$, it follows that

$$f\left(\frac{s_i}{1+s_i}\right) = g(s_i) = \left(\frac{2(1+s_i)^{q-1}}{1+\omega s_i^{q-1}}\right)^{1/q}.$$

for $i = 0, 1$. We now remark that

$$f\left(\frac{s}{1+s}\right) f\left(\frac{1-s}{2}\right) = 2$$

for any $s \in [0,1]$. Then, as was shown in [60], the function $g(s) = f(s/(1+s))$ takes a unique maximal value at s_0 and a unique minimal value at s_1, and hence the function $f((1-s)/2) = 2/f(s/(1+s))$ takes a unique minimal value at s_0 and a unique maximal value at s_1.

s	0		s_0		s_1		1
$f\left(\dfrac{s}{1+s}\right)$		$\nearrow$		$\searrow$		$\nearrow$	
$f\left(\dfrac{1-s}{2}\right)$		$\searrow$		$\nearrow$		$\searrow$	

Hence it follows that $s_1/(1+s_1) = (1-s_0)/2$, which in turn implies that

$$f\left(\frac{s_0}{1+s_0}\right) f\left(\frac{s_1}{1+s_1}\right) = 2. \tag{1.1}$$

This is a key ingredient of our next argument.

Now we shall prove the following result that answers to the question.

Theorem 1.2.7 ([61])**.** *Let $0 < \omega \le 1 < q < \infty$. Then $J(d^{(2)}(\omega, q)^*) = J(d^{(2)}(\omega, q))$.*

Proof. Keep the notation as above. Suppose that $1 < q < 2$ and $(\sqrt{2}-1)^{2-q} < \omega < 1$. Then Theorem 1.2.6 assures that there exists a unique pair of real numbers s_0^*, s_1^* such that

$$\frac{1-\omega}{\omega(1+\omega)} < s_0^* < \omega^{1/(2-q)} < s_1^* < \omega$$

and $(1+s_i^*)^{p-1}(1-\omega^{1-p}s_i^{*p-1}) = \omega^{1-p}(1-s_i^*)^{p-1}(1+\omega^{1-p}s_i^{*p-1})$ for $i = 0, 1$. On the other hand, the pair (s_0, s_1) satisfies that

$$\left(\frac{1-\omega}{\omega(1+\omega)}\right)^{p-1} < s_0 < \omega^{1/(2-q)} < s_1 < 1$$

and $(1+s_i)^{q-1}(1-\omega s_i^{q-1}) = \omega(1-s_i)^{q-1}(1+\omega s_i^{q-1})$ for $i = 0, 1$. Now let $s_i' = \omega s_i^{q-1}$ for $i = 0, 1$. Then it follows that

$$\frac{1-\omega}{1+\omega} < s_0' < \omega^{1/(2-q)} < s_1' < \omega.$$

Furthermore, we have

$$\begin{aligned}
&(1+\omega s_i^{q-1})^{p-1}(1-s_i) - \omega^{1-p}(1-\omega s_i^{q-1})^{p-1}(1+s_i) \\
&= ((1-s_i)^{q-1}(1+\omega s_i^{q-1}))^{p-1} - (\omega^{-1}(1+s_i)^{q-1}(1-\omega s_i^{q-1}))^{p-1} = 0
\end{aligned}$$

for $i = 0, 1$ since $(p-1)(q-1) = 1$. However, the uniqueness of (s_0^*, s_1^*) guarantees that $s_i^* = s_i' = \omega s_i^{q-1}$ for $i = 0, 1$. Since $s_1^* = \omega s_1^{q-1}$, we have

$$\left(\frac{2(1+s_1)^{q-1}}{1+\omega s_1^{q-1}}\right)^{1/q} \left(\frac{2(1+s_1^*)^{p-1}}{1+\omega^{1-p}s_1^{*p-1}}\right)^{1/p} = 2.$$

On the other hand, by (1.1), we also have

$$\left(\frac{2(1+s_0)^{q-1}}{1+\omega s_0^{q-1}}\right)^{1/q} \left(\frac{2(1+s_1)^{q-1}}{1+\omega s_1^{q-1}}\right)^{1/q} = 2.$$

Thus it follows that

$$\left(\frac{2(1+s_0)^{q-1}}{1+\omega s_0^{q-1}}\right)^{1/q} = \left(\frac{2(1+s_1^*)^{p-1}}{1+\omega^{1-p}s_1^{*p-1}}\right)^{1/p}.$$

This together with Theorems 1.2.3 and 1.2.6 proves that $J(d^{(2)}(\omega, q)^*) = J(d^{(2)}(\omega, q))$. The proof is complete. $\square$

1.3 The duality of James constant

As was seen in [39], the equality $J(X^*) = J(X)$ does not hold in general. A counterexample is given by the Day-James ℓ_2-ℓ_1 space, that is, the space $\mathbb{R}^2$ endowed with the norm

$$\|(x,y)\|_{2,1} = \begin{cases} \|(x,y)\|_2 & \text{if} \quad xy \geq 0, \\ \|(x,y)\|_1 & \text{if} \quad xy \leq 0. \end{cases}$$

See [65] for more computations of the James constants of generalized Day-James spaces. We remark that the norm $\|\cdot\|_{2,1}$ is symmetric. Moreover, letting $\|(x,y)\|'_{2,1} = \|(x+y, x-y)\|_{2,1}$ for each (x,y) yields an absolute norm on $\mathbb{R}^2$. Since James constant does not change under isometric isomorphisms, we have obtained counterexamples of two-dimensional normed spaces that are equipped with either symmetric or absolute norms. However, as was shown in this chapter, all the spaces ℓ_p, $(\mathbb{R}^2, \|\cdot\|_\beta)$ and $d^{(2)}(\omega, q)$ satisfy the equality. The common feature of these spaces is that the norm is both symmetric and absolute. In this section, we shall prove that every two-dimensional normed space X endowed with a symmetric absolute norm satisfies $J(X^*) = J(X)$.

In what follows, we denote the normed space $(\mathbb{R}^2, \|\cdot\|_\psi)$ by X_ψ for short. Let

$$f_\psi(t) = \frac{2-2t}{\psi(t)}\psi\left(\frac{1-2t}{2-2t}\right)$$

for each $t \in [0, 1/2]$ and each $\psi \in \Psi_2^S$. Then Lemma 1.1.4 assures that $J(X_\psi) = \max_{0 \leq t \leq 1/2} f_\psi(t)$. The proof is mainly based on an appropriate density of the strictly convex functions in Ψ_2^S and a certain continuity of the map $\psi \mapsto J(X_\psi)$. Some preliminary works are needed. We shall start with some definitions. A finite sequence $(t_i)_{i=0}^n$ of real numbers is said to be a *partition* of the interval $[0, 1/2]$ if $0 = t_0 < t_1 < \cdots < t_n = 1/2$. Any finite subset P of $[0, 1/2]$ including 0 and $1/2$ can be viewed as a partition of $[0, 1/2]$ by taking the strictly increasing rearrangement, and so we can identify the partition $(t_i)_{i=0}^n$ with the set $\{t_i : 0 \leq i \leq n\}$. A function ψ on the interval $[0, 1/2]$ is said to be *piecewise linear* if there exist a partition $(t_i)_{i=0}^n$ of $[0, 1/2]$ and a finite sequence $(a_i)_{i=0}^n$ of real numbers such that

$$\psi(t) = \frac{a_i - a_{i-1}}{t_i - t_{i-1}}t + \frac{a_{i-1}t_i - a_i t_{i-1}}{t_i - t_{i-1}} \tag{1.2}$$

for each $t \in [t_{i-1}, t_i]$. Letting

$$\alpha_i = \frac{a_i - a_{i-1}}{t_i - t_{i-1}} \quad \text{and} \quad \beta_i = \frac{a_{i-1}t_i - a_i t_{i-1}}{t_i - t_{i-1}},$$

one has that $\psi(t) = \alpha_i t + \beta_i$ for each $t \in [t_{i-1}, t_i]$, and that $\psi(t_i) = a_i$ for each $0 \leq i \leq n$. A *corner point* of the function ψ is an element t_i of the partition satisfying $0 < i < n$ and $\alpha_i \neq \alpha_{i+1}$. We remark that every corner point is deduced by determining the intersection point of two distinct (actually, successive) lines. A partition that all the elements $t_1, t_2, \ldots, t_{n-1}$ are corner points is called a *simplified partition*. It is clear that the function ψ is convex if and only if the sequence $(\alpha_i)_{i=1}^n$ is non-decreasing. The piecewise linear function ψ on $[0, 1/2]$ extends to a piecewise linear function on the unit interval $[0, 1]$ by the formula $\psi(t) = \psi(1-t)$ for each $t \in [1/2, 1]$. Then it is not difficult to check that $\psi \in \Psi_2^S$ is equivalent to $a_0 = 1$ and $-1 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n \leq 0$. Henceforth, the function $\psi \in \Psi_2^S$ is said to be piecewise linear if its restriction to $[0, 1/2]$ is piecewise linear.

Now we shall introduce a special class of partitions of $[0, 1/2]$. Let $g(t) = (1 - 2t)/(2 - 2t)$ for each $[0, 1/2]$. Then g is a strictly decreasing function from $[0, 1/2]$ onto itself. Moreover, we have $g(g(t)) = t$ for all $t \in [0, 1/2]$, that is, $g^{-1} = g$. A partition P of $[0, 1/2]$ is said to be *recursive* if $g(P) = P$. We note that every partition P is included in the recursive partition $P \cup g(P)$, the *recursive hull* of P. As will be seen in the following lemma, recursive partitions of $[0, 1/2]$ play a crucial role in our discussion.

Lemma 1.3.1 ([71]). *Let $\psi \in \Psi_2$ be a piecewise linear function defined by* (1.2). *Suppose that the partition $(t_i)_{i=0}^n$ is recursive. Then $J(X_\psi) = \max\{f(t_i) : 0 \leq i \leq n\}$.*

Proof. For each i, let $I_i = [t_{i-1}, t_i]$ and $J_i = [g(t_i), g(t_{i-1})]$. Then $(I_i)_{i=1}^n$ and $(J_i)_{i=1}^n$ are, respectively, covers of the interval $[0, 1/2]$. Moreover, $t \in J_i$ if and only if $g(t) \in I_i$. Now let $K = \{(i, j) : I_i \cap J_j \neq \emptyset\}$. Fix an element $(i, j) \in K$. Then the function f_ψ on $I_i \cap J_j$ is given by

$$\begin{aligned} f_\psi(t) &= \frac{2 - 2t}{\alpha_i t + \beta_i} \left(\alpha_j \left(\frac{1 - 2t}{2 - 2t} \right) + \beta_j \right) \\ &= \frac{-2(\alpha_j + \beta_j)t + \alpha_j + 2\beta_j}{\alpha_i t + \beta_i}. \end{aligned}$$

Since the function f_ψ is monotone on $I_i \cap J_j$, we have

$$f_\psi(t) \leq \max\{f_\psi(t_{i-1}), f_\psi(t_i), f_\psi(g(t_{j-1})), f_\psi(g(t_j))\}.$$

The recursivity of the partition assures that $g(t_j) \in \{t_i : 0 \leq i \leq n\}$ for each j. Thus one obtains

$$\max_{0 \leq t \leq 1/2} f_\psi(t) = \max_{(i,j) \in K} \max_{t \in I_i \cap J_j} f_\psi(t) \leq \max\{f_\psi(t_i) : 0 \leq i \leq n\}.$$

The converse is obvious, and therefore we have the lemma. $\square$

We next provide two technical lemmas. The first one shows the continuity of the function $\psi \mapsto J(X_\psi)$ with respect to the uniform norm $\|\cdot\|_\infty$ on $C[0,1]$.

Lemma 1.3.2 ([71])**.** *The function $\psi \mapsto J(X_\psi)$ is continuous on Ψ_2^S.*

Proof. Let (ψ_n) be a sequence in Ψ_2^S that converges uniformly to $\psi \in \Psi_2^S$. Then it follows from $1/2 \leq \psi_n(t), \psi(t) \leq 1$ that

$$\begin{aligned}\|f_{\psi_n} - f_\psi\|_\infty &= \max_{t\in[0,1/2]} \frac{(2-2t)|\psi_n(g(t))\psi(t) - \psi_n(t)\psi(g(t))|}{\psi_n(t)\psi(t)} \\ &\leq 16\|\psi_n - \psi\|_\infty,\end{aligned}$$

and hence $\|f_{\psi_n} - f_\psi\|_\infty \to 0$ as $n \to \infty$. In particular, we have

$$J(X_\psi) = \|f_\psi\|_\infty = \lim_n \|f_{\psi_n}\|_\infty = \lim_n J(X_{\psi_n}).$$

This completes the proof. $\square$

We now consider the dual function of a piecewise linear function. Let $\psi \in \Psi_2^S$ be a piecewise linear function defined by (1.2). Then the dual function ψ^* is given by

$$\begin{aligned}\psi^*(s) &= \max_{0\leq t\leq 1/2} \frac{(1-s)(1-t)+st}{\psi(t)} \\ &= \max_{1\leq i\leq n} \max_{t\in[t_{i-1},t_i]} \frac{(1-s)(1-t)+st}{\psi(t)} \\ &= \max_{1\leq i\leq n} \max_{t\in[t_{i-1},t_i]} \frac{(2s-1)t+1-s}{\alpha_i t+\beta_i} \\ &= \max_{1\leq i\leq n} \max\left\{\frac{(2s-1)t_{i-1}+1-s}{\alpha_i t_{i-1}+\beta_i}, \frac{(2s-1)t_i+1-s}{\alpha_i t_i+\beta_i}\right\} \\ &= \max_{0\leq i\leq n} \frac{(2t_i-1)s+1-t_i}{a_i}\end{aligned}$$

for each $s \in [0,1/2]$. Hence the dual function is also piecewise linear as the maximum function of $(n+1)$-lines.

To present appropriate density properties in Ψ_2^S, we here note the following useful fact: If a monotone sequence (ψ_n) in Ψ_2^S converges uniformly to $\psi \in \Psi_2^S$, then the sequence (ψ_n^*) also converges uniformly to ψ^*. Indeed, the function $((1-s)(1-t) + st)/\psi_n(s)$ of $s \in [0,1]$ converges uniformly to $((1-s)(1-t)+st)/\psi^*(s)$ for each $t \in [0,1]$, and so

$$\begin{aligned}\psi^*(t) &= \max_{0\le s\le 1}\frac{(1-s)(1-t)+st}{\psi(s)}\\ &= \lim_n \max_{0\le s\le 1}\frac{(1-s)(1-t)+st}{\psi_n(s)}\\ &= \lim_n \psi_n^*(t)\end{aligned}$$

for all $t\in[0,1]$. Moreover, since $\varphi\le\psi$ implies $\varphi^*\ge\psi^*$, the sequence (ψ_n^*) is also monotone. In other words, the monotone sequence (ψ_n^*) converges pointwise to the continuous function ψ^*. Thus the sequence (ψ_n^*) converges uniformly to ψ^* by Dini's theorem.

To prove the second lemma, we need the following results.

Lemma 1.3.3 (Takahashi, Kato and Saito [78]). *Let $\psi\in\Psi_2$. Then X_ψ is strictly convex if and only if ψ is strictly convex.*

Lemma 1.3.4 (Mitani, Saito and Suzuki [59]). *Let $\psi\in\Psi_2$. Then X_ψ is smooth if and only if ψ is differentiable on (0,1), $\psi_R'(0)=-1$ and $\psi_L'(1)=1$, where ψ_R' and ψ_L' are, respectively, the right and left derivative of ψ.*

Let $\psi\in\Psi_2^S$ be a piecewise linear function defined by (1.2). Then $\psi(t)>1-t$ for all $t\in(0,1/2]$ if and only if $\alpha_1>-1$. Now suppose that $\alpha_1>-1$. It follows from $\psi(t)\ge\alpha_1 t+1$ for all $t\in(0,1/2]$ that

$$\psi^*(s)=\max_{0\le t\le 1/2}\frac{(1-s)(1-t)+st}{\psi(t)}\le\max_{0\le t\le 1/2}\frac{(1-s)(1-t)+st}{\alpha_1 t+1}$$

for each $s\in[0,1/2]$. In particular, we have $\psi^*(s)=1-s$ if $0\le s\le(1+\alpha_1)/(2+\alpha_1)$.

Lemma 1.3.5 ([71]). *Let $\psi\in\Psi_2^S$. Then there exists a sequence (ψ_n) of strictly convex elements of Ψ_2^S such that $\|\psi_n-\psi\|_\infty\to 0$ and $\|\psi_n^*-\psi^*\|_\infty\to 0$ as $n\to\infty$.*

Proof. We first show that any the maximum function of two lines can be uniformly approximated by a smooth convex function. Let $\ell_1(t)=at+b$ and $\ell_2=ct+d$ with $a<c$. Without loss of generality, we may assume that the lines intersects at $t_0\ge 0$. Then we have $d\le b$. Take an arbitrary $\varepsilon>0$. Now let

$$\begin{aligned}\alpha_\varepsilon &= \frac{c-a}{4\varepsilon},\\ \beta_\varepsilon &= \frac{a(t_0+\varepsilon)-c(t_0-\varepsilon)}{2\varepsilon},\\ \gamma_\varepsilon &= \frac{(c-a)(t_0-\varepsilon)^2}{4\varepsilon}+b,\end{aligned}$$

and let $h_\varepsilon(t) = \alpha_\varepsilon t^2 + \beta_\varepsilon t + \gamma_\varepsilon$. It follows from $\alpha_\varepsilon > 0$ and $t_0 = (b-d)/(c-a)$ that h_ε is a convex and smooth function satisfying $h'_\varepsilon(t_0 - \varepsilon) = a$, $h'_\varepsilon(t_0+\varepsilon) = c$, $h_\varepsilon(t_0-\varepsilon) = \ell_1(t_0-\varepsilon)$ and $h_\varepsilon(t_0+\varepsilon) = \ell_2(t_0+\varepsilon)$. Hence, letting

$$k_\varepsilon(t) = \begin{cases} \ell_1(t) & \text{if} \quad t \in (-\infty, t_0-\varepsilon], \\ h_\varepsilon(t) & \text{if} \quad t \in [t_0-\varepsilon, t_0+\varepsilon], \\ \ell_2(t) & \text{if} \quad t \in [t_0+\varepsilon, \infty), \end{cases}$$

yields a smooth function. Moreover, since k_ε is convex, we obtain

$$\begin{aligned} \sup_t |k_\varepsilon(t) - (\max\{\ell_1,\ell_2\})(t)| &\le \frac{\ell_1(t_0-\varepsilon) + \ell_2(t_0+\varepsilon)}{2} - (\max\{\ell_1,\ell_2\})(t_0) \\ &= (c-a)(t_0+\varepsilon) + d - b. \end{aligned}$$

Thus the supremum tends to 0 as $\varepsilon \to 0$. We remark that if $a = -c$ then all the functions $\max\{\ell_1,\ell_2\}$, h_ε and k_ε are symmetric at t_0. We also remark that we can take a sequence (ε_n) of positive numbers such that the sequence (k_{ε_n}) is decreasing and converges uniformly to $\max\{\ell_1,\ell_2\}$ since $k_\varepsilon(t) > (\max\{\ell_1,\ell_2\})(t)$ for all $t \in (t_0-\varepsilon, t_0+\varepsilon)$. To see this, consider the tangent line of k_ε at t_0. An appropriate figure might be help.

Next we take a piecewise linear function $\psi \in \Psi_2^S$ such that $\psi(t) > 1-t$ for all $t \in (0,1/2]$. Then the dual function ψ^* is also piecewise linear and $\psi^*(t) = 1-t$ on $[0,t_0]$ for some $t_0 \in (0,1/2]$. Now we can apply the above argument to "remove" the corner points of the function ψ^*, that is, there exists a sequence (ψ_n) in Ψ_2^S such that the sequence (ψ_n^*) is decresing, each the function ψ_n^* is smooth and $\psi_n^*(t) = 1-t$ on $[0,t_n]$ for some $t_n \in (0,1/2)$, and converges uniformly to ψ^*. Then the space $X_{\psi_n^*}$ is smooth by Lemma 1.3.4. However, by Lemma 1.3.3, this shows that the space X_{ψ_n}, and hence the function ψ_n, is strictly convex. Moreover, the sequence (ψ_n) converges uniformly to ψ.

We now remark that every piecewise linear element ψ in Ψ_2^S such that $\psi(t_0) = 1-t_0$ for some $t_0 \in (0,1/2]$ can be approximated uniformly by a decreasing sequence (ψ_n) of piecewise linear functions in Ψ_2^S satisfying $\psi_n(t) > 1-t$ for all $t \in (0,1/2]$. This and the above argument together show the lemma. $\square$

We now ready to prove the main theorem.

Theorem 1.3.6 ([71])**.** *Let $\psi \in \Psi_2^S$. Then $J(X_{\psi^*}) = J(X_\psi)$.*

Proof. We first assume that the function $\psi \in \Psi_2^S$ is piecewise linear with respect to a simplified and recursive partition $(t_i)_{i=0}^n$. Then the dual function ψ^* is also piecewise linear, and so there exist a partition $(s_j)_{j=0}^m$ of $[0,1/2]$ and a sequence of positive numbers $(b_j)_{j=0}^m$ satisfying $t_0 = 0$, $t_n = 1/2$ and

$$\psi^*(s) = \frac{b_j - b_{j-1}}{s_j - s_{j-1}} s + \frac{b_{j-1}s_j - b_j s_{j-1}}{s_j - s_{j-1}}$$

for each $s \in [s_{j-1}, s_j]$. For each pair (i,j) with $i \neq j$, let

$$\gamma_{ij} = \frac{b_i - b_j}{s_i - s_j} \quad \text{and} \quad \delta_{ij} = \frac{b_j s_i - b_i s_j}{s_i - s_j},$$

respectively. Since $\psi = \psi^{**}$, the function ψ is given by

$$\psi(t) = \max_{0 \le i \le n} \frac{(2s_i - 1)t + 1 - s_i}{b_i}$$

for each $t \in [0,1/2]$. Let ℓ_j be the line given by

$$\ell_j(t) = \frac{(2s_i - 1)t + 1 - s_i}{b_i},$$

and let $I_k = \{t \in [0,1/2] : \psi(t) = \ell_k(t)\}$. Take an arbitrary $i \in \{0, 1, \ldots, n\}$. If $i = 0$, then $f_\psi(t_0) = 2\psi(1/2) = 1/\psi^*(1/2) = f_{\psi^*}(1/2)$. Similarly, if $i = n$ then $f_\psi(t_n) = f_{\psi^*}(0)$. So we assume that $0 < i < n$. Since t_i is a corner point, there exist two distinct lines ℓ_i and ℓ_j such that $\psi(t_i) = \ell_i(t_i) = \ell_j(t_i)$. It follows that

$$t_i = \frac{\gamma_{ij} + \delta_{ij}}{\gamma_{ij} + 2\delta_{ij}}.$$

Moreover, one has

$$\psi(t_i) = \ell_i(t_i) = \frac{1}{\gamma_{ij} + 2\delta_{ij}}.$$

Now pick a index k such that $g(t_i) \in I_k$. Then we have

$$\psi(g(t_i)) = \frac{-2(\gamma_{ij} + \delta_{ij})s_k + \gamma_{ij} + 2\delta_{ij}}{2b_k\delta_{ij}},$$

and hence

$$f(t_i) = \frac{-2(\gamma_{ij} + \delta_{ij})s_k + \gamma_{ij} + 2\delta_{ij}}{b_k}.$$

On the other hand, one has

$$\begin{aligned}
\psi^*(s) &= \max_{0 \le t \le 1/2} \frac{(1-s)(1-t) + st}{\psi(t)} \\
&\ge \frac{(1-s)(1-t_i) + st_i}{\psi(t_i)} \\
&= \delta_{ij}(1-s) + (\gamma_{ij} + \delta_{ij})s.
\end{aligned}$$

for each $s \in [0, 1/2]$. In particular, the inequality

$$(2 - 2s_k)\psi^*(g(s_k)) \geq -2(\gamma_{ij} + \delta_{ij})s_k + \gamma_{ij} + 2\delta_{ij}$$

holds. This together with $b_k = \psi^*(s_k)$ imply that $f_\psi(t_i) \leq f_{\psi^*}(s_k)$, which in turn implies that $J(X_\psi) \leq J(X_{\psi^*})$ by Lemma 1.3.1.

Now suppose that ψ is a strictly convex element in Ψ_2^S. For each $n \in \mathbb{N}$, let $(t_i^{(n)})_{i=0}^{k_n}$ be the recursive hull of the dyadic partition $(k/2^n)_{k=0}^{2^{n-1}}$, and let $\psi_n \in \Psi_2^S$ be the piecewise linear function determined by the partition $(t_i^{(n)})_{i=0}^{k_n}$ and the values $(\psi(t_i^{(n)}))_{i=0}^{k_n}$. Since $(k/2^n)_{k=0}^{2^{n-1}} \subset (k/2^{n+1})_{k=0}^{2^n}$, we obtain $(t_i^{(n)})_{i=0}^{k_n} \subset (t_i^{(n+1)})_{i=0}^{k_{n+1}}$, which implies that the sequence (ψ_n) is decreasing. We remark that the function ψ_m coincides with ψ on $\{k/2^n : 0 \leq k \leq 2^{n-1}\}$ whenever $m > n$. This shows that $\|\psi_n - \psi\|_\infty \to 0$, and also that $\|\psi_n^* - \psi^*\|_\infty \to 0$. Since the partition $(t_i^{(n)})_{i=0}^{k_n}$ is simplified and recursive, it follows that $J(X_{\psi_n}) \leq J(X_{\psi_n^*})$ for each n. However, the continuity of the function $\psi \mapsto J(X_\psi)$ assures that $J(X_\psi) \leq J(X_{\psi^*})$.

Finally, let ψ be an arbitrary element in Ψ_2^S. Then, Lemma 1.3.5 provides a sequence (ψ_n) of strictly convex elements of Ψ_2^S such that $\|\psi_n - \psi\|_\infty \to 0$ and $\|\psi_n^* - \psi^*\|_\infty \to 0$ as $n \to \infty$. It follows that $J(X_\psi) = \lim_n J(X_{\psi_n}) \leq \lim_n J(X_{\psi_n^*}) = J(X_{\psi^*})$. Combining this and the fact that $\psi^{**} = \psi$ yields the equality $J(X_{\psi^*}) = J(X_\psi)$, as desired. $\square$

Since James constants are invariant under scaling, we finally have the following result.

Theorem 1.3.7 ([71])**.** *Let X be a two-dimensional real normed space $\mathbb{R}^2$ equipped with a symmetric absolute norm. Then $J(X^*) = J(X)$.*

1.4 Remarks

We shall give some remarks on the results in the chapter. Let x, y be two vectors in a Banach space. Then x is said to be *Roberts orthogonal* to y, denoted by $x \perp_R y$, if $\|x + \alpha y\| = \|x - \alpha y\|$ for all $\alpha \in \mathbb{R}$ (cf. [66]). A two-dimensional normed space X is isometrically isomorphic to the space $\mathbb{R}^2$ equipped with a symmetric absolute norm if and only the space X has a basis $\{x, y\}$ such that $x \perp_R y$ and $x + y \perp_R x - y$. Indeed, if $x \perp_R y$ and $x + y \perp_R x - y$ then the norm on $\mathbb{R}^2$ defined by $\|(a, b)\| = \|ax + by\|$ for each (a,b) is clearly absolute. Moreover, it follows from $x+y \perp_R x-y$ that

$$\begin{aligned}\|(a,b)\| = \|ax+by\| &= \frac{|a+b|}{2}\left\|\frac{2a}{a+b}x + \frac{2b}{a+b}y\right\| \\ &= \frac{|a+b|}{2}\left\|x+y+\frac{a-b}{a+b}(x-y)\right\| \\ &= \frac{|a+b|}{2}\left\|x+y-\frac{a-b}{a+b}(x-y)\right\| \\ &= \frac{|a+b|}{2}\left\|\frac{2b}{a+b}x + \frac{2a}{a+b}y\right\| \\ &= \|bx+ay\| = \|(b,a)\|\end{aligned}$$

whenever $a \neq -b$, that is, the norm $\|\cdot\|$ is also symmetric. The converse is shown by putting $x=(1,0)$ and $y=(0,1)$. Thus we obtain the following consequence of the main theorem.

Corollary 1.4.1 ([71]). *Let X be a two-dimensional real normed space with $J(X^*) \neq J(X)$. Then there is no basis $\{x,y\}$ for X satisfying $x \perp_R y$ and $x+y \perp_R x-y$.*

We conclude this chapter with counterexamples that have hexagonal (and hence asymmetric) norms. Remark that both AN_2 and Ψ_2 are convex, and that the correspondence preserves the convex structure. Namely, the following hold:

(i) If $\|\cdot\|, \|\cdot\|' \in AN_2$, then $\lambda\|\cdot\| + (1-\lambda)\|\cdot\|' \in AN_2$ for all $\lambda \in (0,1)$.

(ii) If $\psi, \psi' \in \Psi_2$, then $\lambda\psi + (1-\lambda)\psi' \in \Psi_2$ for all $\lambda \in (0,1)$.

(iii) $\|\cdot\|_{\lambda\psi+(1-\lambda)\psi'} = \lambda\|\cdot\|_\psi + (1-\lambda)\|\cdot\|_{\psi'}$ for each $\psi, \psi' \in \Psi_2$ and all $\lambda \in (0,1)$.

By (iii), the extreme points of AN_2 and Ψ_2 are essentially the same. Moreover, we have the following result.

Theorem 1.4.2 (Grząślewicz [30]; Komuro, Saito and Mitani [40]). *For each $0 \leq \alpha \leq 1/2 \leq \beta \leq 1$, define the function $\psi_{\alpha,\beta}$ by*

$$\psi_{\alpha,\beta} = \begin{cases} 1-t & \text{if} \quad 0 \leq t \leq \alpha, \\ \dfrac{(\alpha+\beta-1)t+\beta-2\alpha\beta}{\beta-\alpha} & \text{if} \quad \alpha \leq t \leq \beta, \\ t & \text{if} \quad \beta \leq t \leq 1. \end{cases}$$

Then $\mathrm{ext}(\Psi_2) = \{\psi_{\alpha,\beta} : 0 \leq \alpha \leq 1/2 \leq \beta \leq 1\}$.

The James constant of $X_{\psi_{\alpha,\beta}}$ is completely determined by Komuro, Saito and Mitani [41]; see also [42].

Theorem 1.4.3 (Komuro, Saito and Mitani [41]). *Let* $0 \leq \alpha \leq 1/2 \leq \beta \leq 1$ *with* $\alpha < 1 - \beta$.

(i) If $\psi_{\alpha,\beta}(1/2) \leq 1/2(1-\alpha)$, *then*

$$J(X_{\psi_{\alpha,\beta}}) = \frac{1}{\psi_{\alpha,\beta}(1/2)}.$$

(ii) If $1/2(1-\alpha) \leq \psi_{\alpha,\beta}(1/2) \leq c(\alpha,\beta)$, *then*

$$J(X_{\psi_{\alpha,\beta}}) = 1 + \frac{1}{\psi_{\alpha,\beta}(1/2) + (2\beta-1)/(\beta-\alpha)}.$$

(iii) If $c(\alpha,\beta) \leq \psi_{\alpha,\beta}(1/2)$, *then*

$$J(X_{\psi_{\alpha,\beta}}) = 2\psi_{\alpha,\beta}(1/2),$$

where

$$c(\alpha,\beta) = \frac{1}{4}\left(1 - \frac{2\beta-1}{\beta-\alpha} + \sqrt{\left(1 + \frac{2\beta-1}{\beta-\alpha}\right)^2 + 4}\right).$$

Using this result, we can provide new examples of $J(X^*) \neq J(X)$, where X is the space $\mathbb{R}^2$ endowed with an extreme absolute normalized norm on $\mathbb{R}^2$.

Example 1.4.4. The computation is based on Theorem 1.4.3. For each $\beta \in (1/2, 1)$, let ψ_β be an asymmetric element of Ψ_2 given by

$$\psi_\beta(t) = \psi_{0,\beta}(t) = \begin{cases} \dfrac{\beta-1}{\beta}t + 1 & \text{if} \quad t \in [0,\beta], \\ t & \text{if} \quad t \in [\beta,1], \end{cases}$$

and let

$$c(\beta) = c(0,\beta) = \frac{1}{4}\left(\frac{1-\beta}{\beta} + \sqrt{\left(1 + \frac{2\beta-1}{\beta}\right)^2 + 4}\right).$$

Then it follows that $\psi_\beta(1/2) \geq c(\beta)$ if and only if $\beta \geq 2/3$. Hence, by Theorem 1.4.3, we have

$$J(X_{\psi_\beta}) = \begin{cases} \dfrac{6\beta-2}{5\beta-2} & \text{if} \quad \beta \in (1/2, 2/3], \\ \dfrac{3\beta-1}{\beta} & \text{if} \quad \beta \in [2/3, 1). \end{cases}$$

We next consider the dual function of ψ_β. After an easy computation, we obtain

$$\psi_\beta^*(t) = \begin{cases} 1-t & \text{if} \quad t \in [0, (2\beta-1)/(3\beta-1)], \\ \dfrac{2\beta-1}{\beta}t + \dfrac{1-\beta}{\beta} & \text{if} \quad t \in [(2\beta-1)/(3\beta-1), 1]. \end{cases}$$

We note that $\psi_{\beta/(3\beta-1)}(t) = \psi_\beta^*(1-t)$ for each $t \in [0,1]$, which implies that

$$\|(x,y)\|_{\psi_{\beta/(3\beta-1)}} = \|(y,x)\|_{\psi_\beta^*}$$

for each (x,y), that is, the space $X_{\psi_{\beta/(3\beta-1)}}$ is isometrically isomorphic to the space $X_{\psi_\beta^*}$. Hence we have

$$J(X_{\psi_\beta^*}) = \begin{cases} \dfrac{1}{\beta} & \text{if} \quad \beta \in (1/2, 2/3], \\ \dfrac{2}{2-\beta} & \text{if} \quad \beta \in [2/3, 1). \end{cases}$$

Thus, consequently, we obtain $J(X_{\psi_\beta^*}) \neq J(X_{\psi_\beta})$ whenever $\beta \neq 2/3$.

Chapter 2

New geometric properties of Banach spaces

In the study of geometric properties of Banach spaces, norm inequalities often play fundamental roles. One of the most famous results in this direction is the uniform convexity of the L_p-spaces that is easily proved by Clarkson's inequality. Moreover, the notions of p-uniform smoothness and q-uniform convexity have important characterizations in terms of norm inequalities that are useful to prove some duality results.

This chapter is devoted to study new geometric properties of Banach spaces that generalize p-uniform smoothness and q-uniform convexity. The goal of the chapter is to present imitations of the basic results on p-uniform smoothness and q-uniform convexity including characterizations by norm inequalities and duality properties. For this purpose, so-called Beckner's inequality and its generalizations are also investigated. Then we try to formulate new geometric properties.

2.1 An elementary proof of Beckner's inequality

Beckner's inequality that was shown by Beckner in his 1975 paper [12] plays an important role in a characterization of p-uniform smoothness in terms of norm inequalities (cf. [49, Lemma 1.e.14]).

Theorem 2.1.1 (Beckner's inequality)**.** *Suppose that $1 < p \leq q < \infty$. Let $\gamma_{p,q} = \sqrt{(p-1)/(q-1)}$. Then the inequality*

$$\left(\frac{|u+\gamma_{p,q}v|^q + |u-\gamma_{p,q}v|^q}{2}\right)^{1/q} \leq \left(\frac{|u+v|^p + |u-v|^p}{2}\right)^{1/p}$$

holds for all $u, v \in \mathbb{R}$.

It is also known that the constant $\gamma_{p,q}$ in Beckner's inequality is the best possible choice, that is, if $\gamma \geq 0$ and the inequality

$$\left(\frac{|u+\gamma v|^q + |u-\gamma v|^q}{2}\right)^{1/q} \leq \left(\frac{|u+v|^p + |u-v|^p}{2}\right)^{1/p}$$

holds for all $u, v \in \mathbb{R}$, then one has $\gamma \leq \gamma_{p,q}$. We note that the case of $0 \leq \gamma \leq 1$ is essential in this direction. Indeed, letting $u = 0$ and $v = 1$ in the above inequality shows $\gamma \leq 1$. The proof of this fact can be essentially found in [88, Theorem 6].

We shall present an elementary proof of Theorem 2.1.1 and the above fact; see [83], and also [48, 62]. Naturally, the case of $p = q$ is trivial. Hence we assume that $1 < p < q < \infty$, and that $b \in [0, 1]$. Let A_δ be the linear operator from ℓ_p^2 into ℓ_q^2 defined by

$$A_\delta = \begin{pmatrix} 1 & \delta \\ \delta & 1 \end{pmatrix},$$

and let $\|A_\delta\|_{p,q}$ denote the operator norm of A_δ. Define the real-valued function $f_{p,q,\delta}$ on $[0, 1]$ by

$$f_{p,q,\delta}(t) = \left(\left(t^{1/p} + \delta(1-t)^{1/p}\right)^q + \left(\delta t^{1/p} + (1-t)^{1/p}\right)^q\right)^{1/q}.$$

Then we have the following two lemmas.

Lemma 2.1.2. $\|A_\delta\|_{p,q} = \max_{0 \leq t \leq 1/2} f_{p,q,\delta}(t)$.

Proof. Take an arbitrary $(u, v) \in \mathbb{R}^2$ with $\|(u, v)\|_p = 1$. It follows from $|v| = (1 - |u|^p)^{1/p}$ that

$$\begin{aligned}
\left\| \begin{pmatrix} 1 & \delta \\ \delta & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \right\|_q &= (|u + \delta v|^q + |\delta u + v|^q)^{1/q} \\
&\leq ((|u| + \delta|v|)^q + (\delta|u| + |v|)^q)^{1/q} \\
&= \left(\left(|u| + \delta(1 - |u|^p)^{1/p}\right)^q + \left(\delta|u| + (1 - |u|^p)^{1/p}\right)^q\right)^{1/q} \\
&= f_{p,q,\delta}(|u|^p) \\
&\leq \max_{0 \leq t \leq 1} f_{p,q,\delta}(t),
\end{aligned}$$

which implies that $\|A_\delta\|_{p,q} \leq \max_{0 \leq t \leq 1} f_{p,q,\delta}(t)$. On the other hand, putting $x_t = (t^{1/p}, (1-t)^{1/p})$ yields $\|A_\delta\|_{p,q} \geq \|A_\delta x_t\|_q = f_{p,q,\delta}(t)$ for each $t \in [0, 1]$. This proves $\|A_\delta\|_{p,q} = \max_{0 \leq t \leq 1} f_{p,q,\delta}(t)$. However, the fact that $f_{p,q,\delta}(1-t) = f_{p,q,\delta}(t)$ for each $t \in [0, 1]$ assures that $\max_{0 \leq t \leq 1} f_{p,q,\delta}(t) = \max_{0 \leq t \leq 1/2} f_{p,q,\delta}(t)$. This completes the proof. □

Lemma 2.1.3. *Let $\gamma \in [0,1]$, and let $\delta = (1-\gamma)/(1+\gamma)$. Then the following are equivalent:*

(i) *The inequality*

$$\left(\frac{|u+\gamma v|^q + |u-\gamma v|^q}{2}\right)^{1/q} \leq \left(\frac{|u+v|^p + |u-v|^p}{2}\right)^{1/q} \tag{2.1}$$

holds for all $u, v \in \mathbb{R}$.

(ii) $f_{p,q,\delta}(1/2) = \max_{0 \leq t \leq 1/2} f_{p,q,\delta}(t)$.

Proof. We first note that $f_{p,q,\delta}(1/2) = 2^{1/q-1/p}(1+\delta)$. Therefore, the condition (ii) just means that $\|A_\delta\|_{p,q} = 2^{1/q-1/p}(1+\delta)$ by Lemma 2.1.2. Suppose that (i) holds. Take arbitrary $u, v \in \mathbb{R}$. Applying (2.1) for $u_1 = (u+v)/2$ and $v_1 = (u-v)/2$, we have

$$\left(\frac{|(1+\gamma)u + (1-\gamma)v|^q + |(1-\gamma)u + (1+\gamma)v|^q}{2^{q+1}}\right)^{1/q} \leq \left(\frac{|u|^p + |v|^p}{2}\right)^{1/p},$$

or

$$\frac{1+\gamma}{2}\left(\frac{|u+\delta v|^q + |\delta u + v|^q}{2}\right)^{1/q} \leq \left(\frac{|u|^p + |v|^p}{2}\right)^{1/p}.$$

Since $(1+\gamma)(1+\delta) = 2$, it follows that

$$\|A_\delta(u,v)\|_q = (|u+\delta v|^q + |\delta u + v|^q)^{1/q} \leq 2^{1/q-1/p}(1+\delta)\|(u,v)\|_p,$$

which implies that $\|A_\delta\|_{p,q} = 2^{1/q-1/p}(1+\delta)$. Conversely, we assume that (ii) holds. Let $u, v \in \mathbb{R}$. Put $u_2 = u+v$ and $v_2 = u-v$, respectively. Then we obtain

$$\begin{aligned}
\left(\frac{|u+\gamma v|^q + |u-\gamma v|^q}{2}\right)^{1/q} &= \frac{1+\gamma}{2^{1/q+1}}\left(|u_2+\delta v_2|^q + |\delta u_2 + v_2|^q\right)^{1/q} \\
&= \frac{2^{-1/q}}{(1+\delta)}\|A_\delta(u_2,v_2)\|_q \\
&\leq \frac{2^{-1/q}}{(1+\delta)}\|A_\delta\|_{p,q}\|(u_2,v_2)\|_p \\
&= 2^{-1/p}\|(u_2,v_2)\|_p \\
&= \left(\frac{|u+v|^p + |u-v|^p}{2}\right)^{1/p}.
\end{aligned}$$

The proof is complete. $\square$

Now, let

$$\delta_{p,q} = \frac{1-\gamma_{p,q}}{1+\gamma_{p,q}} = \frac{\sqrt{q-1}-\sqrt{p-1}}{\sqrt{q-1}+\sqrt{p-1}} = \frac{p+q-2-2\sqrt{(p-1)(q-1)}}{q-p},$$

and let $\alpha = 1/p$ and $\beta = q-1$, respectively. We note that $0 < \alpha < 1$ and $\beta + 1 > \alpha\beta - \alpha + 1$.

We need the following three lemmas.

Lemma 2.1.4. *For each $\delta \in (0, \delta_{p,q}]$, let $g_{1,\delta}$ be the real-valued function on $[0,1]$ defined by*

$$g_{1,\delta}(u) = -\beta\delta u^2 + (\alpha\beta + \alpha - 1)(1+\delta^2)u - (2\alpha - 1)\beta\delta - 2(1-\alpha)\delta u^{1/(1-\alpha)}.$$

(i) *If $1 < p < 2$, then there exists a real number $u_0 \in (0,1)$ such that $g_{1,\delta}(u_0) = 0$, $g_{1,\delta}(u) < 0$ for all $u \in [0, u_0)$, and $g_{1,\delta}(u) > 0$ for all $u \in (u_0, 1)$.*

(ii) *If $2 \leq p < \infty$, then $g_{1,\delta}(u) > 0$ for all $u \in (0,1)$.*

Proof. Since $0 < \alpha < 1$, we have $1/(1-\alpha) > 1$, which shows that $g_{1,\delta}$ is strictly concave on $[0,1]$. We first note that

$$\begin{aligned} g_{1,\delta}(1) &= (\alpha\beta + \alpha - 1)(1+\delta^2) - 2(\alpha\beta - \alpha + 1)\delta \\ &= \frac{1}{p}\big((q-p)\delta^2 - 2(p+q-2)\delta + q - p\big) \geq 0. \end{aligned}$$

(i) Suppose that $1 < p < 2$. We claim that $g_{1,\delta}(u_1) > 0$ for some $u_1 \in (0,1]$. If $\delta < \delta_{p,q}$, then we have $g_{1,\delta}(1) > 0$. In the case of $\delta = \delta_{p,q}$, the derivative of $g_{1,\delta_{p,q}}$ is given by

$$g'_{1,\delta_{p,q}}(u) = -2\beta\delta_{p,q}u + (\alpha\beta + \alpha - 1)(1+\delta_{p,q}^2) - 2\delta_{p,q}u^{\alpha/(1-\alpha)}.$$

Since $\beta + 1 > \alpha\beta - \alpha + 1$, we have

$$\begin{aligned} g'_{1,\delta_{p,q}}(1) &= (\alpha\beta + \alpha - 1)(1+\delta_{p,q}^2) - 2(\beta+1)\delta_{p,q} \\ &< (\alpha\beta + \alpha - 1)(1+\delta_{p,q}^2) - 2(\alpha\beta - \alpha + 1)\delta_{p,q} = 0. \end{aligned}$$

Therefore $g_{1,\delta_{p,q}}$ is strictly decreasing on $(1-\varepsilon, 1]$ for some $\varepsilon > 0$, and so $g_{1,\delta_{p,q}}(1-\varepsilon) > g_{1,\delta_{p,q}}(1) = 0$. The claim is proved. Since $g_{1,\delta}(0) = -(2\alpha - 1)\beta\delta < 0$ for each $\delta \in (0, \delta_{p,q}]$, by the intermediate value theorem, there exists a real number $u_0 \in (0,1)$ such that $g_{1,\delta}(u_0) = 0$. By the strict concavity of $g_{1,\delta}$, we finally have (i).

(ii) If $2 \leq p < \infty$, then we have $g_{1,\delta}(0) = -(2\alpha - 1)\beta\delta \geq 0$. So we obtain $g_{1,\delta}(u) > 0$ for all $u \in (0,1)$ since $g_{1,\delta}$ is strictly concave. $\square$

Lemma 2.1.5. *For each $\delta \in (0, \delta_{p,q}]$, let $g_{2,\delta}$ be the real-valued function on $[0,1]$ defined by*

$$g_{2,\delta}(s) = (\alpha\beta + \alpha - 1)(1 + \delta^2)s^\alpha - \alpha\beta\delta(s^{2\alpha-1} + s) - (1 - \alpha)\delta(s^{2\alpha} + 1).$$

(i) $g_{2,\delta_{p,q}}(s) \leq 0$ *for all* $s \in [0,1]$.

(ii) *If* $0 < \delta < \delta_{p,q}$, *then there exists a real number* $s_0 \in (0,1)$ *such that* $g_{2,\delta}(s_0) = 0$, $g_{2,\delta}(s) < 0$ *for all* $s \in [0, s_0)$, *and* $g_{2,\delta}(s) > 0$ *for all* $s \in (s_0, 1)$.

Proof. The derivative of $g_{2,\delta}$ is

$$g'_{2,\delta}(s) = \alpha s^{2\alpha-2} g_{1,b}(s^{1-\alpha}).$$

We note that $g_{2,\delta}(0) = -(1-\alpha)\delta < 0$. So by Lemma 2.1.4, the behavior of $g_{2,\delta}$ is as follows: If $1 < p < 2$, putting $u_1 = u_0^{1/(1-\alpha)}$ yields that the function $g_{2,\delta}$ is strictly decreasing on $[0, u_1]$ and strictly increasing on $[u_1, 1]$.

s	0	$\cdots$	u_1	$\cdots$	1
$g'_{2,\delta}$		$-$	0	$+$	
$g_{2,\delta}$	$-$	$\searrow$		$\nearrow$	

If $2 \leq p < \infty$, then the function $g_{2,\delta}$ is strictly increasing on $[0,1]$.

s	0	$\cdots$	1
$g'_{2,\delta}$		$+$	
$g_{2,\delta}$	$-$	$\nearrow$	

On the other hand, we have

$$g_{2,\delta}(1) = (\alpha\beta + \alpha - 1)(1 + \delta^2) - 2(\alpha\beta - \alpha + 1)\delta.$$

Then it follows that $g_{2,\delta_{p,q}}(1) = 0$ and $g_{2,\delta}(1) > 0$ for all $\delta < \delta_{p,q}$. Thus one obtains $g_{2,\delta_{p,q}}(s) \leq 0$ for all $s \in [0,1]$. If $0 < \delta < \delta_{p,q}$, the intermediate value theorem guarantees that there exists a real number $s_0 \in (0,1)$ such that $g_{2,\delta}(s_0) = 0$, $g_{2,\delta}(s) < 0$ for all $s \in [0, s_0)$ and $g_{2,\delta}(s) > 0$ for all $s \in (s_0, 1)$, which shows (ii). $\square$

Lemma 2.1.6. *For each $\delta \in (0, \delta_{p,q}]$, let $g_{3,\delta}$ be the real-valued function on $[0,1]$ defined by*

$$g_{3,\delta}(s) = (s^\alpha + \delta)^\beta(s^{\alpha-1} - \delta) + (\delta s^\alpha + 1)^\beta(\delta s^{\alpha-1} - 1).$$

(i) $g_{3,\delta_{p,q}}(s) \geq 0$ *for all* $s \in [0,1]$.

(ii) *If $0 \leq \delta < \delta_{p,q}$, then there exists a real number $s_1 \in (0,1)$ such that $g_{3,\delta}(s_1) = 0$, $g_{3,\delta}(s) > 0$ for all $s \in [0, s_1)$, and $g_{3,\delta}(s) < 0$ for all $s \in (s_1, 1)$.*

Proof. We put $\delta_0 = \delta^{1/(1-\alpha)}$. Since $s^{\alpha-1} \geq 1$ for each $s \in [0,1]$, we have $s^{\alpha-1} - \delta > 0$. If $0 \leq s \leq \delta_0$, then $\delta s^{\alpha-1} - 1 \geq 0$, and hence $g_{3,\delta}(s) > 0$. Let $g_{4,\delta}$ be the real-valued function on $(\delta_0, 1]$ defined by

$$g_{4,\delta}(s) = \log(s^\alpha + \delta)^\beta (s^{\alpha-1} - \delta) - \log(\delta s^\alpha + 1)^\beta (1 - \delta s^{\alpha-1}).$$

Obviously, $g_{3,\delta}(s) \geq 0$ if and only if $g_{4,\delta}(s) \geq 0$. The derivative of $g_{4,\delta}$ is given by

$$g'_{4,\delta}(s) = \frac{(1-\delta^2)s^{\alpha-2} g_{2,\delta}(s)}{(s^\alpha + \delta)(s^{\alpha-1} - \delta)(\delta s^\alpha + 1)(1 - \delta s^{\alpha-1})}.$$

Thus one has $g'_{4,\delta_{p,q}}(s) \leq 0$ for all $s \in (\delta_0, 1]$ by Lemma 2.1.5 (i), that is, the function $g_{4,\delta_{p,q}}$ is decreasing on $(\delta_0, 1]$, which in turn implies that $g_{4,\delta_{p,q}}(s) \geq g_{4,\delta_{p,q}}(1) = 0$ for all $s \in (\delta_0, 1]$.

If $0 \leq \delta < \delta_{p,q}$, the behavior of $g_{4,\delta}$ is as follows by Lemma 2.1.5 (ii):

s	δ_0	$\cdots$	s_0	$\cdots$	1
$g'_{4,\delta}$		$-$	0	$+$	
$g_{4,\delta}$	∞	$\searrow$	$-$	$\nearrow$	0

Hence, using the intermediate value theorem, it turns out that there exists a real number $s_1 \in (0,1)$ such that $g_{3,\delta}(s_1) = 0$, $g_{4,\delta}(s) > 0$ for all $s \in [0, s_1)$ and $g_{4,\delta}(s) < 0$ for all $s \in (s_1, 1)$. $\square$

We now present an elementary proof. Suppose that $\delta \in [0, \delta_{p,q}]$. Let g_δ be the real-valued function on $[0, 1/2]$ defined by

$$g_\delta(t) = (f_{p,q,\delta}(t))^q = (t^{1/p} + \delta(1-t)^{1/p})^q + (\delta t^{1/p} + (1-t)^{1/p})^q.$$

If $\delta = 0$, then g_0 is clearly decreasing on $[0, 1/2]$, which shows that $f_{p,q,0}(1/2) < \max_{0 \leq t \leq 1/2} f_{p,q,0}(t)$. On the other hand, for each $\delta \in (0, \delta_{p,q}]$, the derivative of g_δ is

$$g'_\delta(t) = \frac{q}{p}(1-t)^{q/p-1} g_{3,\delta}\left(\frac{t}{1-t}\right).$$

By Lemma 2.1.6 (i), we have $g'_{\delta_{p,q}}(t) \geq 0$ for all $t \in [0, 1/2]$. Thus the function $g_{\delta_{p,q}}$ is nondecreasing on $[0, 1/2]$, that is, $g_{\delta_{p,q}}(1/2) = \max_{0 \leq t \leq 1/2} g_{\delta_{p,q}}(t)$. However, this means that $f_{p,q,\delta_{p,q}}(1/2) = \max_{0 \leq t \leq 1/2} f_{p,q,\delta_{p,q}}(t)$, which and Lemma 2.1.3 together show that the inequality

$$\left(\frac{|u + \gamma_{p,q} v|^q + |u - \gamma_{p,q} v|^q}{2}\right)^{1/q} \leq \left(\frac{|u + v|^p + |u - v|^p}{2}\right)^{1/p}$$

holds for all $u, v \in \mathbb{R}$. This proves Theorem 2.1.1.

Finally, we show that $\gamma_{p,q}$ is the best constant for Beckner's inequality. Suppose that $\gamma_{p,q} < \gamma \leq 1$. Let $\delta = (1-\gamma)/(1+\gamma)$. According to Lemma 2.1.3, it is enough to prove that $f_{p,q,\delta}(1/2) < \max_{0 \leq t \leq 1/2} f_{p,q,\delta}(t)$. In the case of $\gamma = 1$, we have $\delta = 0$. For $\gamma_{p,q} < \gamma < 1$, it follows that $0 < \delta < \delta_{p,q}$. Then Lemma 2.1.6 (ii) assures that the function g_δ is strictly increasing on $[0, s_2]$ and strictly decreasing on $[s_2, 1/2]$, where $s_2 = s_1/(1+s_1)$.

t	0	$\cdots$	s_2	$\cdots$	$1/2$
g'_δ		$+$	0	$-$	
g_δ		$\nearrow$		$\searrow$	

This shows that $f_{p,q,\delta}(1/2) < f_{p,q,\delta}(s_2) = \max_{0 \leq t \leq 1/2} f_{p,q,\delta}(t)$, as desired.

2.2 Generalized Beckner's inequalities

We next study generalizations of Beckner's inequality. For this purpose, we make use of symmetric absolute normalized norms on $\mathbb{R}^2$. Recall that the function ψ_p corresponding to the ℓ_p-norm on $\mathbb{R}^2$ is given by

$$\psi_p(t) = \begin{cases} ((1-t)^p + t^p)^{1/p} & \text{if} \quad 1 \leq p < \infty, \\ \max\{1-t, t\} & \text{if} \quad p = \infty. \end{cases}$$

Using the functions ψ_p and ψ_q, Beckner's inequality can be viewed as follows: Let $1 < p \leq q < \infty$, and let $\gamma_{p,q} = \sqrt{(p-1)/(q-1)}$. Then the inequality

$$\frac{\|(u + \gamma_{p,q} v, u - \gamma_{p,q} v)\|_q}{2\psi_q(1/2)} \leq \frac{\|(u+v, u-v)\|_p}{2\psi_p(1/2)}$$

holds for all $u, v \in \mathbb{R}$. Hence our problem is as follows: Let $\varphi, \psi \in \Psi_2^S$, and let $\gamma \in [0,1]$. When does the inequality

$$\frac{\|(u + \gamma v, u - \gamma v)\|_\varphi}{2\varphi(1/2)} \leq \frac{\|(u+v, u-v)\|_\psi}{2\psi(1/2)} \tag{2.2}$$

hold for all $u, v \in \mathbb{R}$? For each $\varphi, \psi \in \Psi_2^S$, let $\Gamma(\varphi, \psi)$ be the set of all $\gamma \in [0,1]$ such that the inequality (2.2) holds for all $u, v \in \mathbb{R}$. Needless to say, the inequality is trivial for $\gamma = 0$. Thus the main purpose of the section is to clarify the condition of $\Gamma(\varphi, \psi) \neq \{0\}$.

The following is an important characterization of absolute norms on $\mathbb{R}^2$. The proof can be found in [13, Proposition IV.1.1] (see, also [70, Lemma 4.1]).

Lemma 2.2.1. *A norm $\|\cdot\|$ on $\mathbb{R}^2$ is absolute if and only if it is monotone, that is, if $|x_1| \le |x_2|$ and $|y_1| \le |y_2|$ then $\|(x_1, y_1)\| \le \|(x_2, y_2)\|$.*

The first task is the following lemma.

Lemma 2.2.2 ([72])**.** *Let $\varphi, \psi \in \Psi_2^S$, and let $\gamma \in [0, 1]$. Then the following are equivalent:*

(i) *The inequality*
$$\frac{\|(u+\gamma v, u-\gamma v)\|_\varphi}{2\varphi(1/2)} \le \frac{\|(u+v, u-v)\|_\psi}{2\psi(1/2)}$$
holds for all $u, v \in \mathbb{R}$.

(ii) *The inequality*
$$\frac{\|(1+\gamma u, 1-\gamma u)\|_\varphi}{2\varphi(1/2)} \le \frac{\|(1+u, 1-u)\|_\psi}{2\psi(1/2)}$$
holds for all $u \in [0, 1]$.

Proof. It is enough to show that (ii) $\Rightarrow$ (i). Suppose that (ii) holds. We first take an arbitrary $u > 1$. Then $|1 \pm \gamma u| \le |u \pm \gamma|$, which and Lemma 2.2.1 imply that

$$\begin{aligned}\frac{\|(1+\gamma u, 1-\gamma u)\|_\varphi}{2\varphi(1/2)} &\le \frac{\|(u+\gamma, u-\gamma)\|_\varphi}{2\varphi(1/2)} \\ &= \frac{u\,\|(1+\gamma u^{-1}, 1-\gamma u^{-1})\|_\varphi}{2\varphi(1/2)} \\ &\le \frac{u\,\|(1+u^{-1}, 1-u^{-1})\|_\psi}{2\psi(1/2)} \\ &= \frac{\|(1+u, 1-u)\|_\psi}{2\psi(1/2)}.\end{aligned}$$

Now let $u \le 0$. Since $\varphi, \psi \in \Psi_2^S$, the assumption and the above inequality show that

$$\begin{aligned}\frac{\|(1+\gamma u, 1-\gamma u)\|_\varphi}{2\varphi(1/2)} &= \frac{\|(1-\gamma u, 1+\gamma u)\|_\varphi}{2\varphi(1/2)} \\ &\le \frac{\|(1-u, 1+u)\|_\psi}{2\psi(1/2)} \\ &= \frac{\|(1+u, 1-u)\|_\psi}{2\psi(1/2)}.\end{aligned}$$

Thus the inequality

$$\frac{\|(1+\gamma u, 1-\gamma u)\|_\varphi}{2\varphi(1/2)} \le \frac{\|(1+u, 1-u)\|_\psi}{2\psi(1/2)}$$

holds for all $u \in \mathbb{R}$. Finally, take arbitrary $u, v \in \mathbb{R}$. If $u = 0$, we have

$$\frac{\|(\gamma v, -\gamma v)\|_\varphi}{2\varphi(1/2)} = \gamma|v| \leq |v| = \frac{\|(v, -v)\|_\psi}{2\psi(1/2)}.$$

So we assume that $u \neq 0$. Then

$$\begin{aligned}
\frac{\|(u + \gamma v, u - \gamma v)\|_\varphi}{2\varphi(1/2)} &= \frac{|u|\,\|(1 + \gamma u^{-1} v, 1 - \gamma u^{-1} v)\|_\varphi}{2\varphi(1/2)} \\
&\leq \frac{|u|\,\|(1 + u^{-1} v, 1 - u^{-1} v)\|_\psi}{2\psi(1/2)} \\
&= \frac{\|(u + v, u - v)\|_\psi}{2\psi(1/2)}.
\end{aligned}$$

This completes the proof. $\square$

We remark that the condition (ii) in the preceding lemma is equivalent to the following statement: The inequality

$$\frac{\varphi((1 - \gamma u)/2)}{\psi((1 - u)/2)} \leq \frac{\varphi(1/2)}{\psi(1/2)}$$

holds for all $u \in [0, 1]$. Thus it follows that

$$\Gamma(\varphi, \psi) = \left\{ \gamma \in [0, 1] : \frac{\varphi((1 - \gamma u)/2)}{\psi((1 - u)/2)} \leq \frac{\varphi(1/2)}{\psi(1/2)} \text{ for all } u \in [0, 1] \right\}.$$

for all $\varphi, \psi \in \Psi_2^S$. Moreover, since the function

$$[0, 1] \ni \gamma \mapsto \frac{\varphi((1 - \gamma u)/2)}{\psi((1 - u)/2)}$$

is continuous and convex for each fixed $u \in [0, 1]$, the set $\Gamma(\varphi, \psi)$ is closed and convex. This means that $\Gamma(\varphi, \psi)$ is a subinterval of $[0, 1]$. Let $\gamma_{\varphi,\psi} = \max \Gamma(\varphi, \psi)$. Then $\gamma_{\varphi,\psi}$ is the best constant for the inequality.

In what follows, we study some conditions for $\gamma_{\varphi,\psi} > 0$. The following is the simplest result in this direction.

Proposition 2.2.3 ([72])**.** *Let $\varphi, \psi \in \Psi_2^S$. Suppose that $\varphi(t) = \varphi(1/2)$ on $[\delta, 1 - \delta]$ for some $0 \leq \delta < 1/2$. Then $\gamma_{\varphi,\psi} > 0$.*

Proof. Let $\gamma = 1 - 2\delta > 0$. Then we have

$$\frac{1}{2} \geq \frac{1 - \gamma u}{2} \geq \frac{1 - \gamma}{2} = \delta$$

for all $u \in [0,1]$, which implies that

$$\frac{\varphi((1-\gamma u)/2)}{\psi((1-u)/2)} = \frac{\varphi(1/2)}{\psi((1-u)/2)} \leq \frac{\varphi(1/2)}{\psi(1/2)}.$$

Thus $\gamma \in \Gamma(\varphi, \psi)$, and so $\gamma_{\varphi,\psi} \geq \gamma > 0$. $\square$

For each $\psi \in \Psi_2^S$, we remark that $\psi'_L(1/2) \leq 0$, and that $\psi'_L(1/2) = 0$ if and only if ψ is differentiable at $1/2$, where ψ'_L denotes the left derivative of ψ. For two functions $\varphi, \psi \in \Psi_2^S$, we consider the following four cases:

(I) $\varphi'_L(1/2) = 0$ and $\psi'_L(1/2) = 0$.

(II) $\varphi'_L(1/2) = 0$ and $\psi'_L(1/2) < 0$.

(III) $\varphi'_L(1/2) < 0$ and $\psi'_L(1/2) = 0$.

(IV) $\varphi'_L(1/2) < 0$ and $\psi'_L(1/2) < 0$.

We first present the following result concerning cases (II), (III) and (IV).

Theorem 2.2.4 ([72]). *Let $\varphi, \psi \in \Psi_2^S$.*

(i) *If $\varphi'_L(1/2) = 0$ and $\psi'_L(1/2) < 0$, then $\gamma_{\varphi,\psi} > 0$.*

(ii) *If $\varphi'_L(1/2) < 0$ and $\psi'_L(1/2) = 0$, then $\gamma_{\varphi,\psi} = 0$.*

(iii) *If $\varphi'_L(1/2) < 0$ and $\psi'_L(1/2) < 0$, then $\gamma_{\varphi,\psi} > 0$.*

In particular, if $\varphi'_L(1/2) < 0$ then

$$\gamma_{\varphi,\psi} \leq \frac{\varphi(1/2)\psi'_L(1/2)}{\psi(1/2)\varphi'_L(1/2)}.$$

Proof. (i) We first remark that

$$\psi(t) \geq \psi(1/2) - \psi'_L(1/2)\left(\frac{1}{2} - t\right)$$

for all $t \in [0, 1/2]$. Since $\varphi'(1/2) = 0$, there exists $t_0 \in [0, 1/2)$ such that

$$\frac{\varphi(1/2) - \varphi(t)}{1/2 - t} \geq \frac{\psi'_L(1/2)}{2},$$

or

$$\varphi(t) \leq \varphi(1/2) - \frac{\psi'_L(1/2)}{2}\left(\frac{1}{2} - t\right)$$

for all $t \in [t_0, 1/2]$. Putting $\gamma = 1 - 2t_0 > 0$, we have

$$\frac{1}{2} \geq \frac{1-\gamma u}{2} \geq \frac{1-\gamma}{2} = t_0$$

for each $u \in [0,1]$. Hence, by an easy calculation, it follows that

$$\frac{\varphi((1-\gamma u)/2)}{\psi((1-u)/2)} \leq \frac{\varphi(1/2) - \psi'_L(1/2)\gamma u/4}{\psi(1/2) - \psi'_L(1/2)u/2} \leq \frac{\varphi(1/2) - \psi'_L(1/2)u/4}{\psi(1/2) - \psi'_L(1/2)u/2} \leq \frac{\varphi(1/2)}{\psi(1/2)}.$$

This means that $\gamma \in \Gamma(\varphi, \psi)$. Thus we obtain $\gamma_{\varphi_\psi} \geq \gamma > 0$.

(ii) and (iii): Assume that $\varphi'_L(1/2) < 0$. Put

$$k_0 = \frac{\varphi(1/2)\psi'_L(1/2)}{\psi(1/2)\varphi'_L(1/2)}.$$

We first show that the inequality $\gamma_{\varphi,\psi} \leq k_0$ holds. Suppose that $k_0 < 1$, and that $k_0 < \gamma \leq 1$. Since

$$\frac{\gamma\psi(1/2)\varphi'_L(1/2)}{\varphi(1/2)} < \psi'_L(1/2),$$

there exists $t_0 \in [0, 1/2)$ such that

$$\frac{\psi(1/2) - \psi(t)}{1/2 - t} > \frac{\gamma\psi(1/2)\varphi'_L(1/2)}{\varphi(1/2)},$$

for all $t \in [t_0, 1/2)$, or

$$\psi(t) < \psi(1/2) - \frac{\gamma\psi(1/2))\varphi'_L(1/2)}{\varphi(1/2)} \left(\frac{1}{2} - t\right)$$

for all $t \in [t_0, 1/2)$. On the other hand, since

$$\varphi(t) \geq \varphi(1/2) - \varphi'_L(1/2) \left(\frac{1}{2} - t\right)$$

for each $t \in [0, 1/2]$, putting $u_0 = 1 - 2t_0$ shows that

$$\begin{aligned}\psi((1-u_0)/2) = \psi(t_0) &< \frac{\psi(1/2)}{\varphi(1/2)} \left(\varphi(1/2) - \gamma\varphi'_L(1/2)\left(\frac{1}{2} - t_0\right)\right) \\ &= \frac{\psi(1/2)}{\varphi(1/2)} \left(\varphi(1/2) - \frac{\gamma u_0 \varphi'_L(1/2)}{2},\right),\end{aligned}$$

and that

$$\varphi((1-\gamma u_0)/2) \geq \varphi(1/2) - \frac{\gamma u_0 \varphi'_L(1/2)}{2}.$$

These proves that

$$\frac{\varphi((1-\gamma u_0)/2)}{\psi((1-u_0)/2)} > \frac{\varphi(1/2)}{\psi(1/2)},$$

that is, $\gamma \notin \Gamma(\varphi, \psi)$. Thus $\gamma_{\varphi,\psi} \le k_0$. The statement (ii) immediately follows from this.

Finally, we shall show (iii). Suppose that $\varphi'_L(1/2) < 0$, and that $\psi'_L(1/2) < 0$. Then we have $k_0 > 0$. Take an arbitrary γ satisfying $0 < \gamma < \min\{k_0, 1\}$. Since

$$\varphi'_L(1/2) > \frac{\varphi(1/2)\psi'_L(1/2)}{\gamma\psi(1/2)},$$

there exists $t_0 \in [0, 1/2)$ such that if $t \in [t_0, 1/2)$ then

$$\frac{\varphi(1/2) - \varphi(t)}{1/2 - t} \ge \frac{\varphi(1/2)\psi'_L(1/2)}{\gamma\psi(1/2)}.$$

Namely, for each $t \in [t_0, 1/2)$, one has

$$\varphi(t) \le \varphi(1/2) - \frac{\varphi(1/2)\psi'_L(1/2)}{\gamma\psi(1/2)}\left(\frac{1}{2} - t\right).$$

We also note that

$$\psi(t) \ge \psi(1/2) - \psi'_L(1/2)\left(\frac{1}{2} - t\right)$$

for all $t \in [0, 1/2]$. Now putting $\gamma_0 = \min\{1 - 2t_0, \gamma\} > 0$, we have

$$\frac{1}{2} \ge \frac{1 - \gamma_0 u}{2} \ge \frac{1 - \gamma_0}{2} \ge t_0$$

for all $u \in [0, 1]$, which implies that

$$\begin{aligned}
\varphi((1 - \gamma_0 u)/2) &\le \varphi(1/2) - \frac{\gamma_0 u \varphi(1/2)\psi'_L(1/2)}{2\gamma\psi(1/2)} \\
&= \frac{\varphi(1/2)}{\psi(1/2)}\left(\psi(1/2) - \frac{\gamma_0 u}{2\gamma}\psi'_L(1/2)\right) \\
&\le \frac{\varphi(1/2)}{\psi(1/2)}\left(\psi(1/2) - \frac{u}{2}\psi'_L(1/2)\right).
\end{aligned}$$

Then, it follows from the inequality

$$\psi((1 - u)/2) \ge \psi(1/2) - \frac{u\psi'_L(1/2)}{2}$$

that

$$\frac{\varphi((1 - \gamma_0 u)/2)}{\psi((1 - u)/2)} \le \frac{\varphi(1/2)}{\psi(1/2)}.$$

for all $u \in [0, 1]$. This shows that $\gamma_0 \in \Gamma(\varphi, \psi)$, and so we have $\gamma_{\varphi,\psi} \ge \gamma_0 > 0$. $\square$

The following is an application of the preceding theorem.

Example 2.2.5 ([72])**.** For each $\alpha \in (1/2, 1)$, let ψ_α be an element of Ψ_2^S defined by

$$\psi_\alpha(t) = \begin{cases} 1 + 2(\alpha - 1)t & \text{if } t \in [0, 1/2], \\ 2\alpha - 1 + 2(1-\alpha)t & \text{if } t \in [1/2, 1]. \end{cases}$$

Suppose that $\alpha, \beta \in (1/2, 1)$, and that $\alpha \leq \beta$. Then

$$k_0 = \frac{\psi_\alpha(1/2)(\psi_\beta)'_L(1/2)}{\psi_\beta(1/2)(\psi_\alpha)'_L(1/2)} = \frac{\alpha(1-\beta)}{\beta(1-\alpha)} \leq 1.$$

On the other hand, for each $u \in [0, 1]$, we have

$$\begin{aligned} \psi_\alpha((1 - k_0 u)/2) &= 1 + (\alpha - 1)(1 - k_0 u) \\ &= \alpha - (\alpha - 1)k_0 u \\ &= \alpha - \frac{\alpha(1-\beta)}{\beta} u \\ &= \frac{\alpha}{\beta}(\beta - (1-\beta)u) \\ &= \frac{\alpha}{\beta}(1 + (\beta - 1)(1 - u)) \\ &= \frac{\alpha}{\beta}\psi_\beta((1-u)/2). \end{aligned}$$

Thus $k_0 \in \Gamma(\psi_\alpha, \psi_\beta)$, which and Theorem 2.2.4 together show that

$$\gamma_{\psi_\alpha, \psi_\beta} = k_0 = \frac{\alpha(1-\beta)}{\beta(1-\alpha)}.$$

Theorem 2.2.4 clarifies whether $\gamma_{\varphi,\psi} > 0$ in the cases (II), (III) and (IV). However, we have had no information about (I) yet. Therefore we next consider several special subcases of (I). Let $\varphi, \psi \in \Psi_2^S$. Suppose that the second derivatives φ'' and ψ'' are continuous on $(\delta, 1-\delta)$ for some $0 \leq \delta < 1/2$. Then we remark that $\varphi''(1/2) \geq 0$ and $\psi''(1/2) \geq 0$ by the convexity. This allows us to consider the following four subcases of (I):

(I-a) $\varphi''(1/2) = 0$ and $\psi''(1/2) = 0$.

(I-b) $\varphi''(1/2) = 0$ and $\psi''(1/2) > 0$.

(I-c) $\varphi''(1/2) > 0$ and $\psi''(1/2) = 0$.

(I-d) $\varphi''(1/2) > 0$ and $\psi''(1/2) > 0$.

Here we do not consider the case (I-a) because of its complexity. For the cases (I-b), (I-c) and (I-d), we have the following result.

Theorem 2.2.6 ([72])**.** *Let $\varphi, \psi \in \Psi_2^S$. Suppose that the second derivatives φ'' and ψ'' are continuous on $(\delta, 1-\delta)$ for some $0 \leq \delta < 1/2$.*

(i) *If $\varphi''(1/2) = 0$ and $\psi''(1/2) > 0$, then $\gamma_{\varphi,\psi} > 0$.*

(ii) *If $\varphi''(1/2) > 0$ and $\psi''(1/2) = 0$, then $\gamma_{\varphi,\psi} = 0$.*

(iii) *If $\varphi''(1/2) > 0$ and $\psi''(1/2) > 0$, then $\gamma_{\varphi,\psi} > 0$.*

In particular, if $\varphi''(1/2) > 0$ then

$$\gamma_{\varphi,\psi} \leq \sqrt{\frac{\varphi(1/2)\psi''(1/2)}{\psi(1/2)\varphi''(1/2)}}.$$

Proof. (i) For each $\gamma \in (0,1]$, define the function $f_\gamma : [0, 1-2\delta) \to \mathbb{R}$ by the formula

$$f_\gamma(u) = \frac{\psi((1-u)/2)}{\psi(1/2)} - \frac{\varphi((1-\gamma u)/2)}{\varphi(1/2)}.$$

Then, the first and second derivative of f_γ are as follows:

$$\begin{aligned} f'_\gamma(u) &= \frac{1}{2}\left(\frac{\gamma\varphi'((1-\gamma u)/2)}{\varphi(1/2)} - \frac{\psi'((1-u)/2)}{\psi(1/2)}\right), \\ f''_\gamma(u) &= \frac{1}{4}\left(\frac{\psi''((1-u)/2)}{\psi(1/2)} - \frac{\gamma^2\varphi''((1-\gamma u)/2)}{\varphi(1/2)}\right). \end{aligned}$$

So we have $f'_a(0) = 0$ and

$$f''_a(0) = \frac{1}{4}\left(\frac{\psi''(1/2)}{\psi(1/2)} - \frac{\gamma^2\varphi''(1/2)}{\varphi(1/2)}\right) = \frac{\psi''(1/2)}{4\psi(1/2)} > 0.$$

From these facts, the function f_a is nonnegative on the interval $[0, u_0]$ for some $u_0 \in (0,1]$. Let $\gamma_0 = \gamma u_0 > 0$. Take an arbitrary $u \in [0,1]$ and put $v = u_0 u$. Then $0 \leq v \leq \min\{u_0, u\}$, and so

$$\frac{1-u}{2} \leq \frac{1-v}{2} \leq \frac{1}{2},$$

which implies that $\psi((1-u)/2) \geq \psi((1-v)/2)$. Hence it follows that

$$\begin{aligned} f_{\gamma_0}(u) &= \frac{\psi((1-u)/2)}{\psi(1/2)} - \frac{\varphi((1-\gamma_0 u)/2)}{\varphi(1/2)} \\ &\geq \frac{\psi((1-v)/2)}{\psi(1/2)} - \frac{\varphi((1-\gamma v)/2)}{\varphi(1/2)} \\ &= f_\gamma(v) \geq 0. \end{aligned}$$

This shows $\gamma_0 \in \Gamma(\varphi, \psi)$, and hence $\gamma_{\varphi,\psi} \geq \gamma_0 > 0$.

We next suppose that $\varphi''(1/2) > 0$. Put

$$k_0 = \sqrt{\frac{\varphi(1/2)\psi''(1/2)}{\psi(1/2)\varphi''(1/2)}}.$$

Then, as in the proof of (i), we have $f_a''(0) < 0$ for each $\gamma > k_0$. It follows that $f_\gamma(u_0) < 0$ for some $u_0 \in (0, 1 - 2\delta)$. This means that $\gamma \notin \Gamma(\varphi, \psi)$, which shows $\gamma_{\varphi,\psi} \leq k_0$. We note that this also proves (ii). In the case of (iii), one has $k_0 > 0$. Moreover, for each γ with $0 < \gamma < \min\{1, k_0\}$, we obtain $f_\gamma''(0) > 0$. Hence the function f_γ is nonnegative on some nontrivial interval $[0, u_0]$. We finally get $\gamma_{\varphi,\psi} > 0$ by an argument similar to that in the first paragraph. This completes the proof. $\square$

Remark 2.2.7. We remark that

$$\sqrt{\frac{\psi_q(1/2)\psi_p''(1/2)}{\psi_p(1/2)\psi_q''(1/2)}} = \sqrt{\frac{p-1}{q-1}} = \gamma_{p,q},$$

where $\gamma_{p,q}$ is the best constant for Beckner's inequality. This gives another aspect of the constant $\gamma_{p,q}$.

We next consider the duality of generalized Beckner's inequalities. The following lemma which is analogous to Lemma 2.1.3 will be needed.

Lemma 2.2.8 ([72])**.** *Suppose that $\varphi, \psi \in \Psi_2^S$. For each $\gamma \in [0, 1]$, let*

$$A_\gamma = \begin{pmatrix} 1+\gamma & 1-\gamma \\ 1-\gamma & 1+\gamma \end{pmatrix}.$$

Then $\gamma \in \Gamma(\varphi, \psi)$ if and only if

$$\big\| A_\gamma : (\mathbb{R}^2, \|\cdot\|_\psi) \to (\mathbb{R}^2, \|\cdot\|_\varphi) \big\| \leq \frac{2\varphi(1/2)}{\psi(1/2)}.$$

Proof. Let $\gamma \in \Gamma(\varphi, \psi)$. Then, by Lemma 2.2.2,

$$\frac{\|(u+\gamma v, u-\gamma v)\|_\varphi}{2\varphi(1/2)} \leq \frac{\|(u+v, u-v)\|_\psi}{2\psi(1/2)}$$

for all $u, v \in \mathbb{R}$. Take arbitrary $u, v \in \mathbb{R}$. Applying the inequality for $u + v$ and $u - v$, we obtain

$$\frac{\|((1+\gamma)u + (1-\gamma)v, (1-\gamma)u + (1+\gamma)v)\|_\varphi}{2\varphi(1/2)} \leq \frac{\|(2u, 2v)\|_\psi}{2\psi(1/2)},$$

or

$$\|A_\gamma(u,v)\|_\varphi \le \frac{2\varphi(1/2)}{\psi(1/2)}\|(u,v)\|_\psi.$$

Thus we have

$$\big\|A : (\mathbb{R}^2, \|\cdot\|_\psi) \to (\mathbb{R}^2, \|\cdot\|_\varphi)\big\| \le \frac{2\varphi(1/2)}{\psi(1/2)}.$$

Conversely, suppose that

$$\big\|A_\gamma : (\mathbb{R}^2, \|\cdot\|_\psi) \to (\mathbb{R}^2, \|\cdot\|_\varphi)\big\| \le \frac{2\varphi(1/2)}{\psi(1/2)}.$$

Let $u, v \in \mathbb{R}$. Putting $u_1 = (u+v)/2$ and $v_1 = (u-v)/2$, we have

$$\begin{aligned}
\|(u+\gamma v, u-\gamma v)\|_\varphi &= \|((1+\gamma)u_1 + (1-\gamma)v_1, (1-\gamma)u_1 + (1+\gamma)v_1)\|_\varphi \\
&= \|A_\gamma(u_1, v_1)\|_\varphi \\
&\le \frac{2\varphi(1/2)}{\psi(1/2)}\|(u_1,v_1)\|_\psi \\
&= \frac{\varphi(1/2)}{\psi(1/2)}\|(u+v, u-v)\|_\psi.
\end{aligned}$$

Hence one has $\gamma \in \Gamma(\varphi, \psi)$. □

We now present the following duality result.

Theorem 2.2.9 ([72]). *Let $\varphi, \psi \in \Psi_2^S$. Then $\gamma_{\varphi,\psi} = \gamma_{\psi^*,\varphi^*}$.*

Proof. Since $\min_{0\le t\le 1}\varphi(t) = \varphi(1/2)$, it follows that

$$\varphi^*(1/2) = \sup_{0\le t\le 1}\frac{(1-t)/2 + t/2}{\varphi(t)} = \frac{1}{2\varphi(1/2)}.$$

We similarly have $\psi^*(1/2) = 1/2\psi(1/2)$, which implies that

$$\frac{\psi^*(1/2)}{\varphi^*(1/2)} = \frac{\varphi(1/2)}{\psi(1/2)}.$$

Now for each $\gamma \in [0,1]$, define the matrix A_γ as in Lemma 2.2.8. We remark that $A_\gamma^* = A_\gamma$, where A_γ^* is the adjoint operator of A_γ. Then Lemma 2.2.8 assures that $\gamma \in \Gamma(\varphi,\psi)$ if and only if

$$\big\|A_\gamma : (\mathbb{R}^2, \|\cdot\|_\psi) \to (\mathbb{R}^2, \|\cdot\|_\varphi)\big\| \le \frac{2\varphi(1/2)}{\psi(1/2)},$$

which happens if and only if

$$\big\|A_\gamma : (\mathbb{R}^2, \|\cdot\|_{\varphi^*}) \to (\mathbb{R}^2, \|\cdot\|_{\psi^*})\big\| \le \frac{2\psi^*(1/2)}{\varphi^*(1/2)}.$$

The last statement is equivalent to $\gamma \in \Gamma(\psi^*, \varphi^*)$. Thus we obtain $\Gamma(\varphi,\psi) = \Gamma(\psi^*,\varphi^*)$, which proves $\gamma_{\varphi,\psi} = \gamma_{\psi^*,\varphi^*}$. □

Finally, we give the norm version of generalized Beckner's inequalities.

Theorem 2.2.10 ([72])**.** *Let X be a Banach space. Suppose that $\varphi, \psi \in \Psi_2^S$, and that $\gamma \in \Gamma(\varphi, \psi)$. Then the inequality*

$$\frac{\|(x+\gamma y, x-\gamma y)\|_\varphi}{2\varphi(1/2)} \le \frac{\|(x+y, x-y)\|_\psi}{2\psi(1/2)}$$

holds for all $x, y \in X$.

Proof. Take arbitrary $x, y \in X$, and put $z = x + y$ and $w = x - y$, respectively. We also put $u = (\|z\| + \|w\|)/2$ and $v = (\|z\| - \|w\|)/2$. Then we have

$$\begin{aligned}
&\frac{\|(x+\gamma y, x-\gamma y)\|_\varphi}{2\varphi(1/2)} \\
&= \frac{\|(2^{-1}(1+\gamma)z + 2^{-1}(1-\gamma)w, 2^{-1}(1-\gamma)z + 2^{-1}(1+\gamma)w)\|_\varphi}{2\varphi(1/2)} \\
&\le \frac{\|(2^{-1}(1+\gamma)\|z\| + 2^{-1}(1-\gamma)\|w\|, 2^{-1}(1-\gamma)\|z\| + 2^{-1}(1+\gamma)\|w\|)\|_\varphi}{2\varphi(1/2)} \\
&= \frac{\|(u+\gamma v, u-\gamma v)\|_\varphi}{2\varphi(1/2)} \\
&\le \frac{\|(u+v, u-v)\|_\varphi}{2\psi(1/2)} \\
&= \frac{\|(x+y, x-y)\|_\psi}{2\psi(1/2)}.
\end{aligned}$$

The proof is complete. □

2.3 ψ-uniform smoothness

It is the time to introduce new geometric properties of Banach spaces that generalize the notion of p-uniform smoothness. We start this section with the definition of ψ-uniform smoothness.

Definition 2.3.1 ([73])**.** Let $\psi \in \Psi_2$. Then a Banach space X is said to be *ψ-uniformly smooth* if there exists $M > 0$ such that $\rho_X(\tau) \le \|(1, M\tau)\|_\psi - 1$ for each $\tau \in [0, 1]$.

The above definition can be weakened as follows.

Lemma 2.3.2 ([73])**.** *Let $\psi \in \Psi_2$. Then a Banach space X is ψ-uniformly smooth if and only if there exist $M > 0$ and $\delta \in (0, 1]$ such that $\rho_X(\tau) \le \|(1, M\tau)\|_\psi - 1$ for each $\tau \in [0, \delta]$.*

Proof. Suppose that there exist $M > 0$ and $\delta \in (0, 1]$ such that $\rho_X(\tau) \leq \|(1, M\tau)\|_\psi - 1$ for each $\tau \in [0, \delta]$. Let

$$M_0 = \max\left\{M, \frac{\delta + 1}{\delta}\right\}.$$

Then for each $\tau \in [\delta, 1]$ we obtain

$$\rho_X(\tau) \leq \tau \leq M_0\tau - 1 \leq \|(1, M_0\tau)\|_\psi - 1.$$

This and Lemma 2.2.1 together show that $\rho_X(\tau) \leq \|(1, M_0\tau)\|_\psi - 1$ for each $\tau \in [0, 1]$. $\square$

As will be seen in the following lemma, p-uniform smoothness is equivalent to ψ_p-uniform smoothness, and hence the notion of ψ-uniform smoothness is a natural generalization of that of p-uniform smoothness. Recall that a Banach space X is said to be *p-uniformly smooth* if there exists $K > 0$ such that $\rho_X(\tau) \leq K\tau^p$ for all $\tau \geq 0$.

Proposition 2.3.3 ([73])**.** *Let X be a Banach space, and let $1 < p \leq 2$. Then X is p-uniformly smooth if and only if it is ψ_p-uniformly smooth.*

Proof. Suppose that X is p-uniformly smooth. Then there exists a $K > 0$ satisfying $\rho_X(\tau) \leq K\tau^p$ for each $\tau > 0$. Since the function f on $[0, 1]$ given by

$$f(\tau) = 1 + pK(1 + K)^{p-1}\tau^p - (1 + K\tau^p)^p$$

is nondecreasing, it follows that $f \geq 0$. Putting $M = p^{1/p}K^{1/p}(1 + K)^{1-1/p}$ we have

$$\begin{aligned}\rho_X(\tau) &\leq 1 + K\tau^p - 1\\ &\leq (1 + pK(1 + K)^{p-1}\tau^p)^{1/p} - 1\\ &= \|(1, M\tau)\|_p - 1\end{aligned}$$

for each $\tau \in [0, 1]$. This shows that X is ψ_p-uniformly smooth.

Conversely, assume that X is ψ_p-uniformly smooth. Let M be a positive real number such that

$$\rho_X(\tau) \leq \|(1, M\tau)\|_p - 1$$

for each $\tau \in [0, 1]$. Then for each $\tau \in [0, 1]$ one has

$$\rho_X(\tau) \leq \|(1, M\tau)\|_p - 1 = (1 + M^p\tau^p)^{1/p} - 1 \leq 1 + \frac{1}{p}M^p\tau^p - 1 = \frac{1}{p}M^p\tau^p.$$

On the other hand, if $\tau \geq 1$ then $\rho_X(\tau) \leq \tau \leq \tau^p$. Hence we obtain

$$\rho_X(\tau) \leq \max\{M^p/p, 1\}\tau^p$$

for each $\tau \geq 0$, that is, the space X is p-uniformly smooth. $\square$

For each $\psi \in \Psi_2$, let ψ'_R denote the right derivative of ψ. The relationship between uniform smoothness and ψ-uniform smoothness is as follows.

Proposition 2.3.4 ([73]). *Suppose that $\psi \in \Psi_2$ and that $\psi'_R(0) = -1$. Then every ψ-uniformly smooth Banach space is uniformly smooth.*

Proof. Let X be a ψ-uniformly smooth Banach space. Then there exists $M > 0$ such that $\rho_X(\tau) \leq \|(1, M\tau)\|_\psi - 1$ for each $\tau \in [0, 1]$. Since the function

$$\tau \mapsto \frac{M\tau}{1 + M\tau}$$

is increasing on $\mathbb{R}^+$, it follows that

$$\begin{aligned}
\lim_{\tau \to 0^+} \frac{\psi(M\tau/(1+M\tau)) - 1}{\tau} &= \lim_{\tau \to 0^+} \frac{\psi(M\tau/(1+M\tau)) - 1}{M\tau/(1+M\tau)} \cdot \frac{M\tau/(1+M\tau)}{\tau} \\
&= M\psi'_R(0) = -M.
\end{aligned}$$

From this one has

$$\begin{aligned}
\lim_{\tau \to 0^+} \frac{\rho_X(\tau)}{\tau} &\leq \lim_{\tau \to 0^+} \frac{\|(1, M\tau)\|_\psi - 1}{\tau} \\
&= \lim_{\tau \to 0^+} \frac{(1+M\tau)\psi(M\tau/(1+M\tau)) - 1}{\tau} \\
&= \lim_{\tau \to 0^+} \frac{M\tau\psi(M\tau/(1+M\tau)) + \psi(M\tau/(1+M\tau)) - 1}{\tau} \\
&= \lim_{\tau \to 0^+} M\psi(M\tau/(1+M\tau)) + \lim_{\tau \to 0^+} \frac{\psi(M\tau/(1+M\tau)) - 1}{\tau} \\
&= M - M = 0.
\end{aligned}$$

Thus X is uniformly smooth. $\square$

We now introduce a technical condition on elements in Ψ_2. A function $\psi \in \Psi_2$ is said to have the property $(*)$ if there exists $M > 0$ satisfying

$$\|(1, \tau)\|_\psi + \tau^2 \leq \|(1, M\tau)\|_\psi$$

for each $\tau \in [0, 1]$. We can prove that the function ψ_p has the property $(*)$ for each $1 \leq p \leq 2$. Indeed, putting $M = (1/p + 1)^{1/p}$ yields

$$\|(1, \tau)\|_p + \tau^2 = (1 + \tau^p)^{1/p} + \tau^2 \leq 1 + \frac{1}{p}\tau^p + \tau^p = 1 + M^p\tau^p.$$

Now we choose a positive number K such that $K^p > \max\{pM^p, (1+M^p)^p - 1\}$. Then the function

$$f(\tau) = \|(1, K\tau)\|_p - (1 + M^p\tau^p)$$

is non-negative on $[0,1]$. Hence it follows that

$$\|(1,\tau)\|_p + \tau^2 \leq 1 + M^p\tau^p \leq \|(1, K\tau)\|_p$$

for each $\tau \in [0,1]$.

The following gives a useful characterization of the property $(*)$.

Lemma 2.3.5 ([73]). *Let $\psi \in \Psi_2$. Then the following are equivalent.*

(i) *The function ψ has the property $(*)$.*

(ii) *For each $K > 0$ and each $\alpha > 0$ there exists $M > 0$ such that*

$$\|(1, K\tau)\|_\psi + \alpha\tau^2 \leq \|(1, M\tau)\|_\psi$$

for each $\tau \in [0,1]$.

(iii) *There exist $M > 0$ and $\delta \in (0,1]$ such that*

$$\|(1,\tau)\|_\psi + \tau^2 \leq \|(1, M\tau)\|_\psi$$

for each $\tau \in [0,\delta]$.

Proof. Suppose that four positive real numbers K, α, δ and M satisfy the inequality

$$\|(1, K\tau)\|_\psi + \alpha\tau^2 \leq \|(1, M\tau)\|_\psi$$

for each $\tau \in [0,\delta]$. Letting

$$M_0 = \max\left\{M, \max_{\tau\in[\delta,1]} \frac{\|(1,K\tau)\|_\psi + \alpha\tau^2}{\tau}\right\}$$

we have

$$\|(1, K\tau)\|_\psi + \alpha\tau^2 \leq \|(1, M_0\tau)\|_\psi$$

for each $\tau \in [0,1]$. This shows (i) $\Leftrightarrow$ (iii).

Now suppose that $K > 0$ and $\alpha > 0$. Put $\beta = \max\{K, \sqrt{\alpha}\}$. Then for each $\tau \in [0, \beta^{-1}]$ one has

$$\begin{aligned}\|(1, K\tau)\|_\psi + \alpha\tau^2 &\leq \|(1, \beta\tau)\|_\psi + \beta^2\tau^2 \\ &\leq \|(1, M\beta\tau)\|_\psi,\end{aligned}$$

which and the above argument together show that (i) $\Leftrightarrow$ (ii). $\square$

The property $(*)$ plays an important role in our argument.

Proposition 2.3.6 ([73]). *Let $\psi \in \Psi_2$ with the property $(*)$. Then every 2-uniformly smooth Banach space is ψ-uniformly smooth.*

Proof. Suppose that a Banach space X is 2-uniformly smooth. Then there exists $K > 0$ such that $\rho_X(\tau) \leq K\tau^2$ for each $\tau \geq 0$. Since ψ has the property $(*)$, the preceding lemma assures that there exists $M > 0$ such that

$$\begin{aligned}\rho_X(\tau) &\leq 1 + K\tau^2 - 1 \\ &\leq \|(1,\tau)\|_\psi + K\tau^2 - 1 \\ &\leq \|(1, M\tau)\|_\psi - 1\end{aligned}$$

for each $\tau \in [0,1]$. Hence X is ψ-uniformly smooth. $\square$

We shall characterize ψ-uniform smoothness in terms of norm inequalities. The following lemma will be needed.

Lemma 2.3.7 ([73]). *Let $\psi \in \Psi_2^S$. Suppose that $M \geq 1$. Then $\|(1, M\tau)\|_\psi \leq \|(M,\tau)\|_\psi$ for each $\tau \in [0,1]$.*

Proof. The case of $\tau = 0$ or 1 is clear. For each $\tau \in (0,1)$, we have

$$\frac{M}{M+\tau} = \frac{M\tau}{M\tau+\tau^2} \geq \frac{M\tau}{1+M\tau} \geq \frac{M\tau}{M^2+M\tau} = \frac{\tau}{M+\tau}.$$

Then it follows that

$$\psi\left(\frac{M\tau}{1+M\tau}\right) \leq \max\left\{\psi\left(\frac{M}{M+\tau}\right), \psi\left(\frac{\tau}{M+\tau}\right)\right\} = \psi\left(\frac{\tau}{M+\tau}\right).$$

On the other hand, since $M \geq 1$, we obtain $M + \tau \geq 1 + M\tau$ for each $\tau \in (0,1)$, which implies that

$$\begin{aligned}\|(M,\tau)\|_\psi &= (M+\tau)\psi\left(\frac{\tau}{M+\tau}\right) \\ &\geq (1+M\tau)\psi\left(\frac{M\tau}{1+M\tau}\right) \\ &= \|(1, M\tau)\|_\psi\end{aligned}$$

for each $\tau \in (0,1)$. $\square$

Theorem 2.3.8 ([73]). *Let X be a Banach space and $\psi \in \Psi_2^S$ with the property $(*)$. Suppose that $\gamma_{\psi,\psi_2} > 0$. Then the following are equivalent*

(i) *The space X is ψ-uniformly smooth.*

(ii) *There exists $M > 0$ such that*

$$\frac{\|(x+y, x-y)\|_\psi}{2\psi(1/2)} \leq \|(x, My)\|_\psi$$

for each $x, y \in X$.

(iii) *For any $\varphi \in \Psi_2^S$ with $\gamma_{\varphi,\psi} > 0$, there exists an $M_\varphi > 0$ such that*

$$\frac{\|(x+y, x-y)\|_\varphi}{2\varphi(1/2)} \leq \|(x, M_\varphi y)\|_\psi$$

for each $x, y \in X$.

(iv) *For some $\varphi \in \Psi_2^S$ with $\gamma_{\varphi,\psi} > 0$, there exists an $M_\varphi > 0$ such that*

$$\frac{\|(x+y, x-y)\|_\varphi}{2\varphi(1/2)} \leq \|(x, M_\varphi y)\|_\psi$$

for each $x, y \in X$.

Proof. Let $\gamma = \gamma_{\psi,\psi_2}$ and

$$F(x,y) = \max\left\{\frac{\|(x+\gamma y, x-\gamma y)\|_\psi}{2\psi(1/2)}, \frac{\|(\gamma x+y, \gamma x-y)\|_\psi}{2\psi(1/2)}\right\}$$

for each $x, y \in X$, for short. Notice that

$$F(x,y) \leq \frac{\|(x+y, x-y)\|_2}{2\psi_2(1/2)}$$

for each $x, y \in X$.

(i) $\Rightarrow$ (ii) Suppose that X is ψ-uniformly smooth. Then there exists a $K > 0$ such that $\rho_X(\tau) \leq \|(1, K\tau)\|_\psi - 1$ for each $\tau \in [0,1]$. Let $x \in S_X$ and $y \in B_X \setminus \{0\}$. It follows from

$$\frac{\|x+y\| + \|x-y\|}{2} - 1 \leq \rho_X(\|y\|) \leq \|(1, K\|y\|)\|_\psi - 1$$

that

$$\frac{\|x+y\| + \|x-y\|}{2} \leq \|(1, K\|y\|)\|_\psi.$$

Putting

$$\alpha = \frac{\|x+y\| + \|x-y\|}{2} \quad \text{and} \quad \alpha\beta = \frac{\|x+y\| - \|x-y\|}{2},$$

we have $\|x+y\|=\alpha+\alpha\beta$ and $\|x-y\|=\alpha-\alpha\beta$. Since $\gamma_{\psi,\psi_2}>0$, one has

$$\begin{aligned}
F(x,y)-\|(1,K\|y\|)\|_\psi &\le \frac{\|(x+y,x-y)\|_2}{2\psi_2(1/2)}-\frac{\|x+y\|+\|x-y\|}{2}\\
&=\frac{\|(\alpha+\alpha\beta,\alpha-\alpha\beta)\|_2}{\sqrt{2}}-\alpha\\
&=\frac{\alpha\|(1+\beta,1-\beta)\|_2}{\sqrt{2}}-\alpha\\
&=\alpha\left(\left(\frac{(1+\beta)^2+(1-\beta)^2}{2}\right)^{1/2}-1\right)\\
&=\alpha(\sqrt{1+\beta^2}-1)\\
&\le \alpha(1+\beta^2-1)\\
&=\alpha\beta^2.
\end{aligned}$$

From the facts that

$$\alpha=\frac{\|x+y\|+\|x-y\|}{2}\ge\frac{\|x+y+x-y\|}{2}=\|x\|=1$$

and

$$(\alpha\beta)^2=\left(\frac{\|x+y\|-\|x-y\|}{2}\right)^2\le\left(\frac{\|x+y-(x-y)\|}{2}\right)^2=\|y\|^2,$$

it follows that

$$\begin{aligned}
F(x,y) &\le \|(1,K\|y\|)\|_\psi+\alpha\beta^2\\
&\le \|(1,K\|y\|)\|_\psi+(\alpha\beta)^2\\
&\le \|(1,K\|y\|)\|_\psi+\|y\|^2\\
&\le \|(1,M\|y\|)\|_\psi\\
&=\|(x,My)\|_\psi
\end{aligned}$$

for some $M>0$ by the property $(*)$. Clearly, it may be assumed that $M\ge 1$.

Now suppose that $x,y\in X\setminus\{0\}$. If $\|x\|\ge\|y\|$ then, as was shown in the above, one has

$$\frac{\|(x+\gamma y,x-\gamma y)\|_\psi}{2\psi(1/2)}\le F(x,y)\le\|(x,My)\|_\psi.$$

On the other hand, if $\|x\|<\|y\|$ then the above inequality and Lemma 2.3.7 guarantee that $F(x,y)\le\|(y,Mx)\|_\psi\le\|(My,x)\|_\psi$, or

$$\frac{\|(x+\gamma y,x-\gamma y)\|_\psi}{2\psi(1/2)}\le\|(x,My)\|_\psi.$$

Thus we finally have
$$\frac{\|(x+y,x-y)\|_\psi}{2\psi(1/2)} \leq \|(x, M\gamma^{-1}y)\|_\psi$$
for each $x, y \in X$, as desired.

(ii) $\Rightarrow$ (iii) Suppose that $\gamma_{\varphi,\psi} > 0$. Then, by Theorem 2.2.10, we have
$$\begin{aligned}\frac{\|(x+\gamma_{\varphi,\psi}y, x-\gamma_{\varphi,\psi}y)\|_\varphi}{2\varphi(1/2)} &\leq \frac{\|(x+y,x-y)\|_\psi}{2\psi(1/2)} \\ &\leq \|(x, My)\|_\psi\end{aligned}$$
for each $x, y \in X$. Putting $M_\varphi = M\gamma_{\varphi,\psi}^{-1} > 0$ and replacing y with $\gamma_{\varphi,\psi}^{-1}y$, we obtain
$$\frac{\|(x+y,x-y)\|_\varphi}{2\varphi(1/2)} \leq \|(x, M_\varphi y)\|_\psi$$
for each $x, y \in X$.

(iii) $\Rightarrow$ (iv) Obvious.

(iv) $\Rightarrow$ (i) Suppose that $\psi \in \Psi_2^S$ and $M_\varphi > 0$ satisfy $\gamma_{\varphi,\psi} > 0$ and
$$\frac{\|(x+y,x-y)\|_\varphi}{2\varphi(1/2)} \leq \|(x, M_\varphi y)\|_\psi$$
for each $x, y \in X$. By [69, Lemma 3], one has
$$\begin{aligned}\frac{\|x+y\|+\|x-y\|}{2} = \frac{\|(\|x+y\|,\|x-y\|)\|_1}{2} &\leq \frac{\|(\|x+y\|,\|x-y\|)\|_\varphi}{2\varphi(1/2)} \\ = \frac{\|(x+y,x-y)\|_\varphi}{2\varphi(1/2)} &\leq \|(x, M_\varphi y)\|_\psi\end{aligned}$$
for each $x, y \in X$. Now let $x, y \in S_X$ and $\tau \in (0, 1]$. Then we obtain
$$\frac{\|x+\tau y\|+\|x-\tau y\|}{2} - 1 \leq \|(1, M_\varphi \tau)\|_\psi - 1,$$
which implies that $\rho_X(\tau) \leq \|(1, M_\varphi\tau)\|_\psi - 1$. Therefore the space X is ψ-uniformly smooth. $\square$

We conclude this section with some examples of functions $\psi \in \Psi_2^S$ that satisfy the assumption of the preceding theorem.

Example 2.3.9. Let $1 \leq p \leq 2$. As was mentioned preceding Lemma 2.3.5, the function ψ_p has the property $(*)$. Moreover, it is easy to check that $\gamma_{\psi_p,\psi_2} = 1$.

Example 2.3.10. Suppose that $0 < \omega 1 < p, q < \infty$, and that with $1/p + 1/q = 1$. Recall that the two-dimensional Lorentz sequence space $d^{(2)}(\omega, q)$ is the space $\mathbb{R}^2$ endowed with the norm

$$\|(x, y)\|_{\omega,q} = (\max\{|x|^q, |y|^q\} + \omega \min\{|x|^q, |y|^q\})^{1/q}.$$

We note $\| \cdot \|_{\omega,q} \in AN_2$ and the function $\psi_{\omega,q}$ associated with this norm is given by

$$\psi_{\omega,q}(t) = \begin{cases} ((1-t)^q + \omega t^q)^{1/q} & \text{if } 0 \le t \le 1/2, \\ (t^q + \omega(1-t)^q)^{1/q} & \text{if } 1/2 \le t \le 1. \end{cases}$$

As in [56], the function $\psi^*_{\omega,q}$ associated with the norm of $d^{(2)}(\omega, q)^*$ is given by

$$\psi^*_{\omega,q}(t) = \begin{cases} ((1-t)^p + \omega^{1-p} t^p)^{1/p} & \text{if } 0 \le t \le \omega/(1+\omega), \\ (1+\omega)^{1/p-1} & \text{if } \omega/(1+\omega) \le t \le 1/(1+\omega), \\ (t^p + \omega^{1-p}(1-t)^p)^{1/p} & \text{if } 1/(1+\omega) \le t \le 1. \end{cases}$$

Then $\psi^*_{\omega,q}$ satisfies the assumption of Theorem 2.3.8 for each $2 \le q < \infty$ and each $0 < \omega < 1$. Indeed, we have $\gamma_{\psi^*_{\omega,q},\psi_2} > 0$ by Proposition 2.2.3. Moreover, for each $\tau \in [0, \omega]$ we obtain

$$\begin{aligned} \|(1, \tau)\|^*_{\omega,q} + \tau^2 &= (1 + \omega^{1-p}\tau^p)^{1/p} + \tau^2 \\ &\le 1 + \frac{1}{p}\omega^{1-p}\tau^p + \tau^p \\ &\le 1 + \omega^{1-p}\left(\frac{1}{p} + \omega^{p-1}\right)\tau^p. \end{aligned}$$

As in the proof of Lemma 2.3.3, it follows that

$$1 + K_0\tau^p \le (1 + pK_0(1 + K_0)^{p-1}\tau^p)^{1/p},$$

where $K_0 = \omega^{1-p}(p^{-1} + \omega^{p-1})$. Hence, for $M = \max\{1, (pK_0(1 + K_0)^{p-1}\omega^{1-p})^{1/p}\}$ and each $\tau \in [0, M^{-1}\omega]$, one has

$$\|(1, \tau)\|^*_{\omega,q} + \tau^2 \le (1 + \omega^{1-p}M^p\tau^p)^{1/p} = \|(1, M\tau)\|^*_{\omega,q}.$$

This together with Lemma 2.3.5 ensures that $\psi^*_{\omega,q}$ has the property $(*)$.

We remark that the set Ψ_2 can be viewed as the subset of $C[0, 1]$, the Banach space of all continuous functions on $[0, 1]$ equipped with the uniform norm. The following result shows the elements of Ψ_2 that satisfy the assumption of Theorem 2.3.8 are rich.

Theorem 2.3.11 ([73]). *Let D be the set of all functions $\psi \in \Psi_2^S$ that have the property $(*)$ and satisfy $\gamma_{\psi,\psi_2} > 0$. Then D is dense in Ψ_2^S.*

Proof. We first note that $\emptyset \neq D$ since $\psi_p \in D \cap \Psi_2^S$ for each $1 \leq p \leq 2$. Let $\psi \in \Psi_2^S \setminus \{\psi_1\}$ and $\psi_0 \in D$. For each $\lambda \in (0, 1]$, let $\varphi_\lambda = (1-\lambda)\psi + \lambda\psi_0 \in \Psi_2^S$. Since ψ_0 has the property $(*)$, Lemma 2.3.5 guarantees that there exists $M > 0$ satisfying

$$\begin{aligned}\|(1,\tau)\|_{\varphi_\lambda} + \tau^2 &= (1-\lambda)\|(1,\tau)\|_\psi + \lambda\|(1,\tau)\|_{\psi_0} + \tau^2 \\ &= (1-\lambda)\|(1,\tau)\|_\psi + \lambda(\|(1,\tau)\|_{\psi_0} + \lambda^{-1}\tau^2) \\ &\leq (1-\lambda)\|(1,\tau)\|_\psi + \lambda\|(1,M\tau)\|_{\psi_0}\end{aligned}$$

for each $\tau \in [0, 1]$. Letting $M_0 = \max\{M, 1\}$ yields

$$\|(1,\tau)\|_{\varphi_\lambda} + \tau^2 \leq \|(1, M_0\tau)\|_{\varphi_\lambda}$$

for each $\tau \in [0, 1]$. Hence the function φ_λ has the property $(*)$ for each $\lambda \in (0, 1]$.

Next, for each $\lambda \in (0, 1]$ and each $\mu \in (\varphi_\lambda(1/2), 1)$, let $\varphi_{\lambda,\mu} = \max\{\varphi_\lambda, \mu\}$. Remark that $\varphi_{\lambda,\mu} \neq \psi_1$. Then there exists $\delta \in (0, 1/2)$ such that $\varphi_{\lambda,\mu} = \varphi_\lambda$ on $[0, \delta] \cup [1-\delta, 1]$ and $\varphi_{\lambda,\mu} = \mu$ on $[\delta, 1-\delta]$. From this we have $\gamma_{\varphi_{\lambda,\mu},\psi_2} > 0$ by Proposition 2.2.3. Moreover, for each $\tau \in [0, \delta/(1-\delta)M_0]$ it follows from $\tau/(1+\tau) \leq M_0\tau/(1+M_0\,\tau) \leq \delta$ that

$$\begin{aligned}\|(1,\tau)\|_{\varphi_{\lambda,\mu}} + \tau^2 &= \|(1,\tau)\|_{\varphi_\lambda} + \tau^2 \\ &\leq \|(1, M_0\tau)\|_{\varphi_\lambda} \\ &= \|(1, M_0\tau)\|_{\varphi_{\lambda,\mu}},\end{aligned}$$

which and Lemma 2.3.5 together show that $\varphi_{\lambda,\mu}$ also has the property $(*)$, that is, $\varphi_{\lambda,\mu} \in D$.

Finally, we note that

$$\|\psi - \varphi_{\lambda,\mu}\|_\infty \leq \|\psi - \varphi_\lambda\|_\infty + \|\varphi_\lambda - \varphi_{\lambda,\mu}\|_\infty = \lambda\|\psi - \psi_0\|_\infty + \mu - \varphi_\lambda(1/2)$$

for each $\lambda \in (0, 1]$ and each $\mu \in (\varphi_\lambda(1/2), 1)$. This proves that D is dense in Ψ_2^S. $\square$

2.4 ψ^*-uniform convexity and duality

In this section, we consider ψ^*-uniform convexity of Banach spaces. As in the case of ψ-uniform smoothness, some characterizations using norm inequalities will be given. As an application it is shown that ψ-uniform smoothness and ψ^*-uniform convexity are the dual properties of each other.

Definition 2.4.1 ([73])**.** Let $\psi \in \Psi_2$. Then a Banach space X is said to be *ψ-uniformly convex* if there exists $K > 0$ such that $\|(1-\delta_X(\varepsilon), K\varepsilon)\|_\psi \le 1$ for each $\varepsilon \in [0,2]$.

The following result shows that the preceding definition is also a natural generalization of the notion of q-uniform convexity. Recall that a Banach space X is said to be *q-uniformly convex* if there exists $C > 0$ such that $\delta_X(\varepsilon) \ge C\varepsilon^q$ for each $\varepsilon \in [0,2]$. Remark that for $1 \le p \le q \le \infty$ with $p^{-1} + q^{-1} = 1$ we have $\psi_q^* = \psi_p$.

Proposition 2.4.2 ([73])**.** *Let $2 \le q < \infty$. Then a Banach space X is q-uniformly convex if and only if it is ψ_q-uniformly convex.*

Proof. Suppose that X is q-uniformly convex. Then there exists $C > 0$ such that $\delta_X(\varepsilon) \ge C\varepsilon^q$ for each $\varepsilon \in [0,2]$. One can easily check that

$$(1-x)^q \le 1 - \frac{x}{2}$$

for each $x \in [0,1]$. Hence, by $0 \le C\varepsilon^q \le \delta_X(\varepsilon) \le 1$, we have

$$(1-\delta_X(\varepsilon))^q \le (1-C\varepsilon^q)^q \le 1 - \frac{C\varepsilon^q}{2}.$$

Putting $K = (C/2)^{1/q}$, we obtain $\|(1-\delta_X(\varepsilon), K\varepsilon)\|_q = (1-\delta_X(\varepsilon))^q + K^q\varepsilon^q \le 1$ for each $\varepsilon \in [0,2]$.

Conversely, assume that there exists $K > 0$ such that $\|(1-\delta_X(\varepsilon), K\varepsilon)\|_q \le 1$ for each $\varepsilon \in [0,2]$. Then $(1-\delta_X(\varepsilon))^q \le 1 - K^q\varepsilon^q$, and so

$$1-\delta_X(\varepsilon) \le (1-K^q\varepsilon^q)^{1/q} \le 1 - \frac{1}{q}K^q\varepsilon^q.$$

Thus, for $C = K^q/q$, we have $\delta_X(\varepsilon) \ge C\varepsilon^q$ for each $\varepsilon \in [0,2]$. This shows X is q-uniformly convex. □

The following proposition shows a basic duality between the functions ψ and ψ^*. We remark that $\psi^{**} = \psi$.

Proposition 2.4.3 ([73])**.** *Let $\psi \in \Psi_2$. Then $\psi'_R(0) = -1$ if and only if $\psi^*(t) > 1-t$ for each $t \in (0,1/2]$.*

Proof. Let $\alpha_0 = \psi'_R(0) > -1$. It follows from $\alpha_0 \le (\psi(t)-1)/t$ that $\psi(t) \ge 1 + \alpha_0 t$ for each $t \in [0,1]$, which implies that

$$\psi^*(t) = \max_{0 \le s \le 1} \frac{(1-s)(1-t)+st}{\psi(s)} \le \max_{0 \le s \le 1} \frac{(1-s)(1-t)+st}{1+\alpha_0 s}$$

for each $t \in [0,1]$. However, the function

$$s \to \frac{(1-s)(1-t)+st}{1+\alpha_0 s}$$

is decreasing on $[0,1]$ if $t \le (1+\alpha_0)/(2+\alpha_0)$, and hence one obtains $\psi^*(t) = 1-t$ on $[0,(1+\alpha_0)/(2+\alpha_0)]$.

Conversely, we assume that $\psi^*(t) = 1-t$ on $[0,\delta]$ for some $0 < \delta \le 1/2$. Then, for each $t \in [0,1]$, we have

$$\begin{aligned}
\psi(t) &= \sup_{0\le s\le 1} \frac{(1-s)(1-t)+st}{\psi^*(s)} \\
&\ge \sup_{0\le s\le \delta} \frac{(1-s)(1-t)+st}{\psi^*(s)} \\
&= \sup_{0\le s\le \delta} \left(1-t+\frac{s}{1-s}t\right) \\
&= 1-t+\frac{\delta}{1-\delta}t \\
&= 1-\frac{1-2\delta}{1-\delta}t.
\end{aligned}$$

This shows that

$$\psi_R'(0) = \lim_{t\to 0^+} \frac{\psi(t)-1}{t} \ge -\frac{1-2\delta}{1-\delta} > -1,$$

which completes the proof. $\square$

This duality provides the following natural implication.

Proposition 2.4.4 ([73]). *Suppose that $\psi \in \Psi_2$ and that $\psi_R'(0) = -1$. Then every ψ^*-uniformly convex Banach space is uniformly convex.*

Proof. Suppose that a Banach space X is ψ^*-uniformly convex. Then there exists $K > 0$ such that $\|(1-\delta_X(\varepsilon), K\varepsilon)\|_{\psi^*} \le 1$ for each $\varepsilon \in [0,2]$. Take an arbitrary $\varepsilon \in (0,2]$, and put

$$t_\varepsilon = \frac{K\varepsilon}{1-\delta_X(\varepsilon)+K\varepsilon}.$$

Since $\psi(t_\varepsilon) > 1-t_\varepsilon$ by the preceding proposition, it follows that

$$\frac{1-\delta_X(\varepsilon)}{1-\delta_X(\varepsilon)+K\varepsilon} = 1-t_\varepsilon < \psi^*(t_\varepsilon) = \psi^*\left(\frac{K\varepsilon}{1-\delta_X(\varepsilon)+K\varepsilon}\right),$$

or $1-\delta_X(\varepsilon) < \|(1-\delta_X(\varepsilon), K\varepsilon)\|_{\psi^*} \le 1$, which in turn implies that $\delta_X(\varepsilon) > 0$. The proof is complete. $\square$

The notion of ψ^*-uniform convexity also has a characterization using norm inequalities. To see this, we need the following two lemmas.

Lemma 2.4.5 ([73])**.** *Let X be a Banach space. Suppose that $\psi \in \Psi_2^S$, and that $K > 0$. Then the following are equivalent.*

(i) *The inequality*

$$\left\| \left(1 - \delta_X(\varepsilon), \frac{K\varepsilon}{2}\right) \right\|_\psi \leq 1.$$

holds for each $\varepsilon \in [0, 2]$.

(ii) *The inequality*

$$\left\| \left(\frac{1}{2}(x+y), \frac{K}{2}(x-y)\right) \right\|_\psi \leq 1.$$

holds for each $x, y \in X$.

Proof. Suppose that (i) holds. Take arbitrary $x, y \in B_X$. Put $\|x - y\| = \varepsilon$. Then, we have

$$\left\| \frac{1}{2}(x+y) \right\| \leq 1 - \delta_X(\varepsilon),$$

and so

$$\left\| \left(\frac{1}{2}(x+y), \frac{K}{2}(x-y)\right) \right\|_\psi \leq \left\| \left(1 - \delta_X(\varepsilon), \frac{K\varepsilon}{2}\right) \right\|_\psi \leq 1.$$

Assume conversely that (ii) holds. Let $\varepsilon \in [0, 2]$. Then for any $\delta > 0$ there exist $x, y \in B_X$ such that $\|x - y\| = \varepsilon$ and

$$1 - \left\| \frac{x+y}{2} \right\| < \delta_X(\varepsilon) + \delta,$$

which implies that

$$1 - \delta_X(\varepsilon) - \delta < \left\| \frac{1}{2}(x+y) \right\|,$$

and so

$$\left\| \left(1 - \delta_X(\varepsilon) - \delta, \frac{K\varepsilon}{2}\right) \right\|_\psi \leq \left\| \left(\frac{1}{2}(x+y), \frac{K}{2}(x-y)\right) \right\|_\psi \leq 1.$$

Hence we have

$$\left\| \left(1 - \delta_X(\varepsilon), \frac{K\varepsilon}{2}\right) \right\|_\psi \leq 1.$$

as $\delta \to 0$. $\square$

Lemma 2.4.6 (Generalized Hölder's inequality [13, 54])**.** *Let $\psi \in \Psi_2$. Then $|\langle x, y \rangle| \leq \|x\|_\psi \|y\|_{\psi^*}$ for each $x, y \in \mathbb{R}^2$.*

To show the characterization, we also make use of the following duality between two norm inequalities concerning with the pairs (φ, ψ) and (φ^*, ψ^*) will be needed.

Lemma 2.4.7 ([73])**.** *Let $\varphi, \psi \in \Psi_2^S$ and $K > 0$. Then the following are equivalent.*

(i) *The inequality*

$$\frac{\|(x+y, x-y)\|_\varphi}{2\varphi(1/2)} \leq \|(x, Ky)\|_\psi.$$

holds for each $x, y \in X$.

(ii) *The inequality*

$$\frac{\|(f+g, f-g)\|_{\varphi^*}}{2\varphi^*(1/2)} \geq \|(f, K^{-1}g)\|_{\psi^*}.$$

holds for each $f, g \in X^*$.

The equivalence remains true even if X is replaced with X^*.

Proof. Suppose that (i) holds. Put $\gamma = K^{-1}$. Replacing y with γy, we obtain

$$\frac{\|(x+\gamma y, x-\gamma y)\|_\varphi}{2\varphi(1/2)} \leq \|(x, y)\|_\psi$$

for each $x, y \in X$. Let

$$A = \begin{pmatrix} 1 & \gamma \\ 1 & -\gamma \end{pmatrix}.$$

Then the above inequality means that

$$\left\| A \begin{pmatrix} x \\ y \end{pmatrix} \right\|_\varphi \leq 2\varphi(1/2) \left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\|_\psi.$$

Hence we have $\|A : X \oplus_\psi X \to X \oplus_\varphi X\| \leq 2\varphi(1/2)$. From this, we also have

$$\|A^* : X^* \oplus_{\varphi^*} X^* \to X^* \oplus_{\psi^*} X^*\| \leq 2\varphi(1/2) = \frac{1}{\varphi^*(1/2)}.$$

Remark that

$$A^* = \begin{pmatrix} 1 & 1 \\ \gamma & -\gamma \end{pmatrix}.$$

It follows that

$$\|(h+k, \gamma(h-k))\|_{\psi^*} = \left\| A^* \begin{pmatrix} h \\ k \end{pmatrix} \right\|_{\psi^*} \leq \frac{\|(h,k)\|_{\varphi^*}}{\varphi^*(1/2)}$$

for each $h, k \in X^*$. Now suppose that $f, g \in X^*$. Putting $h = f + g$ and $k = f - g$ we obtain

$$\|(2f, 2\gamma g)\|_{\psi^*} \leq \frac{\|(f+g, f-g)\|_{\varphi^*}}{\varphi^*(1/2)},$$

and so

$$\|(f, K^{-1}g)\|_{\psi^*} \leq \frac{\|(f+g, f-g)\|_{\varphi^*}}{2\varphi^*(1/2)}.$$

Conversely, assume that (ii) holds. Then one has

$$\|A^* : X^* \oplus_{\varphi^*} X^* \to X^* \oplus_{\psi^*} X^*\| \leq \frac{1}{\varphi^*(1/2)} = 2\varphi(1/2),$$

or $\|A : X \oplus_\psi X \to X \oplus_\varphi X\| \leq 2\varphi(1/2)$. It follows that

$$\|(x + \gamma y, x - \gamma y)\|_\varphi = \left\| A \begin{pmatrix} x \\ y \end{pmatrix} \right\|_\varphi \leq 2\varphi(1/2)\|(x, y)\|_\psi.$$

Finally, replacing y with Ky, we obtain

$$\frac{\|(x+y, x-y)\|_\varphi}{2\varphi(1/2)} \leq \|(x, Ky)\|_\psi$$

for all $x, y \in X$.

The same argument is still valid even if X is replaced with X^*. □

We now present the characterization.

Theorem 2.4.8 ([73])**.** *Let X be a Banach space and $\psi \in \Psi_2^S$ with the property $(*)$. Suppose that $\gamma_{\psi,\psi_2} > 0$. Then the following are equivalent*

(i) *The space X is ψ^*-uniformly convex.*

(ii) *There exists $M > 0$ such that*

$$\frac{\|(x+y, x-y)\|_{\psi^*}}{2\psi(1/2)} \geq \|(x, My)\|_{\psi^*}$$

for each $x, y \in X$.

(iii) *For any $\varphi \in \Psi_2^S$ with $\gamma_{\varphi,\psi} > 0$, there exists an $M_\varphi > 0$ such that*

$$\frac{\|(x+y, x-y)\|_{\varphi^*}}{2\varphi^*(1/2)} \geq \|(x, M_\varphi y)\|_{\psi^*}$$

for each $x, y \in X$.

(iv) *For some* $\varphi \in \Psi_2^S$ *with* $\gamma_{\varphi,\psi} > 0$, *there exists an* $M_\varphi > 0$ *such that*

$$\frac{\|(x+y, x-y)\|_{\varphi^*}}{2\varphi^*(1/2)} \geq \|(x, M_\varphi y)\|_{\psi^*}$$

for each $x, y \in X$.

Proof. (i) $\Rightarrow$ (ii) Suppose that X is ψ^*-uniformly convex. Then there exists $K > 0$ such that $\|(1-\delta_X(\varepsilon), K\varepsilon)\|_{\psi^*} \leq 1$ for each $\varepsilon \in [0,2]$. Let $\tau \in (0,1]$ and $f, g \in S_{X^*}$. For any $\varepsilon > 0$ there exist $x, y \in B_X$ such that $(f+\tau g)(x), (f-\tau g)(y) \geq 0$, $\|f+\tau g\| < (f+\tau g)(x) + \varepsilon$ and $\|f - \tau g\| < (f-\tau g)(y) + \varepsilon$. By Lemmas 2.4.5 and 2.4.6, we have

$$\begin{aligned}
\frac{\|f+\tau g\| + \|f-\tau g\|}{2} - 1 &< \frac{(f+\tau g)(x) + (f-\tau g)(y)}{2} - 1 + \varepsilon \\
&= \left|\frac{(f+\tau g)(x) + (f-\tau g)(y)}{2}\right| - 1 + \varepsilon \\
&= \left|\frac{f(x+y) + \tau g(x-y)}{2}\right| - 1 + \varepsilon \\
&\leq \frac{\|x+y\| + \tau\|x-y\|}{2} - 1 + \varepsilon \\
&= \left\|\frac{1}{2}(x+y)\right\| + \frac{\tau}{K}\left\|\frac{K}{2}(x-y)\right\| - 1 + \varepsilon \\
&\leq \left\|\left(1, \frac{\tau}{K}\right)\right\|_\psi \left\|\left(\frac{1}{2}(x+y), \frac{K}{2}(x-y)\right)\right\|_{\psi^*} - 1 + \varepsilon \\
&\leq \left\|\left(1, \frac{\tau}{K}\right)\right\|_\psi - 1 + \varepsilon.
\end{aligned}$$

As $\varepsilon \to 0$, one has that

$$\frac{\|f+\tau g\| + \|f-\tau g\|}{2} - 1 \leq \left\|\left(1, \frac{\tau}{K}\right)\right\|_\psi - 1,$$

which implies that

$$\rho_{X^*}(\tau) \leq \left\|\left(1, \frac{\tau}{K}\right)\right\|_{\psi^*} - 1.$$

Thus X^* is ψ-uniformly smooth. However, then Theorem 2.3.8 assures that there exists $M > 0$ such that

$$\frac{\|(f+g, f-g)\|_\psi}{2\psi(1/2)} \leq \|(f, Mg)\|_\psi$$

for each $f, g \in X^*$. This and Lemma 2.4.7 together show that

$$\frac{\|(x+y, x-y)\|_{\psi^*}}{2\psi^*(1/2)} \geq \|(x, M^{-1}y)\|_{\psi^*}$$

for each $x, y \in X$.

(ii) $\Rightarrow$ (iii) Suppose that $\varphi \in \Psi_2^S$ with $\gamma_{\psi^*,\varphi^*} = \gamma_{\varphi,\psi} > 0$. Then, by Theorem 2.2.10, we have

$$\begin{aligned}\|(x, M\gamma_{\varphi,\psi}y)\|_{\psi^*} &\le \frac{\|(x+\gamma_{\psi^*,\varphi^*}y, x-\gamma_{\psi^*,\varphi^*}y)\|_{\psi^*}}{2\psi^*(1/2)} \\ &\le \frac{\|(x+y, x-y)\|_{\varphi^*}}{2\varphi^*(1/2)}\end{aligned}$$

for each $x, y \in X$. Putting $M_\varphi = M\gamma_{\varphi,\psi} > 0$, one has (iii).

(iii) $\Rightarrow$ (iv): Obvious.

(iv) $\Rightarrow$ (i): Suppose that there exist $\varphi \in \Psi_2^S$ with $\gamma_{\varphi,\psi} > 0$ and $M_\varphi > 0$ such that

$$\frac{\|(x+y, x-y)\|_{\varphi^*}}{2\varphi^*(1/2)} \ge \|(x, M_\varphi y)\|_{\psi^*}$$

for each $x, y \in X$. As in the proof of [69, Lemma 2], the function $t \mapsto \varphi(t)/(1-t)$ is non-decreasing, which implies that $\|\cdot\|_{\varphi^*} \le 2\varphi^*(1/2)\|\cdot\|_\infty$. Hence we have

$$\begin{aligned}\frac{\|(x+y, x-y)\|_{\varphi^*}}{2\varphi^*(1/2)} &= \frac{\|(\|x+y\|, \|x-y\|)\|_{\varphi^*}}{2\varphi^*(1/2)} \\ &\le \|(\|x+y\|, \|x-y\|)\|_\infty = \max\{\|x+y\|, \|x-y\|\},\end{aligned}$$

or $\|(x, M_\varphi y)\|_{\psi^*} \le \max\{\|x+y\|, \|x-y\|\}$ for each $x, y \in X$. In particular, for each $x, y \in B_X$ one has

$$\left\|\left(\frac{1}{2}(x+y), \frac{M_\varphi}{2}(x-y)\right)\right\|_{\psi^*} \le \max\{\|x\|, \|y\|\} \le 1,$$

which together with Lemma 2.4.5 prove that $\|(1-\delta_X(\varepsilon), M_\varphi\varepsilon/2)\|_{\psi^*} \le 1$ for each $\varepsilon \in [0,2]$. This shows that X is ψ^*-uniformly convex. □

The preceding theorem together with Lemma 2.4.7 and Theorems 2.3.8 also show the duality between ψ-uniform smoothness and ψ^*-uniform convexity.

Corollary 2.4.9 ([73]). *Let X be a Banach space and $\psi \in \Psi_2^S$ with the property $(*)$. Suppose that $\gamma_{\psi,\psi_2} > 0$.*

(i) *The space X is ψ-uniformly smooth if and only if X^* is ψ^*-uniformly convex.*

(ii) *The space X^* is ψ-uniformly smooth if and only if X is ψ^*-uniformly convex.*

We conclude this chapter with the following consequence of Corollary 2.4.9.

Corollary 2.4.10 ([73]). *Let $\psi \in \Psi_2$ with the property $(*)$. Suppose that $\gamma_{\psi,\psi_2} > 0$. Then every 2-uniformly convex Banach space is ψ^*-uniformly convex.*

Chapter 3

Recent progress in Tingley's problem

The purpose of this chapter is to describe some recent results on Tingley's problem. As the first result in this topic, Tingley [84] proved that every surjective isometry between the unit spheres of two finite dimensional normed spaces maps the pairs of antipodal points to such pairs. From this, we have a feeling that, at least in the two-dimensional case, spherical isometries preserve the extreme points. We shall consider this problem in a bit more general settings. Specifically, we make use of the frame of the unit ball as a natural generalization of the set of the k-extreme points. Then it is shown that every spherical isometries preserves the frames of the unit balls.

A usual way to attack Tingley's problem is to show that the natural extension of a spherical isometry is linear. However, in this chapter, we try to present a new geometric approach to the two-dimensional Tingley problem that does not rely on the natural extensions. As applications, we present various new examples including the two-dimensional Lorentz sequence space $d^{(2)}(\omega, q)$ and its dual $d^{(2)}(\omega, q)^*$ by simple arguments.

3.1 The frame of the unit ball

We start this section with the definition of the frame of the unit ball of a Banach space. If A and B are two subsets of a Banach space with $A \subset B$, let $\mathrm{Int}_r(A \cap B)$ and $\partial_r(A \cap B)$ denote the relative interior and relative boundary of A with respect to B, respectively. For each $x \in S_X$, let $\nu(x)$ be the subset of S_{X^*} whose members are the support functionals for B_X that support B_X at x, that is, $\nu(x) = \{f \in S_{X^*} : f(x) = 1\}$.

The map $\nu : S_X \to S_{X^*}$ is called the *spherical image map* for S_X; see, for example, [53, Definition 5.4.23]. Let $\mathrm{SF}(B_X)$ denote the set of all support functionals f for B_X satisfying $\|f\| = 1$. Then $\mathrm{SF}(B_X) = \bigcup_{x \in S_X} \nu(x)$. For each $f \in \mathrm{SF}(B_X)$, the closed convex subset $F(f) = f^{-1}(\{1\}) \cap B_X$ of S_X is called the *exposed face* of B_X with respect to f. We remark that if $x \in S_X$, then $x \in F(f)$ if and only if $f \in \nu(x)$. Since $\nu(x) \neq \emptyset$ for all $x \in S_X$ by the Hahn-Banach theorem, we also remark that $S_X = \bigcup\{F(f) : f \in \mathrm{SF}(B_X)\}$. The notion of exposed face of B_X is an important tool for the study of Banach space geometry. The readers interested in this topic are referred to Aizpuru and García-Pacheco [1, 2], and García-Pacheco [28, 29]. The frame of the unit ball is defined as follows:

Definition 3.1.1 ([64])**.** For each $f \in \mathrm{SF}(B_X)$, define the *edge* $E(f)$ of B_X with respect to f by $E(f) = \partial_r(F(f) \cap f^{-1}(\{1\}))$. Then the *frame of the unit ball* $\mathrm{frm}(B_X)$ is given by

$$\mathrm{frm}(B_X) = \bigcup\{E(f) : f \in \mathrm{SF}(B_X)\}.$$

In [64], a simple relationship between the frame and the extreme points of B_X was given. Let $\mathrm{ext}(B_X)$ denote the set of all extreme points of B_X.

Theorem 3.1.2 ([64])**.** *Let X be a Banach space. Then $\mathrm{ext}(B_X) \subset \mathrm{frm}(B_X)$. In particular, $\mathrm{frm}(B_X) = \mathrm{ext}(B_X)$ if $\dim X = 2$.*

Hence, the frame of the unit ball is closely related to the set of all extreme points. However, if $\dim X \geq 3$, the equality in the preceding theorem cannot be expected. So it is natural to ask what kind of set equals $\mathrm{frm}(B_X)$ when $\dim X \geq 3$.

We recall the notion of k-extreme points. The definition adopted here can be found in [51, 92] (cf. [10]).

Definition 3.1.3. Let X be a Banach space, and let k be a positive integer such that $\dim X \geq k + 1$. An element $x \in S_X$ is said to be a *k-extreme point* of B_X if $\{x_1, x_2, \ldots, x_{k+1}\} \subset S_X$ and

$$x = \frac{x_1 + x_2 + \cdots + x_{k+1}}{k + 1}$$

imply the linear dependence of $\{x_1, x_2, \ldots, x_{k+1}\}$. The set of all k-extreme points of B_X is denoted by $\mathrm{ext}_k(B_X)$.

The notion of k-extreme points is a natural generalization of that of extreme points. We remark that 1-extreme points are just extreme points, that is, $\mathrm{ext}_1(B_X) = \mathrm{ext}(B_X)$.

We first present a characterization of the frame of the unit ball using k-extreme points. The following lemma can be found in [53, Theorem 1.3.14]. We give the proof of the first lemma only for the sake of completeness.

Lemma 3.1.4. *Every closed, convex, absorbing subset of a Banach space includes a neighborhood of the origin.*

Proof. Let C be a closed, convex, absorbing subset of a Banach space X, and let $D = C \cap (-C)$. Then, the subset D of C is also closed, convex, and absorbing. From this, it follows that $X = \bigcup_{n\in\mathbb{N}} nD$. The Baire category theorem assures that $\operatorname{Int} D \neq \emptyset$, which implies that

$$0 \in \frac{1}{2} \operatorname{Int} D + \frac{1}{2} \operatorname{Int}(-D) \subset \frac{1}{2} D + \frac{1}{2}(-D) = D \subset C.$$

This proves the lemma. □

We now give the characterization mentioned in the above.

Theorem 3.1.5 ([80]). *Let X be an n-dimensional Banach space. Then* $\operatorname{frm}(B_X) = \operatorname{ext}_{n-1}(B_X)$.

Proof. Suppose that $x \in S_X \setminus \operatorname{ext}_{n-1}(B_X)$. Then, there exist n-elements $x_1, x_2, \ldots, x_n$ of S_X such that $\{x_i\}_{i=1}^n$ is linearly independent and

$$x = \frac{1}{n} \sum_{i=1}^{n} x_i.$$

Take an arbitrary $f \in \nu(x)$. Since $f(x_i) \leq 1$ for each i and

$$\frac{1}{n} \sum_{i=1}^{n} f(x_i) = f(x) = 1,$$

we have $f(x_i) = 1$ for all $i = 1, 2, \ldots, n$, that is, $\{x_i\}_{i=1}^n \subset F(f)$. It follows that $\{x_i - x\}_{i=1}^{n-1} \subset \ker f$. Then we claim that $\ker f = \langle \{x_i - x\}_{i=1}^{n-1} \rangle$. To see this, it is enough to prove that $\{x_i - x\}_{i=1}^{n-1}$ is linearly independent since $\dim \ker f = n-1$. Suppose that $\{\alpha_i\}_{i=1}^{n-1} \subset \mathbb{R}$ and $\sum_{i=1}^{n-1} \alpha_i (x_i - x) = 0$. Then, it follows that

$$\sum_{i=1}^{n-1} \alpha_i x_i = \left(\sum_{i=1}^{n-1} \alpha_i \right) x = \frac{1}{n} \left(\sum_{i=1}^{n-1} \alpha_i \right) \sum_{i=1}^{n} x_i.$$

However, from the fact that $\{x_i\}_{i=1}^n$ is linearly independent, we obtain $\alpha_j = n^{-1} \sum_{i=1}^{n-1} \alpha_i$ for all $j = 1, 2, \ldots, n-1$ and $n^{-1} \sum_{i=1}^{n-1} \alpha_i = 0$. This proves $\alpha_1 = \alpha_2 = \cdots = \alpha_{n-1} = 0$, that is, $\{x_i - x\}_{i=1}^{n-1}$ is linearly independent.

Next, we put $A = \mathrm{co}(\{x_i - x\}_{i=1}^n)$. Remark that

$$0 = \frac{1}{n}\sum_{i=1}^{n}(x_i - x) \in A.$$

Let $y \in \ker f \setminus \{0\}$. Then, there exists $\{\alpha_i\}_{i=1}^{n-1} \subset \mathbb{R}$ with $\sum_{i=1}^{n-1} |\alpha_i| > 0$ such that $y = \sum_{i=1}^{n-1} \alpha_i(x_i - x)$. Put $I = \{i \in \{1, 2, \ldots, n-1\} : \alpha_i \geq 0\}$ and $J = \{1, 2, \ldots, n-1\} \setminus I$, respectively. Since $x_i - x = -\sum_{j \neq i}(x_j - x)$ for each i, we obtain

$$\begin{aligned}
y &= \sum_{i \in I} \alpha_i(x_i - x) + \sum_{i \in J} \alpha_i(x_i - x) \\
&= \sum_{i \in I} |\alpha_i|(x_i - x) + \sum_{i \in J} (-\alpha_i) \sum_{j \neq i}(x_j - x) \\
&= \sum_{i \in I} |\alpha_i|(x_i - x) + \sum_{i \in J} |\alpha_i| \sum_{j \neq i}(x_j - x) \\
&= \sum_{i \in I} |\alpha_i|(x_i - x) + \sum_{i \in J} (n-1)|\alpha_i| \left(\frac{1}{n-1} \sum_{j \neq i}(x_j - x) \right).
\end{aligned}$$

Letting $K = \sum_{i \in I} |\alpha_i| + \sum_{i \in J}(n-1)|\alpha_i|$ yields $K > 0$ and $K^{-1}y \in A$. Then, we also have $t^{-1}y \in A$ for all $t > K$ since $0 \in A$. Therefore, there exists a positive real number r such that $rB_{\ker f} \subset A$ by Lemma 3.1.4, which implies that

$$x + rB_{\ker f} \subset x + A = x + \mathrm{co}(\{x_i - x\}_{i=1}^n) = \mathrm{co}(\{x_i\}_{i=1}^n) \subset F(f).$$

This shows that $x \in F(f) \setminus E(f)$, and hence $x \notin \mathrm{frm}(B_X)$.

Conversely, we assume that $x \in S_X \setminus \mathrm{frm}(B_X)$. Let $f \in \nu(x)$. Then $x \in F(f) \setminus E(f)$, and so there exists a positive real number r such that $x + rB_{\ker f} \subset F(f)$. Since $\dim X = n$, we have $\dim \ker f = n - 1$. Let $\{e_i\}_{i=1}^{n-1}$ be a basis for $\ker f$. Putting $e_n = -\sum_{i=1}^{n-1} e_i$, then $e_n \neq 0$ and $\sum_{i=1}^n e_i = 0$. We also put

$$u_i = \frac{re_i}{\max_{1 \leq i \leq n} \|e_i\|}$$

for each $i = 1, 2, \ldots, n$. Since $\|u_i\| \leq r$, it follows that $u_i \in rB_{\ker f}$, that is, $x + u_i \in F(f)$ for all $i = 1, 2, \ldots, n$. We remark that

$$\frac{1}{n}\sum_{i=1}^{n}(x + u_i) = x + \frac{1}{n}\sum_{i=1}^{n} u_i = x.$$

Now suppose that $\{\alpha_i\}_{i=1}^n \subset \mathbb{R}$ and $\sum_{i=1}^n \alpha_i(x + u_i) = 0$. Then, we have

$$\left(\sum_{i=1}^{n} \alpha_i\right) x + \sum_{i=1}^{n} \alpha_i u_i = 0.$$

In particular,

$$\sum_{i=1}^{n} \alpha_i = f\left(\left(\sum_{i=1}^{n} \alpha_i\right) x + \sum_{i=1}^{n} \alpha_i u_i\right) = 0.$$

On the other hand, it follows from $u_n = -\sum_{i=1}^{n-1} u_i$ that

$$\begin{aligned} 0 = \sum_{i=1}^{n} \alpha_i u_i &= \sum_{i=1}^{n-1} \alpha_i u_i + \alpha_n u_n \\ &= \sum_{i=1}^{n-1} \alpha_i u_i - \alpha_n \sum_{i=1}^{n-1} u_i \\ &= \sum_{i=1}^{n-1} (\alpha_i - \alpha_n) u_i. \end{aligned}$$

Since $\{e_i\}_{i=1}^{n-1}$ is linearly independent, we obtain $\alpha_1 = \alpha_2 = \cdots = \alpha_n$. This and $\sum_{i=1}^{n} \alpha_i = 0$ together imply that $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$. Thus, the set $\{x + u_i\}_{i=1}^{n}$ is linearly independent, and so $x \notin \mathrm{ext}_{n-1}(B_X)$. This completes the proof. □

From the fact that $\mathrm{ext}_1(B_X) = \mathrm{ext}(B_X)$, one obtains the latter half of Theorem 3.1.2. The preceding theorem shows that $\mathrm{frm}(B_X)$ is a natural generalization of the set of all (the weakest) k-extreme points of the unit ball.

In 1960, Singer [74] introduced the notion of k-strict convexity of Banach spaces. For $k \in \mathbb{N}$, a Banach space X is said to be *k-strictly convex* if for any $k+1$ elements $x_1, x_2, \ldots, x_{k+1} \in X$ the equality

$$\left\| \sum_{i=1}^{k+1} x_i \right\| = \sum_{i=1}^{k+1} \|x_i\|$$

implies the linear dependence of $\{x_1, x_2, \ldots, x_{k+1}\}$. For the complex version of this notion, the readers are referred to Liu and Zhuang [51] and Zhuang [92]. We shall recall Singer's characterization of k-strict convexity (cf. [74, Theorem 1] and [75, Lemma 4.2]). The proof can be simplified by using the sharp triangle inequality for n-elements; see Kato, Saito and Tamura [45]. For any nonzero elements $x_1, x_2, \ldots, x_n$ of a Banach space, the sharp triangle inequality

$$\begin{aligned}
&\sum_{i=1}^{n}\|x_i\| - \left(n - \left\|\sum_{i=1}^{n}\frac{x_i}{\|x_i\|}\right\|\right)\max_{1\le i\le n}\|x_i\| \\
&\le \left\|\sum_{i=1}^{n} x_i\right\| \\
&\le \sum_{i=1}^{n}\|x_i\| - \left(n - \left\|\sum_{i=1}^{n}\frac{x_i}{\|x_i\|}\right\|\right)\min_{1\le i\le n}\|x_i\|
\end{aligned}$$

holds.

Theorem 3.1.6 (Singer [74])**.** *A Banach space X is k-strictly convex if and only if* $\mathrm{ext}_k(B_X) = S_X$*.*

Proof. It is enough to prove that $\mathrm{ext}_k(B_X) = S_X$ implies the k-strict convexity of X. Let $x_1, x_2, \ldots, x_{k+1} \in X \setminus \{0\}$ such that

$$\left\|\sum_{i=1}^{k+1} x_i\right\| = \sum_{i=1}^{k+1}\|x_i\|.$$

Then, by the sharp triangle inequality, we have

$$\left(k+1 - \left\|\sum_{i=1}^{k+1}\frac{x_i}{\|x_i\|}\right\|\right)\min_{1\le i\le k+1}\|x_i\| \le \sum_{i=1}^{k+1}\|x_i\| - \left\|\sum_{i=1}^{k+1} x_i\right\| = 0,$$

which implies that

$$\left\|\sum_{i=1}^{k+1}\frac{x_i}{\|x_i\|}\right\| = k+1.$$

However, then $\{x_1, x_2, \ldots, x_{k+1}\}$ is linearly dependent since $\mathrm{ext}_k(B_X) = S_X$. The proof is complete. □

As a consequence of Theorems 3.1.5 and 3.1.6, we have the following result.

Corollary 3.1.7. *An n-dimensional Banach space X is $(n-1)$-strictly convex if and only if* $\mathrm{frm}(B_X) = S_X$*.*

We next present a characterization of frames using the extreme points of the unit balls of two-dimensional subspaces. To do this, some preparations are needed. The proof of the following lemma can be found in [64, Lemma 3.3].

Lemma 3.1.8 ([64]). *Suppose that X is a Banach space, that $x \in S_X$ and $f \in \nu(x)$, and that $y \in \ker f \setminus \{0\}$. Then there exist $t_1, t_2 \in \mathbb{R}$ such that $t_1 \leq 0 \leq t_2$, $\{t \in \mathbb{R} : \|x + ty\| = 1\} = [t_1, t_2]$, and $x + t_1 y, x + t_2 y \in E(f)$. In particular, if $t_1 t_2 = 0$ then $x \in E(f)$.*

As an easy consequence of Lemma 3.1.8, we have the following.

Proposition 3.1.9 ([80]). $B_X = \mathrm{co}(\mathrm{frm}(B_X))$.

Proof. It is enough to prove that $S_X \subset \mathrm{co}(\mathrm{frm}(B_X))$. Let $x \in S_X$, and let $f \in \nu(x)$. Take an arbitrary $y \in \ker f \setminus \{0\}$. By the preceding lemma, there exist $t_1, t_2 \in \mathbb{R}$ such that $t_1 \leq 0 \leq t_2$, $\{t \in \mathbb{R} : \|x + ty\| = 1\} = [t_1, t_2]$, and $x + t_1 y, x + t_2 y \in E(f)$. If $t_1 t_2 = 0$ then $x \in E(f)$, and so we assume that $t_1 < 0 < t_2$. Then, it follows that

$$x = \frac{t_2}{t_2 - t_1}(x + t_1 y) + \frac{-t_1}{t_2 - t_1}(x + t_2 y) \in \mathrm{co}(\mathrm{frm}(B_X)).$$

This shows the proposition. □

The famous Krein-Milman theorem states that every nonempty compact convex subset of a Hausdorff locally convex space is the closed convex hull of the set of its extreme points. In particular, if X is a reflexive Banach space, then $B_X = \overline{\mathrm{co}}(\mathrm{ext}(B_X))$. However, it is well known that the unit ball of the space c_0 has no extreme points, where c_0 is the Banach space of all sequences of scalars that converge to 0. Thus, it is rather interesting to compare this fact to the preceding proposition.

The frame of the unit ball is closely related to the notion of Birkhoff orthogonality. Let X be a Banach space, and let $x, y \in X$. Then x is said to be *Birkhoff orthogonal* to y, denoted by $x \perp_B y$, if $\|x + ty\| \geq \|x\|$ for all $t \in \mathbb{R}$. It is known that Birkhoff orthogonality is not symmetric in general, that is, $x \perp_B y$ does not imply $y \perp_B x$. More details about Birkhoff orthogonality can be found in Birkhoff [14], Day [19, 20] and James [33, 34, 35].

In [34], James gave the following characterization of Birkhoff orthogonality.

Lemma 3.1.10 (James [34]). *Let X be a real normed linear space and let x and y be two elements of X. Then $x \perp_B y$ if and only if there exists $f \in X^* \setminus \{0\}$ such that $|f(x)| = \|f\|\|x\|$ and $f(y) = 0$.*

For the sake of convenience, we introduce the following notion: An element x is said to be *strongly Birkhoff orthogonal* to y in the positive real line, denoted by $x \perp_B^+ y$, if $x \perp_B y$ and $\|x + ty\| > \|x\|$ for all $t > 0$.

Some basic properties of the frame of the unit ball are collected in the following proposition.

Proposition 3.1.11 ([80]). *Let X be a Banach space.*

(i) *Suppose that $x \in S_X$, and that $f \in \nu(x)$. Then $x \in E(f)$ if and only if there exists $y \in \ker f \setminus \{0\}$ such that $\|x + ty\| > 1$ for all $t > 0$.*

(ii) *Suppose that $x \in S_X$. Then $x \in \mathrm{frm}(B_X)$ if and only if there exists $y \in X \setminus \{0\}$ such that $x \perp_B^+ y$.*

(iii) *The set $\mathrm{frm}(B_X)$ is symmetric, that is, $\mathrm{frm}(B_X) = -\mathrm{frm}(B_X)$.*

(iv) *Let M be a closed subspace of X. Then $\mathrm{frm}(B_M) \subset \mathrm{frm}(B_X) \cap M$.*

Proof. (i) First, we assume that $x \in E(f)$. To show that there exists $y \in X \setminus \{0\}$ satisfying $\|x + ty\| > 1$ for all $t > 0$, suppose contrary that for any $y \in \ker f \setminus \{0\}$, there exists $t_y > 0$ such that $\|x + t_y y\| = 1$. Then, we remark that $x + ty \in F(f)$ for all $t \in [0, t_y]$. Indeed, for each $t \in [0, t_y]$, we obtain

$$\begin{aligned} 1 = f(x+ty) &\leq \|x+ty\| \\ &= \left\| \left(1 - \frac{t}{t_y}\right) x + \frac{t}{t_y}(x + t_y y) \right\| \\ &\leq \left(1 - \frac{t}{t_y}\right) \|x\| + \frac{t}{t_y}\|x + t_y y\| \\ &= 1. \end{aligned}$$

Putting $A = -x + F(f)$, then A is closed, convex subset of $\ker f$ and $0 \in A$. Furthermore, since $y \in sA$ for all $s > t_y^{-1}$, the set A is absorbing in $\ker f$. Thus, we have $0 \in \mathrm{Int}_r(A \cap \ker f)$ by Lemma 3.1.4, which implies that

$$x + rB_{\ker f} \subset x + A = F(f)$$

for some $r > 0$. It follows that $x \in F(f) \setminus E(f)$, a contradiction.

Conversely, suppose that there exists $y \in \ker f \setminus \{0\}$ such that $\|x + ty\| > 1$ for all $t > 0$. Then, it directly follows from Lemma 3.1.8 that $x \in E(f)$.

(ii) This follows from (i) and Lemma 3.1.10.

(iii) Since $x \perp_B^+ y$ implies $-x \perp_B^+ -y$, we have (iii).

(iv) This is an immediate consequence of (ii). $\square$

As a consequence of Theorem 3.1.5 and (iv) of the preceding proposition, we have the following corollary.

Corollary 3.1.12 ([80])**.** *Let X be an infinite dimensional Banach space. Then*

$$\bigcup_{k\in\mathbb{N}} \mathrm{ext}_k(B_X) \subset \mathrm{frm}(B_X).$$

We now present another characterization of frames.

Theorem 3.1.13 ([80])**.** *Let X be a Banach space. Then*

$$\mathrm{frm}(B_X) = \bigcup\{\mathrm{ext}(B_M) : M \text{ is a two-dimensional subspace of } X\}.$$

Proof. Let $x \in \mathrm{frm}(B_X)$. Then, there exists $y \in X \setminus \{0\}$ such that $x \perp_B^+ y$ by Proposition 3.1.11 (ii). Putting $M = [\{x, y\}]$, then $\dim M = 2$ since the set $\{x, y\}$ is linearly independent. Applying Theorem 3.1.2 and Proposition 3.1.11 (ii), one has that $x \in \mathrm{frm}(B_M) = \mathrm{ext}(B_M)$. Conversely, let M be a two-dimensional subspace of X. Then, by Proposition 3.1.2 and Proposition 3.1.11 (iv), we have $\mathrm{ext}(B_M) = \mathrm{frm}(B_M) \subset \mathrm{frm}(B_X) \cap M$. This completes the proof. $\square$

We shall study the topological properties of the frame of the unit ball. The following lemmas are needed.

Lemma 3.1.14 ([80])**.** *Let X be a Banach space, and let f and g be two distinct elements of* $\mathrm{SF}(B_X)$*. Then $F(f) \cap F(g) = E(f) \cap E(g)$. In particular, if $x \notin \mathrm{frm}(B_X)$ then $\nu(x)$ must be a singleton.*

Proof. It is clear that $F(f) \cap F(g) \supset E(f) \cap E(g)$. So we assume that $F(f) \cap F(g) \neq \emptyset$. Let $x \in F(f) \cap F(g)$. Then, we remark that $\ker f \setminus \ker g \neq \emptyset$ and $\ker g \setminus \ker f \neq \emptyset$ since $f \neq g$ and $f(x) = g(x) = 1$. Suppose that $y \in \ker g \setminus \ker f$. Putting $z = x - f(y)^{-1}y$, then $z \in \ker f \setminus \{0\}$ and

$$\begin{aligned} \|x + tz\| &= \left\| (1+t)x - \frac{t}{f(y)} y \right\| \\ &\geq g\left((1+t)x - \frac{t}{f(y)} y \right) \\ &= 1 + t > 1 \end{aligned}$$

for all $t > 0$. Therefore, $x \in E(f)$ by Proposition 3.1.11 (i). Similarly, one can show that $x \in E(g)$. $\square$

Lemma 3.1.15 ([80])**.** *Let X be a Banach space, and let $x \in S_X$. If $x \in F(f) \setminus E(f)$ for some $f \in \nu(x)$, then $\nu(x) = \{f\}$ and $x \notin \mathrm{frm}(B_X)$.*

Proof. This directly follows from the preceding lemma. $\square$

The frame of the unit ball is always closed.

Theorem 3.1.16 ([80])**.** *Let X be a Banach space. Then* $\mathrm{frm}(B_X)$ *is closed in* X*.*

Proof. It is enough to prove that $X \setminus \mathrm{frm}(B_X)$ is open. Let $x \in S_X \setminus \mathrm{frm}(B_X)$. Then, by Lemma 3.1.14, the set $\nu(x)$ is a singleton. Putting $f = \nu(x)$, then $x \in F(f) \setminus E(f)$, and so there exists a positive real number r such that $x + rB^\circ_{\ker f} \subset F(f)$, where $B^\circ_{\ker f} = \{x \in \ker f : \|x\| < 1\}$. Since the set $x + rB^\circ_{\ker f}$ is open in $F(f)$, it follows that $x + rB^\circ_{\ker f} \subset F(f) \setminus E(f)$. Applying Lemma 3.1.15, we have $x + rB^\circ_{\ker f} \subset X \setminus \mathrm{frm}(B_X)$. Now, let

$$B = x + \frac{r}{2+r} B^\circ_X,$$

where $B^\circ_X = \{x \in X : \|x\| < 1\}$. Take an arbitrary $y \in B$. Then,

$$y = x + \frac{r}{2+r} z$$

for some $z \in B^\circ_X$. Hence, we have

$$\begin{aligned}
(2+r)y &= (2+r)x + rz \\
&= (2 + r + rf(z))x + r(z - f(z)x) \\
&= (2 + r + rf(z)) \left(x + \frac{r}{2+r+rf(z)} (z - f(z)x) \right).
\end{aligned}$$

We put

$$w = \frac{1}{2+r+rf(z)} (z - f(z)x).$$

Then, we obtain $w \in \ker f$. Moreover, since $z \in B^\circ_X$, it follows that $2 + r + rf(z) \geq 2$ and $\|z - f(z)x\| \leq \|z\| + |f(z)|\|x\| < 2$, which implies that

$$\|w\| = \frac{\|z - f(z)x\|}{2+r+rf(z)} < 1.$$

This shows that

$$\begin{aligned}
y = \frac{2+r+rf(z)}{2+r}(x + rw) &\in \frac{2+r+rf(z)}{2+r}(x + rB^\circ_{\ker f}) \\
&\subset \frac{2+r+rf(z)}{2+r} F(f) \\
&\subset \frac{2+r+rf(z)}{2+r} S_X.
\end{aligned}$$

Thus we have $y \notin S_X$ whenever $f(z) \neq 0$, whence $y \in X \setminus \mathrm{frm}(B_X)$. On the other hand, if $f(z) = 0$, then $y \in x + rB^\circ_{\ker f} \subset X \setminus \mathrm{frm}(B_X)$. These proves that $B \subset X \setminus \mathrm{frm}(B_X)$, that is, $x \in \mathrm{Int}(X \setminus \mathrm{frm}(B_X))$. Therefore the set $X \setminus \mathrm{frm}(B_X)$ is open. $\square$

As an application of the preceding theorem, we next show that the frame of the unit ball is connected if the dimension of the space is not less than three. For this, we need the following three lemmas.

Lemma 3.1.17 ([80])**.** *Let X be a Banach space with $\dim X \geq 3$. Suppose that $f \in \mathrm{SF}(B_X)$, and that $F(f) \setminus E(f) \neq \emptyset$. Then $E(f)$ is a connected subset of $\mathrm{frm}(B_X)$.*

Proof. Let $x_0 \in F(f) \setminus E(f)$. Then, there exists a positive real number r such that $x_0 + rB_{\ker f} \subset F(f)$. Putting $A = -x_0 + F(f)$, then A is a closed, convex, absorbing subset of $\ker f$. Let p_A be the Minkowski functional of A, that is, let $p_A(x) = \inf\{t > 0 : x \in tA\}$ for each $x \in \ker f$. Then, p_A is a sublinear functional on $\ker f$ and

$$\{x \in \ker f : p_A(x) < 1\} \subset A \subset \{x \in \ker f : p_A(x) \leq 1\}.$$

It follows from this inclusion that $A = \{x \in \ker f : p_A(x) \leq 1\}$. Indeed, if $x \in X$ and $p_A(x) = 1$, then

$$p_A\left(\left(1 - \frac{1}{n}\right)x\right) = \left(1 - \frac{1}{n}\right)p_A(x) < 1,$$

for each $n \geq 2$. This shows $(1-n^{-1})x \in A$ for all $n \geq 2$, which implies that $x \in A$ since A is closed. We first see the continuity of p_A on $\ker f$. Since $rB_{\ker f} \subset A \subset 2B_{\ker f}$, one can easily have

$$\frac{1}{2}\|x\| \leq p_A(x) \leq \frac{1}{r}\|x\|$$

for all $x \in \ker f$. On the other hand, by the sublinearity of p_A, we obtain

$$p_A(x) - p_A(y) \leq p_A(x-y) \leq \frac{1}{r}\|x-y\|$$

for all $x, y \in \ker f$. Replacing x with y, we have

$$|p_A(x) - p_A(y)| \leq \frac{1}{r}\|x-y\|$$

for all $x, y \in \ker f$. This proves that p_A is continuous on $\ker f$.

Next, we show that $\mathrm{Int}_r(A \cap \ker f) = \{x \in \ker f : p_A(x) < 1\}$. From the preceding paragraph, it follows that $\{x \in \ker f : p_A(x) < 1\} = p_A^{-1}((-\infty, 1))$ is open in $\ker f$, which implies that $\{x \in \ker f : p_A(x) < 1\} \subset \mathrm{Int}_r(A \cap \ker f)$. Conversely, let $x \in \mathrm{Int}_r(A \cap \ker f)$. We may assume that $x \neq 0$. Then, $x + sB_{\ker f} \subset A$ for some $s > 0$. In particular,

$$\left(1 + \frac{s}{\|x\|}\right)x \in A.$$

Hence, we have $p_A(x) \leq (1 + s\|x\|^{-1})^{-1} < 1$. Thus, $\mathrm{Int}_r(A \cap \ker f) = \{x \in \ker f : p_A(x) < 1\}$, which in turn implies that $\partial_r(A \cap \ker f) = \{x \in \ker f : p_A(x) = 1\}$.

Suppose that x and y are two distinct elements of $\partial_r(A \cap \ker f)$ satisfying $\|y\|x + \|x\|y \neq 0$. Let

$$\kappa(t) = \frac{(1-t)x + ty}{p_A((1-t)x + ty)}$$

for each $t \in [0, 1]$. Then, one has that κ is a path from x to y in $\partial_r(A \cap \ker f)$.

We shall omit the assumption that $\|y\|x + \|x\|y \neq 0$. If $\|y\|x + \|x\|y = 0$, then there exists $z \in \ker f$ such that $\{x, z\}$ is linearly independent since $\dim \ker f \geq 2$. Putting $w = p_A(z)^{-1}z$, we have that $w \in \partial_r(A \cap \ker f)$, and that $\{x, w\}$ is also linearly independent. Since $\|w\|x + \|x\|w \neq 0$ and $\|y\|w + \|w\|y \neq 0$, as in the preceding paragraph, there are two paths λ, μ such that λ joins x to w and μ joins w to y. Putting

$$\kappa(t) = \begin{cases} \lambda(2t) & (t \in [0, 1/2]), \\ \mu(2t - 1) & (t \in [1/2, 1]), \end{cases}$$

then κ is a path from x to y. Thus, the set $\partial_r(A \cap \ker f)$ is path-connected, whence it is a connected subset of A.

Finally, since the map $x \mapsto x_0 + x$ is a homeomorphism from $\ker f$ onto $f^{-1}(\{1\})$, and it maps A onto $F(f)$, we obtain that $E(f) = x_0 + \partial_r(A \cap \ker f)$ is also a connected subset of $\mathrm{frm}(B_X)$. $\square$

Lemma 3.1.18 ([80])**.** *Suppose that X is a Banach space. Let $x, y \in S_X$ with $x + y \neq 0$, and let $z(t) = (1-t)x + ty$ for each $t \in \mathbb{R}$. Then the following are equivalent:*

(i) $\|z(t_0)\| = \min_{t \in \mathbb{R}} \|z(t)\|$.

(ii) $z(t_0) \perp_B x - y$.

Proof. Since $z(t_0) + t(x - y) = z(t_0 - t)$ for all $t \in \mathbb{R}$, it follows that $\|z(t_0)\| = \min_{t \in \mathbb{R}} \|z(t)\|$ if and only if $z(t_0) \perp_B x - y$. $\square$

Lemma 3.1.19 ([80])**.** *Let X be a Banach space, let x and y be two elements of $\mathrm{frm}(B_X)$ such that $\{x, y\}$ is linearly independent, and let $z(t) = (1-t)x + ty$ for each $t \in \mathbb{R}$. Suppose that $\|z(t)\|^{-1}z(t) \notin \mathrm{frm}(B_X)$ for all $t \in (0, 1)$. Then there exists a functional $f \in S_{X^*}$ such that $x, y \in E(f)$ and $F(f) \setminus E(f) \neq \emptyset$.*

Proof. Let $M = [\{x, y\}]$, and let $\alpha = \min_{t\in\mathbb{R}} \|z(t)\| > 0$. First, we show that $\alpha = 1$. Suppose to the contrary that $0 < \alpha < 1$. Let A be the closed subset of the unit interval $[0, 1]$ defined by

$$A = \{t \in \mathbb{R} : \|z(t)\| = \alpha\}.$$

Putting $t_0 = \min A$, we have $t_0 \in (0, 1)$ since the function $t \mapsto \|z(t)\|$ is convex. Then, Lemma 3.1.18 assures that $z(t_0) \perp_B x - y$, and so there exists a functional $f \in S_{X^*}$ such that $f(\alpha^{-1}z(t_0)) = 1$ and $f(x - y) = 0$ by Lemma 3.1.10. On the other hand, for each $t > 0$, we obtain $\|z(t_0) + t(x - y)\| = \|z(t_0 - t)\| > \alpha$, that is,

$$\left\|\frac{z(t_0)}{\alpha} + t\frac{x - y}{\alpha}\right\| > 1.$$

Thus, we have $\alpha^{-1}z(t_0) \in E(f)$ by Proposition 3.1.11 (i), a contradiction which proves $\alpha = 1$. Now, we remark that $\|z(t)\| = 1$ for all $t \in [0, 1]$ since $\alpha = 1$. Let $f \in \nu(z(1/2))$. Then, it follows from $z(1/2) \notin \mathrm{frm}(B_X)$ that $z(1/2) \in F(f) \setminus E(f)$. Moreover, from the fact that $f((x + y)/2) = f(z(1/2)) = 1$, one can easily check that $f(x) = f(y) = 1$. Hence, by Lemma 3.1.15, we also have $x, y \in E(f)$. □

We now ready to prove the connectedness of frames.

Theorem 3.1.20 ([80])**.** *Let X be a Banach space with $\dim X \geq 3$. Then $\mathrm{frm}(B_X)$ is connected in X.*

Proof. To see the connectivity of $\mathrm{frm}(B_X)$, suppose contrary that $\mathrm{frm}(B_X)$ is disconnected. Then, there exist two open sets U and V in X such that $\mathrm{frm}(B_X) \subset U \cup V$, $U \cap V \cap \mathrm{frm}(B_X) = \emptyset$, $U \cap \mathrm{frm}(B_X) \neq \emptyset$, and $V \cap \mathrm{frm}(B_X) \neq \emptyset$. For each $x \in \mathrm{frm}(B_X)$, let $C(x)$ be the connected component of x in $\mathrm{frm}(B_X)$, that is,

$$C(x) = \bigcup\{C : C \text{ is a connected subset of } \mathrm{frm}(B_X) \text{ such that } x \in C\}.$$

Put $A = U \cap \mathrm{frm}(B_X)$ and $B = V \cap \mathrm{frm}(B_X)$, respectively. First, we show that A and B are closed subset of X. Let $\{x_n\}_{n=1}^{\infty}$ be an arbitrary sequence in A that converges to some $x \in X$. Then, $x \in \mathrm{frm}(B_X)$ since $\mathrm{frm}(B_X)$ is closed by Theorem 3.1.16. It follows from $U \cap V \cap \mathrm{frm}(B_X) = \emptyset$ that $A \subset X \setminus V$, which implies that $\overline{A} \subset X \setminus V$ since $X \setminus V$ is closed in X. This and $x \in \mathrm{frm}(B_X) \subset U \cup V$ together imply that $x \in U$. Therefore, we have $x \in A$. This shows that A is a closed subset of X. Similarly, one has that B is also closed.

Suppose that $x \in A$ and $y \in B$, and that $z(t) = (1 - t)x + ty$ for each $t \in [0, 1]$. We may assume that $\{x, y\}$ is linearly independent. Indeed, by Proposition 3.1.9, we have

$\mathrm{co}(\mathrm{frm}(B_X)) = B_X$. So there exists $z \in \mathrm{frm}(B_X) \setminus [\{x, y\}]$ since $\dim X \geq 3$. Replacing x or y with z if necessary, we have that $\{x, y\}$ is linearly independent. Now, let

$$C = \left\{ t \in [0,1] : \frac{z(t)}{\|z(t)\|} \in A \right\},$$
$$D = \left\{ t \in [0,1] : \frac{z(t)}{\|z(t)\|} \in B \right\},$$

respectively. Then, C and D are closed subsets of $[0, 1]$ since A and B are closed in X, and so are compact in $[0, 1]$, which and $C \cap D = \emptyset$ together imply that

$$d(C, D) = \min\{|t_1 - t_2| : t_1 \in C,\ t_2 \in D\} > 0.$$

Let $t_1 \in C$ and $t_2 \in D$ such that $|t_1 - t_2| = d(C, D)$, and let $u_i = \|z(t_i)\|^{-1} z(t_i)$ for $i = 1, 2$. Then, we obtain $\|z(t)\|^{-1} z(t) \notin \mathrm{frm}(B_X)$ for all $t \in \{(1-\lambda)t_1 + \lambda t_2 : \lambda \in (0, 1)\}$. On the other hand, let $w(s) := (1 - s)u_1 + s u_2 = k_s z(t_s)$ for each $s \in [0, 1]$, where

$$k_s = \frac{s\|z(t_1)\| + (1-s)\|z(t_2)\|}{\|z(t_1)\|\|z(t_2)\|},$$
$$t_s = \frac{(1-s)\|z(t_2)\|}{s\|z(t_1)\| + (1-s)\|z(t_2)\|} t_1 + \frac{s\|z(t_1)\|}{s\|z(t_1)\| + (1-s)\|z(t_2)\|} t_2.$$

Then, we have

$$\frac{w(s)}{\|w(s)\|} = \frac{z(t_s)}{\|z(t_s)\|}$$

for all $s \in [0, 1]$. In particular, if $s \in (0, 1)$ then $t_s \in \{(1 - \lambda)t_1 + \lambda t_2 : \lambda \in (0, 1)\}$, and so $\|w(s)\|^{-1} w(s) \notin \mathrm{frm}(B_X)$ for all $s \in (0, 1)$. Thus, by Lemma 3.1.19, there exists a linear functional $f \in S_{X^*}$ such that $u_1, u_2 \in E(f)$ and $F(f) \setminus E(f) \neq \emptyset$, and then, Lemma 3.1.17 assures that $E(f)$ is a connected subset of $\mathrm{frm}(B_X)$, which implies that $u_2 \in C(u_1)$. However this means that $C(u_1) \subset \mathrm{frm}(B_X) = A \cup B$, $A \cap B \cap C(u_1) = \emptyset$, $A \cap C(u_1) \neq \emptyset$ and $B \cap C(u_1) \neq \emptyset$, which contradicts the connectedness of $C(u_1)$ since A and B are relatively open in $\mathrm{frm}(B_X)$. Thus, $\mathrm{frm}(B_X)$ must be connected in X. □

Finally, we shall give some examples.

Theorem 3.1.21 ([80])**.** $\mathrm{frm}(B_{c_0}) = \{(a_n)_{n=1}^\infty \in S_{c_0} : |a_m a_n| = 1$ for some $m, n \in \mathbb{N}$ with $m \neq n\}$.

Proof. We first note that for any $(a_n)_{n=1}^\infty \in S_{c_0}$ there exists at least one $m \in \mathbb{N}$ such that $|a_m| = 1$. For each $m \in \mathbb{N}$, let x_m^* be the m-th coordinate functional on c_0, that is, $x_m^*((a_n)_{n=1}^\infty) = a_m$ for all $(a_n)_{n=1}^\infty \in c_0$. Then, one has that $x_m^* \in \mathrm{SF}(B_{c_0})$ for all $m \in \mathbb{N}$

since $\|x_m^*\| = 1$ and it supports B_{c_0} at e_m, where e_m is the m-th standard coordinate vector. Let $(a_n)_{n=1}^\infty$ be an element of S_{c_0} such that $|a_n| = 1$ only at $n = m$. Since $\mathrm{frm}(B_{c_0})$ is symmetric by Proposition 3.1.11 (iii), we may assume that $a_n = 1$. Then, we have $x_m^* \in \nu((a_n)_{n=1}^\infty)$. As mentioned above, we remark that

$$k = \|(a_1, \ldots, a_{m-1}, 0, a_{m+1}, \ldots)\|_\infty < 1.$$

Since the kernel of x_m^* is given by

$$\ker x_m^* = \{(b_n)_{n=1}^\infty \in c_0 : b_m = 0\},$$

it follows that

$$\begin{aligned}
&\left\| (a_n)_{n=1}^\infty + \frac{1-k}{\|(b_n)_{n=1}^\infty\|_\infty} (b_n)_{n=1}^\infty \right\|_\infty \\
&= \sup_{n \in \mathbb{N}} \left| a_n + \frac{(1-k)b_n}{\|(b_n)_{n=1}^\infty\|_\infty} \right| \\
&\leq \max \left\{ \left| a_m + \frac{(1-k)b_m}{\|(b_n)_{n=1}^\infty\|_\infty} \right|, \sup_{n \neq m} \left(|a_n| + \frac{(1-k)|b_n|}{\|(b_n)_{n=1}^\infty\|_\infty} \right) \right\} \\
&\leq 1.
\end{aligned}$$

for each $(b_n)_{n=1}^\infty \in \ker x_m^* \setminus \{0\}$. By Proposition 3.1.11 (i), we have $(a_n)_{n=1}^\infty \in F(x_m^*) \setminus E(x_m^*)$. Hence, one obtain $(a_n)_{n=1}^\infty \notin \mathrm{frm}(B_{c_0})$ by Lemma 3.1.15.

Conversely, suppose that $(a_n)_{n=1}^\infty \in S_{c_0}$, and that $|a_m a_n| = 1$ for some $m, n \in \mathbb{N}$ with $m \neq n$. Then, one has that $(a_n)_{n=1}^\infty \in F(a_m x_m^*) \cap F(a_n x_n^*)$. Since $a_m x_m^* \neq a_n x_n^*$, it follows that $(a_n)_{n=1}^\infty \in \mathrm{frm}(B_{c_0})$ by Lemma 3.1.14. $\square$

Next, we deal with the space ℓ_p with $1 \leq p < \infty$.

Theorem 3.1.22 ([80]). *Let* $1 \leq p < \infty$. *Then* $\mathrm{frm}(B_{\ell_p}) = S_{\ell_p}$.

Proof. In the case of $1 < p < \infty$, the space ℓ_p is uniformly convex, and so

$$S_{\ell_p} = \mathrm{ext}(S_{\ell_p}) \subset \mathrm{frm}(B_{\ell_p}) \subset S_{\ell_p}$$

by Corollary 3.1.12. Thus, we have $\mathrm{frm}(B_{\ell_p}) = S_{\ell_p}$. So, we suppose that $p = 1$.

Let c_{00} be the vector space of finitely nonzero sequences. Then, c_{00} is a dense subspace of the space ℓ_1. Take an arbitrary $x \in S_{c_{00}}$. Then, x has the form

$$x = (a_1, \ldots, a_m, 0, 0, \ldots).$$

Suppose that $f \in \nu(x)$. Since $\ell_1^* = \ell_\infty$, there exists $y = (b_n)_{n=1}^\infty \in S_{\ell_\infty}$ such that

$$f((c_n)_{n=1}^\infty) = \sum_{n=1}^\infty b_n c_n$$

for all $(c_n)_{n=1}^\infty \in \ell_1$. We may assume that y also has the form

$$y = (b_1, \ldots, b_m, 0, 0, \ldots).$$

Then, the kernel of f is given by

$$\ker f = \left\{ (c_n)_{n=1}^\infty : \sum_{i=1}^m b_i c_i = 0 \right\}.$$

In particular, we have $e_{m+1} \in \ker f \setminus \{0\}$. On the other hand, since

$$\|x + te_{m+1}\|_1 = \sum_{i=1}^m |a_i| + t = 1 + t > 1$$

for all $t > 0$, it follows that $x \in \mathrm{frm}(B_{\ell_1})$ by Proposition 3.1.11 (i). Therefore, one has that $S_{c_{00}} \subset \mathrm{frm}(B_{\ell_1})$. However, $\mathrm{frm}(B_{\ell_1})$ is closed by Theorem 3.1.16, and thus

$$S_{\ell_1} = \overline{S_{c_{00}}} \subset \mathrm{frm}(B_{\ell_1}).$$

This shows $\mathrm{frm}(B_{\ell_1}) = S_{\ell_1}$ □

From the case $p = 1$ of the preceding theorem, it turns out that the equality $\mathrm{frm}(B_M) = \mathrm{frm}(B_X) \cap M$ does not hold in general; see Proposition 3.1.11 (iv). Indeed, let M be the two-dimensional subspace of ℓ_1 given by

$$M = \{(a_1, a_2, 0, 0, \ldots) : a_1, a_2 \in \mathbb{R}\}.$$

Then, M is identified with the space $(\mathbb{R}^2, \|\cdot\|_1)$ in a natural manner. Hence, we have

$$\begin{aligned}
\mathrm{frm}(B_M) &= \{(\pm 1, 0, 0, \ldots), (0, \pm 1, 0, 0, \ldots)\} \\
&\subsetneq \{(a_1, a_2, 0, 0, \ldots) : |a_1| + |a_2| = 1\} \\
&= \mathrm{frm}(B_{\ell_1}) \cap M
\end{aligned}$$

by Proposition 3.1.2.

Finally, we determine the set $\mathrm{frm}(B_{\ell_\infty})$.

Theorem 3.1.23 ([80]). *Let*

$$A = \{(a_n)_{n=1}^{\infty} \in S_{\ell_\infty} : |a_m a_n| = 1 \text{ for some } m, n \in \mathbb{N} \text{ with } m \neq n\} \text{ and}$$
$$B = \left\{(a_n)_{n=1}^{\infty} \in S_{\ell_\infty} : \limsup_{n\to\infty} |a_n| = 1\right\},$$

respectively. Then $\mathrm{frm}(B_{\ell_\infty}) = A \cup B$.

Proof. Let $(a_n)_{n=1}^{\infty}$ be an element of $S_{\ell_\infty} \setminus (A \cup B)$. Then, one can easily show that there exists a unique $m \in \mathbb{N}$ such that $|a_m| = 1$, and that

$$k = \|(a_1, \ldots, a_{m-1}, 0, a_{m+1}, \ldots)\|_\infty < 1.$$

Hence, as in the proof of Theorem 3.1.21, we have $(a_n)_{n=1}^{\infty} \notin \mathrm{frm}(B_{\ell_\infty})$.

For the converse, take an arbitrary $(a_n)_{n=1}^{\infty} \in A \cup B$. In the case of $(a_n)_{n=1}^{\infty} \in A$, an argument similar to that in the latter half of the proof of Theorem 3.1.21 shows $(a_n)_{n=1}^{\infty} \in \mathrm{frm}(B_{\ell_\infty})$. So, we suppose that $(a_n)_{n=1}^{\infty} \in B$. Then, there exists a subsequence $(a_{n_k})_{k=1}^{\infty}$ of $(a_n)_{n=1}^{\infty}$ such that $\lim_{k\to\infty} |a_{n_k}| = 1$. We may assume that $a_{n_k} \neq 0$ for all $k \in \mathbb{N}$. For each subsequence $N = (n_{k_l})_{l=1}^{\infty}$ of $(n_k)_{k=1}^{\infty}$, let $e(N) = (e_{N,n})_{n=1}^{\infty}$ where $e_{N,n}$ is given by

$$e_{N,n} = \begin{cases} |a_n|^{-1} a_n & (n \in N), \\ 0 & (n \notin N), \end{cases}$$

for all $n \in \mathbb{N}$. Then, one has that $e(N) \in \ell_\infty \setminus \{0\}$ for all subsequence N of $(n_k)_{k=1}^{\infty}$.

Let f be an element of $\nu((a_n)_{n=1}^{\infty})$. First, we assume that $f(e(N)) = 0$ for some subsequence $N = (n_{k_l})_{l=1}^{\infty}$ of $(n_k)_{k=1}^{\infty}$. It follows from $\lim_{l\to\infty} |a_{n_{k_l}}| = 1$ that for any $t > 0$ there exists $l \in \mathbb{N}$ such that $|a_{n_{k_l}}| > 1 - t$. Therefore, we have

$$\begin{aligned} \|(a_n)_{n=1}^{\infty} + te(N)\|_\infty &\geq \left| a_{n_{k_l}} + te_{N,n_{k_l}} \right| \\ &= \left| a_{n_{k_l}} + t\frac{a_{n_{k_l}}}{|a_{n_{k_l}}|} \right| \\ &= |a_{n_{k_l}}| + t > 1. \end{aligned}$$

Since $e(N) \in \ker f \setminus \{0\}$, we obtain $(a_n)_{n=1}^{\infty} \in \mathrm{frm}(B_{\ell_\infty})$ by Proposition 3.1.11 (i).

Next, suppose that $f(e(N)) \neq 0$ for any subsequence N of $(n_k)_{k=1}^{\infty}$. Let $N_1 = (n_{2k-1})_{k=1}^{\infty}$ and $N_2 = (n_{2k})_{k=1}^{\infty}$, respectively. Then, N_1 and N_2 are subsequences of $(n_k)_{k=1}^{\infty}$ such that $N_1 \cap N_2 = \emptyset$. Putting

$$u = e(N_1) - \frac{f(e(N_1))}{f(e(N_2))} e(N_2),$$

we have $u = (u_n)_{n=1}^\infty \in \ker f \setminus \{0\}$ and $u_{n_{2k-1}} = e_{N_1, n_{2k-1}}$ for all $k \in \mathbb{N}$. From the fact that $\lim_{k\to\infty} |a_{n_{2k-1}}| = 1$, as in the preceding paragraph, we have $(a_n)_{n=1}^\infty \in \mathrm{frm}(B_{\ell_\infty})$. This completes the proof. $\square$

Remark 3.1.24. In a sense, the space ℓ_1 is similar to the space ℓ_∞. For example, the spaces ℓ_1 and ℓ_∞ are neither strictly convex nor smooth. Moreover, each extreme point of the unit balls of ℓ_1 or ℓ_∞ is one of its vertices, respectively. Namely, one has that

$$\begin{aligned} \mathrm{ext}(B_{\ell_1}) &= \{\pm e_m : m \in \mathbb{N}\}, \\ \mathrm{ext}(B_{\ell_\infty}) &= \{(\varepsilon_n)_{n=1}^\infty : \varepsilon_n = \pm 1 \text{ for all } n \in \mathbb{N}\}, \end{aligned}$$

where e_m is the m-th standard coordinate vector. However, Theorems 3.1.22 and 3.1.23 show that the shape of $\mathrm{frm}(B_{\ell_1})$ is completely different from that of $\mathrm{frm}(B_{\ell_\infty})$.

3.2 A further property of spherical isometries

Here we will show that every spherical isometries preserves the frames of the unit balls. We start this section with the following simple characterization of $\mathrm{frm}(B_X)$.

Theorem 3.2.1 ([81])**.** *Let X be a Banach space, and let $x \in S_X$. Then $x \notin \mathrm{frm}(B_X)$ if and only if $(x + tB_X) \cap S_X$ is convex for some $t > 0$.*

Proof. Suppose that $x \notin \mathrm{frm}(B_X)$. Then x is a smooth point of B_X by [80, Lemma 4.1], and we have $x + rB_{\ker \nu(x)} \subset F(\nu(x))$ for some $r > 0$. Putting $t = r/(2+r)$, it follows that $(x + tB_X) \cap S_X = x + tB_{\ker \nu(x)}$. Indeed, for each $y \in x + tB_X$, one has $y = x + tz$ for some $z \in B_X$, or

$$\begin{aligned} (2+r)y &= (2+r)x + rz \\ &= (2 + r + r\langle z, \nu(x)\rangle)x + r(z - \langle z, \nu(x)\rangle x) \\ &= (2 + r + r\langle z, \nu(x)\rangle)\left(x + \frac{r}{2 + r + r\langle z, \nu(x)\rangle}(z - \langle z, \nu(x)\rangle x)\right). \end{aligned}$$

We now remark that

$$\frac{1}{2 + r + r\langle z, \nu(x)\rangle}(z - \langle z, \nu(x)\rangle x) \in B_{\ker \nu(x)}$$

since $\|z - \langle z, \nu(x)\rangle x\| \le 2$ and $2 + r + r\langle z, \nu(x)\rangle \ge 2$, which implies that

$$y \in \frac{2 + r + r\langle z, \nu(x)\rangle}{2+r}(x + rB_{\ker \nu(x)}) \subset \frac{2 + r + r\langle z, \nu(x)\rangle}{2+r} S_X.$$

This shows $y \in S_X$ if and only if $\langle z, \nu(x) \rangle = 0$, and hence we obtain $(x + tB_X) \cap S_X \subset x + tB_{\ker \nu(x)}$. The other inclusion is obvious.

Conversely, we assume that $(x + tB_X) \cap S_X$ is convex for some $t > 0$. Suppose that there exists $y \in S_X$ such that $x \perp_B^+ y$. Let $z(r) = \|x + ry\|^{-1}(x + ry)$ for all $r \in \mathbb{R}$. Then we have $\max\{\|z(r) - x\|, \|z(-r) - x\|\} \le t$ for some $r > 0$, that is, $z(r), z(-r) \in (x + tB_X) \cap S_X$. However, putting $k = (\|x + ry\| + \|x - ry\|)/2$ and $\lambda = \|x+ry\|/(\|x+ry\|+\|x-ry\|)$, one obtains $k > 1$ and $x = k((1-\lambda)z(r)+\lambda z(-r))$. This is a contradiction, which together with Theorem 3.1.11 (ii) prove the theorem. □

As a consequence, we obtain another formulation of $\mathrm{frm}(B_X)$.

Corollary 3.2.2 ([81])**.** *Let X be a Banach space. Then* $\mathrm{frm}(B_X) = \{x \in S_X : (x + tB_X) \cap S_X$ is not convex for all $t > 0\}$.

For our purpose, we need three lemmas which can be essentially found in Cheng and Dong [16]; see also Holmes [31, Exercise 2.18] for the first one. The proofs are given only for the sake of completeness, and based on the original ones except for the former half of the third one.

For each $x \in S_X$, let $\mathrm{st}(x, S_X) = \{y \in S_X : \|x + y\| = 2\}$. Then we remark that $C \subset \mathrm{st}(x, S_X)$ whenever C is a convex subset of S_X and $x \in C$.

Lemma 3.2.3 (Holmes [31]; Cheng and Dong [16])**.** *Let X be a separable Banach space. Suppose that C is a maximal convex subset of S_X. Then $C = \mathrm{st}(x, S_X)$ for some $x \in C$.*

Proof. Let $\{x_n\}_{n=1}^\infty$ be a dense subset of C, and let $x_0 = \sum_n 2^{-n} x_n$. Then, for each $f \in \nu(x_0)$, we have $f(x_n) = 1$ for all $n \in \mathbb{N}$, which implies that $C \subset F(f)$. This and the maximality of C together imply that $C = F(f)$. Now, take an arbitrary $y \in \mathrm{st}(x_0, S_X)$. For an element g of $\nu(2^{-1}(x_0 + y))$, one has $g(x_0) = g(y) = 1$, or $g \in \nu(x_0)$ and $y \in F(g) = C$. This completes the proof. □

Lemma 3.2.4 (Cheng and Dong [16])**.** *Let X and Y be Banach spaces, and let A be a separable subset of X. Suppose that $T_0 : S_X \to S_Y$ is a surjective isometry. Then there exist separable closed subspaces $X_0 \subset X$ and $Y_0 \subset Y$ such that $A \subset X_0$ and $T_0(S_{X_0}) = S_{Y_0}$.*

Proof. Let $M_1 = [A]$ and $N_1 = [T_0(S_{M_1})]$, respectively. Define the closed subspaces $M_k \subset X$ and $N_k \subset Y$ inductively by $M_k = [T_0^{-1}(S_{N_{k-1}})]$ and $N_k = [T_0(S_{M_k})]$ for all $k \ge 2$. Then one can show that $M_0 = \overline{\bigcup_{k \in \mathbb{N}} M_k}$ and $N_0 = \overline{\bigcup_{k \in \mathbb{N}} N_k}$ have the desired properties. □

Lemma 3.2.5 (Cheng and Dong [16]). *Let X and Y be Banach spaces. Suppose that $T_0 : S_X \to S_Y$ is a surjective isometry. If C is a maximal convex subset of S_X, then $T_0(C)$ is a maximal convex subset of S_Y.*

Proof. Once it has been proved that the lemma is true for separable Banach spaces, then one can prove the general case from it. Indeed, applying the preceding lemma to each finite subset $A \subset C$, we have separable subspaces $M_A \subset X$ and $N_A \subset Y$ such that $A \subset M_A$ and $T_0(S_{M_A}) = S_{N_A}$. Let $C_A = C \cap M_A$, and let K_A be a maximal convex subset of S_{M_A} such that $C_A \subset K_A$. Then, the separability of M_A ensures that $T_0(K_A)$ is a maximal convex subset of S_{N_A}, or $\operatorname{co} T_0(C_A) \subset S_{N_A}$, which in turn implies that $\operatorname{co} T_0(C) \subset S_Y$. Let K be a maximal convex subset of S_Y such that $T_0(C) \subset K$. Using the above argument for T_0^{-1} and K, we also have $C \subset T_0^{-1}(K) \subset \operatorname{co} T_0^{-1}(K) \subset S_X$. This shows $T_0(C) = K$, as desired.

Now, suppose that both X and Y are separable. Then Lemma 3.2.3 assures that $C = \operatorname{st}(x, S_X)$ for some $x \in C$. Let K be a maximal convex subset of S_Y such that $T_0 x \in K$. Then, similarly by Lemma 3.2.3, there exists $y \in K$ such that $K = \operatorname{st}(y, S_Y)$. We observe that $x \in T_0^{-1}(K) = \operatorname{st}(-T_0(-y_0), S_X)$ means $-T_0(-y_0) \in C$. This and the convexity of C guarantee that $C \subset T_0^{-1}(K)$, or $\operatorname{co} T_0(C) \subset S_Y$. The proof is completed by an argument similar to that at the end of the preceding paragraph. □

Remark 3.2.6. The finite-dimensional case of the preceding lemma is due to Tingley [84, Lemma 13].

We now present a further geometric property of spherical isometries.

Theorem 3.2.7 ([81]). *Let X and Y be Banach spaces. Suppose that $T_0 : S_X \to S_Y$ is a surjective isometry. Then $T_0(\operatorname{frm}(B_X)) = \operatorname{frm}(B_Y)$.*

Proof. Let $x \notin \operatorname{frm}(B_X)$. Then Theorem 3.2.1 guarantees that $(x + tB_X) \cap S_X$ is convex for some $t > 0$. Let C be a maximal convex subset of S_X such that $(x + tB_X) \cap S_X \subset C$. From the identity $(x + tB_X) \cap S_X = ((x + tB_X) \cap S_X) \cap C$, we obtain

$$\begin{aligned}(T_0 x + tB_Y) \cap S_Y &= T_0((x + tB_X) \cap S_X)\\ &= T_0(((x + tB_X) \cap S_X) \cap C)\\ &= ((T_0 x + tB_Y) \cap S_Y) \cap T_0(C)\\ &= (T_0 x + tB_Y) \cap T_0(C).\end{aligned}$$

Applying Lemma 3.2.5, one has that $T_0(C)$ is also a maximal convex subset of S_Y. Thus the set $(T_0 x + tB_Y) \cap S_Y$ is convex, which together with Theorem 3.2.1 imply that $T_0x \notin \mathrm{frm}(B_Y)$. Finally, since T_0^{-1} is also a surjective isometry, we have $T_0(S_X \setminus \mathrm{frm}(B_X)) = S_Y \setminus \mathrm{frm}(B_Y)$, and the theorem follows from the bijectivity of T_0. $\square$

By Theorems 3.1.5 and 3.2.7, we immediately have the following corollary.

Corollary 3.2.8 ([81]). *Let X and Y be n-dimensional Banach spaces. Suppose that $T_0 : S_X \to S_Y$ is a surjective isometry. Then $T_0(\mathrm{ext}_{n-1}(B_X)) = \mathrm{ext}_{n-1}(B_Y)$.*

We wonder whether Theorem 3.2.7 and Corollary 3.2.8 remain true for the sets of all stronger k-extreme points. Namely, does every spherical isometry T_0 satisfy $T_0(\mathrm{ext}_k(B_X)) = \mathrm{ext}_k(B_Y)$ for all $k \in \mathbb{N}$? As a remark, it is known that $\mathrm{ext}(B_X) \neq \emptyset$ if and only if $\mathrm{ext}_k(B_X) \neq \emptyset$ for some $k \in \mathbb{N}$. Indeed, Let X be a Banach space, and let $k \in \mathbb{N}$. We first show that $x \notin \mathrm{ext}_k(B_X)$ if and only if there exists a subspace M such that $\dim M \geq k$ and $x + tB_M \subset S_X$ for some $t > 0$. Suppose that $x \notin \mathrm{ext}_k(B_X)$. Then $x = (k+1)^{-1}\sum_{i=1}^{k+1} x_i$ for some linearly independent subset $\{x_i\}_{i=1}^{k+1}$ of S_X. We remark that $\mathrm{co}(\{x_i\}_{i=1}^{k+1}) \subset F(f)$ whenever $f \in \nu(x)$. Let $M = [\{-x + x_i\}_{i=1}^{k+1}]$. It is easy to see that $\{-x+x_i\}_{i=1}^{k}$ is linearly independent, and so $\dim M = k$. Moreover, it follows from the identity $x = (k+1)^{-1}\sum_{i=1}^{k+1} x_i$ that $k^{-1}(x - x_j) = k^{-1}\sum_{i\neq j}(-x+x_i) \in -x+F(f)$ for all $1 \leq j \leq k$, which together with $0 \in -x + F(f)$ imply that $k^{-1}\,\mathrm{aco}(\{-x + x_i\}_{i=1}^{k}) \subset -x+F(f)$. This shows $tB_M \subset -x+F(f)$ for some $t > 0$, or $x+tB_M \subset S_X$. Conversely, assume that there exists a subspace M of X such that $\dim M \geq k$ and $x + tB_M \subset S_X$ for some $t > 0$. We remark that $x \notin M$ from the assumption. Let $\{e_i\}_{i=1}^{k}$ be a linearly independent subset of M, and let $e_{k+1} = -\sum_{i=1}^{k} e_i$. Putting $L = t^{-1}\max_{1\leq i\leq k+1}\|e_i\|$, one can show that $\{x + L^{-1}e_i\}_{i=1}^{k+1} \subset S_X$ is also linearly independent and that $x = (k+1)^{-1}\sum_{i=1}^{k+1}(x + L^{-1}e_i)$. Hence it follows that $x \notin \mathrm{ext}_k(B_X)$. We note that this equivalence easily shows $\mathrm{ext}_k(B_X) \subset \mathrm{ext}_{k+1}(B_X)$ for all $k \in \mathbb{N}$.

Now, suppose that $\mathrm{ext}_k(B_X) \neq \emptyset$ for some $k \in \mathbb{N}$. Take an arbitrary $x \in \mathrm{ext}_k(B_X)$. We assume that $x \notin \mathrm{ext}(B_X)$. Then there exist two distinct elements $y, z \in S_X$ and $s \in (0, 1)$ such that $x = (1-s)y + sz$. Let $z(t) = (1-t)y + tz$ for all $t \in \mathbb{R}$. Since the function $t \to \|z(t)\|$ is convex, it follows that $\{t \in \mathbb{R} : z(t) \in S_X\} = [t_1, t_2]$ for some $t_1 \leq 0$ and $1 \leq t_2$. Without loss of generality, we may assume that $t_1 = 0$ and $t_2 = 1$. It is enough to prove that $y \in \mathrm{ext}_{k-1}(B_X)$. To this end, suppose contrary that $y \notin \mathrm{ext}_{k-1}(B_X)$. As was shown in above, there exists a subspace M of X such that $\dim M \geq k-1$ and $y + tB_M \subset S_X$ for some $t > 0$. Let $\{e_i\}_{i=1}^{k-1}$ be a linearly independent subset of S_M. Then one has $x \pm (1-s)te_i \in F(f)$ for all $1 \leq i \leq k-1$ whenever $f \in \nu(x)$. Putting $r = \min\{1-s, s, (1-s)t\}$ and $e_k = z - y$, it follows that $r > 0$ and $\mathrm{aco}(\{re_i\}_{i=1}^{k}) \subset -x + F(f)$. This gurantees there exists $r_0 > 0$ such that $x + r_0 B_N \subset S_X$, where $N = [\{e_i\}_{i=1}^k]$. Since it can be shown that $\dim N = k$, we have $x \notin \mathrm{ext}_k(B_X)$ by the argument in the preceding paragraph, a contradiction which proves $y \in \mathrm{ext}_{k-1}(B_X)$. Thus $\mathrm{ext}(B_X) \neq \emptyset$ follows by an induction, and so we should assume that $\mathrm{ext}(B_X) \neq \emptyset$ when considering the above problem.

We finally mention two problems which naturally arise from Theorem 3.2.7. The first one is a Mazur-Ulam type problem.

Problem 3.2.9. Let X and Y be Banach spaces. Suppose that $T_0 : \mathrm{frm}(B_X) \to \mathrm{frm}(B_Y)$ is a surjective isometry. Then, does T_0 have a linear isometric extension $T : X \to Y$?

Needless to say, this is more difficult than Tingley's problem unless the following is solved positively.

Problem 3.2.10. Let X and Y be Banach spaces. Suppose that $T_0 : \mathrm{frm}(B_X) \to \mathrm{frm}(B_Y)$ is a surjective isometry. Then, does T_0 have an isometric extension $\widetilde{T_0} : S_X \to S_Y$?

Remark 3.2.11. If no assumptions are added, both Problems 3.2.9 and 3.2.10 have negative answers in the case $\dim X = \dim Y = 2$. Indeed, let $X = Y = \ell_\infty^2$. Then $\mathrm{frm}(B_X) = \mathrm{ext}(B_X) = \{(1,1),(1,-1),(-1,1),(-1,-1)\}$. Define an operator T_0 on $\mathrm{frm}(B_X)$ by $T_0(1,1) = (1,1)$, $T_0(1,-1) = (1,-1)$, $T_0(-1,1) = (-1,-1)$ and $T_0(-1,-1) = (-1,1)$. This is a counter example of the problems since T_0 does not map antipodal pairs of points to such pairs. Hence, in the case $\dim X = \dim Y = 2$, we at least need an assumption which implies $T_0(-x) = -T_0 x$ for all $x \in \mathrm{frm}(B_X)$.

3.3 Another approach to Tingley's problem

In this section, we construct new methods for Tingley's problem on two-dimensional spaces. We first recall the following result of Tingley [84].

Lemma 3.3.1 (Tingley [84])**.** *Let X and Y be finite dimensional normed spaces. Suppose that $T_0 : S_X \to S_Y$ is a surjective isometry. Then $T_0(-x) = -T_0 x$ for all $x \in S_X$.*

It is known that if there exists a surjective isometry between the unit spheres of two finite dimensional normed spaces then the dimensions of the spaces coincide. Though this is in fact a topological result, we here give another proof for the two-dimensional case by using isosceles orthogonality. Recall that an element x of a Banach space is said to be isosceles orthogonal to another element y, denoted by $x \perp_I y$, if $\|x+y\| = \|x-y\|$.

Recall also that isosceles orthogonality has the uniqueness property; see Lemma 1.1.3. Let X be a normed space. For each $x \in S_X$, define a subset $I(x)$ of S_X by $I(x) = \{y \in S_X : x \perp_I y\}$.

Lemma 3.3.2. *Let X be a two-dimensional normed space, and let Y be a normed space. If there exists a surjective isometry $T_0 : S_X \to S_Y$, then* $\dim Y = 2$.

Proof. First we note that S_Y is compact since it is a continuous image of the compact set S_X. Then it follows that Y is finite dimensional, and hence, we have $\dim Y \geq 2$ by Lemma 3.3.1. Indeed, if $\{u, v\}$ is a basis for X then $\|u \pm v\| > 0$. Since $T_0(-x) = -T_0 x$ for all $x \in S_X$, we also have $\|T_0 u \pm T_0 v\| > 0$, which implies that $\{T_0 u, T_0 v\}$ is linearly independent.

Take an arbitrary $x \in S_X$. Then we have $T_0(I(x)) = I(T_0 x)$, that is, $|I(x)| = |I(T_0 x)|$. To show that $\dim Y = 2$, suppose contrary that $\dim Y > 2$. Let $\{T_0 x, e_1, e_2\}$ be a linearly independent set of Y. For each $\theta \in [0, \pi)$, define a two-dimensional subspace M_θ of Y by $M_\theta = \langle\{T_0 x, \cos\theta e_1 + \sin\theta e_2\}\rangle$. Then Lemma 1.1.3 assures that there exists $y_\theta \in S_{M_\theta}$ such that $y_\theta \in I(T_0 x)$. From the fact that $M_{\theta_1} \cap M_{\theta_2} = \langle\{T_0 x\}\rangle$ for $\theta_1, \theta_2 \in [0, \pi)$ with $\theta_1 \neq \theta_2$, we know that $y_{\theta_1} \neq y_{\theta_2}$, which in turn implies that $|I(T_0 x)| = \infty$. However, this is impossible since $|I(x)| = 2$ by the uniqueness property of isosceles orthogonality. This completes the proof. $\square$

To prove key lemmas, we make use of the following result of Alonso and Martín [6].

Lemma 3.3.3 (Alonso and Martín [6])**.** *Let $(\mathbb{R}^2, \|\cdot\|)$ be a normed space, and let*

$$x(\theta) = \frac{(\cos\theta, \sin\theta)}{\|(\cos\theta, \sin\theta)\|}$$

for all $\theta \in \mathbb{R}$. Suppose that $\theta_0 \in \mathbb{R}$. Then the functions $\theta \to \|x(\theta) + x(\theta_0)\|$ and $\theta \to \|x(\theta) - x(\theta_0)\|$ are, respectively, decreasing and increasing on $[\theta_0, \theta_0 + \pi]$.

For each $x, y \in S_X$ with $x + y \neq 0$, let $A(x, y)$ be the arc of S_X from x to y, that is,

$$A(x, y) = \left\{ \frac{(1-t)x + ty}{\|(1-t)x + ty\|} : t \in [0, 1] \right\}.$$

Then we have the following lemma.

Lemma 3.3.4 ([82])**.** *Let X be a two-dimensional normed space. Suppose that $x, y \in S_X$, and that $x \pm y \neq 0$. Then there exists an element $z \in A(x, y)$ such that $\|z - x\| = \|z - y\| \leq \|x - y\|$. Furthermore, such an element is unique in $A(x, y)$.*

Proof. For each $t \in \mathbb{R}$, let $z(t) = (1-t)x + ty$ and $w(t) = \|z(t)\|^{-1}z(t)$, respectively. Define two functions $f, g : [0,1] \to \mathbb{R}$ by the formulas $f(t) = \|w(t) - x\|$ and $g(t) = \|w(t) - y\|$. Then the preceding lemma assures that f and g are, respectively, non-decreasing and non-increasing on $[0,1]$. Define a function $h : [0,1] \to \mathbb{R}$ by $h(t) = f(t) - g(t)$. Then h is continuous and

$$h(0) = -\|x - y\| < 0 < \|x - y\| = h(1).$$

Therefore the intermediate value theorem assures that there exists $t_0 \in (0,1)$ such that $h(t_0) = 0$, that is, $\|w(t_0) - x\| = \|w(t_0) - y\| \leq \|x - y\|$. So what remains is to prove the uniqueness of such an element. If $\|z(t)\| = 1$ for some $t \in (0,1)$, then it follows that $\|z(t)\| = 1$ for all $t \in [0,1]$. The uniqueness is obvious in this case. So we assume that $\|z(t)\| < 1$ for all $t \in (0,1)$. To see the uniqueness, suppose contrary that there exist $t_1, t_2 \in (0,1)$ such that $t_1 < t_2$ and $h(t_1) = h(t_2) = 0$. Then we have

$$0 = h(t_2) - h(t_1) = f(t_2) - f(t_1) + g(t_1) - g(t_2).$$

By the preceding paragraph, it follows that $f(t_1) = f(t_2) = g(t_2) = g(t_1) = k > 0$. For $i = 1, 2$, put

$$x_i = \frac{w(t_i) - x}{k} \quad \text{and} \quad y_i = \frac{w(t_i) - y}{k},$$

respectively. Then $x_i, y_i \in S_X$ for $i = 1, 2$. Since

$$t_1 z(t_2) - \|z(t_2)\| t_1 x = \|z(t_1)\| t_2 w(t_1) - (t_2 - (1 - \|z(t_2)\|)t_1)x,$$

we first note that $x_1 = \alpha((1-\lambda)x_2 + \lambda w(t_1))$, where

$$\begin{aligned} \alpha &= \frac{(1 - \|z(t_1)\|)t_2 - (1 - \|z(t_2)\|)t_1 + k\|z(t_2)\|t_1}{k(t_2 - (1 - \|z(t_2)\|)t_1)}, \\ \lambda &= \frac{(1 - \|z(t_1)\|)t_2 - (1 - \|z(t_2)\|)t_1}{(1 - \|z(t_1)\|)t_2 - (1 - \|z(t_2)\|)t_1 + k\|z(t_2)\|t_1}. \end{aligned}$$

Then we remark that $\lambda \in [0,1]$. Indeed, since

$$t_1 = \left(1 - \frac{t_1}{t_2}\right) \cdot 0 + \frac{t_1}{t_2} \cdot t_2,$$

one has that

$$\begin{aligned} \|z(t_1)\| &\leq \left(1 - \frac{t_1}{t_2}\right)\|z(0)\| + \frac{t_1}{t_2}\|z(t_2)\| \\ &= 1 - \frac{(1 - \|z(t_2)\|)t_1}{t_2}, \end{aligned}$$

which implies that $(1-\|z(t_1)\|)t_2-(1-\|z(t_2)\|)t_1 \geq 0$. This shows that $\lambda \in [0,1]$. Therefore we obtain

$$1=\|x_1\|=\alpha\|(1-\lambda)x_2+\lambda w(t_1)\| \leq \alpha,$$

and so

$$\begin{aligned}
&(1-k-\|z(t_1)\|)t_2-(1-k-\|z(t_2)\|)t_1 \\
&=(1-\|z(t_1)\|)t_2-(1-\|z(t_2)\|)t_1+k\|z(t_2)\|t_1-k(t_2-(1-\|z(t_2)\|)t_1) \\
&\geq 0.
\end{aligned}$$

We next remark that $w(t_2)=\beta((1-\mu)x_2+\mu w(t_1))$, where

$$\beta=\frac{\|z(t_1)\|t_2+k(t_2-t_1)}{\|z(t_2)\|t_1+t_2-t_1} \quad \text{and} \quad \mu=\frac{\|z(t_1)\|t_2}{\|z(t_1)\|t_2+k(t_2-t_1)}.$$

It follows from $\mu \in [0,1]$ that $\beta \geq 1$, which implies that

$$\begin{aligned}
&(1-k-\|z(t_2)\|)t_1-(1-k-\|z(t_1)\|)t_2 \\
&=\|z(t_1)\|t_2+k(t_2-t_1)-(\|z(t_2)\|t_1+t_2-t_1) \geq 0.
\end{aligned}$$

Hence one has $(1-k-\|z(t_1)\|)t_2-(1-k-\|z(t_2)\|)t_1=0$ and $\alpha=\beta=1$.

Similarly, putting

$$\gamma=\frac{\|z(t_2)\|(1-t_1)+k(t_2-t_1)}{\|z(t_1)\|(1-t_2)+t_2-t_1} \quad \text{and} \quad \nu=\frac{\|z(t_2)\|(1-t_1)}{\|z(t_2)\|(1-t_1)+k(t_2-t_1)}$$

we have $w(t_1)=\gamma((1-\nu)y_1+\nu w(t_2))$. Since $\nu \in [0,1]$, one has that $\gamma \geq 1$. Therefore we obtain

$$\begin{aligned}
0 &\leq \|z(t_2)\|(1-t_1)+k(t_2-t_1)-(\|z(t_1)\|(1-t_2)+t_2-t_1) \\
&=(1-k-\|z(t_2)\|)t_1-(1-k-\|z(t_1)\|)t_2+\|z(t_2)\|-\|z(t_1)\| \\
&=\|z(t_2)\|-\|z(t_1)\|.
\end{aligned}$$

On the other hand, since

$$\begin{aligned}
&(1-t_2)z(t_1)-\|z(t_1)\|(1-t_2)y \\
&=\|z(t_2)\|(1-t_1)w(t_2)-(t_2-t_1+\|z(t_1)\|(1-t_2))y,
\end{aligned}$$

it follows that $y_2=\delta((1-\xi)y_1+\xi w(t_2))$, where

$$\delta=\frac{(1-\|z(t_1)\|)t_2-(1-\|z(t_2)\|)t_1+\|z(t_1)\|-\|z(t_2)\|+k\|z(t_1)\|(1-t_2)}{k(t_2-t_1+\|z(t_1)\|(1-t_2))},$$

$$\xi=\frac{(1-\|z(t_1)\|)t_2-(1-\|z(t_2)\|)t_1+\|z(t_1)\|-\|z(t_2)\|}{(1-\|z(t_1)\|)t_2-(1-\|z(t_2)\|)t_1+\|z(t_1)\|-\|z(t_2)\|+k\|z(t_1)\|(1-t_2)}.$$

Form the identity

$$t_2 = \frac{1-t_2}{1-t_1} \cdot t_1 + \frac{t_2-t_1}{1-t_1} \cdot 1,$$

we have

$$\begin{aligned}\|z(t_2)\| &\le \frac{1-t_2}{1-t_1}\|z(t_1)\| + \frac{t_2-t_1}{1-t_1}\|z(1)\| \\ &= 1 - \frac{(1-\|z(t_1)\|)(1-t_2)}{1-t_1},\end{aligned}$$

which implies that

$$\frac{1-\|z(t_1)\|}{1-t_1} \le \frac{1-\|z(t_2)\|}{1-t_2}.$$

Since $\|z(t)\| < 1$ for all $t \in [0,1]$, we obtain

$$\frac{1-t_2}{1-\|z(t_2)\|} \le \frac{1-t_1}{1-\|z(t_1)\|}.$$

Then it follows that

$$\frac{t_1-\|z(t_1)\|}{1-\|z(t_1)\|} = 1 - \frac{1-t_1}{1-\|z(t_1)\|} \le 1 - \frac{1-t_2}{1-\|z(t_2)\|} = \frac{t_2-\|z(t_2)\|}{1-\|z(t_2)\|}.$$

This shows

$$\begin{aligned}&(1-\|z(t_1)\|)t_2 - (1-\|z(t_2)\|)t_1 + \|z(t_1)\| - \|z(t_2)\| \\ &= (t_2-\|z(t_2)\|)(1-\|z(t_1)\|) - (t_1-\|z(t_1)\|)(1-\|z(t_2)\|) \ge 0.\end{aligned}$$

Thus we have $\xi \in [0,1]$ and $\delta \ge 1$. Moreover, form the fact that $\delta \ge 1$, one has

$$\begin{aligned}0 &\le (1-\|z(t_1)\|)t_2 - (1-\|z(t_2)\|)t_1 + \|z(t_1)\| - \|z(t_2)\| + k\|z(t_1)\|(1-t_2) \\ &\qquad - k(t_2 - t_1 + \|z(t_1)\|(1-t_2)) \\ &= (1-k-\|z(t_1)\|)t_2 - (1-k-\|z(t_2)\|)t_1 + \|z(t_1)\| - \|z(t_2)\| \\ &= \|z(t_1)\| - \|z(t_2)\|,\end{aligned}$$

which in turn implies that $\|z(t_1)\| = \|z(t_2)\| = \ell \in (0,1)$ and $\gamma = \delta = 1$. This also means $(1-k-\ell)t_2 = (1-k-\ell)t_1$. However, since $t_1 < t_2$, it follows that $1-k-\ell = 0$.

Now, putting

$$z_1 = \frac{1+\ell}{2}y_1 - \frac{1-\ell}{2}x_2 \quad \text{and} \quad z_2 = \frac{1+\ell}{2}x_2 - \frac{1-\ell}{2}y_1,$$

then $z_1, z_2 \in B_X$ and $z_2 - z_1 = x_2 - y_1 = m(y-x)$, where

$$m = \frac{1}{k}\left(1 + \frac{t_2-t_1}{\ell}\right).$$

We remark that $m > 1$ since $k \in (0, 1)$. Furthermore, put

$$\tau = \frac{mk^2 + 2(\ell - t_1)}{2mk}.$$

Then we obtain $x = (1-\tau)z_1 + \tau z_2$ since $1-k-\ell = 0$. Therefore one has that $z_i = z(s_i)$ for $i = 1, 2$, where $s_1 = -\tau m$ and $s_2 = s_1 + m$, respectively. It follows from $m > 1$ that $s_1 < 0$ or $1 < s_2$.

We first assume that $s_1 < 0$. Then $x = (1 - \kappa_1)z(t_1) + \kappa_1 z(s_1)$, where $\kappa_1 = t_1/(t_1 - s_1) \in (0, 1)$. However, we have

$$\begin{aligned} 1 = \|x\| &\leq (1 - \kappa_1)\|z(t_1)\| + \kappa_1\|z(s_1)\| \\ &\leq (1 - \kappa_1)\ell + \kappa_1 < 1 \end{aligned}$$

since $\ell = 1 - k \in (0, 1)$. This is a contradiction.

Finally, suppose that $1 < s_2$. Putting $\kappa_2 = (1 - t_2)/(s_2 - t_2)$, we have $\kappa_2 \in (0, 1)$ and $y = (1 - \kappa_2)z(t_2) + \kappa_2 z(s_2)$. Then, similarly, we obtain

$$\begin{aligned} 1 = \|y\| &\leq (1 - \kappa_2)\|z(t_2)\| + \kappa_2\|z(s_2)\| \\ &\leq (1 - \kappa_2)\ell + \kappa_2 < 1, \end{aligned}$$

a contradiction, which proves the lemma. □

The following three technical lemmas will be needed.

Lemma 3.3.5 ([82])**.** *Let X be a two-dimensional normed space. Suppose that $x, y \in S_X$, and that $x \pm y \neq 0$. If $z \in S_X$, $\|z - x\| = \|z - y\| \leq \min\{\|z + x\|, \|z + y\|\}$, and $\|z - x\| = \|z - y\| < \|x - y\|$, then $z \in A(x, y)$.*

Proof. Without loss of generality, we may assume that $X = \mathbb{R}^2$, and that $x = (1, 0)$ and $y = (0, 1)$ since the set $\{x, y\}$ is linearly independent. Define $x(\theta)$ for all $\theta \in \mathbb{R}$ as in the preceding lemma. We remark that $A(x, y) = \{x(\theta) : \theta \in [0, \pi/2]\}$. Let $\theta \in [0, 2\pi)$ such that $\|x(\theta) - x\| = \|x(\theta) - y\| \leq \min\{\|x(\theta) + x\|, \|x(\theta) + y\|\}$ and $\|x(\theta) - x\| = \|x(\theta) - y\| < \|x - y\|$. We show that $\theta \in [0, \pi/2]$.

We first assume that $\theta \in [\pi/2, \pi]$. Then by Lemma 3.3.3, it follows that

$$\|x - y\| = \|x(0) - x(\pi/2)\| \leq \|x(0) - x(\theta)\| = \|x(\theta) - x\| < \|x - y\|,$$

a contradiction. Similarly, if $\theta \in [3\pi/2, 2\pi)$, we obtain

$$\|x - y\| > \|x(\theta) - y\| = \|x(\theta) + x(3\pi/2)\| \geq \|x(2\pi) + x(3\pi/2)\| = \|x - y\|.$$

So we assume that $\theta \in [\pi, 3\pi/2]$. Then we have

$$\|x(\theta) + x\| = \|x(\theta) + x(2\pi)\| \leq \|x(\theta) + x(3\pi/2)\| = \|x(\theta) - y\|$$

and

$$\|x(\theta) + y\| = \|x(\theta) - x(3\pi/2)\| \leq \|x(\theta) - x(2\pi)\| = \|x(\theta) - x\|.$$

These imply that

$$\begin{aligned}\|x(\theta) - x\| = \|x(\theta) - y\| &\leq \min\{\|x(\theta) + x\|, \|x(\theta) + y\|\} \\ &\leq \max\{\|x(\theta) + x\|, \|x(\theta) + y\|\} \\ &\leq \|x(\theta) - x\| = \|x(\theta) - y\|.\end{aligned}$$

Hence we have $\|x(\theta)-x\| = \|x(\theta)-y\| = \|x(\theta)+x\| = \|x(\theta)+y\|$. However, this means that $x(\theta) \perp_I x$ and $x(\theta) \perp_I y$, which implies that $x = \pm y$. This is a contradiction. Thus θ must be in $[0, \pi/2]$. $\square$

Lemma 3.3.6 ([82])**.** *Let X be a two-dimensional normed space. Suppose that $x, y \in S_X$, and that $x \pm y \neq 0$. If $z \in A(x, y)$ and $\|z - x\| = \|z - y\|$, then $\|z - x\| = \|z - y\| \leq \min\{\|z + x\|, \|z + y\|\}$.*

Proof. We assume that $X = \mathbb{R}^2$ and define $x, y, x(\theta)$ as in the preceding lemma. Suppose that $z \in A(x, y)$ and that $\|z - x\| = \|z - y\|$. Let θ_0 be an element of $[0, \pi/2]$ such that $z = x(\theta_0)$. Then, By Lemma 3.3.3, it follows that

$$\|x(\theta_0) - x\| = \|x(\theta_0) + x(\pi)\| \leq \|x(\theta_0) + x(\pi/2)\| = \|x(\theta_0) + y\|$$

and

$$\|x(\theta_0) - y\| = \|x(\theta_0) - x(\pi/2)\| \leq \|x(\theta_0) - x(\pi)\| = \|x(\theta_0) + x\|.$$

Thus we have $\|x(\theta_0) - x\| = \|x(\theta_0) - y\| \leq \min\{\|x(\theta_0) + x\|, \|x(\theta_0) + y\|\}$. $\square$

Lemma 3.3.7 ([82])**.** *Let X be a two-dimensional normed space. Suppose that $x, y \in S_X$, that $x \pm y \neq 0$, and that $\|x - y\| = 2$. If $z \in A(x, y)$ and $\|z - x\| = \|z - y\|$, then $\|z - x\| = \|z - y\| < 2$.*

Proof. Keep the notations as in Lemma 3.3.6. If $\|z - x\| = \|z - y\| = 2$, we have $\|z - x\| = \|z - y\| = \|z + x\| = \|z + y\| = 2$ by the preceding lemma. Then, as in the proof of Lemma 3.3.5, we obtain $x = \pm y$. This is a contradiction. Thus one has $\|z - x\| = \|z - y\| < 2$. $\square$

The following is the second key ingredient for our approach.

Lemma 3.3.8 ([82])**.** *Let X be a two-dimensional normed space, and let Y be a normed space. Suppose that $T_0 : S_X \to S_Y$ is a surjective isometry. Then $T_0(A(x,y)) = A(T_0x, T_0y)$ whenever $x, y \in S_X$ and $x \pm y \neq 0$.*

Proof. We first assume that $\|x - y\| < 2$. Then we have

$$\operatorname{diam}(T_0(A(x,y))) = \operatorname{diam}(A(x,y)) \leq \|x - y\| < 2.$$

Moreover, since $A(x,y)$ is connected in S_X, it follows that $T_0(A(x,y))$ is also a connected subset of S_Y. On the other hand, $\dim Y = 2$ by Lemma 3.3.2, and hence $T_0(A(x,y)) = A(T_0x, T_0y)$ or $T_0(A(x,y)) = (S_Y \setminus A(T_0x, T_0y)) \cup \{T_0x, T_0y\}$. However, it follows from $T_0x, -T_0x \in (S_Y \setminus A(T_0x, T_0y)) \cup \{T_0x, T_0y\}$ that $\operatorname{diam}((S_Y \setminus A(T_0x, T_0y)) \cup \{T_0x, T_0y\}) = 2$. Thus we have $T_0(A(x,y)) = A(T_0x, T_0y)$.

Suppose next that $\|x - y\| = 2$. Let z be the element of $A(x,y)$ such that $\|z - x\| = \|z - y\|$. Then Lemmas 3.3.6 and 3.3.7 assure that $\|z - x\| = \|z - y\| \leq \min\{\|z + x\|, \|z + y\|\}$ and $\|z - x\| = \|z - y\| < 2$. Since T_0 is an isometry, one has that $T_0z \in A(T_0x, T_0y)$ by Lemma 3.3.5. Furthermore, as was shown in the preceding paragraph, it follows that $T_0(A(x,z)) = A(T_0x, T_0z)$ and $T_0(A(z,y)) = A(T_0z, T_0y)$. Thus we have

$$\begin{aligned} T_0(A(x,y)) &= T_0(A(x,z) \cup A(z,y)) \\ &= A(T_0x, T_0z) \cup A(T_0z, T_0y) \\ &= A(T_0x, T_0y). \end{aligned}$$

This completes the proof. □

Remark 3.3.9. Wang [86] proved Lemma 3.3.8 under the additional assumption that both X and Y are two-dimensional and strictly convex. However, we have shown that those assumptions are redundant except $\dim X = 2$.

We now present a new method for Tingley's problem on two-dimensional spaces.

Theorem 3.3.10 ([82])**.** *Let X be a two-dimensional normed space, and let Y be a normed space. Suppose that $T_0 : S_X \to S_Y$ is a surjective isometry. If there exists an isometric isomorphism $T : X \to Y$ such that $T_0x = Tx$ and $T_0y = Ty$ for some $x, y \in S_X$ with $x \pm y \neq 0$, then $T_0 = T|_{S_X}$.*

Proof. Let $x, y \in S_X$ such that $x \pm y \neq 0$, $T_0x = Tx$, and $T_0y = Ty$. As in the proof of Lemma 3.3.5, we may assume that $X = \mathbb{R}^2$, and that $x = (1,0)$ and $y = (0,1)$. Define $x(\theta)$ as in Lemma 3.3.3.

Let I be a subset of $[0, \pi]$ defined by $I = \{\theta \in [0, \pi] : T_0x(\theta) = Tx(\theta)\}$. It is enough to prove that $I = [0, \pi]$. We first take arbitrary $\theta_1, \theta_2 \in I$ such that $0 < \theta_2 - \theta_1 < \pi$. Then there exists $\theta \in (\theta_1, \theta_2)$ such that $\|x(\theta) - x(\theta_1)\| = \|x(\theta) - x(\theta_2)\|$ by Lemma 3.3.4, which implies that $\|T_0x(\theta) - T_0x(\theta_1)\| = \|T_0x(\theta) - T_0x(\theta_2)\|$ and $\|Tx(\theta) - T_0x(\theta_1)\| = \|Tx(\theta) - T_0x(\theta_2)\|$ since $T_0x(\theta_i) = Tx(\theta_i)$ for $i = 1, 2$. However, according to Lemma 3.3.8, both $T_0x(\theta)$ and $Tx(\theta)$ must be in $A(T_0x(\theta_1), T_0x(\theta_2))$. These and Lemma 3.3.4 show that $T_0x(\theta) = Tx(\theta)$.

Now, to see $I = [0, \pi]$, suppose to the contrary that $I \neq [0, \pi]$. Let $\varphi \in [0, \pi] \setminus I$. Since I is closed, there exists a positive number ε such that $(\varphi - \varepsilon, \varphi + \varepsilon) \subset [0, \pi] \setminus I$. From the compactness of $[0, \varphi] \cap I$ and $[\varphi, \pi] \cap I$, there exist $\varphi_1 \in [0, \varphi] \cap I$ and $\varphi_2 \in [\varphi, \pi] \cap I$ satisfying

$$\varphi_2 - \varphi_1 = \min\{\theta_2 - \theta_1 : \theta_1 \in [0, \varphi] \cap I,\ \theta_2 \in [\varphi, \pi] \cap I\}.$$

We remark that $0 < 2\varepsilon \leq \varphi_2 - \varphi_1 \leq \pi/2$ since $0, \pi/2, \pi \in I$. Then, by the preceding paragraph, there exists a $\varphi_3 \in (\varphi_1, \varphi_2)$ such that $\varphi_3 \in I$. However, this contradicts to the choice of φ_1 and φ_2. Thus the set I must coincide with $[0, \pi]$. This completes the proof. $\square$

Suppose that T is a map from a set C into itself. Then an element $x \in C$ is said to be a fixed point of T if $Tx = x$. The set of all fixed points of T is denoted by $F(T)$. Applying the preceding theorem, we immediately have the following result.

Corollary 3.3.11 ([82]). *Let X be a two-dimensional normed space. Suppose that $T_0 : S_X \to S_X$ is a surjective isometry. If there exist $x, y \in S_X \cap F(T_0)$ such that $x \pm y \neq 0$, then $T_0 = I|_{S_X}$, where I is the identity map on X.*

3.4 Applications of geometric constants

Finally, we present some new sufficient conditions for Tingley's problem on symmetric absolute normalized norms on $\mathbb{R}^2$ as applications of the results in the preceding section. We first note the following two properties.

Lemma 3.4.1 ([82]). *Let $\psi \in \Psi_2^S$. Then $\|\cdot\|_\psi$ is $\pi/4$ rotation invariant if and only if*

$$\frac{2 - 2t}{\psi(t)} \psi\left(\frac{1}{2 - 2t}\right) = \sqrt{2}$$

for all $t \in [0, 1/2]$.

Proof. Let

$$R(\pi/4) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Then we have

$$\begin{aligned} \left\| R(\pi/4) \frac{1}{\psi(t)} \begin{pmatrix} 1-t \\ \pm t \end{pmatrix} \right\|_\psi &= \frac{1}{\sqrt{2}\psi(t)} \left\| \begin{pmatrix} 1-t \mp t \\ 1-t \pm t \end{pmatrix} \right\|_\psi \\ &= \frac{1}{\sqrt{2}\psi(t)} \left\| \begin{pmatrix} 1-2t \\ 1 \end{pmatrix} \right\|_\psi \\ &= \begin{cases} \dfrac{2-2t}{\sqrt{2}\psi(t)} \psi\left(\dfrac{1}{2-2t}\right) & \text{if } t \in [0, 1/2], \\ \dfrac{2t}{\sqrt{2}\psi(t)} \psi\left(\dfrac{1}{2t}\right) & \text{if } t \in [1/2, 1], \end{cases} \end{aligned}$$

since $\| \cdot \|_\psi$ is symmetric. Moreover, it follows from $\psi \in \Psi_2^S$ that

$$\frac{2-2t}{\psi(t)} \psi\left(\frac{1}{2-2t}\right) = \sqrt{2}$$

for all $t \in [0, 1/2]$ if and only if

$$\frac{2t}{\psi(t)} \psi\left(\frac{1}{2t}\right) = \sqrt{2}$$

for all $t \in [1/2, 1]$. Thus

$$\left\| R(\pi/4) \frac{1}{\psi(t)} \begin{pmatrix} 1-t \\ \pm t \end{pmatrix} \right\|_\psi = 1$$

for all $t \in [0, 1]$ if and only if

$$\frac{2-2t}{\psi(t)} \psi\left(\frac{1}{2-2t}\right) = \sqrt{2}$$

for all $t \in [0, 1/2]$. This shows the lemma. $\square$

Lemma 3.4.2 ([82])**.** *Let $\psi \in \Psi_2^S$. Suppose that T_0 is an isometry from the unit sphere of $(\mathbb{R}^2, \| \cdot \|_\psi)$ onto itself. If $\| \cdot \|_\psi$ is not $\pi/4$ rotation invariant, then $T_0(1,0) \neq \psi(1/2)^{-1}(1/2, 1/2)$.*

Proof. Put $x(t) = \psi(t)^{-1}(1-t,t)$ and $y(t) = \psi(t)^{-1}(1-t,-t)$ for all $t \in [0,1]$. We remark that $I(x(t)) = \{y \in S_X : x(t) \perp_I y\} = \{\pm y(1-t)\}$ for all $t \in [0,1]$. To see $T_0x(0) \neq x(1/2)$, suppose contrary that $T_0x(0) = x(1/2)$. Since $x \perp_I y$ if and only if $T_0x \perp_I T_0y$, we obtain $T_0x(1) = \pm y(1/2)$. From the symmetry of $\|\cdot\|_\psi$, it may be assumed that $T_0x(1) = y(1/2)$. Then Lemmas 3.3.4 and 3.3.8 imply that $T_0x(1/2) = x(0)$, which guarantees there exists a strictly decreasing continuous function κ on $[0,1/2]$ such that $T_0x(t) = x(\kappa(t))$ for all $t \in [0,1/2]$.

We now consider the function f given by

$$f(t) = \|x(t) \pm y(1-t)\|_\psi = \frac{2-2t}{\psi(t)}\psi\left(\frac{1}{2-2t}\right)$$

for all $t \in [0,1/2]$. Then one easily has

$$f(t)f\left(\frac{1-2t}{2-2t}\right) = 2$$

for all $t \in [0,1/2]$, which implies that

$$\min_{0\le t\le 1/2} f(t) \max_{0\le t\le 1/2} f(t) = 2.$$

Hence, from the assumption, we have

$$\min_{0\le t\le 1/2} f(t) < \sqrt{2} < \max_{0\le t\le 1/2} f(t).$$

Let

$$t_1 = \min\left\{t \in [0,1] : f(t) = \min_{0\le t\le 1/2} f(t)\right\},$$
$$t_2 = \min\left\{t \in [0,1] : f(t) = \max_{0\le t\le 1/2} f(t)\right\},$$

respectively. Then it follows that

$$\frac{1-2t_2}{2-2t_2} = \max\left\{t \in [0,1] : f(t) = \min_{0\le t\le 1/2} f(t)\right\},$$
$$\frac{1-2t_1}{2-2t_1} = \max\left\{t \in [0,1] : f(t) = \max_{0\le t\le 1/2} f(t)\right\}.$$

since the function $t \mapsto (1-2t)/(2-2t)$ is strictly decreasing and continuous.

On the other hand, since T_0 is an isometry, we obtain

$$\begin{aligned} f(t) &= \|x(t) - y(1-t)\|_\psi \\ &= \|T_0x(t) - T_0y(1-t)\|_\psi \\ &= \|x(\kappa(t)) \pm y(1-\kappa(t))\|_\psi \\ &= f(\kappa(t)) \end{aligned}$$

for all $t \in [0, 1/2]$, which implies that

$$\kappa(t_1) = \max\left\{t \in [0,1] : f(t) = \min_{0\le t\le 1/2} f(t)\right\},$$
$$\kappa(t_2) = \max\left\{t \in [0,1] : f(t) = \max_{0\le t\le 1/2} f(t)\right\},$$

that is,

$$\kappa(t_1) = \frac{1-2t_2}{2-2t_2} \quad \text{and} \quad \kappa(t_2) = \frac{1-2t_1}{2-2t_1}.$$

This is a contradiction. The proof is complete. $\square$

To present sufficient conditions for Tingley's problem, the following two geometric constants of a normed space X play important roles.

$$C'_{NJ}(X) = \sup\left\{\frac{\|x+y\|^2 + \|x-y\|^2}{4} : x, y \in S_X\right\},$$
$$c'_{NJ}(X) = \inf\left\{\frac{\|x+y\|^2 + \|x-y\|^2}{4} : x, y \in S_X\right\}.$$

These constants were introduced by Gao [25], and are naturally strongly related to the von Neumann-Jordan constant $C_{NJ}(X)$. In particular, the constant $C'_{NJ}(X)$ is called the modified von Neumann-Jordan constant, and has been studied in [7, 27, 63].

Define a partial order $\le$ on Ψ_2 by declaring that $\varphi \le \psi$ if $\varphi(t) \le \psi(t)$ for all $t \in [0,1]$. We need the following two propositions.

Proposition 3.4.3 (Mizuguchi and Saito [63]). *Let $\psi \in \Psi_2$. If $\psi \le \psi_2$, then*

$$C'_{NJ}((\mathbb{R}^2, \|\cdot\|_\psi)) = \max_{0\le t\le 1} \frac{\psi_2(t)^2}{\psi(t)^2}.$$

Proposition 3.4.4 ([82]). *Let $\psi \in \Psi_2$. If $\psi \ge \psi_2$, then*

$$c'_{NJ}((\mathbb{R}^2, \|\cdot\|_\psi)) = \min_{0\le t\le 1} \frac{\psi_2(t)^2}{\psi(t)^2}.$$

Proof. We first note that $m\|\cdot\|_\psi \le \|\cdot\|_2 \le \|\cdot\|_\psi$, where $m = \min_{0\le t\le 1} \psi_2(t)/\psi(t)$. Take arbitrary $x, y \in \mathbb{R}^2$ such that $\|x\|_\psi = \|y\|_\psi = 1$. Then we have

$$\begin{aligned}\|x+y\|_\psi^2 + \|x-y\|_\psi^2 &\ge \|x+y\|_2^2 + \|x-y\|_2^2 \\ &= 2(\|x\|_2^2 + \|y\|_2^2) \\ &\ge 4m^2,\end{aligned}$$

which implies that $c'_{NJ}((\mathbb{R}^2, \|\cdot\|_\psi)) \geq m^2$. On the other hand, let t_0 be an element of $[0,1]$ satisfying $m = \psi_2(t_0)/\psi(t_0)$. If we put $x = \psi(t_0)^{-1}(1-t_0, t_0)$ and $y = \psi(t_0)^{-1}(1-t_0, -t_0)$, then it follows that $\|x\|_\psi = \|y\|_\psi = 1$ and

$$\|x+y\|_\psi^2 + \|x-y\|_\psi^2 = 4m^2.$$

This completes the proof. □

We now present the following two sufficient conditions.

Theorem 3.4.5 ([82]). *Let $\psi \in \Psi_2^S$. Then Tingley's problem is solved positively if $X = Y = (\mathbb{R}^2, \|\cdot\|_\psi)$ and either of the following statements holds.*

(i) *$\psi \leq \psi_2$ and the function ψ_2/ψ on $[0,1/2]$ takes the maximum only at $t_0 \in (0,1/2]$.*

(ii) *$\psi \geq \psi_2$ and the function ψ_2/ψ on $[0,1/2]$ takes the minimum only at $t_0 \in (0,1/2]$.*

Proof. We denote $(\mathbb{R}^2, \|\cdot\|_\psi)$ by X_ψ for short. Put $x(t)$ and $y(t)$ as in the proof of Lemma 3.4.2. Let T_0 be an isometry from S_{X_ψ} onto itself. Suppose first that (i) holds. Define the subset A of S_{X_ψ} by

$$A = \left\{ x \in S_{X_\psi} : \frac{\|x+y\|^2 + \|x-y\|^2}{4} = C'_{NJ}(X_\psi) \text{ for some } y \in S_{X_\psi} \right\}.$$

Putting $M = \max_{0 \leq t \leq 1} \psi_2(t)/\psi(t) = \psi_2(t_0)/\psi(t_0)$, we have $C'_{NJ}(X_\psi) = M^2$ by Proposition 3.4.3. We remark that $A = \{\pm x(t_0), \pm x(1-t_0), \pm y(t_0), \pm y(1-t_0)\}$. Indeed, it is obvious that $A \supset \{\pm x(t_0), \pm x(1-t_0), \pm y(t_0), \pm y(1-t_0)\}$. For the converse, note that the inequality

$$\begin{aligned}\|x+y\|_\psi^2 + \|x-y\|_\psi^2 &\leq \|x+y\|_2^2 + \|x-y\|_2^2 \\ &= 2(\|x\|_2^2 + \|y\|_2^2) \\ &\leq 4M^2\end{aligned}$$

holds for all $x, y \in S_{X_\psi}$, which guarantees $\|x\|_2 = M$ if $x \in A$. We now take an arbitrary $x \in A$. Then there exists $t \in [0,1]$ such that $x = \pm\psi(t)^{-1}(1-t, \pm t)$. It follows from the assumption and $M = \|x\|_2 = \psi_2(t)/\psi(t)$ that $t = t_0, 1-t_0$. This shows $A \subset \{\pm x(t_0), \pm x(1-t_0), \pm y(t_0), \pm y(1-t_0)\}$.

We next prove that T_0 has a linear isometric extension. Remark that $T_0(A) = A$. Without loss of generality, we may assume that $T_0 x(t_0) = x(t_0)$. Indeed, for any $x \in A$, there exists an isometric isomorphism S from X_ψ onto itself such that $Sx = x(t_0)$

since the norm is absolute and symmetric. In particular, if S is such an operator for $T_0x(t_0) \in A$ then it follows that ST_0 is an isometry from S_{X_ψ} onto itself satisfying $ST_0x(t_0) = x(t_0)$. Once it has been proved that ST_0 has a linear isometric extension U, we have an affirmative answer $T_0 = S^{-1}U|_{S_{X_\psi}}$.

As in the proof of Lemma 3.4.2, we have $T_0y(1-t_0) = \pm y(1-t_0)$. If $t_0 = 1/2$, one can easily check that T_0 has a linear isometric extension since $\psi \in \Psi_2^S$. Let $t_0 \in (0, 1/2)$. If $T_0y(1-t_0) = y(1-t_0)$, one has $T_0 = I|_{S_{X_\psi}}$ by Corollary 3.3.11. So we consider the case of $T_0y(1-t_0) = -y(1-t_0)$. Then it follows that

$$\begin{aligned} T_0y(t_0) \in T_0(A \cap A(x(t_0), y(1-t_0))) &= T_0(A) \cap T_0(A(x(t_0), y(1-t_0))) \\ &= A \cap A(T_0x(t_0), T_0y(1-t_0)) \\ &= A \cap A(x(t_0), -y(1-t_0)) \\ &= \{x(1-t_0)\} \end{aligned}$$

by Lemma 3.3.8. Therefore we have $T_0y(t_0) = x(1-t_0)$. Moreover, since $x(0) \in A(x(t_0), y(t_0))$ and $\|x(0) - x(t_0)\|_\psi = \|x(0) - y(t_0)\|_\psi$, one has

$$\begin{aligned} \|T_0x(0) - x(t_0)\|_\psi &= \|T_0x(0) - T_0x(t_0)\|_\psi \\ &= \|x(0) - x(t_0)\|_\psi \\ &= \|x(0) - y(t_0)\|_\psi \\ &= \|T_0x(0) - T_0y(t_0)\|_\psi \\ &= \|T_0x(0) - x(1-t_0)\|_\psi, \end{aligned}$$

and $T_0x(0) \in A(x(t_0), x(1-t_0))$. However, from the facts that $x(1/2) \in A(x(t_0), x(1-t_0))$ and $\|x(1/2) - x(t_0)\|_\psi = \|x(1/2) - x(1-t_0)\|_\psi$, we obtain $T_0x(0) = x(1/2)$ by Lemma 3.3.4. This shows the case of $T_0y(1-t_0) = -y(1-t_0)$ does not occur by Lemma 3.4.2 if the norm $\|\cdot\|_\psi$ is not $\pi/4$ rotation invariant.

Finally, suppose that $\|\cdot\|_\psi$ is $\pi/4$ rotation invariant. Since $T_0x(0) = x(1/2)$, one has $T_0x(1) = \pm y(1/2)$. It may be assumed that $T_0x(1) = -y(1/2)$. Then we obtain $T_0x(0) = R(\pi/4)x(0)$ and $T_0x(1) = R(\pi/4)x(1)$, which in turn implies that $T_0 = R(\pi/2)$ by Theorem 3.3.10. This shows the sufficiency of the condition (i).

The proof of the case (ii) is omitted since it is shown by an argument similar to that in the above. □

Applying this theorem, we can obtain many examples easily. Though the following three examples are special cases of known results, they are shown by extremely simple arguments.

Example 3.4.6 (Ding [21, 22, 23])**.** Let $1 \le p \le \infty$ such that $p \ne 2$. Then Tingley's problem is solved positively if $X = Y = \ell_p^2$.

Example 3.4.7 (Kadets and Martín [37])**.** Let $1/2 < \beta \le 1/\sqrt{2}$, and let $\psi_{\beta,1-\beta}(t) = \max\{1-t, t, \beta\}$. Then Tingley's problem is solved positively if $X = Y = (\mathbb{R}^2, \|\cdot\|_{\psi_{\beta,1-\beta}})$.

Example 3.4.8 (Kadets and Martín [37])**.** Let $1/\sqrt{2} \le \alpha < 1$, and let $\psi_{\alpha,\alpha}(t) = \max\{1 - 2(1-\alpha)t, 2\alpha - 1 + 2(1-\alpha)t\}$. Then Tingley's problem is solved positively if $X = Y = (\mathbb{R}^2, \|\cdot\|_{\psi_{\alpha,\alpha}})$.

We show several new examples below.

Example 3.4.9 ([82])**.** Let $1 \le p < q \le \infty$, and let $2^{1/q-1/p} < \lambda < 1$. Suppose that $\psi_{p,q,\lambda} = \max\{\lambda\psi_p, \psi_q\}$. Then it is easy to check that the function ψ_q/ψ_p is strictly decreasing on $[0, 1/2]$, and so there exists a unique $t_\lambda \in [0, 1/2]$ such that

$$\psi_{p,q,\lambda}(t) = \begin{cases} \psi_q(t) & \text{if } t \in [0, t_\lambda], \\ \lambda\psi_p(t) & \text{if } t \in [t_\lambda, 1 - t_\lambda], \\ \psi_q(t) & \text{if } t \in [1 - t_\lambda, 1]. \end{cases}$$

In the case of $2 < p < q \le \infty$, we have $\psi_{p,q,\lambda} \le \psi_2$ and $\psi_2/\psi_{p,q,\lambda}$ is strictly increasing on $[0, 1/2]$. On the other hand, if $1 \le p < q < 2$, it follows that $\psi_{p,q,\lambda} \ge \psi_2$, and that the function $\psi_2/\psi_{p,q,\lambda}$ is strictly decreasing on $[0, 1/2]$. Thus Tingley's problem is solved positively if $X = Y = (\mathbb{R}^2, \|\cdot\|_{\psi_{p,q,\lambda}})$ by Theorem 3.4.5.

Example 3.4.10 ([82])**.** Define the function $\varphi_p \in \Psi_2^S$ by

$$\varphi_p(t) = \begin{cases} \max\{2^{1/2-1/p}\psi_p(t), 1 - t, t\} & \text{if } 1 \le p < 2, \\ \max\{\psi_p(t), 1/\sqrt{2}\} & \text{if } 2 < p \le \infty. \end{cases}$$

Remark that $\varphi_p \le \psi_2$ for all p. We first consider the case of $1 \le p < 2$. Let t_p be the unique solution of the equation $2^{1/2-1/p}\psi_p(t) = 1 - t$ such that $t_p \in [0, 1/2]$. Then the function ψ_2/φ_p on $[0, 1/2]$ takes the maximum only at t_p.

In the case of $2 < p < \infty$, suppose that t_p is the unique solution of the equation $\psi_p = 1/\sqrt{2}$ such that $t_p \in [0, 1/2]$. Then it is easy to see that the function ψ_2/φ_p on $[0, 1/2]$ takes the maximum only at t_p. Thus we have an affirmative answer to Tingley's problem in the case of $X = Y = (\mathbb{R}^2, \|\cdot\|_{\varphi_p})$.

Example 3.4.11 ([82])**.** Let $0 \le t_1 < t_2 \le 1/2$, and let φ_{t_1,t_2} be the function given by

$$\varphi_{t_1,t_2}(t) = \max\{\psi_2(t), \alpha t + \beta, \alpha(1-t) + \beta\}$$

where

$$\alpha = \frac{\psi_2(t_2) - \psi_2(t_1)}{t_2 - t_1} \quad \text{and} \quad \beta = \frac{t_2\psi_2(t_1) - t_1\psi_2(t_2)}{t_2 - t_1}$$

Then we have $\varphi_{t_1,t_2} \geq \psi_2$ and

$$\varphi_{t_1,t_2}(t) = \begin{cases} \psi_2(t) & \text{if } t \in [0, t_1], \\ \alpha t + \beta & \text{if } t \in [t_1, t_2], \\ \psi_2(t) & \text{if } t \in [t_2, 1 - t_2], \\ \alpha(1-t) + \beta & \text{if } t \in [1 - t_2, 1 - t_1], \\ \psi_2(t) & \text{if } t \in [1 - t_1, 1]. \end{cases}$$

Hence there exists a unique element $t_0 \in [0, 1/2]$ such that $t_1 < t_0 < t_2$ and the function $\psi_2/\varphi_{t_1,t_2}$ on $[0, 1/2]$ takes the minimum only at t_0. This shows that if $X = Y = (\mathbb{R}^2, \|\cdot\|_{\varphi_{t_1,t_2}})$, Tingley's problem is solved positively.

Up to this time, we only have considered functions that are comparable with ψ_2. In what follows, we present sufficient conditions for incomparable cases.

Proposition 3.4.12 (Mizuguchi and Saito [63])**.** *Let $\psi \in \Psi_2^S$. If ψ_2/ψ takes the minimum at $1/2$, then*

$$C'_{NJ}((\mathbb{R}^2, \|\cdot\|_\psi)) = \frac{\psi(1/2)^2}{\psi_2(1/2)^2} \max_{0 \leq t \leq 1} \frac{\psi_2(t)^2}{\psi(t)^2}.$$

Proposition 3.4.13 ([82])**.** *Let $\psi \in \Psi_2^S$. If ψ_2/ψ takes the maximum at $1/2$, then*

$$c'_{NJ}((\mathbb{R}^2, \|\cdot\|_\psi)) = \frac{\psi(1/2)^2}{\psi_2(1/2)^2} \min_{0 \leq t \leq 1} \frac{\psi_2(t)^2}{\psi(t)^2}.$$

Proof. Let $m = \min_{0 \leq t \leq 1} \psi_2(t)/\psi(t)$ and $M = \psi_2(1/2)/\psi(1/2)$. Then $m\|\cdot\|_\psi \leq \|\cdot\|_2 \leq M\|\cdot\|_\psi$. For each $x, y \in \mathbb{R}^2$ such that $\|x\|_\psi = \|y\|_\psi = 1$, we have

$$\begin{aligned} \|x+y\|_\psi^2 + \|x-y\|_\psi^2 &\geq M^{-2}(\|x+y\|_2^2 + \|x-y\|_2^2) \\ &= 2M^{-2}(\|x\|_2^2 + \|y\|_2^2) \\ &\geq 4M^{-2}m^2. \end{aligned}$$

Hence we obtain $c'_{NJ}((\mathbb{R}^2, \|\cdot\|_\psi)) \geq M^{-2}m^2$. On the other hand, let $t_0 \in [0, 1/2]$ such that $\psi_2(t_0)/\psi(t_0) = m$. Putting $x = \psi(t_0)^{-1}(1 - t_0, t_0)$ and $y = \psi(t_0)^{-1}(t_0, 1 - t_0)$, one has

$$\|x+y\|_\psi^2 + \|x-y\|_\psi^2 = 4M^{-2}m^2,$$

which shows the proposition. $\square$

Using these results, we obtain the following theorem that presents sufficient conditions for incomparable cases. The proof is almost the same as that of Theorem 3.4.5, and so is omitted.

Theorem 3.4.14 ([82]). *Let $\psi \in \Psi_2^S$. Then Tingley's problem is solved positively if $X = Y = (\mathbb{R}^2, \|\cdot\|_\psi)$ and either of the following statements holds.*

(i) *The function ψ_2/ψ on $[0, 1/2]$ takes the minimum at $1/2$ and the maximum only at $t_0 \in (0, 1/2]$.*

(ii) *The function ψ_2/ψ on $[0, 1/2]$ takes the maximum at $1/2$ and the minimum only at $t_0 \in (0, 1/2]$.*

The rest of this paper is devoted to giving further examples.

Example 3.4.15 ([82]). Let $1 \leq p < 2 < q \leq \infty$, and let $2^{1/q-1/p} < \lambda < 1$. Then the function $\psi_2/\psi_{p,q,\lambda}$ is increasing on $[0, t_\lambda]$, and decreasing on $[t_\lambda, 1/2]$. Hence it takes the maximum only at t_λ. We remark that $\psi_{p,q,\lambda} \leq \psi_2$ if and only if $2^{1/q-1/p} < \lambda \leq 2^{1/2-1/p}$. If $2^{1/2-1/p} < \lambda < 1$, it turns out that $\psi_2/\psi_{p,q,\lambda}$ takes the minimum at $1/2$. Thus, in both cases, Tingley's problem is affirmative if $X = Y = (\mathbb{R}^2, \|\cdot\|_{\psi_{p,q,\lambda}})$.

Example 3.4.16 ([82]). Let $0 < \omega < 1$ and $1 < q < \infty$. Recall that the two-dimensional Lorentz sequence space $d^{(2)}(\omega, q)$ is defined as the space $\mathbb{R}^2$ endowed with the norm

$$\|(x, y)\|_{\omega,q} = (\max\{|x|^q, |y|^q\} + \omega \min\{|x|^q, |y|^q\})^{1/q}.$$

Then $\|\cdot\|_{\omega,q}$ is a symmetric absolute normalized norm on $\mathbb{R}^2$, and the function $\psi_{\omega,q}$ associated with this norm is given by

$$\psi_{\omega,q}(t) = \begin{cases} ((1-t)^q + \omega t^q)^{1/q} & \text{if } 0 \leq t \leq 1/2, \\ (t^q + \omega(1-t)^q)^{1/q} & \text{if } 1/2 \leq t \leq 1. \end{cases}$$

We now consider the function $\psi_2/\psi_{\omega,q}$ on $[0, 1/2]$. Then the first derivative is given by

$$\left(\frac{\psi_2}{\psi_{\omega,q}}\right)'(t) = \frac{((1-t)^q + \omega t^q)^{1/q-1}(t(1-t)^{q-1} - \omega t^{q-1}(1-t))}{\psi_2(t)\psi_{\omega,q}(t)^2}$$

for all $t \in (0, 1/2)$. From this, one can easily check that the function $\psi_{\omega,q}$ satisfies the assumption of Theorems 3.4.5 or 3.4.14. Thus, we have an affirmative answer to Tingley's problem in the case of $X = Y = d^{(2)}(\omega, q)$.

Example 3.4.17 ([82]). Let $0 < \omega < 1 < q < \infty$. By Proposition 1.2.4, the space $d^{(2)}(\omega, q)^*$ is isometrically isomorphic to the space $\mathbb{R}^2$ endowed with the norm $\|\cdot\|_{\omega,q}^*$ defined by

$$\|(x,y)\|_{\omega,q}^* = \begin{cases} (|x|^p + \omega^{1-p}|y|^p)^{1/p} & \text{if } |y| \leq \omega|x|, \\ (1+\omega)^{1/p-1}(|x|+|y|) & \text{if } \omega|x| \leq |y| \leq \omega^{-1}|x|, \\ (\omega^{1-p}|x|^p + |y|^p)^{1/p} & \text{if } \omega^{-1}|x| \leq |y|, \end{cases}$$

where $1/p + 1/q = 1$. The norm $\|\cdot\|_{\omega,q}^*$ is symmetric, absolute and normalize, and the corresponding function $\psi_{\omega,q}^*$ is given by

$$\psi_{\omega,q}^*(t) = \begin{cases} ((1-t)^p + \omega^{1-p}t^p)^{1/p} & \text{if } 0 \leq t \leq \omega/(1+\omega), \\ (1+\omega)^{1/p-1} & \text{if } \omega/(1+\omega) \leq t \leq 1/(1+\omega), \\ (t^p + \omega^{1-p}(1-t)^p)^{1/p} & \text{if } 1/(1+\omega) \leq t \leq 1. \end{cases}$$

We can conclude that Tingley's problem is solved positively if $X = Y = d^{(2)}(\omega, q)^*$ by an argument similar to that in the preceding example.

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