The Alexandrov-Toponogov comparison theorem for radial curvature and its applications

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THE ALEXANDROV-TOPONOGOV COMPARISON THEOREM FOR RADIAL CURVATURE AND ITS APPLICATIONS

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1. Introduction

We discuss the Alexandrov-Toponogov comparison theorem and the sphere theorem under the conditions of radial curvature of a pointed manifold (M, o) with reference surface of revolution $(\widetilde{M}, \tilde{o})$. There are two obstructions to make the comparison theorem for a triangle one of whose vertices is a base point o. One is the cut points of another vertex $\tilde{p} \neq \tilde{o}$ of a comparison triangle in $\widetilde{M}$. The other is the cut points of the base point o in M. We find a condition under which the comparison theorem is valid for any geodesic triangle with a vertex at o in M. In addition, we introduce a new constant for the surfaces of revolution homeomorphic to the 2-sphere. We prove a sphere theorem for radial curvature, assuming an inequality in the constant and the ratio of the difference of the maximal distance to the base point from the diameter of the reference surface and the injectivity radius of the base point. Namely, if a compact pointed Riemannian n-manifold which is referred to a surface of revolution satisfies the inequality, then it is topologically an n-sphere.

The Alexandrov-Toponogov comparison theorem (shortly ATCT) has been very useful in the study of geometry of geodesics including Riemannian geometry. In the present paper we discuss ATCT and the sphere theorem under the conditions of radial curvature of a pointed manifold (M, o). The notion of radial curvature was first introduced by Klingenberg [9]. In 2003, Itokawa-Machigashira-Shiohama prove the Toponogov comparison theorem for radial curvature where the reference surface is a von Mangoldt surface of revolution (see [8]). In 2007, Sinclair and Tanaka prove the Toponogov comparison theorem for radial curvature where the reference surface is a 2-sphere of revolution whose cut locus is a subarc of a meridian or a single point (see [19]).

Many attempts have been made to extend the classical (diameter) sphere theorems. The topological sphere theorems for pointed manifolds with positive radial curvature have been investigated in [15] and [14]. When the reference surface of a compact pointed manifold is a compact surface of revolution, Lee [12], [13] proved topological sphere theorems under certain restrictions on the diameter of M and on the supremum of the distance from the base point. Indeed, the restriction on the diameter was needed, since the Alexandrov-Toponogov comparison theorem for spherical model surfaces was not established for every geodesic triangle but only for narrow triangles. Kondo-Ohta [10] proved it under the assumptions that a compact surface is a von Mangoldt surface of revolution and that there exists a point $x \in M$ such that the base point is a critical point of the distance function to x. The Toponogov comparison theorem has been established in [6] under a certain restrictions on the cut locus and the critical points of distance function to the base point.

The one purpose of the present paper is to establish ATCT under the conditions of radial curvature of a pointed manifold. The other purpose is to establish a new topological sphere theorem for pointed compact manifolds whose reference surfaces are compact surfaces of revolution. In the proof of the sphere theorem, the Toponogov comparison theoprem plays an important role.

Reference space is a smooth rotationally symmetric surface homeomorphic to either the plane $\mathbb{R}^2$ or the 2-sphere $\mathbb{S}^2$, namely a surface of revolution $(\widetilde{M}, \tilde{o})$ with a geodesic polar coordinate system (r, θ) around $\tilde{o}$ such that its metric is given by

$$ds^2 = dr^2 + f(r)^2 d\theta^2.$$

where $f(r) > 0$, $0 < r < \ell \leq \infty$, $f(0) = 0$, $f'(0) = 1$ and $\theta \in \mathbb{S}^1$. In the sphere theorem, reference space is particularly a surface homeomorphic to 2-sphere $\mathbb{S}^2$. Throughout the paper $(\widetilde{M}, \tilde{o})$ denotes a reference surface of revolution.

Let $p \in M$ be fixed and set $\tilde{p} = (d(o,p), 0) \in \widetilde{M}$. We may find a point $\tilde{q} \in \widetilde{M}$ satisfying

$$d(\tilde{o}, \tilde{q}) = d(o, q) \qquad \text{and} \qquad d(\tilde{p}, \tilde{q}) = d(p, q)$$

for any point $q \in M$. The precise definition of $\tilde{q}$ will be given after (2.2) in §2. We call $\tilde{q}$ the *reference point* of q and the map $\Phi : M \to \widetilde{M}$ given by $q \mapsto \tilde{q}$ the *reference map* of M into $\widetilde{M}$. It is not certain whether or not every point $q \in M$ has a reference point and every geodesic triangle $\triangle opq$, $p, q \in M$, admits the corresponding geodesic triangle $\triangle \tilde{o}\tilde{p}\tilde{q}$, $\tilde{p}, \tilde{q} \in \widetilde{M}$. This problem will be resolved affirmatively under our condition to be proposed (see (2.6)).

There are two obstructions to ATCT. In the present paper we show that ATCT is established if the obstructions do not occur simultaneously at any point in M.

Let $T(p, q)$ denote a minimizing geodesic segment connecting p and q in M and set $\widetilde{T}(p, q) = \Phi(T(p, q))$. From the definition (see (2.2)), $\widetilde{T}(p, q)$ is a continuous curve in $\widetilde{M}$. The main part of our study is to see the positional relation between $\widetilde{T}(p, q)$ and $T(\tilde{p}, \tilde{q})$. Actually, ATCT is valid if the reference curve $\widetilde{T}(p, q)$ and the minimizing geodesic segment $T(\tilde{p}, \tilde{q})$ satisfy the good positional relation (see (2.7)). One obstruction to establish it is the appearance of cut points $Cut(\tilde{p})$ to $\tilde{p}$ in the reference curve $\widetilde{T}(p,q)$ of $\triangle opq$. In fact, we do not know what relation there is between the positions of the reference curve $\widetilde{T}(p,q)$ and $T(\tilde{p}, \tilde{q})$ if the reference curve $\widetilde{T}(p,q)$ intersects $Cut(\tilde{p})$. The cases treated in all papers for ATCT referred to a surface of revolution are free from the obstruction on the cut loci. Actually, it is proved in [8] that ATCT holds for any geodesic triangle with a vertex o if there are no cut points in a open half part of $\widetilde{M}$ whose boundary consists of two meridians with angle π at the vertex $\tilde{o}$.

In general, the composition $r \circ \Phi$ of the reference map Φ with r-coordinate function may have a local maximum. In fact, the local maximum points of the distance function to o in M attain a local maximum of $r \circ \Phi$. This may cause a bad positional relation between $\widetilde{T}(p,q)$ and $T(\tilde{p}, \tilde{q})$. This is the other obstruction, namely the existence of the local maximum points of the distance function to o in M, which is restricted to an ellipsoid with foci at o and p (see (2.4)). Let $E(p) \subset M$ be the set of all such local maximum points (see just after (2.4)). Then $E(p)$ is a subset of the cut points to the base point o (see Lemma 3.4). The reference points of $E(p)$ may be in the boundary of the image of the reference map of M locally, so there possibly exists a minimizing geodesic segment whose endpoints are reference points but containing a non-reference point in $\widetilde{M}$. This situation shows that ATCT may not be true.

It is natural to ask what condition verifies the existence of a minimizing geodesic segment $T(\tilde{p}, \tilde{q})$ in $\widetilde{M}$ which has the good positional relation with $\widetilde{T}(p,q)$. We assume that the reference points of all points in $E(p)$ are not cut points of $\tilde{p}$ (see (2.6)). Under this assumption we prove that ATCT holds for any geodesic triangle $\triangle opq$.

We say that (M, o) is *referred* to the reference surface $(\widetilde{M}, \tilde{o})$ if every radial sectional curvature at $p \in M$ is bounded below by $K(d(o,p))$, where $K(t) = -\frac{f''(t)}{f(t)}$ is the Gauss curvature of $\widetilde{M}$ at the points in the parallel t-circle. Namely a pointed Riemannian manifold (M, o) is said to be *referred to* a model surface $(\widetilde{M}, \tilde{o})$ if and only if all the radial sectional curvatures K_M of M satisfy

$$K_M(\Pi) \geq K(d(o,x)), \quad x \in M. \tag{1.1}$$

A triple of minimizing geodesic segments $T(o,p) \cup T(o,q) \cup T(p,q)$ joining points $o, p, q \in M$ is called a *geodesic triangle* and denoted by $\triangle opq$. A geodesic triangle $\triangle \tilde{o}\tilde{p}\tilde{q} \subset \widetilde{M}$ is called a *comparison triangle corresponding to* $\triangle opq \subset M$ if the corresponding edges have the same lengths, namely,

$$d(o,p) = d(\tilde{o}, \tilde{p}), \qquad d(o,q) = d(\tilde{o}, \tilde{q}), \qquad d(p,q) = d(\tilde{p}, \tilde{q}).$$

The Alexandrov convexity and the Toponogov comparison theorem are stated as follows.

Theorem 1.1 (The Alexandrov convexity)**.** *Assume that a complete pointed Riemannian manifold* (M, o) *is referred to* $(\widetilde{M}, \tilde{o})$*. If a point* $p \in M$ *satisfies* $\Phi(E(p)) \cap Cut(\tilde{p}) = \emptyset$*, then for any geodesic triangle* $\triangle opq \subset M$ *with* $q \neq o, p$ *there exists its comparison triangle* $\triangle \tilde{o}\tilde{p}\tilde{q} \subset \widetilde{M}$ *such that* $d(\tilde{o}, \tilde{x}) \leq d(o, x)$ *for any point* $\tilde{x} \in T(\tilde{p}, \tilde{q})$ *and* $x \in T(p, q)$ *with* $d(\tilde{p}, \tilde{x}) = d(p, x)$*. Here* $T(p, q)$ *and* $T(\tilde{p}, \tilde{q})$ *are the bases of geodesic triangles* $\triangle opq$ *and* $\triangle \tilde{o}\tilde{p}\tilde{q}$*, respectively.*

Theorem 1.2 (The Toponogov comparison theorem)**.** *Under the same assumptions as in Theorem 1.1, every geodesic triangle* $\triangle opq \subset M$ *admits its comparison triangle* $\triangle \tilde{o}\tilde{p}\tilde{q} \subset \widetilde{M}$ *such that*

$$\angle opq \geq \angle \tilde{o}\tilde{p}\tilde{q}, \qquad \angle oqp \geq \angle \tilde{o}\tilde{q}\tilde{p}, \qquad \angle poq \geq \angle \tilde{p}\tilde{o}\tilde{q}.$$

It should be emphasized that those properties are restated by the positional relation (2.7) of $\widetilde{T}(p, q)$ and $T(\tilde{p}, \tilde{q})$ in Theorem 2.1.

As an application of these theorems we are allowed to define a pointed Alexandrov space (M, o) with radial curvature bounded below by a function K (see Remark 2.6 and sequent paragraphs).

Corollary 1.3. *Let* (M, o) *be a compact Riemannian manifold which is an Alexandrov space with a base point at* o *with radial curvature bounded below by the function* K*. Here* $K : [0, \ell] \to \mathbb{R}$*,* $\ell < \infty$*, is the radial curvature function of* $(\widetilde{M}, \tilde{o})$*. Then, the perimeters of all geodesic triangles* $\triangle opq$ *in* M *are less than or equal to* 2ℓ *and the diameter of* M *is less than or equal to* ℓ*. Moreover, if there exists a geodesic triangle* $\triangle opq$ *in* M *whose perimeter is* 2ℓ*, then* M *is isometric to the warped product manifold whose warping function is* K*. In particular, the same conclusion holds for* M *if the diameter of* M *is* ℓ*.*

The following corollary is proved in [10] when M is a noncompact pointed Riemannian manifold with radial curvature bounded below by the function K which is monotone non-increasing. We call such a surface of revolution with monotone non-increasing curvature function a *von Mangoldt surface.* There are no cut points in an open half part of a von Mangoldt surface $\widetilde{M}$ whose boundary consists of two meridians with angle π at the vertex $\tilde{o}$ (see [21]).

Corollary 1.4. *Let* (M, o) *be a noncompact Alexandrov space with a base point at* o *with radial curvature bounded below by the function* K*. Here* $K : [0, \infty) \to \mathbb{R}$ *is the radial curvature function of* $(\widetilde{M}, \tilde{o})$*. If the total curvature of* $\widetilde{M}$ *is positive, then* M *has one end and has no straight line.*

The (classical) diameter sphere theorem is stated as follows. Our sphere theorem is similar to this.

Theorem 1.5. *Let M be a connected, complete Riemannian manifold with sectional curvature $K_M(\Pi) \geq \lambda > 0$ and diameter $\operatorname{diam}(M) > \pi/(2\sqrt{\lambda})$ for all $x \in M$, where Π is the 2-dimensional subspace of T_xM.*

These results will be stated more precisely in §2 after introducing some definitions and notations.

The idea of the proof of Theorem 1.1 and 1.2 is this. The good positional relation (2.7) is equivalent to the Alexandrov convexity and the Toponogov angle comparison (see Remark 2.2). Therefore, we study what positional relation holds between the reference curve $\widetilde{T}(p,q)$ of every minimizing geodesic segment $T(p,q)$ in M and a minimizing geodesic segment $T(\tilde{p},\tilde{q})$ in $\widetilde{M}$. To do this we use the partial order $\leq$ in the set of all curves which are parameterized by the angle coordinate θ in $\widetilde{M}$.

A set C is said to be *parameterized by the angle coordinate* θ if $C \cap [\theta = a]$ contains at most one point where $[\theta = a] = \{\tilde{x} \in \widetilde{M} \mid \theta(\tilde{x}) = a\}$. Let two sets C_1 and C_2 be parameterized by the angle coordinate. We then define the positional relation between C_1 and C_2 by $C_1 \leq C_2$ if $r(C_1 \cap [\theta = a]) \leq r(C_2 \cap [\theta = a])$ for all $a \in \mathbb{R}$ with $C_1 \cap [\theta = a] \neq \emptyset$ and $C_2 \cap [\theta = a] \neq \emptyset$.

Let $U(\tilde{p},\tilde{q})$ and $L(\tilde{p},\tilde{q})$ denote the minimizing geodesic segments connecting $\tilde{p}$ and $\tilde{q}$ in $\widetilde{M}$ such that $L(\tilde{p},\tilde{q}) \leq T(\tilde{p},\tilde{q}) \leq U(\tilde{p},\tilde{q})$ for any minimizing geodesic segment $T(\tilde{p},\tilde{q})$, namely all minimizing geodesic segments $T(\tilde{p},\tilde{q})$ lie in the biangle domain bounded by $L(\tilde{p},\tilde{q}) \cup U(\tilde{p},\tilde{q})$ in $\widetilde{M}$ (see (6.1)).

Let $\widetilde{M}^+_{\tilde{p}}$ denote the half part of $\widetilde{M}$ bounded by the union of the meridians through $\tilde{p}$ and opposite to $\tilde{p}$.

Let r_0 be the supremum of the set of all $r_1 > d(o,p)$ satisfying the following properties: If $q \in E(o,p\,;r)$ for every $r \in (d(o,p), r_1)$, then

- (C1) there exists a minimizing geodesic segment $T(p,q)$ such that $T(p,q)$ is contained in the set $\Phi^{-1}(\widetilde{M}^+_{\tilde{p}})$ and $\widetilde{T}(p,q) \geq U(\tilde{p},\tilde{q})$,
- (C2) every minimizing geodesic segment $T(p,q)$ is contained in the set $\Phi^{-1}(\widetilde{M}^+_{\tilde{p}})$ and satisfies $\widetilde{T}(p,q) \geq L(\tilde{p},\tilde{q})$.

It follows from [7], [8] and [11] that the set of those parameters is not empty, and, hence, $r_0 > d(o,p)$ (see Lemma 5.1). We then prove that every geodesic triangle $\triangle opq$ in M for $q \in E(o,p\,;r)$, $d(o,p) < r < r_0$, has a comparison triangle $\triangle\tilde{o}\tilde{p}\tilde{q}$ in $\widetilde{M}$ satisfying (2.8) (Assertion 7.1). Also all points in $E(o,p\,;r_0)$ satisfy (C1) and (C2) (Assertion 7.2). We see, from the assumption of our theorems, that even when $q \in E(o,p\,;r_0)$ with $q \notin Cut(p)$ and $\tilde{q} \in Cut(\tilde{p})$, the reference curve $\widetilde{T}_e(p,q)$ of the maximal minimizing geodesic $T_e(p,q)$ through p and q crosses $Cut(\tilde{p})$ from the far side to the near side from $\tilde{o}$ in $\widetilde{M}$. This fact shows that there exists an $r' > r_0$ such that (C1) and (C2) are true for any r, $d(o,p) < r < r'$ (see Assertions 7.3, 7.4 and 7.5). This means that $r_0 = \sup\{d(o,x) + d(p,x) \,|\, x \in M\}$, and, therefore, $\cup_{d(o,p)\le r\le r_0} E(o,p\,;r) = M$.

The rest of this article is organized as follows. In §2 we state our results precisely with giving some notions we need. In §3 we give some properties of circles and ellipses in a reference surface of revolution. Moreover we give a sufficient condition for a point $q \in M$ not being contained in $E(p)$, and an example showing the property of ellipses which is essentially different from circles. From §4 we start studying reference curves. In §4 we give the fundamental properties of the reference curves and the reference reverse curves. In §5 we treat the case that the reference curves from $\tilde{p}$ do not meet $Cut(\tilde{p})$ in $\widetilde{M}$. In §6 we study the reference curves from $\tilde{p}$ meeting $Cut(\tilde{p})$ from the far side to the near side from $\tilde{o}$. In those cases we have the good positional relation between $\widetilde{T}(p,q)$ and $T(\tilde{p},\tilde{q})$. In §7 we give the proof of our ATCT for radial curvature. Our assumption (2.6) ensures that $\widetilde{T}(p,q)$ crosses $Cut(\tilde{p})$ from the far side to the near side from $\tilde{o}$. The assumption is used only in the proofs of Assertions 7.2. In §8 we show some corollaries concerning the maximal perimeter and diameter as applications of our ATCT. Those are the Riemannian version of Corollary 1.3, and we give the proof of Corollary 1.4. In §9 we give the proof of our sphere theorem for radial cirvature where (M,o) is a compact pointed Riemannian manifold. Roughly speaking, our constant μ measure the symmetry of the reference surface of revolution $(\widetilde{M},\tilde{o})$. In §10 we show some examples of our constants c_1, r_1 $c_2(r)$, $c_3(r)$ and μ. These constants are determined by the metric on $\widetilde{M}$.

Basic tools in Riemannian Geometry are referred to [2], [3] and [20].

2. Notations and statements

Let $(\widetilde{M},\tilde{o})$ be a reference surface of revolution with a geodesic polar coordinate system (r,θ) around $\tilde{o}$. Its metric is of class C^2 and given by

$$ds^2 = dr^2 + f(r)^2 d\theta^2, \quad (r,\theta) \in (0,\ell) \times \mathbf{S}^1, \ell \le \infty. \tag{2.1}$$

Here, $r : \widetilde{M} \to \mathbf{R}$ is the distance function to $\tilde{o}$, and $f : (0,\ell) \to \mathbf{R}$ the warping function which is positive smooth and satisfies the Jacobi equation:

$$f'' + Kf = 0, \quad f(0) = 0, \quad f'(0) = 1.$$

Here, $K : [0,\ell] \to \mathbf{R}$ is the Gaussian curvature of $\widetilde{M}$. In addition,

$$f(\ell) = 0, \quad f'(\ell) = -1 \qquad \text{if} \qquad \ell < \infty.$$

The function K is called the *radial curvature function* of $\widetilde{M}$.

Let (M, o) be a complete Riemannian manifold with a base point at o. A *radial plane* $\Pi \subset T_pM$ at a point $p \in M$ is by definition a plane containing a vector tangent to a minimizing geodesic segment emanating from o where T_pM is the tangent space of M at p. A *radial sectional curvature* $K_M(\Pi)$ is by definition a sectional curvature with respect to a radial plane Π. We say that (M, o) is *referred* to $(\widetilde{M}, \tilde{o})$ if every radial sectional curvature at $p \in M$ is bounded below by $K(d(o,p))$, namely, $K_M(\Pi) \geq K(d(o,p))$ where $d(o,p)$ is by definition the distance between o and p.

In the reference surface of revolution $\widetilde{M}$ every geodesic triangle $\triangle\tilde{o}\tilde{p}\tilde{q}$ bounds a region because $\widetilde{M}$ is homeomorphic to either the plane $\mathbb{R}^2$ or the 2-sphere $\mathbb{S}^2$ and $\triangle\tilde{o}\tilde{p}\tilde{q}$ is a simple closed curve. The region is called a *triangle domain* and denoted by the same symbol $\triangle\tilde{o}\tilde{p}\tilde{q}$.

With respect to a point $\tilde{p} \in \widetilde{M}$, we divide $\widetilde{M}$ into two parts as follows:

$$\widetilde{M}^+_{\tilde{p}} = [\theta(\tilde{p}) \leq \theta \leq \theta(\tilde{p}) + \pi], \qquad \widetilde{M}^-_{\tilde{p}} = [\theta(\tilde{p}) - \pi \leq \theta \leq \theta(\tilde{p})].$$

Here we set

$$[a \leq \theta \leq b] = \{\tilde{x} \in \widetilde{M} \mid a \leq \theta(\tilde{x}) \leq b\}.$$

The pair of distance functions $\tilde{x} \mapsto (d(\tilde{o}, \tilde{x}), d(\tilde{p}, \tilde{x}))$, $\tilde{x} \in \widetilde{M}$, defines a Lipschitz chart on the interiors $\mathrm{Int}(\widetilde{M}^\pm_{\tilde{p}})$ of $\widetilde{M}^\pm_{\tilde{p}}$ respectively (see Lemma 3.1 (1)). For an arbitrary fixed point $p \in M$ and $\tilde{p} \in \widetilde{M}$ we define the maps :

$$F_p : M \to \mathbb{R}^2, \qquad F_p(x) = (d(o,x), d(p,x)), \quad x \in M,$$

and

$$\widetilde{F}_{\tilde{p}} : \widetilde{M}^+_{\tilde{p}} \to \mathbb{R}^2, \qquad \widetilde{F}_{\tilde{p}}(\tilde{x}) = (d(\tilde{o}, \tilde{x}), d(\tilde{p}, \tilde{x})), \quad \tilde{x} \in \widetilde{M}^+_{\tilde{p}},$$

$$\widetilde{G}_{\tilde{p}} : \widetilde{M}^-_{\tilde{p}} \to \mathbb{R}^2, \qquad \widetilde{G}_{\tilde{p}}(\tilde{x}) = (d(\tilde{o}, \tilde{x}), d(\tilde{p}, \tilde{x})), \quad \tilde{x} \in \widetilde{M}^-_{\tilde{p}}.$$

Then, F_p, $\widetilde{F}_{\tilde{p}}$ and $\widetilde{G}_{\tilde{p}}$ are Lipschitz continuous, both $\widetilde{F}_{\tilde{p}}$ and $\widetilde{G}_{\tilde{p}}$ are injective and their inverse maps are locally Lipschitz continuous in $\widetilde{F}_{\tilde{p}}(\mathrm{Int}(\widetilde{M}^+_{\tilde{p}}))$ and $\widetilde{G}_{\tilde{p}}(\mathrm{Int}((\widetilde{M}^-_{\tilde{p}}))$, respectively. This is because every minimizing geodesic segment $T(\tilde{p}, \tilde{x})$ is not tangent to the meridians. In other words, it crosses the meridians from the left (resp., right) hand side to the right (resp., left) hand side in $\mathrm{Int}(\widetilde{M}^+_{\tilde{p}})$ (resp., $\mathrm{Int}(\widetilde{M}^-_{\tilde{p}})$) (cf. Lemma 3.1 (1)).

A unit speed minimizing geodesic segment from p to q is denoted by $T(p,q)(t)$, $0 \leq t \leq d(p,q)$, where $T(p,q)(0) = p$ and $T(p,q)(d(p,q)) = q$. Also $T(p,q)$ is identified with its image $\{T(p,q)(t) \,|\, 0 \leq t \leq d(p,q)\}$.

For an arbitrary fixed point $p \in M$ let $\tilde{p}$ be the point $(d(o,p), 0)$ in the geodesic polar coordinates on $\widetilde{M}$. If $T(p,q) \subset {F_p}^{-1}(\widetilde{F}_{\tilde{p}}(\widetilde{M}^+_{\tilde{p}}))$, we then define a curve $\widetilde{T}(p,q)$ in $\widetilde{M}$ such that

$$\widetilde{T}(p,q)(t) = \widetilde{F}_{\tilde{p}}^{-1} \circ F_p(T(p,q)(t)), \quad 0 \le t \le d(p,q). \tag{2.2}$$

Obviously we have $\widetilde{T}(p,q)(0) = \tilde{p}$. If $\tilde{p}_1 = \tau_\theta(\tilde{p})$ for some rotation τ_θ of $\widetilde{M}$ around $\tilde{o}$, then $\widetilde{F}_{\tilde{p}_1}^{-1} \circ F_p(T(p,q)(t)) = \tau_\theta(\widetilde{F}_{\tilde{p}}^{-1} \circ F_p(T(p,q)(t)))$. Therefore, if $\Phi_{\tilde{p}} = \widetilde{F}_{\tilde{p}}^{-1} \circ F_p$ and $\Phi_{\tilde{p}_1} = \widetilde{F}_{\tilde{p}_1}^{-1} \circ F_p$, we then have $\Phi_{\tilde{p}_1} = \tau_\theta \circ \Phi_{\tilde{p}}$. The reference map Φ used in §1 is determined up to the rotation around $\tilde{o}$.

It is convenient to use the expression $\widetilde{F}_{\tilde{p}}^{-1} \circ F_p$ for defining the reference reverse curve. Setting $\tilde{q} = \widetilde{T}(p,q)(d(p,q))$, we have the *reference reverse curve* $\widetilde{R}(p,q)$ of $T(p,q)$ which is given by

$$\widetilde{R}(p,q)(t) = \widetilde{G}_{\tilde{q}}^{-1} \circ F_q(T(p,q)(d(p,q)-t)), \quad 0 \le t \le d(p,q). \tag{2.3}$$

We then have $\widetilde{R}(p,q)(0) = \tilde{q}$, $\widetilde{R}(p,q)(d(p,q)) = \tilde{p}$. Both $\widetilde{T}(p,q)$ and $\widetilde{R}(p,q)$ are curves connecting $\tilde{p}$ and $\tilde{q}$ in $\widetilde{M}_{\tilde{p}}^+$. Notice that $\widetilde{T}(p,q) \neq \widetilde{R}(p,q)$, in general, as point sets in $\widetilde{M}$ (see Corollary 5.3).

We say that a point $\tilde{q}$ in $\widetilde{M}$ is a *cut point* of $\tilde{p}$ if any extension of a minimizing geodesic segment $T(\tilde{p},\tilde{q})$ is not minimizing. Let $Cut(\tilde{p})$ denote the set of all cut points of $\tilde{p} \in \widetilde{M}$. It is known from [20] that $Cut(\tilde{p})$ carries the structure of a tree in $\widetilde{M}$. All edges of $Cut(\tilde{p}) \cap \mathrm{Int}(\widetilde{M}_{\tilde{p}}^{\pm})$ and all non-meridian geodesics in $\widetilde{M}_{\tilde{p}}^+$ are parameterized by the angle coordinate. This is because every minimizing geodesic segment $T(\tilde{p},\tilde{x})$ does not hit orthogonally to the parallel circle through $\tilde{x} \in \mathrm{Int}(\widetilde{M}_{\tilde{p}}^+) \cup \mathrm{Int}(\widetilde{M}_{\tilde{p}}^-)$ (cf. Lemma 3.1 (1)).

For $r > d(o,p)$ we define an *ellipsoid* with foci at o and p and with radius $r > d(o,p)$ in M by

$$E(o,p\,;r) = \{x \in M \mid d(o,x) + d(p,x) = r\} \tag{2.4}$$

and the distance function to o restricted to $E(o,p\,;r)$ by $d_r(x) = d(o,x)$, $x \in E(o,p\,;r)$. Let $E_p(r)$ be the set of all points where d_r attains local maximums. We will have

$$E(p) := \bigcup_{r>d(o,p)} E_p(r) \subset \mathrm{Cut}(o) \tag{2.5}$$

(see Lemma 3.4).

2.1. **ATCT for radial curvature.** By using these notations we will prove the following theorem which is a restatement of Theorem 1.1.

Theorem 2.1. *Assume that a complete pointed Riemannian manifold (M,o) is referred to a surface of revolution $(\widetilde{M},\tilde{o})$. Let $p \in M$. Suppose*

$$F_p(E(p)) \cap \widetilde{F}_{\tilde{p}}(Cut(\tilde{p}) \cap \mathrm{Int}(\widetilde{M}_{\tilde{p}}^+)) = \emptyset. \tag{2.6}$$

Then, there exists a minimizing geodesic segment $T(\tilde{p},\tilde{q})$ in $\widetilde{M}^+_{\tilde{p}}$ such that

$$\widetilde{T}(p,q) \geq T(\tilde{p},\tilde{q}) \qquad \textit{and} \qquad \widetilde{R}(p,q) \geq T(\tilde{p},\tilde{q}). \tag{2.7}$$

holds for every minimizing geodesic segment $T(p,q)$, $q \in M$.

Remark 2.2. The relation (2.7) is nothing but the Alexandrov convexity property. Namely, we have from (2.7)

$$d(o, T(p,q)(t)) \geq d(\tilde{o}, T(\tilde{p},\tilde{q})(t)), \quad d(o, T(q,p)(t)) \geq d(\tilde{o}, T(\tilde{q},\tilde{p})(t))$$

for all $t \in [0, d(p,q)]$ (see Lemma 4.3 (2)). Then the angle estimates at the corners p and q of $\triangle opq$ are obtained by the above relations (see Lemma 4.3 (3)).

Moreover, the angle estimate at o is obtained, also. The following theorem is the refined statement of Theorem 1.2.

Theorem 2.3. *Under the same assumptions as in Theorem 2.1, every geodesic triangle $\triangle opq \subset M$ admits its comparison triangle $\triangle\tilde{o}\tilde{p}\tilde{q} \subset \widetilde{M}$ such that*

$$\angle opq \geq \angle\tilde{o}\tilde{p}\tilde{q}, \qquad \angle oqp \geq \angle\tilde{o}\tilde{q}\tilde{p}, \qquad \angle poq \geq \angle\tilde{p}\tilde{o}\tilde{q}. \tag{2.8}$$

We emphasize that (2.8) is obtained under the radial curvature assumption with respect to o.

Remark 2.4. Under the same assumptions as in Theorem 2.1, if $\widetilde{T}(p,q) \cap T(\tilde{p},\tilde{q}) \neq \{\tilde{p},\tilde{q}\}$ for a minimizing geodesic segment $T(p,q)$ in M, then $\widetilde{T}(p,q) = T(\tilde{p},\tilde{q})$ and a geodesic triangle $\triangle opq$ in M bounds a totally geodesic 2-dimensional submanifold which is isometric to a comparison triangle domain $\triangle\tilde{o}\tilde{p}\tilde{q}$ in $\widetilde{M}$ corresponding to $\triangle opq$ (see Lemma 5.5).

Remark 2.5. If $\widetilde{M}$ is the standard 2-sphere, the flat plane or the Poincaré disk, then every point $\tilde{o} \in \widetilde{M}$ is viewed as a base point of $\widetilde{M}$ and any point $p \in M$ satisfies $Cut(\tilde{p}) \cap \mathrm{Int}(\widetilde{M}^+_{\tilde{p}}) = \emptyset$. We say that a surface of revolution $\widetilde{M}$ is a *von Mangoldt surface* if its radial curvature function is monotone non-increasing. Every point on a von Mangoldt surface of revolution $\widetilde{M}$ satisfies $Cut(\tilde{p}) \cap \mathrm{Int}(\widetilde{M}^+_{\tilde{p}}) = \emptyset$ (see [21]). Thus, Theorem 2.3 implies that (2.8) holds for every geodesic triangle $\triangle opq$. This result was first obtained in [8]. For a triangle $\triangle opq$ in a sector without cut points, the angle estimate of $\angle poq$ has been obtained by Kondo and Tanaka [11]

Remark 2.6. Assume that $o \in M$ is a pole of M. Namely, the exponential map $\exp_o : T_oM \to M$ at o is a diffeomorphism. Then $E(p)$ for every point $p \neq o$ is the subray from p of the ray from o passing through p (see Lemma 3.4). We then have (2.6) for every $p \in M$, $p \neq o$, and (2.8) for every $\triangle opq$. The same fact holds for a compact

Riemannian manifold M if $Cut(o)$ consists of a single point which is the first conjugate point to o along any unit speed geodesic emanating from o.

We define a pointed Alexandrov space (M, o) with radial curvature bounded below by a function K as follows. Let $(\widetilde{M}, \tilde{o})$ be a reference surface of revolution with radial curvature function K. We say that an Alexandrov space (M, o) with curvature locally bounded below is a *pointed Alexandrov space with radial curvature bounded below by the function* K if the following condition is satisfied:

(1) Every geodesic triangle $\triangle opq$ in M admits its comparison triangle $\triangle \tilde{o}\tilde{p}\tilde{q}$ in $\widetilde{M}$ satisfying (2.7).

(2) Conversely, for every geodesic triangle $\triangle \tilde{o}\tilde{p}\tilde{q}$ whose vertices $\tilde{p}$ and $\tilde{q}$ in $\widetilde{M}$ are the reference points p and q in M, respectively, there exists a geodesic triangle $\triangle opq$ in M satisfying (2.7).

The definition expects that if K_1 is any function less than or equal to K, then (M, o) is also a pointed Alexandrov space with radial curvature bounded below by the function K_1. This is proved as follows: Let (M, o) be a pointed Alexandrov space with radial curvature bounded below by the function K, $p \in M$ and let $\Phi : M \to \widetilde{M}$ be the reference map with respect to p. If $(\widetilde{M}_1, \tilde{o}_1)$ is the reference surface of revolution with curvature function K_1, we then have, from Remark 2.6, the reference map $\Psi : \widetilde{M} \to \widetilde{M}_1$ with respect to $\Phi(p)$. The reference map Φ_1 from M into $\widetilde{M}_1$ is $\Psi \circ \Phi$. Since the positional relation is invariant under the reference map Ψ, we have, from Remark 2.6,

$$\Phi_1(T(p, q)) \geq \Psi(T(\tilde{p}, \tilde{q})) \geq T(\tilde{p}_1, \tilde{q}_1)$$

where $\tilde{p} = \Phi(p)$, $\tilde{q} = \Phi(q)$, $\tilde{p}_1 = \Psi(\tilde{p}) = \Phi_1(p)$ and $\tilde{q}_1 = \Psi(\tilde{q}) = \Phi_1(q)$. This positional relation shows that (M, o) is a pointed Alexandrov space with radial curvature bounded below by the function K_1.

Using this notion, we have Corollaries 1.3 and 1.4 in §1.

2.2. **The sphere theorem for radial curvature.** A compact model surface $(\widetilde{M}, \tilde{o})$ is by definition a compact Riemannian 2-manifold whose metric ds^2 is expressed in terms of the polar coordinates around the base point $\tilde{o}$ as:

$$ds^2 = dr^2 + f(r)^2 d\theta^2,$$

where $f(r) > 0$, $0 < r < \ell < \infty$, $\theta \in S^1$ and $f : [0, \ell) \to \mathbb{R}$ satisfies the Jacobi equation

$$f'' + Kf = 0, \qquad f(0) = f(\ell) = 0, \quad f'(0) = -f'(\ell) = 1.$$

Let $\tilde{o}_1 := (\ell, 0)$ be the farthest point from $\tilde{o}$ in $\widetilde{M}$. For an arbitrary fixed point $\tilde{p} \neq \tilde{o}, \tilde{o}_1$ in $\widetilde{M}$, we set $\theta(\tilde{p}) := 0$. We divide $(\widetilde{M}, \tilde{o})$ by a simple closed geodesic consisting of two meridians $\theta^{-1}(\{0\}) \cup \theta^{-1}(\{\pi\})$ into $\widetilde{M}^+_{\tilde{p}}$ and $\widetilde{M}^-_{\tilde{p}}$, where $\widetilde{M}^+_{\tilde{p}} := \theta^{-1}([0, \pi])$.

Let (M, o) be a pointed compact Riemannian n-manifold. A 2-plane $\Pi \subset M_x$ at a point $x \in M$ is called a *radial plane* iff it contains a vector tangent to a minimizing geodesic joining o to x. The sectional curvature $K_M(\Pi)$ of M with respect to a radial plane is called a *radial curvature of* (M, o). A compact pointed manifold (M, o) is said to be *referred to* a compact model surface $(\widetilde{M}, \tilde{o})$ if and only if all the radial sectional curvatures of M satisfy

$$K_M(\Pi) \geq K(d(o, x)), \quad x \in M. \tag{2.9}$$

Let $i : M,\ \widetilde{M} \to \mathbf{R}$ be the injectivity radius function of the exponential map on $\widetilde{M}$ and M respectively. Let $B(\tilde{x}, a) \subset \widetilde{M}$ be the open metric a-ball with center at $\tilde{x}$. Let $\mathrm{Cut}(\tilde{p})$ denote the cut locus of $\tilde{p} \in \widetilde{M}$.

For the statement of our theorem, we define some constants given on $\widetilde{M}$. Let c_1 be the supremum of those $c > 0$ which satisfy

(1) $f^{-1}(\{t\})$ consists of two points for all $t \in [0, c)$,
(2) $[f^{-1}]'(t) \neq 0$ for all $t \in [0, c)$.

Let r_1 and $r_1{}^*$ be such that $0 < r_1 \leq \ell - r_1{}^* < \ell$ and $f(r_1) = f(\ell - r_1{}^*) = c_1$. Then, $f'(r) > 0$ for $r \in [0, r_1)$ and $f'(r) < 0$ for $r \in (\ell - r_1{}^*, \ell]$.

We define a constant $c_2(r)$, $r > 0$, by

$$\begin{aligned} & c_2(r) \\ := \ & \sup\{r^* \mid \mathrm{Cut}(\tilde{p}) \cap \mathrm{Int}(\widetilde{M}^+_{\tilde{p}}) \subset B(\tilde{o}, r) \text{ for } \tilde{p} \text{ with } r(\tilde{p}) > \ell - r^*\}. \end{aligned}$$

Here $\mathrm{Int}(\widetilde{M}^+_{\tilde{p}})$ is by definition the interior of the set $\widetilde{M}^+_{\tilde{p}}$.

We say that a minimizing geodesic segment T is *maximal* if any extension of T is not minimizing. Let $\Gamma(\tilde{p})$ be the maximal minimizing geodesic segment emanating from $\tilde{p}$ and tangent to the parallel $r^{-1}(\{r(\tilde{p})\})$ at $\tilde{p}$. The terminal point of $\Gamma(\tilde{p})$ is denoted by $\Gamma(\tilde{p})_e$. For every $r \in (0, r_1)$, we define $c_3(r)$ by

$$c_3(r) := \sup\{r^* \mid \Gamma(\tilde{p})_e \in B(\tilde{o}, r) \text{ for any } \tilde{p} \text{ with } r(\tilde{p}) > \ell - r^*\}.$$

Both $c_2(r)$ and $c_3(r)$ are monotone and non-decreasing in $r \in (0, r_1)$. Finally, we define the constant $\mu = \mu(\widetilde{M})$ by

$$\mu = \inf_{r \in (0, r_1)} \frac{\min\{c_2(r), c_3(r)\}}{r}.$$

Notice that the constants c_1, r_1, $c_2(r)$, $c_3(r)$ and μ are determined by the metric on $\widetilde{M}$. With this notation we state our theorem:

Theorem 2.7. *Assume that a compact pointed n-dimensional Riemannian manifold (M, o) is referred to $(\widetilde{M}, \tilde{o})$ with $\mu > 0$. Then, M is homeomorphic to an n-sphere, if there exists a point $p \in M$ such that*

$$\mu > \frac{\ell - d(o, p)}{\min\{r_1, i(o)\}}. \tag{2.10}$$

It has been proved in [7] that if M is referred to $\widetilde{M}$, then $\ell \geq \max\{d(o,x) \,|\, x \in M\}$, equality holding if and only if M is isometric to the warped product $[0,\ell] \times_f S^{n-1}(1)$ with warping function f. Here $S^{n-1}(1)$ denotes the unit sphere with dimension $n-1$. Hence, we note $\ell - d(o,p) \geq 0$.

Our theorem is thought of as a new version of the classical diameter sphere theorem [5], in which two assumptions are settled. No further assumption is needed. We will mention the difference of their assumptions in Remark 10.2.

The importance of injectivity radii in studying sphere theorems are seen in [1], [3].

3. Circles and ellipses

Let $(\widetilde{M}, \tilde{o})$ be a reference surface of revolution with vertex $\tilde{o}$. Throughout this section, we do not assume that a complete pointed Riemannian manifold (M,o) is referred to $(\widetilde{M}, \tilde{o})$.

Let $S(\tilde{p}\,; a) = \{\tilde{x} \in \widetilde{M} \,|\, d(\tilde{p},\tilde{x}) = a\}$ be the metric a-circle centered at $\tilde{p}$ and $S(\tilde{p},a)^{\pm} = S(\tilde{p}\,;a) \cap \widetilde{M}^{\pm}_{\tilde{p}}$. Let

$$I^{\pm}_{\tilde{p},a} = \{u \in [r(\tilde{p}) - a, r(\tilde{p}) + a] \,|\, S(\tilde{p}\,;a)^{\pm} \cap [r = u] \neq \emptyset\}$$

and

$$S^{\pm}_{\tilde{p},a}(u) = S(\tilde{p}\,;a)^{\pm} \cap [r = u], \quad u \in I^{\pm}_{\tilde{p},a}.$$

We will show that $S^{\pm}_{\tilde{p},a} : I^{\pm}_{\tilde{p},a} \to \widetilde{M}^{\pm}_{\tilde{p}}$ is a union of curves. Obviously, $I^{+}_{\tilde{p},a} = I^{-}_{\tilde{p},a} =: I_{\tilde{p},a}$. In general, $S(\tilde{p}\,;a)$ is not necessarily connected, and then $I_{\tilde{p},a}$ is the union of some intervals and points contained in $[r(\tilde{p}) - a, r(\tilde{p}) + a]$ as seen in Lemma 3.1 below. When $\ell < \infty$, we set $\tilde{o}_1$ to be the antipodal vertex of $\tilde{o}$, namely $r(\tilde{o}_1) = \ell$. Obviously, $S(\tilde{o}\,;a) = S(\tilde{o}_1\,;\ell - a) = [r = a]$, $0 \leq a \leq \ell$.

Lemma 3.1. *Let $\tilde{p} \in \widetilde{M}$, $\tilde{p} \neq \tilde{o}$, $\tilde{o}_1$. The metric circles in $\widetilde{M}$ satisfy the following properties.*

(1) *If an r_1-parallel circle $c = [r = r_1] \subset \widetilde{M}^{+}_{\tilde{p}}$ is parameterized as*

$$c(\varphi) = [r = r_1] \cap [\theta = \varphi]$$

for any φ, then $d(\tilde{p}, c(\varphi))$ is strictly increasing in $\varphi \in [\theta(\tilde{p}), \theta(\tilde{p}) + \pi]$.

(2) *Each of $S^{+}_{\tilde{p},a}(u)$ and $S^{-}_{\tilde{p},a}(u)$ consists of only one point for any $u \in I_{\tilde{p},a}$. In particular, each of $S^{+}_{\tilde{p},a}$ and $S^{-}_{\tilde{p},a}$ is the union of curves in $\widetilde{M}$ with parameter $u \in I_{\tilde{p},a}$.*

(3) *Let $\tilde{q} \in \widetilde{M}^{+}_{\tilde{p}}$ and let $s > 0$ and $t > 0$ satisfy $s + t = d(\tilde{p},\tilde{q})$. If $S(\tilde{p},s)^{+} \cap S(\tilde{q},t)^{-} \neq \emptyset$, then there exists a unique minimizing geodesic segment $T(\tilde{p},\tilde{q})$ passing through $\tilde{z} \in S(\tilde{p},s)^{+} \cap S(\tilde{q},t)^{-}$.*

Proof. Since all meridians $[\theta = \varphi]$ are geodesics which intersect the parallel circles $[r = r_1]$ orthogonally and hence the minimizing geodesic segments from $\tilde{p}$ intersect the parallel circle $c = [r = r_1]$ with the angles less than $\pi/2$, we have (1) from the first variation formula.

This fact (1) implies that each of $S^+_{\tilde{p},a}(u)$ and $S^-_{\tilde{p},a}(u)$ consists of at most one point for any $u \in [r(\tilde{p}) - a, r(\tilde{p}) + a]$. This proves (2).

Then (3) follows from the fact

$$d(\tilde{p}, \tilde{z}) + d(\tilde{z}, \tilde{q}) = s + t = d(\tilde{p}, \tilde{q}).$$

This completes the proof. $\square$

Let $\dot{T}(p, q)(0)$ denote the tangent vector of the curve $T(p, q)(t)$ at $t = 0$. We define a map $g : [0, \pi] \to S(\widetilde{p}\,; a)^+ \cup \{\varphi\}$ as follows:

(1) If $\omega \in [0, \pi]$ is the angle of $\dot{T}(\tilde{p}, \tilde{o})(0)$ with $\dot{T}(\tilde{p}, \tilde{q})(0)$ for some point $\tilde{q} \in S(\widetilde{p}\,; a)$, then $g(\omega) = \tilde{q}$.

(2) If ω satisfies $\omega_1 \le \omega \le \omega_2$ for some ω_1 and ω_2 with $g(\omega_1) = g(\omega_2)$, then $g(\omega) = g(\omega_1)$.

(3) Otherwise, $g(\omega) = \varphi$ where φ is the dummy.

The connected components of $[0, \pi] \setminus g^{-1}(\varphi)$ correspond to those of $I_{\tilde{p},a}$ If $\bar{g} : [0, \pi] \setminus g^{-1}(\varphi) \to I_{\tilde{p},a}$ is the map given by $g(\omega) = S^+_{\tilde{p},a}(\bar{g}(\omega))$, then $\bar{g}$ is monotone nondecreasing in each connected component.

Let $B(\tilde{o}, \tilde{p}\,; a) \subset \widetilde{M}$ for $a > d(\tilde{o}, \tilde{p})$ be the domain given by

$$B(\tilde{o}, \tilde{p}\,; a) = \{\tilde{q} \,|\, d(\tilde{o}, \tilde{q}) + d(\tilde{p}, \tilde{q}) \le a\}.$$

When $\ell < \infty$, the function $d(\tilde{p}, \tilde{q}) + d(\tilde{o}, \tilde{q})$, $\tilde{q} \in \widetilde{M}$, attains the maximum $2\ell - d(\tilde{p}, \tilde{o})$ at $\tilde{o}_1$. We therefore have $B(\tilde{o}, \tilde{p}\,; a) \supset \widetilde{M}$ for every $a \ge 2\ell - d(\tilde{p}, \tilde{o})$. If $a < 2\ell - d(\tilde{p}, \tilde{o})$, then $\tilde{o}_1 \notin B(\tilde{o}, \tilde{p}\,; a)$.

Lemma 3.2. *Let $\tilde{p} \neq \tilde{o}$, $\tilde{o}_1$. The ellipses $E(\tilde{o}, \tilde{p}\,; a)$, $d(\tilde{o}, \tilde{p}) < a < 2\ell - d(\tilde{o}, \tilde{p})$, in $\widetilde{M}$ has the following properties.*

(1) *$B(\tilde{o}, \tilde{p}\,; a)$ is star-shaped around $\tilde{p}$ and $\tilde{o}$. Namely, $T(\tilde{p}, \tilde{q}) \subset B(\tilde{o}, \tilde{p}\,; a)$ and $T(\tilde{o}, \tilde{q}) \subset B(\tilde{o}, \tilde{p}\,; a)$ for any $\tilde{q} \in B(\tilde{o}, \tilde{p}\,; a)$. If $\tilde{q} \in E(\tilde{o}, \tilde{p}\,; a)$, then $T(\tilde{p}, \tilde{q}) \cap E(\tilde{o}, \tilde{p}\,; a) = \{\tilde{q}\}$ and $T(\tilde{o}, \tilde{q}) \cap E(\tilde{o}, \tilde{p}\,; a) = \{\tilde{q}\}$. Furthermore, $\angle \tilde{p}\tilde{q}\tilde{o} \neq \pi$.*

(2) *The intersection $E(\tilde{o}, \tilde{p}\,; a) \cap [\theta = \varphi]$ is a single point for all $\varphi \in [\theta(\tilde{p}) - \pi, \theta(\tilde{p}) + \pi]$. If $(r(\varphi), \varphi) = E(\tilde{o}, \tilde{p}\,; a) \cap [\theta = \varphi]$, then $r(\theta(\tilde{p}) - \varphi) = r(\theta(\tilde{p}) + \varphi)$ for $\varphi \in [0, \pi]$. Moreover, $r(\varphi)$ is monotone increasing for $\varphi \in [\theta(\tilde{p}) - \pi, \theta(\tilde{p})]$ and monotone decreasing for $\varphi \in [\theta(\tilde{p}), \theta(\tilde{p}) + \pi]$. In particular, $r(\theta(\tilde{p})) = (a + d(\tilde{o}, \tilde{p}))/2$ is the maximum and $r(\theta(\tilde{p}) \pm \pi)$ is the minimum.*

(3) *If $e^{\pm}(u) = E(\tilde{o}, \tilde{p}\,; a) \cap \widetilde{M}^{\pm}_{\tilde{p}} \cap [r = u]$ for $u \in [r(\theta(\tilde{p}) + \pi), r(\theta(\tilde{p}))]$, then the function $d(\tilde{p}, e^{\pm}(u))$ is monotone decreasing in $u \in [r(\theta(\tilde{p}) + \pi), r(\theta(\tilde{p}))]$.*

(4) *Set $\tilde{q}_1 = (r(\theta(\tilde{p}) \pm \pi), \theta(\tilde{p}) \pm \pi)$. Let $b \in (a - r(\theta(\tilde{p})), d(\tilde{p}, \tilde{q}_1))$. Then, $S(\tilde{p}, b)$ crosses $E(\tilde{o}, \tilde{p}\,; a)$ once in each of $\widetilde{M}^+_{\tilde{p}}$ and $\widetilde{M}^-_{\tilde{p}}$.*

If $b = (a - d(\tilde{o}, \tilde{p}))/2$, then $S(\tilde{p}, b) \subset B(\tilde{o}, \tilde{p}\,; a)$ and $S(\tilde{p}, b) \cap E(\tilde{o}, \tilde{p}\,; a) = \{(r(\theta(\tilde{p})), \theta(\tilde{p}))\}$. If $b = d(\tilde{o}, \tilde{q}_1)$, then $S(\tilde{o}, b) \subset B(\tilde{o}, \tilde{p}\,; a)$ and $S(\tilde{o}, b) \cap E(\tilde{o}, \tilde{p}\,; a) = \{\tilde{q}_1\}$.

To be seen in Example 3.5, the third statement of (1) is not true, in general, for a complete pointed Riemannian manifold (M, o). Namely, $T(p, q) \cup T(q, o)$ may be a geodesic segment from p to o via q with $q \in E(o, p\,; a)$.

Proof. Let $\tilde{q} \in B(\tilde{o}, \tilde{p}\,; a)$ and let $\tilde{q}' \in T(\tilde{p}, \tilde{q})$. We then have

$$\begin{aligned} d(\tilde{o}, \tilde{q}') + d(\tilde{p}, \tilde{q}') &= d(\tilde{o}, \tilde{q}') + d(\tilde{p}, \tilde{q}) - d(\tilde{q}, \tilde{q}') \\ &\leq d(\tilde{o}, \tilde{q}) + d(\tilde{p}, \tilde{q}) \leq a. \end{aligned}$$

This means that $\tilde{q}' \in B(\tilde{o}, \tilde{p}\,; a)$, and, hence, $T(\tilde{p}, \tilde{q}) \subset B(\tilde{o}, \tilde{p}\,; a)$. In the same way we have $T(\tilde{o}, \tilde{q}) \subset B(\tilde{o}, \tilde{p}\,; a)$. These prove the first part of (1).

Suppose there exists a point $\tilde{q}' \in E(o, p\,; a) \cap T(\tilde{p}, \tilde{q}) \setminus \{\tilde{q}\}$. Then the above inequality shows that $d(\tilde{o}, \tilde{q}') = d(\tilde{o}, \tilde{q}) + d(\tilde{q}, \tilde{q}')$. From this we have $T(\tilde{p}, \tilde{q}) \cap T(\tilde{o}, \tilde{q}) \supset T(\tilde{q}', \tilde{q})$, meaning that $T(\tilde{p}, \tilde{q}) \cup T(\tilde{q}, \tilde{o})$ is a geodesic connecting $\tilde{p}$ and $\tilde{o}$ which is different from the meridian passing through $\tilde{p}$, a contradiction. This proves the second part of (1).

The third part of (1) is obvious. In fact, if $\angle \tilde{p}\tilde{q}\tilde{o} = \pi$, then $d(\tilde{p}, \tilde{q}) + d(\tilde{q}, \tilde{o}) + d(\tilde{o}, \tilde{p}) = 2\ell$, meaning that $a = 2\ell - d(\tilde{o}, \tilde{p})$, a contradiction.

Notice that the function $f(r) = d(\tilde{p}, (r, \varphi)) + r$ is monotone increasing in $r \in (0, \ell)$ with $f(r) > r(\tilde{p})$ because of the first variation formula. Since $\sup\{f(r) \mid r \in (0, \ell)\} = 2\ell - d(\tilde{o}, \tilde{p})$, we have the first part of (2).

Let φ_1 and φ_2 be such that $\theta(\tilde{p}) \leq \varphi_1 < \varphi_2 \leq \theta(\tilde{p}) + \pi$. If $r(\varphi_1) = r(\varphi_2)$, then $d(\tilde{p}, (r(\varphi_1), \varphi_1)) = d(\tilde{p}, (r(\varphi_2), \varphi_2))$, which contradicts Lemma 3.1 (1). By the triangle inequality,

$$\begin{aligned} r(\varphi) &\leq d(\tilde{o}, \tilde{p}) + d(\tilde{p}, (r(\varphi), \varphi)) \\ &= d(\tilde{o}, \tilde{p}) + a - r(\varphi), \end{aligned}$$

and, hence, we have $r(\varphi) \leq (a + d(\tilde{o}, \tilde{p}))/2$ where the equality holds if and only if $\varphi = \theta(\tilde{p})$. These imply the other parts of (2) and (3).

If $b \in (a - r(\theta(\tilde{p})), a - r(\theta(\tilde{p}) + \pi))$, then each curve $(r(\varphi), \varphi)$ for $\varphi \in [\theta(\tilde{p}) - \pi, \theta(\tilde{p})]$ and $[\theta(\tilde{p}), \theta(\tilde{p}) + \pi]$ moves from the outside of $S(\tilde{p}, b)$ to its inside and from its inside to its outside, respectively. The property (2) of this lemma implies that the crossing point is unique in each curve, which proves the first part of (4).

In order to prove the second part of (4), let $b = (a - d(\tilde{o}, \tilde{p}))/2$ and $\tilde{q} \in S(\tilde{p}, b)$. We then have

$$d(\tilde{o}, \tilde{q}) + d(\tilde{p}, \tilde{q}) \leq d(\tilde{o}, \tilde{p}) + 2d(\tilde{p}, \tilde{q}) = a$$

and equality holding if and only if $d(\tilde{o}, \tilde{p}) + d(\tilde{p}, \tilde{q}) = d(\tilde{o}, \tilde{q})$. This means that $S(\tilde{p}, b) \subset B(\tilde{o}, \tilde{p}\,; a)$ and $S(\tilde{p}, b) \cap E(\tilde{o}, \tilde{p}\,; a) = \{(r(\theta(\tilde{p})), \theta(\tilde{p}))\}$.

Let $b = d(\tilde{o}, \tilde{q}_1)$ and $\tilde{q} \in S(\tilde{o}, b)$, namely $d(\tilde{o}, \tilde{q}) = d(\tilde{o}, \tilde{q}_1)$. We then have, from Lemma 3.1 (1),

$$d(\tilde{o}, \tilde{q}) + d(\tilde{p}, \tilde{q}) \le d(\tilde{o}, \tilde{q}_1) + d(\tilde{p}, \tilde{q}_1) = a,$$

and equality holding if and only if $\tilde{q} = \tilde{q}_1$. This proves the third part of (4). $\square$

Let $\tilde{p} \ne \tilde{o}$, $\tilde{o}_1$ and $d(\tilde{o}, \tilde{p}) < a < 2\ell - d(\tilde{o}, \tilde{p})$. The reference curves will be made in $\widetilde{M}^+_{\tilde{p}}$, so we work in $\widetilde{M}^+_{\tilde{p}}$. From Lemma 3.2 the set $\Omega(\tilde{o}, \tilde{p}\,; a)^+$ of all minimizing geodesic segments $T(\tilde{p}, \tilde{q})$ from $\tilde{p}$ to points $\tilde{q} \in E(\tilde{o}, \tilde{p}\,; a)^+$ is a totally ordered set with respect to the binary relation $\le$ in the set of curves in $\widetilde{M}^+_{\tilde{p}}$. The minimizing geodesic segments $T(\tilde{p}, \tilde{q})$, $\tilde{q} \in E(\tilde{o}, \tilde{p}\,; a)^+$, divide $B(\tilde{o}, \tilde{p}\,; a)^+$ into two domains. Let $U(\tilde{p}, \tilde{q})$ denote the greatest minimizing geodesic segment connecting $\tilde{p}$ and $\tilde{q}$ and $L(\tilde{p}, \tilde{q})$ the least one. Namely $L(\tilde{p}, \tilde{q}) \le T(\tilde{p}, \tilde{q}) \le U(\tilde{p}, \tilde{q})$ for every minimizing geodesic segment $T(\tilde{p}, \tilde{q})$. If $\tilde{q} \notin Cut(\tilde{p})$, then $U(\tilde{p}, \tilde{q}) = L(\tilde{p}, \tilde{q})$. If $U(\tilde{p}, \tilde{q}) \ne L(\tilde{p}, \tilde{q})$ for a point $\tilde{q} \in Cut(\tilde{p})^+$, then $B(\tilde{o}, \tilde{p}\,; a)^+$ is divided into three domains B_1, B_0 and B_2. Here B_1 is the domain bounded by the meridian $[\theta = \theta(\tilde{p})]$, $E(\tilde{o}, \tilde{p}\,; a)^+$ and $U(\tilde{p}, \tilde{q})$, B_0 is the biangle domain bounded by $U(\tilde{p}, \tilde{q})$ and $L(\tilde{p}, \tilde{q})$, B_2 is the domain bounded by the meridians $[\theta = \theta(\tilde{p})] \cup [\theta = \theta(\tilde{p}) + \pi]$, $E(\tilde{o}, \tilde{p}\,; a)^+$ and $L(\tilde{p}, \tilde{q})$.

Lemma 3.3. *Let $\tilde{p} \ne \tilde{o}$, $\tilde{o}_1$ and $d(\tilde{o}, \tilde{p}) < a < 2\ell - d(\tilde{o}, \tilde{p})$. Let $\tilde{q} \in E(\tilde{o}, \tilde{p}\,; a)^+$. Let $\tilde{q}'$ be a sequence of points in $E(\tilde{o}, \tilde{p}\,; a)^+$ such that $r(\tilde{q}') > r(\tilde{q})$ (resp., $r(\tilde{q}') < r(\tilde{q})$) and it converges to $\tilde{q}$. Then the sequence of segments $T(\tilde{p}, \tilde{q}')$ converges to $U(\tilde{p}, \tilde{q})$ (resp., $L(\tilde{p}, \tilde{q})$).*

Proof. A subsequence of the sequence $T(\tilde{p}, \tilde{q}')$ converges to a minimizing geodesic segment $T(\tilde{p}, \tilde{q})$. Since B_1 and B_2 are star-shaped around $\tilde{p}$, it follows that $T(\tilde{p}, \tilde{q}')$ is contained in either B_1 or B_2, depending on $r(\tilde{q}') > r(\tilde{q})$ or $r(\tilde{q}') < r(\tilde{q})$. From the definition of $U(\tilde{p}, \tilde{q})$ and $L(\tilde{p}, \tilde{q})$, it follows that $T(\tilde{p}, \tilde{q})$ is one of $U(\tilde{p}, \tilde{q})$ and $L(\tilde{p}, \tilde{q})$. This shows that the sequence of minimizing geodesic segments $T(\tilde{p}, \tilde{q}')$ converges to either $U(\tilde{p}, \tilde{q})$ or $L(\tilde{p}, \tilde{q})$. $\square$

We now discuss the property of ellipses in M, which includes new ideas and play an important role. The following lemma gives a sufficient condition for $q \notin E_p(r)$, namely q is not a local maximum point of the distance function d_r to o restricted to $E(o, p\,; r)$.

Lemma 3.4. *Let $q \in E(o, p\,; r) \subset M$. If there exists a point u such that $d(p, u) + d(o, u) > r$ and $d(p, u) - d(p, q) < d(o, u) - d(o, q)$, then there exists a point $q' \in E(o, p\,; r)$ such that $d(o, q') > d(o, q)$. In particular, if $q \notin Cut(o)$ and $p \notin T(o, q)$, then q is not a local maximum point of d_r on $E(o, p\,; r)$.*

We observe that the assumption $d(p,u) + d(o,q) < d(p,q) + d(o,u)$ means that $d(o, T(q,u)(t))$ increase further than $d(p, T(q,u)(t))$ for $t \in [0, d(q,u)]$.

Proof. We first prove that the set $E(o, p\,; r) \cap T(p,u)$ consists of a single point, say q'. Suppose there exists a point $q'' \in E(o, p\,; r) \cap T(p,u)$ with $q'' \neq q'$. Assume without loss of generality that p, q', q'', u are in this order in $T(p,u)$. Since

$$\begin{aligned} d(p,q') + d(o,q') &= r \\ &= d(p,q'') + d(o,q'') \\ &= d(p,q') + d(q',q'') + d(o,q''), \end{aligned}$$

we have $d(o,q') = d(q',q'') + d(o,q'')$. This means that the minimizing geodesic segment $T(q',q'')$ is contained in both segments $T(o,q')$ and $T(p,u)$. In particular, $u \in T(o,q')$. This is a contradiction, because

$$\begin{aligned} r < d(p,u) + d(o,u) &= d(p,q'') + d(q'',u) + d(o,u) \\ &= d(p,q'') + d(o,q'') = r. \end{aligned}$$

We should note that $q' \neq q$. In fact, if $q' = q$, we then have

$$\begin{aligned} d(q',u) &= d(p,u) - d(p,q') \\ &= d(p,u) - d(p,q) \\ &< d(o,u) - d(o,q') \leq d(q',u), \end{aligned}$$

a contradiction.

We next prove that $d(o,q') > d(o,q)$. If $d(p,q') < d(p,q)$, we have

$$\begin{aligned} d(o,q') &= r - d(p,q') \\ &> r - d(p,q) \\ &= d(o,q). \end{aligned}$$

Thus we suppose $d(p,q') \geq d(p,q)$. We then have

$$\begin{aligned} d(o,q') &\geq d(o,u) - d(u,q') \\ &= d(o,u) - d(p,u) + d(p,q') \\ &\geq d(o,u) - d(p,u) + d(p,q) \\ &> d(o,u) - d(o,u) + d(o,q) \\ &= d(o,q). \end{aligned}$$

This completes the proof of the first part of the lemma.

Assume that $q \notin Cut(o)$ and $p \notin T(o,q)$. Let u be a point such that $q \in T(o,u)$ with $u \neq q$. We have

$$\begin{aligned} d(p,u) + d(o,u) &= d(p,u) + d(u,q) + d(q,o) \\ &\geq d(p,q) + d(o,q) = r, \end{aligned}$$

where the equality holds if and only if $d(p,q) = d(p,u) + d(u,q)$. By the triangle inequality, we have

$$d(p,u) - d(p,q) \leq d(q,u) = d(o,u) - d(o,q),$$

where the equality holds if and only if $d(p,u) = d(p,q) + d(q,u)$.

In order to prove that q is not a local maximum point of d_r on $E(o,p\,;r)$, we have to discuss the equality cases. Suppose first that $d(p,u) + d(u,q) = d(p,q)$. Then, $T(u,q) \subset T(p,q) \cap T(o,u)$, which means that $T(u,q) \subset E(o,p\,;r)$. Namely, $T(p,q) \cup T(u,o)$ is a geodesic in M connecting p and o which is not minimizing such that the subsegment from u to q is contained in $E(o,p\,;r)$. Such a geodesic will be seen in Example 3.5. Since every point $q' \in T(q,u) \setminus \{q\}$ satisfies that $d(o,q') > d(o,q)$, the point q is not a local maximal point of the function d_r.

We next suppose $d(p,u) = d(p,q)+d(q,u)$. Then, $T(q,u) \subset T(p,u) \cap T(o,u)$, Since $p \notin T(o,q)$, we have $o \in T(p,q) \subset T(p,u)$. Since $q \notin Cut(p)$, the set $S(p,d(p,q)) = \{u' \mid d(p,u') = d(p,q)\}$ contains a set U around q which is homeomorphic to a disk with dimension $\dim M - 1$ and any point $u' \in U$ with $u' \neq q$ satisfies $d(o,u') > d(o,q)$. In fact, $d(o,u') > |d(p,u') - d(p,o)| = |d(p,q) - d(p,o)| = d(o,q)$. Thus, we can find a point u' near q satisfying the assumption in the first part of the lemma, namely $d(p,u') + d(o,u') > r$ and $d(p,u') - d(p,q) < d(o,u') - d(o,q)$. From these arguments we may assume without loss of generality that there exists a point u' near q satisfying the assumption in the first part.

It remains to find a point $q'' \in E(o,p\,;r)$ near q such that $d(o,q'') > d(o,q)$. Let u' be a sequence of points satisfying the assumption in the first part of the lemma and converging to q. Let $q'(u') = E(o,p\,;r) \cap T(p,u')$ which satisfies $d(o,q'(u')) > d(o,q)$. The sequence of minimizing geodesic segments $T(q'(u'),u')$ converges to the point q or it contains a subsequence converging to a minimizing geodesic segment $T(q',q)$ contained in $E(o,p\,;r)$ as u' goes to q. When the first case occurs, the existence of $q'(u')$ shows that q is not a local maximum point of d_r. Suppose the second case happens. If $q'' \in T(q',q)$, we then have

$$\begin{aligned} d(o,q'') &= r - d(p,q'') \\ &= r - (d(p,q) - d(q'',q)) \\ &> r - d(p,q) = d(o,q). \end{aligned}$$

This implies that q is not a local maximum point of d_r on $E(o,p\,;r)$. This completes the proof. $\square$

The following example is helpful to understand what happens on ellipses when they get larger. It should be noted that there exists a point $q \in E(o,p\,;r)$ which cannot be an accumulation point of interior points of $B(o,p\,;r)$.

Example 3.5. We study how ellipses change in a flat cylinder when they get larger. Let $M = \{(x, y, z) \in \mathbb{E}^3 \,|\, x^2 + z^2 = 1\}$. Let $o = (1, 0, 0)$ and $p = (0, 2, -1)$. Then $Cut(o) = \{(-1, y, 0) \,|\, y \in \mathbb{R}\}$ and $Cut(p) = \{(0, y, 1) \,|\, y \in \mathbb{R}\}$. We identify M with $\mathbb{E}^2/\Gamma$ where $\mathbb{E}^2 = \{(x, y) \,|\, x, y \in \mathbb{R}\}$ and Γ is the isometry group generated by a translation τ such that $\tau((x, y)) = (x, y + 2\pi)$. The universal covering space $\pi : \mathbb{E}^2 \to M$ is given by $\pi((y, \theta)) = (\cos\theta, y, \sin\theta)$. The tangent plane M_o is identified with $\mathbb{E}^2$ also. Then $\widetilde{C}(o) = \{(x, \pm\pi) \,|\, x \in \mathbb{R}\}$ is the tangent cut locus of o and $U = \{(x, y) \,|\, x \in \mathbb{R}, -\pi \le y \le \pi\}$ is the lift of the normal coordinate neighborhood of o, namely $\exp_o : U \setminus \widetilde{C}(o) \longrightarrow M \setminus Cut(o)$ is a diffeomorphism. If $\varphi = \exp_o | U$, then $\varphi^{-1}(o) = (0, 0) =: o_0$ and $\varphi^{-1}(p) = (2, -\pi/2) =: p_0$ by this identification. Set $p_1 = (2, 3\pi/2)$. Further, $\varphi^{-1}(Cut(p)) = \{(x, \pi/2) \,|\, x \in \mathbb{R}\}$. Let $E(o, p\,; r) = \{w \,|\, F(w) := d(o, w) + d(p, w) = r\}$ and $B(o, p\,; r) = \{w \,|\, F(w) \le r\}$ for each $r > d(o, p)$.

Set $r_0 = \min\{F(w) \,|\, w \in Cut(o)\}$, $\{a\} = E(o_0, p_0\,; r_0) \cap T(o_0, p_1)$, $\{q_1\} = \widetilde{C}(o) \cap T(o_0, p_1)$. Let ∂X denote the boundary of a subset X. Then $\varphi^{-1}(E(o, p\,; r))$ changes with r as follows.

(1) $\varphi^{-1}(E(o, p\,; r)) = E(o_0, p_0; r)$ if r satisfies $d(o, p) < r < r_0$.
(2) $\varphi^{-1}(E(o, p\,; r_0)) = E(o_0, p_0\,; r_0) \cup T(a, q_1)$.
(3) $\varphi^{-1}(E(o, p\,; r)) = \partial(B(o_0, p_0; r) \cup B(o_0, p_1\,; r)) \cap U$ if r satisfies $r > r_0$.

Let $q_2 = \tau^{-1}(q_1)$. If $q \in Cut(o)$ satisfies $F(q) = \min F \,|\, Cut(o)$, then $\varphi^{-1}(q) = \{q_1, q_2\}$. Moreover, $\varphi(T(o_0, q_1) \cup T(q_2, p_0))$ is a geodesic connecting o and p in M. The geodesic reflecting against $Cut(o)$ at q in M is identified with $\varphi(T(o_0, q_2) \cup T(q_2, p_0))$.

It should be remarked that no sequence of points $q_j' \in E(o, p\,; r_j)$ for $r_j < r_0$ with $r_j \to r_0$ can converge to a point in $\varphi(T(a, q_1) \setminus \{a, q_1\}) \subset E(o, p\,; r_0)$. Thus, we notice that there exists a geodesic triangle $\triangle opq$ with $q \in E(o, p\,; r_0)$ such that it admits no sequence of geodesic triangles $\triangle opq_j$ with $q_j \in E(o, p\,; r_j)$, $r_j < r_0$, converging to itself.

4. Reference curves

Let $(\widetilde{M}, \tilde{o})$ be a reference surface of revolution with vertex $\tilde{o}$. Throughout this section, we do not assume that (M, o) is referred to $(\widetilde{M}, \tilde{o})$ and $F_p(E(p)) \cap \widetilde{F}_{\tilde{p}}(Cut(\tilde{p}) \cap \mathrm{Int}(\widetilde{M}^+_{\tilde{p}})) = \emptyset$. However, we assume that every minimizing geodesic segment $T(p, q)$ in consideration is contained in ${F_p}^{-1}(\widetilde{F}_{\tilde{p}}(\widetilde{M}^+_{\tilde{p}}))$. Therefore, $\widetilde{T}(p, q)(t)$ is defined for all $t \in [0, d(p, q)]$.

Lemma 4.1. *Let $\tilde{q}(t) = \widetilde{T}(p, q)(t)$, $0 \le t \le d(p, q)$. Let I denote the set of all parameters $t \in [0, d(p, q)]$ such that $\tilde{q}(t) \in \mathrm{Int}(\widetilde{M}^+_{\tilde{p}})$. The curves $\widetilde{T}(p, q)$ and $\widetilde{R}(p, q)$ satisfy the following properties.*

(1) *I is an interval. $\theta(\tilde{q}(t))$ is monotone increasing for $t \in I$. More precisely, if $t_0 := \max\{t \in [0, d(p,q)] \,|\, \theta(\tilde{q}(t)) = \theta(\tilde{p})\} > 0$, we then have two possibilities: If $\tilde{q}(t_0) \in T(\tilde{p}, \tilde{o})$, then $T(p,q)$ is contained in the maximal minimizing geodesic segment $T_e(p,o)$ from p through o and $\widetilde{T}(p,q)$ is contained in the union of meridians $[\theta = \theta(\tilde{p})] \cup [\theta = \theta(\tilde{p}) + \pi]$. If $\tilde{p} \in T(\tilde{o}, \tilde{q}(t_0))$, then $T(p, q(t_0)) = T(o, q(t_0)) \cap T(p,q)$ and $\widetilde{T}(p,q)([0,t_0])$ is contained in the meridian through $\tilde{p}$. In addition, if $t_0' := \min\{t \in [0, d(p,q)] \,|\, \theta(\tilde{q}(t)) = \theta(\tilde{p}) + \pi\} < d(p,q)$, we then have the similar results as above by using q and $\tilde{q}$ instead of p and $\tilde{p}$.*

(2) *$\widetilde{R}(p,q)(t)$ is defined for all $t \in [0, d(p,q)]$.*

(3) *$d(\tilde{p}, \widetilde{T}(p,q)(t)) = t$, $d(\tilde{q}, \widetilde{R}(p,q)(t)) = t$, $0 \le t \le d(p,q)$.*

(4) *$d(\tilde{p}, \widetilde{T}(p,q)(t)) + d(\tilde{q}, \widetilde{R}(p,q)(d(p,q) - t)) = d(p,q) = d(\tilde{p}, \tilde{q})$.*

(5) *$r(\widetilde{T}(p,q)(t)) = r(\widetilde{R}(p,q)(d(p,q) - t))$.*

(6) *$\widetilde{T}(p,q) \supset \widetilde{T}(p,q')$ and $\widetilde{R}(p,q) \supset \widetilde{R}(q',q)$ for any point $q' \in T(p,q)$.*

Proof. Let $q(t) = T(p,q)(t)$. We first prove that if there exist two parameters t_1 and t_2 such that $t_1 < t_2$ and $\theta(\tilde{q}(t_1)) = \theta(\tilde{q}(t_2))$ or $\tilde{q}(t_1) = \tilde{o}$ (or $\tilde{q}(t_1) = \tilde{o}_1$ if $\ell < \infty$), then $d(\tilde{p}, \tilde{q}(t_2)) = d(\tilde{p}, \tilde{q}(t_1)) + d(\tilde{q}(t_1), \tilde{q}(t_2))$, namely $\tilde{q}(t_1) \in T(\tilde{p}, \tilde{q}(t_2))$. In fact, since $\theta(\tilde{q}(t_1)) = \theta(\tilde{q}(t_2))$ implies that $|r(\tilde{q}(t_2)) - r(\tilde{q}(t_1))| = d(\tilde{q}(t_1), \tilde{q}(t_2))$, we have

$$\begin{aligned}
d(\tilde{p}, \tilde{q}(t_2)) - d(\tilde{p}, \tilde{q}(t_1)) &= d(p, q(t_2)) - d(p, q(t_1)) \\
&= d(q(t_1), q(t_2)) \\
&\ge |d(o, q(t_2)) - d(o, q(t_1))| \\
&= |r(\tilde{q}(t_2)) - r(\tilde{q}(t_1))| \\
&= d(\tilde{q}(t_1), \tilde{q}(t_2)) \\
&\ge d(\tilde{p}, \tilde{q}(t_2)) - d(\tilde{p}, \tilde{q}(t_1)),
\end{aligned}$$

meaning that $\tilde{q}(t_1) \in T(\tilde{p}, \tilde{q}(t_2))$. Thus, $T(\tilde{p}, \tilde{q}(t_2))$ is contained in the union of the meridians $[\theta = \theta(\tilde{p})] \cup [\theta = \theta(\tilde{p}) + \pi]$. Therefore, $\theta(\tilde{q}(t))$ is monotone increasing in the interval $I \subset [0, d(p,q)]$ such that $\tilde{q}(I) \subset \mathrm{Int}(\widetilde{M}_{\tilde{p}}^{+})$.

Suppose $t_0 > 0$. We have to treat two cases; $\tilde{q}(t_0) \in T(\tilde{p}, \tilde{o})$ and $\tilde{p} \in T(\tilde{o}, \tilde{q}(t_0))$. In the first case, we have

$$\begin{aligned}
d(p, q(t_0)) + d(q(t_0), o) &= d(\tilde{p}, \tilde{q}(t_0)) + d(\tilde{q}(t_0), \tilde{o}) \\
&= d(\tilde{p}, \tilde{o}) \\
&= d(p, o),
\end{aligned}$$

meaning that $T(p, q(t_0)) \subset T(p, o)$. Therefore, $T(p,q) \subset T_e(p,o)$, and therefore $\widetilde{T}(\tilde{p}, \tilde{q})$ is contained in the union of the meridian through $\tilde{p}$

and the meridian opposite to $\tilde{p}$. In the second case, we have

$$\begin{aligned} d(o,p)+d(p,q(t_0)) &= d(\tilde{o},\tilde{p})+d(\tilde{p},\tilde{q}(t_0)) \\ &= d(\tilde{o},\tilde{q}(t_0)) \\ &= d(o,q(t_0)), \end{aligned}$$

meaning that $T(p,q(t_0)) \subset T(o,q(t_0))$. Therefore, $\widetilde{T}(p,q)([0,t_0])$ is contained in the meridian through $\tilde{p}$.

It remains to prove that $\theta(\tilde{q}(t)) = \theta(\tilde{p}) + \pi$ for $t > t_0'$ in case of $\theta(\tilde{q}(t_0')) = \theta(\tilde{p}) + \pi$. Suppose that there exists a parameter $t \in (t_0', d(p,q))$ such that $\theta(\tilde{p}) < \theta(\tilde{q}(t)) < \theta(\tilde{p}) + \pi$. We then find a parameter $t_3 \in (0, t_0')$ such that $\theta(\tilde{q}(t_3)) = \theta(\tilde{q}(t))$ or $\tilde{q}(t_3) = \tilde{o}$ (or $\tilde{q}(t_3) = \tilde{o}_1$). By the same argument as above, we have a contradiction. In particular, $\theta(q(t_0')) = \theta(\tilde{q}(d(p,q))) = \theta(\tilde{p}) + \pi$. From the argument above, it follows that $T(\tilde{p},\tilde{q})$ is contained in the union of the meridians $[\theta = \theta(\tilde{p})] \cup [\theta = \theta(\tilde{p}) + \pi]$. We have proved (1).

In order to prove (2) we suppose that $\theta(\tilde{p}(t_0)) = \theta(\tilde{p})$ for some $t_0 \in [0, d(p,q))$ where $\tilde{p}(t) = \widetilde{R}(p,q)(t)$, $0 \le t \le d(p,q)$. Then, we have

$$\begin{aligned} d(p,q) &= t_0 + d(p,q) - t_0 \\ &= d(\tilde{q},\tilde{p}(t_0)) + d(\tilde{p},\tilde{q}(d(p,q)-t_0)) \\ &\ge d(\tilde{q},\tilde{p}(t_0)) + d(\tilde{p},\tilde{p}(t_0)) \\ &\ge d(\tilde{p},\tilde{q}) \\ &= d(p,q), \end{aligned}$$

since $r(\tilde{p}(t_0)) = r(\tilde{q}(d(p,q)-t_0)) = d(o, q(d(p,q)-t_0))$ and $\tilde{p}(t_0)$ lies in the meridian through $\tilde{p}$ and because of Lemma 3.1 (1). As before, $\widetilde{R}([t_0, d(p,q)])$ lies on the meridian through $\tilde{p}$. This shows (2).

Since

$$\begin{aligned} & (d(\tilde{o},\widetilde{T}(p,q)(t)), d(\tilde{p},\widetilde{T}(p,q)(t))) \\ = \;& \widetilde{F}_{\tilde{p}}(\widetilde{T}(p,q)(t)) \\ = \;& F_p(T(p,q)(t)) \\ = \;& (d(o,T(p,q)(t)), d(p,T(p,q)(t))), \end{aligned}$$

we have

$$r(\widetilde{T}(p,q)(t)) = d(o,T(p,q)(t)), \quad d(\tilde{p},\widetilde{T}(p,q)(t)) = t.$$

Since

$$\begin{aligned} & (d(\tilde{o},\widetilde{R}(p,q)(t)), d(\tilde{q},\widetilde{R}(p,q)(t))) \\ = \;& \widetilde{G}_{\tilde{q}}(\widetilde{R}(p,q)(t)) \\ = \;& F_q(T(p,q)(d(p,q)-t)) \\ = \;& (d(o,T(p,q)(d(p,q)-t)), d(q,T(p,q)(d(p,q)-t))), \end{aligned}$$

we have

$$r(\widetilde{R}(p,q)(t)) = d(o, T(p,q)(d(p,q)-t)), \quad d(\tilde{q}, \widetilde{R}(p,q)(t)) = t.$$

Thus we have (3) and

$$d(\tilde{p}, \widetilde{T}(p,q)(t)) + d(\tilde{q}, \widetilde{R}(p,q)(d(p,q)-t)) = d(p,q)$$

which proves (4). Then (5) follows from

$$\begin{aligned} r(\widetilde{T}(p,q)(t)) &= d(o, T(p,q)(t)) \\ &= r(\widetilde{R}(p,q)(d(p,q)-t)). \end{aligned}$$

Obviously, (6) follows from the definition of the reference curves and the reference reverse curves □

Let $q(t) = T(p,q)(t)$ and $\tilde{q}(t) = \widetilde{T}(p,q)(t)$, $0 \le t \le d(p,q)$. Lemma 4.1 (3) shows that a geodesic triangle $\triangle\tilde{o}\tilde{p}\tilde{q}(t)$ in $\widetilde{M}$ is a comparison triangle corresponding to $\triangle opq(t)$ in M.

Let $\theta(t) = \theta(\widetilde{R}(p,q)(d(p,q)-t)) - \theta(\widetilde{T}(p,q)(t))$, $0 \le t \le d(p,q)$. The following lemma shows the difference between $\widetilde{T}(p,q)(t)$ and $\widetilde{R}(p,q)(t)$ in terms of $\theta(t)$.

Lemma 4.2. *The reference curves $\tilde{q}(t) = \widetilde{T}(p,q)(t)$ and $\widetilde{R}(p,q)(t)$ satisfy the following properties.*

(1) *$\theta(t) \ge 0$ for all $t \in [0, d(p,q)]$. Moreover, if $\theta(t) \ne 0$ at $t \in (0, d(p,q))$, namely $\widetilde{T}(p,q)(t) \ne \widetilde{R}(p,q)(d(p,q)-t)$, then $T(\tilde{p},\tilde{q})$ does not cross the subarc of the parallel $[r = r(\tilde{q}(t))]$ in $\widetilde{M}^+_{\tilde{p}}$ joining $\widetilde{T}(p,q)(t)$ and $\widetilde{R}(p,q)(d(p,q)-t)$.*

(2) *$\theta(t) = 0$ if and only if $\widetilde{T}(p,q)(t) = \widetilde{R}(p,q)(d(p,q)-t)$. Then the point is in a minimizing geodesic segment $T(\tilde{p},\tilde{q})$.*

(3) *If there exists a point $\tilde{q}' \in \widetilde{T}(p,q) \cap T(\tilde{p},\tilde{q}) \setminus \{\tilde{p},\tilde{q}\}$ (resp., $\widetilde{R}(p,q) \cap T(\tilde{p},\tilde{q}) \setminus \{\tilde{p},\tilde{q}\}$), then $\widetilde{R}(p,q)(d(p,q) - d(\tilde{p},\tilde{q}')) = \tilde{q}'$ (resp., $\widetilde{T}(p,q)(d(p,q) - d(\tilde{p},\tilde{q}')) = \tilde{q}'$).*

(4) *$\widetilde{T}(p,q) \ge T(\tilde{p},\tilde{q})$ if and only if $\widetilde{R}(p,q) \ge T(\tilde{p},\tilde{q})$.*

(5) *$\widetilde{T}(p,q) \le T(\tilde{p},\tilde{q})$ if and only if $\widetilde{R}(p,q) \le T(\tilde{p},\tilde{q})$.*

Proof. It follows that $\theta(\tilde{p}) < \theta(\widetilde{T}(p,q)(t))$ and $\theta(\tilde{p}) < \theta(\widetilde{R}(p,q)(d(p,q)-t))$, $0 < t < d(p,q)$. Suppose the first part of (1) is false. Then Lemma 4.1 (5) and Lemma 3.1 (1) show that

$$d(\tilde{p}, \widetilde{T}(p,q)(t)) > d(\tilde{p}, \widetilde{R}(p,q)(d(p,q)-t))$$

for some t. This contradicts Lemma 4.1 (4), since

$$\begin{aligned} d(\tilde{p},\tilde{q}) &= d(\tilde{p}, \widetilde{T}(p,q)(t)) + d(\tilde{q}, \widetilde{R}(p,q)(d(p,q)-t) \\ &> d(\tilde{p}, \widetilde{R}(p,q)(d(p,q)-t)) + d(\tilde{q}, \widetilde{R}(p,q)(d(p,q)-t) \\ &\ge d(\tilde{p},\tilde{q}). \end{aligned}$$

We prove the second part of (1). Suppose that there exists a point $T(\tilde{p},\tilde{q})(t_0)$ lying on the parallel circle joining $\tilde{q}(t)$ and $\widetilde{R}(p,q)(d(p,q)-t)$ for some $t \in [0,d(p,q)]$. We then have $r(T(\tilde{p},\tilde{q})(t_0)) = r(\tilde{q}(t))$. Since $\theta(\widetilde{T}(p,q)(t))$ is monotone increasing in $t \in [0,d(p,q)]$, we have $\theta(\tilde{p}) \le \theta(\tilde{q}(t))$ and $\theta(\widetilde{R}(p,q)(d(p,q)-t)) \le \theta(\tilde{q})$. Since $\theta(\tilde{q}(t)) < \theta(T(\tilde{p},\tilde{q})(t_0) < \theta(\widetilde{R}(p,q)(d(p,q)-t))$, it follows from Lemma 3.1 (1) that $t < t_0$ and $d(\tilde{p},\tilde{q})-t_0 > d(p,q)-t$. Hence, we have $d(\tilde{p},\tilde{q}) > d(p,q)$, a contradiction.

If $\theta(t) = 0$, then the equality holds in the above inequalities, and hence $\widetilde{T}(p,q)(t) = \widetilde{R}(p,q)(d(p,q)-t)$. The converse is trivial. The second part of (2) follows from Lemma 4.1 (4).

We prove (3). Let $q' \in T(p,q)$ correspond to $\tilde{q}'$, namely, $d(p,q') = d(\tilde{p},\tilde{q}')$. Recall that $\tilde{p}' := \widetilde{R}(p,q)(d(p,q)-d(p,q'))$ is the point in $\widetilde{M}_{\tilde{q}}^-$ such that $d(\tilde{o},\tilde{p}') = d(o,q')$ and $d(\tilde{q},\tilde{p}') = d(q,q')$. Since $d(q,q') = d(p,q)-d(p,q') = d(\tilde{p},\tilde{q})-d(\tilde{p},\tilde{q}') = d(\tilde{q},\tilde{q}')$ and $d(\tilde{o},\tilde{q}') = d(o,q')$, we have $\tilde{p}' = \tilde{q}'$. In the same way we can prove the other case.

For the proof of (4) and (5), we suppose for indirect proof that $\widetilde{T}(p,q) \ge T(\tilde{p},\tilde{q})$ and $\widetilde{R}(p,q)(s) < T(\tilde{p},\tilde{q})$ for some $s \in (0,d(p,q))$. Then, there exists a point $\tilde{z} \in T(\tilde{p},\tilde{q})$ such that $\theta(\widetilde{T}(p,q)(d(p,q)-s)) \le \theta(\tilde{z}) < \theta(\widetilde{R}(p,q)(s))$, contradicting the second part of (1). The remainder cases are proved in the same way. $\square$

Lemma 4.3. *Let $q(t) = T(p,q)(t)$ be a minimizing geodesic segment in M. Assume that $\widetilde{T}(p,q(t)) \ge T(\tilde{p},\tilde{q}(t))$ for any $t \in [0,d(p,q)]$. Set $\tilde{p}(t) = \widetilde{R}(p,q)(t)$. Then, we have*

1. *$T(\tilde{p},\tilde{q}(t)) \ge T(\tilde{p},\tilde{q}(s))$ and $T(\tilde{p}(t),\tilde{q}) \ge T(\tilde{p}(s),\tilde{q})$ for any $t < s$.*
2. *$r(\tilde{q}(t)) \ge r(T(\tilde{p},\tilde{q})(t))$ and $r(\tilde{p}(t)) \ge r(T(\tilde{p},\tilde{q})(d(p,q)-t))$, $0 \le t \le d(p,q)$.*
3. *$\angle opq \ge \angle\tilde{o}\tilde{p}\tilde{q}$ and $\angle oqp \ge \angle\tilde{o}\tilde{q}\tilde{p}$.*

Proof. Let $\theta_0(t) = \theta(\tilde{q}(t))$. Let $(r_1(\theta),\theta)$ and $(r_2(\theta),\theta)$, $\theta \in [0,\theta_0(t)]$, be the parametrization of $\widetilde{T}(p,q(t))$ and $T(\tilde{p},\tilde{q}(t))$ by the angle coordinate θ, respectively. Set $\Omega = \{(r,\theta)\,|\,r_2(\theta) \le r \le r_1(\theta), 0 \le \theta \le \theta_0(t)\}$. We see that $T(\tilde{p},\tilde{q}(s))$ cannot pass through any interior point of Ω. In fact, if $T(\tilde{p},\tilde{q}(s))$ contains an interior point in Ω, then $T(\tilde{p},\tilde{q}(s))$ meets $\widetilde{T}(p,q(t))$ at $\widetilde{T}(p,q(t))(t_0)$ for some $t_0 \in (0,t]$, because $T(\tilde{p},\tilde{q}(t)) \cap T(\tilde{p},\tilde{q}(s)) = \{\tilde{p}\}$ and $\tilde{q}(s) \notin \Omega$. Since $\widetilde{T}(p,q(t))$ is a subarc of $\widetilde{T}(p,q(s))$, we have $\widetilde{T}(p,q(t)) \ge \widetilde{T}(p,q(s))$ and, hence, $\widetilde{T}(p,q(t)) \ge T(\tilde{p},\tilde{q}(s))$. This means that the last parameter t_0 where $T(\tilde{p},\tilde{q}(s))$ meets $\widetilde{T}(p,q(t))$ must be t. This contradicts $T(\tilde{p},\tilde{q}(t)) \cap T(\tilde{p},\tilde{q}(s)) = \{\tilde{p}\}$ again. Since $\widetilde{T}(p,q(t)) \ge T(\tilde{p},\tilde{q}(s))$, we conclude that $T(\tilde{p},\tilde{q}(t)) \ge T(\tilde{p},\tilde{q}(s))$. By Lemma 4.2 (4), we have the same inequality for the reference reverse curves. This completes the proof of (1).

Given t, $0 \le t \le d(p,q)$, we set $c(s) = T(\tilde{p}, \tilde{q}(s))(t) = S(\tilde{p}, t) \cap T(\tilde{p}, \tilde{q}(s))$ for any $s \in (t, d(p,q)]$. Then, (1) implies that $r(c(s))$ is monotone nonincreasing for $s > t$. We then have

$$r(\tilde{q}(t)) \ge r(c(s)) \ge r(c(d(p,q))) = r(T(\tilde{p}, \tilde{q})(t)).$$

In the same way, we have $r(\tilde{p}(t)) \ge r(T(\tilde{p}, \tilde{q})(d(p,q) - t))$ for any t. This completes the proof of (2).

In order to prove (3) we recall that

$$\cos \angle opq = \lim_{t \to +0} \frac{d(p, q(t))^2 + d(o,p)^2 - d(o, q(t))^2}{2d(p, q(t))d(o,p)}.$$

Therefore, we have from (2)

$$\begin{aligned} \cos \angle opq &= \lim_{t \to +0} \frac{t^2 + r(\tilde{p})^2 - r(\tilde{q}(t))^2}{2tr(\tilde{p})} \\ &\le \lim_{t \to +0} \frac{t^2 + r(\tilde{p})^2 - r(T(\tilde{p}, \tilde{q})(t))^2}{2tr(\tilde{p})} = \cos \angle \tilde{o}\tilde{p}\tilde{q}. \end{aligned}$$

Using the reference reverse curve $\widetilde{R}(p,q)$, we have $\angle oqp \ge \angle \tilde{o}\tilde{q}\tilde{p}$ in the same way. This completes the proof of (3). $\square$

5. Reference curves meeting no cut point

In this section, we assume that a complete pointed Riemannian manifold (M, o) is referred to a reference surface of revolution $(\widetilde{M}, \tilde{o})$. When $\ell < \infty$, it has been proved in [7] that M is isometric to the warped product manifold whose warping function is the radial curvature function of $\widetilde{M}$ if there exists a point $p \in M$ such that $d(o,p) = \ell$. Hence, there is nothing to study for the comparison theorems on those manifolds anymore. Therefore, we may assume that $d(o,p) < \ell$ for all points $p \in M$. We study the global positional relation between reference curves $\widetilde{T}(p,q)$ and minimizing geodesic segments $T(\tilde{p}, \tilde{q})$. We start from the following lemma, showing the local relation, which is proved in [7], [8] and [11].

Lemma 5.1. *Let p be a point in M such that $p \ne o$. There exists an $r_p > d(o,p)$ such that any geodesic triangle $\triangle opq$ in M with $d(o,q) + d(p,q) < r_p$ has a comparison triangle $\triangle \tilde{o}\tilde{p}\tilde{q}$ in $\widetilde{M}$ satisfying* (2.7) *and* (2.8)*. Moreover, if one of the equalities holds in* (2.7) *and* (2.8)*, then $\triangle opq$ bounds a totally geodesic 2-dimensional submanifold in M which is isometric to a comparison triangle domain $\triangle \tilde{o}\tilde{p}\tilde{q}$ corresponding to $\triangle opq$.*

Proof. As was seen in Lemma 3.2 (2), the set of all ellipses $E(\tilde{o}, \tilde{p}\,; a)$, $a > d(\tilde{o}, \tilde{p})$, gives a foliation of $\widetilde{M} \setminus T(\tilde{o}, \tilde{p})$. Namely, for any point $\tilde{q} \in \widetilde{M} \setminus T(\tilde{o}, \tilde{p})$ there exists the unique ellipse $E(\tilde{o}, \tilde{p}\,; a)$ passing through $\tilde{q}$. When $r(\tilde{p}) < \ell$, there exists a positive δ such that the δ-neighborhood

$D(\delta)$ of $T(\tilde{o}, \tilde{p})$ does not contain any cut point of $\tilde{p}$. Observe that the proof of the comparison theorems in [7], [8] and [11] is valid if the domain is free from $Cut(\tilde{p})$. Hence, if we set $r_p = \max\{a \mid E(\tilde{o}, \tilde{p}\,; a) \subset D(\delta)\}$, then it satisfies this lemma. $\square$

It follows from (2.7) and the third inequality of (2.8) that the reference curves and the comparison triangle $\triangle\tilde{o}\tilde{p}\tilde{q}$ actually lie in $\widetilde{M}^{+}_{\tilde{p}}$ for all points q with $d(o, q) + d(p, q) < r_p$.

Corollary 5.2. *Let p and q be points in M other than o. Assume that a minimizing geodesic segment $T(p, q)$ is contained in ${F_p}^{-1}(\widetilde{F}_{\tilde{p}}(\widetilde{M}^{+}_{\tilde{p}}))$. If $\widetilde{T}(p, q) = T(\tilde{p}, \tilde{q})$ as a set, then $\triangle opq$ bounds a totally geodesic 2-dimensional submanifold in M which is isometric to a comparison triangle domain $\triangle\tilde{o}\tilde{p}\tilde{q}$ corresponding to $\triangle opq$.*

Proof. As before, set $q(t) = T(p, q)(t)$ and $\tilde{q}(t) = \widetilde{T}(p, q)(t)$, $0 \le t \le d(p, q)$. Since $d(\tilde{p}, \tilde{q}(t)) = t$, $0 \le t \le d(p, q)$, and $\widetilde{T}(p, q) = T(\tilde{p}, \tilde{q})$, we have $\tilde{q}(t) = T(\tilde{p}, \tilde{q})(t)$ for all t. In fact, if $\tilde{q}(t) \neq T(\tilde{p}, \tilde{q})(t)$ for some $t \in (0, d(p, q))$, then there exists t_0 such that $t_0 \neq t$ and $\tilde{q}(t) = T(\tilde{p}, \tilde{q})(t_0)$. We then have $t = d(p, q(t)) = d(\tilde{p}, \tilde{q}(t)) = d(\tilde{p}, T(\tilde{p}, \tilde{q})(t_0)) = t_0$, a contradiction. Hence, if $0 \le t < s \le d(p, q)$, we then have $\widetilde{T}(q(t), q(s))(s - t) = T(\tilde{q}(t), \tilde{q}(s))(s - t)$, since $d(\tilde{q}(t), \tilde{q}(s)) = s - t = d(q(t), q(s))$, $r(q(t)) = r(\tilde{q}(t))$ and $r(q(s)) = r(\tilde{q}(s))$. Hence, we have $\angle oq(t)q(s) = \angle\tilde{o}\tilde{q}(t)\tilde{q}(s)$. It follows from Lemma 5.1 that there exists a $\delta > 0$ such that if $|s - t| < \delta$, then $\triangle oq(t)q(s)$ bounds a totally geodesic 2-dimensional submanifold in M which is isometric to a comparison triangle domain $\triangle\tilde{o}\tilde{q}(t)\tilde{q}(s)$ corresponding to $\triangle oq(t)q(s)$. This shows that there exist a totally geodesic 2-dimensional submanifold $\triangle$ bounded by $\triangle opq$ and an isometry from $\triangle$ onto the domain bounded by $\triangle\tilde{o}\tilde{p}\tilde{q}$ in $\widetilde{M}$. $\square$

Corollary 5.3. *Let M, p, q and $T(p, q)$ be as in Lemma 5.2. In addition, we assume that both $r(\widetilde{T}(p, q)(t)$ and $\theta(\widetilde{T}(p, q)(t)$ are strictly monotone in $t \in [0, d(p, q)]$. If $\widetilde{T}(p, q) = \widetilde{R}(p, q)$ as a set, then $\triangle opq$ bounds a totally geodesic 2-dimensional submanifold in M which is isometric to a comparison triangle domain $\triangle\tilde{o}\tilde{p}\tilde{q}$ corresponding to $\triangle opq$.*

Proof. We have to prove that $\widetilde{T}(p, q) = T(\tilde{p}, \tilde{q})$. Since $r(\widetilde{T}(p, q)(t)$ is monotone in $t \in [0, d(p, q)]$, from Lemma 4.1 (5), $r(\widetilde{R}(p, q)(t)$ is strictly monotone also. From $\widetilde{T}(p, q) = \widetilde{R}(p, q)$ and Lemma 4.1 (5), we have $\theta(t) = \theta(\widetilde{R}(p, q)(d(p, q) - t)) - \theta(\widetilde{T}(p, q)(t)) = 0$ for all $t \in [0, d(p, q)]$. From Lemma 4.2 (2), we have the minimizing geodesic segment $T(\tilde{p}, \tilde{q})$ which equals $\widetilde{T}(p, q)$. $\square$

Remark 5.4. Our proof technique to be employed in the theorems makes it complicated to treat the case where $\widetilde{T}(p, q) = T(\tilde{p}, \tilde{q})$. In order to avoid the case, we employ the same ideas developed in Chapter 2 in [3].

Let $K(r)$, $r \in [0, \ell)$, denote the Gauss curvature of $\widetilde{M}$ on the parallel r-circle. For a sufficiently small $\delta > 0$ we consider a differential equation

$$f''(r) + (K(r) - \delta)f(r) = 0.$$

We denote by $f_\delta(r)$ its solution with $f_\delta(0) = 0$ and $f_\delta'(0) = 1$. Then, $f_\delta(r) > f(r)$ for any $r \in (0, \ell)$. By defining a metric to be

$$ds^2 = dr^2 + f_\delta(r)^2 d\theta^2,$$

we have a reference surface of revolution $\widetilde{M}_\delta$ such that M is referred to $\widetilde{M}_\delta$. When $\ell < \infty$, the coefficient $K(r) - \delta$ and the solution $f_\delta(r)$ are extended on an interval $[0, \ell']$ containing $[0, \ell]$ properly and we do not assume that $f_\delta(\ell') = 0$ and $f_\delta'(\ell') = -1$. To avoid the confusing case where some equality holds in (2.7) or (2.8), we employ $\widetilde{M}_\delta$ instead of $\widetilde{M}$. We prove our results by thinking of $\widetilde{M}_\delta$ as the reference surface, and then conclude the proof by letting $\delta \to 0$. More precisely, we choose $\delta = \delta(R)$ for each R with $0 < (R + d(o,p))/2 < \ell$ such that (2.6) holds in the insides of $E(o,p\,;R)$ and $E(\tilde{o},\tilde{p}\,;R)$ and such that $\delta(R)$ converges to 0 as $(R+d(o,p))/2 \to \ell$. We prove (2.7) and (2.8) in the interior of $B(o,p\,;R)$ and $B(\tilde{o},\tilde{p}\,;R)$, and then take $(R+d(o,p))/2$ to ℓ. The most important fact is that $\widetilde{T}(p,q) = T(\tilde{p},\tilde{q})$ does not occur in $\widetilde{M}_\delta$ for any points $q \neq p$ in $E(o,p\,;R)$. This property simplifies our discussion.

The following lemma is proved in [8]. The proof here is different from theirs. Moreover, the method in the proof will be used when we prove Theorems in §7.

Lemma 5.5. *Assume that a point $q \in M$ admits a minimizing geodesic segment $T(p,q)$ contained in $F_p{}^{-1}(\widetilde{F}_{\tilde{p}}(\widetilde{M}_{\tilde{p}}^{+}))$. If there exists a point $q_1 \in T(p,q)$ such that $\widetilde{T}(p,q') \geq T(\tilde{p},\tilde{q}')$ for all $q' \in T(p,q_1)$, $\tilde{q}_1 \notin Cut(\tilde{p})$ and $((\widetilde{T}(p,q) \setminus \{\tilde{q}\}) \setminus \widetilde{T}(p,q_1)) \cap Cut(\tilde{p}) = \emptyset$, then there exists a minimizing geodesic segment $T(\tilde{p},\tilde{q})$ such that $\widetilde{T}(p,q) \geq T(\tilde{p},\tilde{q})$. In addition, if $\widetilde{T}(p,q) \cap T(\tilde{p},\tilde{q})$ contains a point $\tilde{q}'$ other than $\tilde{p}$ and $\tilde{q}$, then $\triangle opq$ bounds a totally geodesic 2-dimensional submanifold in M which is isometric to a comparison triangle domain $\triangle\tilde{o}\tilde{p}\tilde{q}$ corresponding to $\triangle opq$ in $\widetilde{M}$.*

The point is if $\tilde{q}_1 \notin Cut(\tilde{p})$ or not. In case of $\tilde{q}_1 \notin Cut(\tilde{p})$ the reference curve can be extended, keeping the positional relation to a minimizing geodesic connecting its end points.

Proof. In order to prove the first part, we work in $\widetilde{M}_\delta$ to avoid the case where a reference curve is identified with a minimizing geodesic segment connecting its endpoints. For convenience we set $q(t) = T(p,q)(t)$ and

$\tilde{q}(t) = \widetilde{T}(p,q)(t)$ for any $t \in (0, d(p,q))$. Let t_0 be the least upper bound of the set of all $t_2 \leq d(p,q)$ so that there exists a minimizing geodesic segment $T(\tilde{p}, \tilde{q}(t))$ with $\widetilde{T}(p, q(t)) \geq T(\tilde{p}, \tilde{q}(t))$ for all $t \in (0, t_2)$. If t_1 is the parameter such that $q_1 = q(t_1)$, we then have $t_0 \geq t_1$ because of the assumption.

Suppose for indirect proof that $t_0 < d(p,q)$. Since we assume that $\tilde{q}(t_0) \notin Cut(\tilde{p})$, there exists a neighborhood V of $\tilde{q}(t_0)$ such that $T(\tilde{p}, \tilde{x})$ is the unique minimizing geodesic segment connecting $\tilde{p}$ and $\tilde{x} \in V$. Since the minimizing geodesic segment $T(\tilde{p}, \tilde{q}(t_0))$ is unique, $\widetilde{T}(p, q(t)) \geq T(\tilde{p}, \tilde{q}(t))$ for all $t \in (0, t_0)$ implies that $\widetilde{T}(p, q(t_0)) \geq T(\tilde{p}, \tilde{q}(t_0))$.

We will prove that there exists an $\varepsilon > 0$ such that $\widetilde{T}(p, q(t_0 + t)) \geq T(\tilde{p}, \tilde{q}(t_0 + t))$, $0 \leq t \leq \varepsilon$. Suppose that there exists a monotone decreasing sequence t_j converging to 0 such that no minimizing geodesic segment $T(\tilde{p}, \tilde{q}(t_0 + t_j))$ satisfies $\widetilde{T}(p, q(t_0 + t_j)) \geq T(\tilde{p}, \tilde{q}(t_0 + t_j))$. We then have either

$$\widetilde{T}(p, q(t_0 + t_j)) \leq T(\tilde{p}, \tilde{q}(t_0 + t_j))$$

or

$$\widetilde{T}(p, q(t_0 + t_j)) \cap T(\tilde{p}, \tilde{q}(t_0 + t_j)) \neq \{\tilde{p}, \tilde{q}(t_0 + t_j)\}.$$

Suppose the first is true. We then have $\widetilde{T}(p, q(t_0)) = T(\tilde{p}, \tilde{q}(t_0))$. In fact, since $T(\tilde{p}, \tilde{q}(t_0 + t_j))$ converges to $T(\tilde{p}, \tilde{q}(t_0))$ which is the unique minimizing geodesic segment connecting $\tilde{p}$ and $\tilde{q}(t_0)$, we have $\widetilde{T}(p, q(t_0)) \leq T(\tilde{p}, \tilde{q}(t_0))$. Combining $\widetilde{T}(p, q(t_0)) \geq T(\tilde{p}, \tilde{q}(t_0))$, we conclude $\widetilde{T}(p, q(t_0)) = T(\tilde{p}, \tilde{q}(t_0))$. Since we employ $\widetilde{M}_\delta$, this yields a contradiction because of Corollary 5.2.

Suppose the second is true. Let $\tilde{q}_j$ be a point in $\widetilde{T}(p, q(t_0 + t_j)) \cap T(\tilde{p}, \tilde{q}(t_0 + t_j))$ such that it is different from $\tilde{p}$, $\tilde{q}(t_0 + t_j)$ and $T(\tilde{q}_j, \tilde{q}(t_0 + t_j)) \not\leq \widetilde{T}(p, q(t_0 + t_j))$. Let $q_j \in T(p, q(t_0 + t_j))$ be the point with $F_p(q_j) = \widetilde{F}_{\tilde{p}}(\tilde{q}_j)$. If $\tilde{q}_j$ does not converge to the point $\tilde{q}(t_0)$, then there exists an accumulation point $\tilde{q}' \neq \tilde{q}(t_0)$ such that $\tilde{q}' \in T(\tilde{p}, \tilde{q}(t_0))$. This situation implies that $\widetilde{T}(p, q(t_0)) \geq T(\tilde{p}, \tilde{q}(t_0))$ and $\widetilde{T}(p, q(t_0)) \cap T(\tilde{p}, \tilde{q}(t_0)) \supset \{\tilde{p}, \tilde{q}', \tilde{q}(t_0)\}$, which is the assumption of the second part of this lemma, to be proved in the next paragraph. This is impossible because we now work in $\widetilde{M}_\delta$. We have proved that $\tilde{q}_j$ converges to $\tilde{q}(t_0)$. We then have $\tilde{q}(t_0 + t_j)$ such that $\widetilde{T}(p, q_j) \geq T(\tilde{p}, \tilde{q}_j)$ and $\widetilde{T}(p, q(t_0 + t_j)) \setminus \widetilde{T}(p, q_j) \not\geq T(\tilde{q}_j, \tilde{q}(t_0 + t_j))$, since there exists the unique minimizing geodesic segment $T(p, q_j)$ which is a subsegment of $T(p, q(t_0 + t_j))$. On the other hand, for sufficiently large j, it follows from Lemma 4.2 (3) that $\widetilde{R}(p, q(t_0 + t_j))$ passes through $\tilde{q}_j$. From Lemma 4.1 (6) the reference reverse curve $\widetilde{R}(q_j, q(t_0 + t_j))$ is a subarc of $\widetilde{R}(p, q(t_0 + t_j))$ from $\tilde{q}(t_0 + t_j)$ to $\tilde{q}_j$ which lies in the same side as the subarc of $\widetilde{T}(p, q(t_0 + t_j))$ from $\tilde{q}_j$ to $\tilde{q}(t_0 + t_j)$ (see Lemma 4.2 (4) and (5)). Thus, we have the positional relation

$$\widetilde{R}(q_j, q(t_0 + t_j)) \leq T(\tilde{q}_j, \tilde{q}(t_0 + t_j)).$$

However, this contradicts Lemma 5.1 near the point $\tilde{q}(t_0 + t_j)$. We conclude that $t_0 = d(p, q)$ by employing $\widetilde{M}_\delta$. Letting $\delta \to 0$ we complete the proof of the first part.

We prove the second part. If $\widetilde{T}(p, q) \setminus \widetilde{T}(p, q') \not\subset T(\tilde{p}, \tilde{q})$, then there exists a point $q'' \in T(q', q)$ such that $\tilde{q}''$ does not lie in $T(\tilde{p}, \tilde{q})$ and hence $\tilde{q}'' > T(\tilde{p}, \tilde{q})$. Therefore, $\tilde{q}' < T(\tilde{p}, \tilde{q}'')$, contradicting that $\widetilde{T}(p, q'') \geq T(\tilde{p}, \tilde{q}'')$. Thus, we have $\widetilde{T}(p, q) \setminus \widetilde{T}(p, q') \subset T(\tilde{p}, \tilde{q})$, in other words, $\widetilde{R}(q', q) \subset T(\tilde{p}, \tilde{q})$. Let u and u' be points in $T(p, q)$ such that they are near q' and p, u, q' and u' lie in this order in $T(p, q)$. If $\tilde{u} \notin T(\tilde{p}, \tilde{q})$, then we have a contradiction from Lemma 5.1 and the same argument above. Thus, the segment $T(q', q)$ satisfying $\widetilde{R}(q', q) \subset T(\tilde{p}, \tilde{q})$ can be extended until q' reaches p. Hence, we have $\widetilde{R}(p, q) = T(\tilde{p}, \tilde{q})$, and, equivalently, $\widetilde{T}(p, q) = T(\tilde{p}, \tilde{q})$. It follows from Corollary 5.2 that $\triangle opq$ bounds a totally geodesic 2-dimensional submanifold in M which is isometric to a comparison triangle domain $\triangle \tilde{o}\tilde{p}\tilde{q}$ corresponding to $\triangle opq$ in $\widetilde{M}$. $\square$

6. Reference curves meeting cut points

In this section, we assume that a complete pointed Riemannian manifold (M, o) is referred to a reference surface of revolution $(\widetilde{M}, \tilde{o})$.

For two points $\tilde{x}$, $\tilde{y} \in \widetilde{M}^+_{\tilde{p}}$ with $\theta(\tilde{x}) \neq \theta(\tilde{y})$, let $U(\tilde{x}, \tilde{y})$ and $L(\tilde{x}, \tilde{y})$ denote the minimizing geodesic segments joining $\tilde{x}$ to $\tilde{y}$ such that

$$U(\tilde{x}, \tilde{y}) \geq T(\tilde{x}, \tilde{y}) \geq L(\tilde{x}, \tilde{y}), \quad \text{for all } T(\tilde{x}, \tilde{y}) \tag{6.1}$$

Notice that $U(\tilde{x}, \tilde{y}) = L(\tilde{x}, \tilde{y})$ if and only if $\tilde{y} \notin Cut(\tilde{x})$ or $\tilde{y} \in Cut(\tilde{x})$ is an end point of $Cut(\tilde{x})$ such that $\tilde{y}$ is an isolated conjugate point to $\tilde{x}$ along the unique minimizing geodesic. The following lemma is a consequence of Lemma 3.3 and it plays an important role for the proof of our Theorems.

Assume that $\tilde{y} \in Cut(\tilde{x})$ is not an end point of $Cut(\tilde{x})$. Then there exists $\varepsilon > 0$ such that the connected component C of $B(\tilde{y}, \varepsilon) \cap Cut(\tilde{x})$ containing $\tilde{y}$ divides $B(\tilde{y}, \varepsilon)$ into finitely many connected components where $B(\tilde{y}, \varepsilon) = \{\tilde{z} \in \widetilde{M} \mid d(\tilde{y}, \tilde{z}) \leq \varepsilon\}$. Set

$$H_f = \{\tilde{z} \in B(\tilde{y}, \varepsilon) \mid r(\tilde{z}) > r(\tilde{w}) \text{ for any } \tilde{w} \in C \text{ with } \theta(\tilde{z}) = \theta(\tilde{w}).\}$$

and

$$H_n = \{\tilde{z} \in B(\tilde{y}, \varepsilon) \mid r(\tilde{z}) < r(\tilde{w}) \text{ for any } \tilde{w} \in C \text{ with } \theta(\tilde{z}) = \theta(\tilde{w}).\}.$$

We call H_f (resp., H_n) the *far (resp., near) side of* $Cut(\tilde{x})$ *at* $\tilde{y}$ *from* $\tilde{o}$.

Lemma 6.1. *Assume that* $B(o,p\,;r) \subset F_p^{-1}(\widetilde{F}_{\tilde{p}}(\widetilde{M}_{\tilde{p}}^{+}))$*. If* $q \in E(o,p\,;r)$ *is not a local maximum point of* d_r *on* $E(o,p\,;r)$*, then there exists a sequence of points* $q_j \in E(o,p\,;r)$ *converging to* q *such that* $T(\tilde{p},\tilde{q}_j)$ *converges to* $U(\tilde{p},\tilde{q})$ *as* $j \to \infty$*. In particular, if* $U(\tilde{p},\tilde{q}) \neq L(\tilde{p},\tilde{q})$ *and* $\tilde{q}$ *is not an end point of* $Cut(\tilde{p})$*, then any extension of* $U(\tilde{p},\tilde{q})$ *crosses* $Cut(\tilde{p})$ *from the far side of* $Cut(\tilde{p})$ *at* $\tilde{q}$ *from* $\tilde{o}$ *to the near side.*

Proof. Let q_j be a sequence of points in $E(o,p\,;r)$ converging to q such that $d(o,q_j) > d(o,q)$ for all j. Then we have

$$\begin{aligned} d(p,q_j) &= r - d(o,q_j) \\ &< r - d(o,q) = d(p,q). \end{aligned}$$

In view of Lemma 3.2, we observe that $T(\tilde{p},\tilde{q}_j)$ does not cross $[\theta(\tilde{q}) \leq \theta] \cap [r = d(o,q)]$. This means that $T(\tilde{p},\tilde{q}_j) \setminus \{\tilde{p}\} > T(\tilde{p},\tilde{q}) \setminus \{\tilde{p}\}$ for every minimizing geodesic segment $T(p,q)$. Therefore, $T(\tilde{p},\tilde{q}_j)$ converges to $U(\tilde{p},\tilde{q})$ as $j \to \infty$. $\square$

We observe from Lemma 5.5 that $\widetilde{T}(p,q) \geq U(\tilde{p},\tilde{q})$ if $\widetilde{T}(p,q) \setminus \{\tilde{q}\} \cap Cut(\tilde{p}) = \emptyset$ and $q \in E(o,p\,;r)$ is not a local maximum point of d_r.

In the proof of the following lemma, we need an orientation of the intersection points of curves and $Cut(\tilde{p})$. Let $\tilde{x} \in Cut(\tilde{p})$. A curve $c(\theta)$, $\tilde{x} = c(\theta_0)$, parameterized by angle coordinate θ is said to intersect $Cut(\tilde{p})$ *positively* (resp., *negatively*) at a point $\tilde{x} = c(\theta_0)$ if there is a small neighborhood Ω around $\tilde{x}$ such that $c \cap \Omega \geq Cut(\tilde{p}) \cap \Omega$ for $\theta \leq \theta_0$, (resp., $c \cap \Omega \leq Cut(\tilde{p}) \cap \Omega$ for $\theta \leq \theta_0$). Intuitively, "intersecting positively" means that c meets $Cut(\tilde{p})$ from the far side with respect to $\tilde{o}$.

Lemma 6.2. *Let* $q \in M$ *and let* $T(p,q)$ *be a minimizing geodesic segment. Assume that* $T(p,q) \subset F_p^{-1}(\widetilde{F}_{\tilde{p}}(\widetilde{M}_{\tilde{p}}^{+}))$*. Suppose all intersection points of* $\widetilde{T}(p,q)$ *and* $Cut(\tilde{p})$ *are positive. Then, we have*

$$\widetilde{T}(p,q(t)) \geq U(\tilde{p},\tilde{q}(t)), \quad 0 \leq t < d(p,q).$$

Here we set $q(t) = T(p,q)(t)$*,* $0 \leq t \leq d(p,q)$*.*

Notice again that if $q \notin Cut(p)$, then there exists a unique minimizing geodesic segment $T(p,q)$, and hence, the reference curve $\widetilde{T}(p,q)$ is uniquely determined. However, this does not mean that the number of the reference curves connecting $\tilde{p}$ and $\tilde{q}$ is one, because $F_p^{-1}(\widetilde{F}_{\tilde{p}}(\tilde{q}))$ may not be a single point. If $\tilde{q} \notin Cut(\tilde{p})$, then there exists a unique minimizing geodesic segment $T(\tilde{p},\tilde{q})$, and hence, $T(\tilde{p},\tilde{q}) = U(\tilde{p},\tilde{q}) = L(\tilde{p},\tilde{q})$. However, if $\tilde{q} \in Cut(\tilde{p})$, then there may be many minimizing geodesic segments $T(\tilde{p},\tilde{q})$. So the positional relation between $\widetilde{T}(p,q)$ and $U(\tilde{p},\tilde{q})$ is unknown, in general. These facts are often used without notice.

Proof. We work in $\widetilde{M}_\delta$ instead of the reference surface $\widetilde{M}$ (see Remark 5.4). We choose δ to be sufficiently small so that $\widetilde{M}_\delta$ satisfies the assumption in this lemma. Let t_0 be the least upper bound of the set of all $t_1 \in (0, d(p,q))$ such that $\widetilde{T}(p, q(t)) \geq U(\tilde{p}, \tilde{q}(t))$ for all $t \in (0, t_1)$. We already know that $t_0 > 0$. Suppose for indirect proof that $t_0 < d(p,q)$. If $\tilde{q}(t_0) \notin Cut(\tilde{p})$, then, from Lemma 5.5, there exists an $\varepsilon > 0$ such that $\widetilde{T}(p, q(t_0+t)) \geq U(\tilde{p}, \tilde{q}(t_0+t))$ for all $t \in (0, \varepsilon)$. This contradicts the choice of t_0.

Suppose $\tilde{q}(t_0) \in Cut(\tilde{p})$. Since the minimizing geodesic segment $T(p, q(t_0))$ is unique and $\tilde{q}(t_0)$ is a positive cut point, it follows that $\widetilde{T}(p, q(t_0)) \geq U(\tilde{p}, \tilde{q}(t_0))$. We prove that there exists an $\varepsilon > 0$ such that $\widetilde{T}(p, q(t_0+t)) \leq Cut(\tilde{p})$ for all $t \in (0, \varepsilon)$. In fact, suppose this is not true. Then, there exist a sufficiently small neighborhood Ω around $\tilde{q}(t_0)$ and a sequence $t_j > t_0$ such that t_j converges to t_0 and $\tilde{q}(t_j)$ is contained in the subdomain of Ω bounded below by $U(\tilde{p}, \tilde{q}(t_0)) \cup Cut(\tilde{p})$. Let T_j be a minimizing geodesic segment connecting $\tilde{p}$ and $\tilde{q}(t_j)$. Then, we know that $T_j \cap \widetilde{T}(p, q(t_j)) \neq \{\tilde{p}, \tilde{q}(t_j)\}$, since T_j converges to $U(\tilde{p}, \tilde{q}(t_0))$ and $\widetilde{T}(p,q)([0,t_0]) \neq T(\tilde{p}, \tilde{q}(t_0))$. If $\tilde{q}(t'_j) = \widetilde{T}(p,q)(t'_j) \in T_j \cap \widetilde{T}(p, q(t_j))$ and $\tilde{q}(t'_j) \notin \{\tilde{p}, \tilde{q}(t_j)\}$, then $\tilde{q}(t'_j)$ converges to $\tilde{q}(t_0)$ and it follows that

$$\widetilde{T}(p,q)([t'_j, t_j]) \leq T(\tilde{q}(t'_j), \tilde{q}(t_j)).$$

However, this contradicts Lemma 5.1 and Lemma 4.2. Thus we see that there exists an $\varepsilon > 0$ such that $\widetilde{T}(p, q(t_0+t)) \leq Cut(\tilde{p})$ for all $t \in (0, \varepsilon)$, and, therefore, $\tilde{q}(t_0+t) \notin Cut(\tilde{p})$ for all small $t > 0$. Since $a := d(o, q(t_0)) + d(p, q(t_0)) < d(o, q(t_0+t)) + d(p, q(t_0+t))$, the point $\tilde{q}(t_0+t)$ is outside $E(o, p\,; a)$. Therefore, $\tilde{q}(t_0+t)$ is in the subdomain of Ω bounded above by $L(\tilde{p}, \tilde{q}(t_0)) \cup Cut(\tilde{p})$. Since a sequence of unique minimizing geodesic segments $T(\tilde{p}, \tilde{q}(t_0+t))$ converges to $L(\tilde{p}, \tilde{q})$ as $t \to 0$, it follows that $\widetilde{T}(p, q(t_0+t)) \geq U(\tilde{p}, \tilde{q}(t_0+t))$ for all small $t > 0$. This contradicts the choice of t_0. $\square$

7. Proof of Theorems 2.1 and 2.3

We are ready to prove Theorems 2.1 and 2.3. Let (M, o) and $(\widetilde{M}, \tilde{o})$ be as in Theorem 2.1.

Proof of Theorems 2.1 and 2.3. Let r_0 be the least upper bound of the set of all $r_1 > d(o,p)$ satisfying the following properties: Let $r \in (d(o,p), r_1)$ and $q \in E(o, p\,; r)$. Then,

- (C1) there exists a minimizing geodesic segment $T(p,q)$ such that $T(p,q)$ is contained in the set ${F_p}^{-1}(\widetilde{F}_{\tilde{p}}(\widetilde{M}^+_{\tilde{p}}))$ and $\widetilde{T}(p,q) \geq U(\tilde{p}, \tilde{q})$,

and

- (C2) every minimizing geodesic segment $T(p,q)$ is contained in the set ${F_p}^{-1}(\widetilde{F}_{\tilde{p}}(\widetilde{M}^+_{\tilde{p}}))$ and satisfies $\widetilde{T}(p,q) \geq L(\tilde{p}, \tilde{q})$.

As was seen in Lemma 5.1, we have $r_0 > d(o,p)$. Let r be such that $d(o,p) < r < r_0$. Let $q \in E(o,p\,;r)$, $q_1(t) = T(o,q)(t)$ and $\tilde{q}_1(t) = \widetilde{T}(o,q)(t)$ for any $t \in [0, d(o,q)]$. Then, we have

$$\begin{aligned} & d(o,q_1(t)) + d(p,q_1(t)) \\ = \;& d(o,q) - d(q,q_1(t)) + d(p,q_1(t)) \\ \leq \;& d(o,q) + d(p,q) = r < r_0 \end{aligned}$$

for any $t \in (0, d(o,q))$, and hence, from the condition (C2), every $\triangle opq_1(t)$ in M has a comparison triangle $\triangle \tilde{o}\tilde{p}\tilde{q}_1(t)$ in $\widetilde{M}^+_{\tilde{p}}$ satisfying (2.7). Moreover, from the condition (C1), there exists a minimizing geodesic segment $T(p,q_1(t))$ such that $\widetilde{T}(p,q_1(t)) \geq U(\tilde{p},\tilde{q}_1(t))$.

Assertion 7.1. *Let $q \in E(o,p\,;r)$ with $d(o,p) < r < r_0$. Assume that the minimizing geodesic segments $T(p,q)$ and $T(\tilde{p},\tilde{q})$ satisfy $\widetilde{T}(p,q) \geq T(\tilde{p},\tilde{q})$. Then for any minimizing geodesic segments $T(o,p)$ and $T(o,q)$ the geodesic triangle $\triangle opq = T(o,p) \cup T(p,q) \cup T(o,q)$ has the comparison triangle $\triangle \tilde{o}\tilde{p}\tilde{q}$ with edge $T(\tilde{p},\tilde{q})$ which satisfies* (2.8)*.*

Proof. It follows from Lemmas 4.3 (3), 5.1 and Corollary 5.2 that $\angle opq \geq \angle \tilde{o}\tilde{p}\tilde{q}$ and $\angle oqp \geq \angle \tilde{o}\tilde{q}\tilde{p}$ where one of two equalities holds if and only if the geodesic triangle $\triangle opq$ bounds a totally geodesic 2-dimensional submanifold in M which is isometric to a comparison triangle domain $\triangle \tilde{o}\tilde{p}\tilde{q}$ corresponding to $\triangle opq$ in $\widetilde{M}$, because $\widetilde{T}(p,q) = T(\tilde{p},\tilde{q})$.

In order to show $\angle poq \geq \angle \tilde{p}\tilde{o}\tilde{q}$, we employ $\widetilde{M}_{\delta(r_0)}$ instead of $\widetilde{M}$ and prove that $\theta(t) := \theta(\tilde{q}_1(t)) = \angle \tilde{p}\tilde{o}\tilde{q}_1(t)$ is monotone non-increasing in $t \in [0, d(\tilde{o},\tilde{q})]$. Let $g_t(s) = d(p, q_1(t+s))$ and $\tilde{g}_t(s) = d(\tilde{p}, T(\tilde{o},\tilde{q}_1(t))(t+s))$ for sufficiently small $s > 0$. Here, since $T(\tilde{o},\tilde{q}_1(t))$ lies in the meridian through $\tilde{q}_1(t)$, we can define the point $T(\tilde{o},\tilde{q}_1(t))(t+s)$ for any $s \in [-t, \ell - t]$). It follows from the conditions (C1), (C2) and Lemma 4.3 that $\pi \geq \angle oq_1(t)p > \angle \tilde{o}\tilde{q}_1(t)\tilde{p}$ for every $t \in (0, d(o,q))$. Let ${g_t}'_+(0)$ denote the right hand derivative at $s = 0$, namely,

$${g_t}'_+(0) = \lim_{h \to 0+0} \frac{g_t(h) - g_t(0)}{h}.$$

If $\alpha(t)$ is the angle of $T(\tilde{o},\tilde{q}_1(t))$ with $U(\tilde{p},\tilde{q}_1(t))$, then the first variation formula implies that ${\tilde{g}_t}{}'_+(0) = \cos\alpha(t)$ for every $t \in (0, d(o,q))$. Hence, we have ${g_t}'_+(0) < {\tilde{g}_t}{}'_+(0)$ because of the condition (C1) and Lemma 4.2 (4). There exists an $\varepsilon > 0$ such that

$$\begin{aligned} d(\tilde{p}, \tilde{q}_1(t+s)) &= d(p, q_1(t+s)) = g_t(s) \\ &< \tilde{g}_t(s) = d(\tilde{p}, T(\tilde{o},\tilde{q}_1(t))(t+s)) \end{aligned}$$

for all $s \in (0,\varepsilon)$. Since $r(\tilde{q}_1(t+s)) = r(T(\tilde{o},\tilde{q}_1(t))(t+s)) = t+s$, $\theta(T(\tilde{o},\tilde{q}_1(t))(t+s)) = \theta(T(\tilde{o},\tilde{q}_1(t))(t)) =: \theta(t)$, and $\theta(t+s) := \theta(\tilde{q}_1(t+$

$s)$), it follows from Lemma 11 (2) that $\theta(t) > \theta(t+s)$ for all $s \in (0, \varepsilon)$. Thus, we have

$$\angle poq = \angle \tilde{p}\tilde{o}\tilde{q}_1(0) > \angle \tilde{p}\tilde{o}\tilde{q}_1(d(o,q)) = \angle \tilde{p}\tilde{o}\tilde{q}.$$

Thus, we have $\angle poq \geq \angle \tilde{p}\tilde{o}\tilde{q}$, employing the reference surface $\widetilde{M}$ as $\delta(r_0)$ goes to 0. Here, the equality holds if and only if there exists a geodesic triangle $\triangle opq$ such that it bounds a totally geodesic 2-dimensional submanifold in M which is isometric to a comparison triangle domain $\triangle \tilde{o}\tilde{p}\tilde{q}$ corresponding to $\triangle opq$ in $\widetilde{M}$ with edge $L(\tilde{p}, \tilde{q})$. $\square$

Assertion 7.2. *If $E(o, p\,; r_0) \neq \emptyset$, then every point $q \in E(o, p\,; r_0)$ satisfies that the conditions* (C1) *and* (C2)*. In particular, $B(o, p\,; r_0) \subset F_p^{-1}(\widetilde{F}_{\tilde{p}}(\widetilde{M}_{\tilde{p}}^{+}))$.*

Proof. We first prove that for every point $q \in E(o, p\,; r_0)$ any minimizing geodesic segment $T(p, q)$ satisfies $\widetilde{T}(p, q) \geq L(\tilde{p}, \tilde{q})$. Suppose $\angle oqp \neq \pi$. Let $q_j \in T(p, q) \setminus \{p, q\}$ be a sequence of points converging to q. Then, $r(q_j) < r_0$ is satisfied. Hence, it follows from the definition of r_0 that $\widetilde{T}(p, q) \geq L(\tilde{p}, \tilde{q})$ holds as the limit of $\widetilde{T}(p, q_j) \geq L(\tilde{p}, \tilde{q}_j)$.

When $\angle oqp = \pi$, it is possible that there is no sequence of points q_j with $r(q_j) < r_0$ such that $q_j \to q$ (see Example 3.5). In this case, there exists a cut point p' (resp., o') of o (resp., p) in $T(p, q) \setminus \{p, q\}$ (resp., $T(o, q) \setminus \{o, q\}$). In particular, there exists the unique minimizing geodesic segment $T(p, q)$ connecting p and q and $T(p, q) \supset T(p, p')$. Since any point $q_j \in T(p, p') \setminus \{p, p'\}$ satisfies $d(o, q_j) + d(p, q_j) < r_0$, we have $\widetilde{T}(p, p') \geq L(\tilde{p}, \tilde{p}')$. We notice that $\widetilde{T}(p, q)$ is the union of $\widetilde{T}(p, p')$ and the subarc of $E(\tilde{o}, \tilde{p}\,; r_0)$ from $\tilde{p}'$ to $\tilde{q}$. Therefore, we have $\widetilde{T}(p, q) \geq U(\tilde{p}, \tilde{q}) \geq L(\tilde{p}, \tilde{q})$ (see Lemma 3.3).

We next prove that there exists for every point $q \in E(o, p\,; r_0)$ a minimizing geodesic segment $T(p, q)$ such that $\widetilde{T}(p, q) \geq U(\tilde{p}, \tilde{q})$. This is the condition (C1). If $q \notin F_p{}^{-1}(\widetilde{F}_{\tilde{p}}(Cut(\tilde{p}) \cap \mathrm{Int}(\widetilde{M}_{\tilde{p}}^{+})))$, then it is clear that there exists a minimizing geodesic segment $T(p, q)$ such that $\widetilde{T}(p, q) \geq U(\tilde{p}, \tilde{q})$, since there is the unique minimizing geodesic segment $U(\tilde{p}, \tilde{q}) = L(\tilde{p}, \tilde{q})$ connecting $\tilde{p}$ and $\tilde{q}$.

If $q \in F_p{}^{-1}(\widetilde{F}_{\tilde{p}}(Cut(\tilde{p}) \cap \mathrm{Int}(\widetilde{M}_{\tilde{p}}^{+})))$, then $q \notin E_p(r_0)$ follows from the assumption of Theorems. The assumption (2.6) is used only at this point. Hence, q is not a local maximum point of the distance function to o restricted to $E(o, p\,; r_0)$, namely $d_{r_0} : E(o, p\,; r_0) \to \mathbb{R}$. Therefore, there exists a sequence of points $q_j \in E(o, p\,; r_0)$ such that $q_j \to q$ with $d(o, q_j) > d(o, q)$. Since the sequence of minimizing geodesic segments $T(\tilde{p}, \tilde{q}_j)$ and the sequence of curves $\widetilde{T}(p, q_j)$ converges to $U(\tilde{p}, \tilde{q})$ (see Lemma 3.3) and a curve $\widetilde{T}(p, q)$, respectively, it follows that there exists a minimizing geodesic segment $T(p, q)$ such that $\widetilde{T}(p, q) \geq U(\tilde{p}, \tilde{q})$. $\square$

Up to this point we have proved (2.7) and (2.8) for all points $q \in M$ with $d(o,q) + d(p,q) \leq r_0$. In order to prove that $M \subset B(o,p\,;r_0)$, we suppose $M \setminus B(o,p\,;r_0) \neq \emptyset$ and derive a contradiction.

When we employ $\widetilde{M}_{\delta(R)}$, $R > r_0$, as the reference surface of M (see Remark 5.4) and make the same arguments as in the proofs of Assertions 7.1 and 7.2, we have $r_0(R)$ instead of r_0. We prove $M \subset B(o,p\,;r_0(R))$ for all $R > r_0$ which contradicts $M \setminus B(o,p\,;r_0) \neq \emptyset$ because $\delta(R) \to 0$ as $R \to \ell$.

Assertion 7.3. *Suppose that $M \setminus B(o,p\,;r_0(R)) \neq \emptyset$. Then, there exists an $r_1 > r_0$ such that all points $x \in M$ with $d(o,x) + d(p,x) < r_1$ belong to ${F_p}^{-1}(\widetilde{F}_{\tilde{p}}(\widetilde{M}^{+}_{\delta(R)\tilde{p}}))$.*

Proof. Since $B(o,p\,;r_0(R))$ is compact, it suffices to find an open set U containing $E(o,p\,;r_0(R))$ such that every point $q \in U$ and every minimizing geodesic segment $T(p,q)$ have the reference point $\tilde{q}$ and the reference curve $\widetilde{T}(p,q)$, respectively. Let $q \in E(o,p\,;r_0(R))$. As was seen in the proof of Assertion 7.1, $\theta(\tilde{u})$ is monotone non-increasing as u moves from o to q along $T(o,q)$. We use this fact to determine the location of the reference curve $\widetilde{T}(o,q)$ and to study its property.

The complicated case is that the angle of $T(\tilde{o},\tilde{p})^{\cdot}(0)$ with $T(\tilde{o},\tilde{q})^{\cdot}(0)$ is π. Suppose that $\angle\tilde{p}\tilde{o}\tilde{q} = \pi$. It follows from Assertions 7.1 and 7.2 that $\angle poq' = \pi$ for all q' whose reference point is $\tilde{q}' = \tilde{q}$. Hence, we have $q' = q$ if $\tilde{q}' = \tilde{q}$. Moreover, $\angle oup = \angle\tilde{o}\tilde{u}\tilde{p}$ for all $u \in T(o,q)$. If there exists a point $u \in T(o,q)$ such that $\angle oup \neq 0$, then there exists a minimizing geodesic segment $T(p,u)$ such that $\triangle oup$ bounds a totally geodesic 2-dimensional submanifold which is isometric to the comparison triangle domain $\triangle\tilde{o}\tilde{u}\tilde{p}$ in $\widetilde{M}^{+}_{\delta(R)\tilde{p}}$. Since the radial sectional curvature of M is greater than or equal to K, this contradicts that the curvature of $\widetilde{M}^{+}_{\delta(R)\tilde{p}}$ is $K - \delta(R)$. Thus, we obtain $\angle oup = \angle\tilde{o}\tilde{u}\tilde{p} = 0$ for all $u \in T(o,q)$. Therefore, both $T = T(p,o) \cup T(o,q)$ and $\widetilde{T} = T(\tilde{p},\tilde{o}) \cup T(\tilde{o},\tilde{q})$ are minimizing geodesic segments.

In addition to $\angle\tilde{p}\tilde{o}\tilde{q} = \pi$, suppose $\tilde{q} \in Cut(\tilde{p})$. From the present curvature assumption, $\tilde{q}$ is not a point conjugate to $\tilde{p}$ along $\widetilde{T}$. Hence, $\tilde{q}$ is not an end point of $Cut(\tilde{p})$. There exist at least two minimizing geodesic segments connecting $\tilde{p}$ and $\tilde{q}$. In particular, $U(\tilde{p},\tilde{q})$ is different from $\widetilde{T}$. Since $q \notin Cut(o)$, Lemmas 3.4 and 6.1, there exists a minimizing geodesic segment $T(p,q)$ such that $\widetilde{T}(p,q) \geq U(\tilde{p},\tilde{q})$. Thus, we conclude that q is a cut point of p, since $T(p,q)$ is different from T.

Let W' be a neighborhood of q which is foliated by minimizing geodesic segments from o. Then, from the present curvature assumption, there exist a neighborhood $W \subset W'$ of q and an $\varepsilon > 0$ such that, choosing the appropriate geodesic triangles, $\angle oq'p - \angle\tilde{o}\tilde{q}'\tilde{p} > \varepsilon$ for all points $q' \in W \cap E(o,p\,;r_0(R))$. Using this property and the same method as in the proof of Assertion 7.1, we can have a neighborhood V'_q of q such that all points in $V'_q \setminus T(o,q)$ have their reference points in $\mathrm{Int}(\widetilde{M}^+_{\delta(R)\tilde{p}})$.

We next suppose $\tilde{q} \notin Cut(\tilde{p})$, in addition to $\angle\tilde{p}\tilde{o}\tilde{q} = \pi$. Let $T_e(p,o)$ be the maximal minimizing geodesic from p through o. If q lies in $T_e(p,o)$ but not the endpoint, then the minimizing geodesic segment $T(p,q)$ is unique and $T(p,q) \subset T_e(p,o)$. Even if q is the endpoint of $T_e(p,o)$, then $T_e(p,o) = T(p,o) \cup T(o,q)$ is a minimizing geodesic segment. Therefore, we have $\widetilde{T}(p,q) = T(\tilde{p},\tilde{o}) \cup T(\tilde{o},\tilde{q})$ as its reference curve, because $\angle oqp = 0$ and (2.8).

Let N be the normal neighborhood around o, namely the domain around o bounded by $Cut(o)$. Obviously, $T(p,q) = T(p,o) \cup T(o,q) \subset N$. Because $\tilde{q} \notin Cut(\tilde{p})$, we can have a neighborhood $\widetilde{U}_{\tilde{q}}$ of $\tilde{q}$ in $\widetilde{M}_{\delta(R)}$ such that for all $\tilde{x} \in \widetilde{U}_{\tilde{q}} \cap \widetilde{M}^+_{\delta(R)\tilde{p}}$, if we write $T(\tilde{p},\tilde{x})(t) = (r(t), \theta(\tilde{p}) + \theta(t))$ for all $t \in [0, d(\tilde{p},\tilde{x})]$, then $\exp_o(r(t)(\cos\theta(t)u + \sin\theta(t)v)) \in N$, $0 \le t \le d(\tilde{p},\tilde{x})$. Here $\exp_o : T_oM \to M$ is the exponential map and v is an arbitrary unit tangent vector such that v is perpendicular to $u := T(o,p)^{\cdot}(0)$ in T_oM.

We prove that there exists a neighborhood V_q of q so that $V_q \subset N$ and the reference curve $\widetilde{T}(p,x)$ is defined for any $x \in V_q$.

Let $x \in N$. Let θ_x denote the angle of $T(o,x)^{\cdot}(0)$ with $T(o,p)^{\cdot}(0)$. Then θ_x is continuous for $x \in N$. Define a map $\Psi : N \to \widetilde{M}^+_{\delta(R)\tilde{p}}$ by $\Psi(x) = (d(o,x), \theta(\tilde{p}) + \theta_x)$. Since Ψ is continuous, there exists a neighborhood V'_q of q such that $\Psi(V'_q) \subset \widetilde{U}_{\tilde{q}} \cap \widetilde{M}^+_{\delta(R)\tilde{p}}$.

We claim that all points $x \in V'_q$ have their reference points. Let $x \in V'_q \setminus T_e(o,q)$ where $T_e(o,q)$ denotes the maximal minimizing geodesic from o through q. Since $x \notin T_e(o,q)$, we have $\theta_x \neq \pi$. Let $r(t)$ and $\theta(t)$ satisfy the equation $T(\tilde{p},\Psi(x))(t) = (r(t), \theta(\tilde{p}) + \theta(t))$, $0 \le t \le d(\tilde{p},\Psi(x))$. Let $u := T(o,p)^{\cdot}(0)$ and $v_x = T(o,x)^{\cdot}(0)$. Set $v = (v_x - \cos\theta_x u)/\sin\theta_x$ which is the unit tangent vector perpendicular to u and contained in the subspace spanned by $\{u, v_x\}$. Then we define a curve $c(t) = \exp_o(r(t)v(t))$, $0 \le t \le d(\tilde{p},\Psi(x))$, where $v(t) = \cos\theta(t)u + \sin\theta(t)v$. The curve c connects p and x and its length is less than $d(\tilde{p},\Psi(x))$ because of the curvature condition and the Rauch comparison theorem (see [3]). Therefore, we have $d(p,x) < d(\tilde{p},\Psi(x))$. Thus, we can define the reference point $\tilde{x}$ of x in $\widetilde{M}^+_{\delta(R)\tilde{p}}$ because $r(\tilde{x}) = r(\Psi(x)) = d(o,x)$, $\theta(\tilde{x}) \le \theta_x$ and Lemma 3.1 (1). Since all points $x \in V'_q \cap T_e(o,q)$ are accumulation points of $V'_q \setminus T_e(o,q)$, every point $x \in V'_q$ has its reference point $\tilde{x}$ in $\widetilde{M}^+_{\delta(R)\tilde{p}}$.

From V'_q, we can have a neighborhood V_q of q mentioned above. Suppose for indirect proof that there exists a sequence of points x_j converging to q such that some point $y_j \in T(p,x_j)$ defines the reference curve $\widetilde{T}(p,y_j)$ and some point in $T(y_j,x_j)$ close to y_j does not have any reference point. Since those points y_j's satisfy $d(o, y_j) + d(p, y_j) \geq r_0$, $q \in E(o, p\,; r_0)$ and Lemma 3.2 (1), the sequence $d(y_j, x_j)$ goes to zero, and, hence, the sequence of the points y_j converges to q. Thus, $T(y_j, x_j) \subset V_q'$ for a sufficiently large j, contradicting that all points $x \in V_q'$ have their reference points. Therefore, we have the neighborhood V_q as required.

Suppose that $\angle \tilde{p}\tilde{o}\tilde{q} < \pi$. Let $\widetilde{U}_{\tilde{q}} \subset \mathrm{Int}(\widetilde{M}^+_{\delta(R)})$ be a neighborhood of $\tilde{q}$. Then there exists a neighborhood V_q' of q in M such that $\widetilde{F}_{\tilde{p}}^{-1} \circ F_p(V_q') \subset \widetilde{U}_{\tilde{q}}$. As the argument above, we can have a neighborhood V_q of q as required.

Thus, we have found the set $U = \bigcup_{q \in E(o,p\,;r_0)} V_q$ which is a neighborhood around $E(o, p\,; r_0(R))$ such that $U \subset {F_p}^{-1}(\widetilde{F}_{\tilde{p}}(\widetilde{M}^+_{\delta(R)\tilde{p}}))$. □

Assertion 7.4. *There exists an r_2 with $r_0(R) < r_2 \leq r_1$ such that the condition* (C1) *is true for any point $q \in E(o, p\,; r)$, $r_0(R) < r < r_2$.*

Proof. Suppose for indirect proof that (C1) is not true for any $r > r_0(R)$, namely there exists a sequence of $r_j > r_0(R)$ such that r_j converges to $r_0(R)$ and there are no minimizing geodesic segments $T(p, q_j)$ with $\widetilde{T}(p, q_j) \geq U(\tilde{p}, \tilde{q}_j)$ for some $q_j \in E(o, p\,; r_j)$. Suppose without loss of generality that q_j converges to $q_0 \in E(o, p\,; r_0(R))$. We then have either

$$\widetilde{T}(p, q_j) \leq U(\tilde{p}, \tilde{q}_j) \qquad \text{or} \qquad \widetilde{T}(p, q_j) \cap U(\tilde{p}, \tilde{q}_j) \neq \{\tilde{p}, \tilde{q}_j\}.$$

Let $q_j' = T(p, q_j) \cap E(o, p\,; r_0(R))$. Then, we have $\widetilde{T}(p, q_j') \subset \widetilde{T}(p, q_j)$, since $T(p, q_j') \subset T(p, q_j)$. It follows from the choice of $r_0(R)$ and the condition (C1) that $\widetilde{T}(p, q_j') \geq U(\tilde{p}, \tilde{q}_j')$. If the first inequality is true, we then have $\widetilde{T}(p, q_0) = U(\tilde{p}, \tilde{q}_0)$ as its limit. From Lemma 5.4, this is impossible because the curvature of $\widetilde{M}^+_{\delta(R)\tilde{p}}$ is $K - \delta(R)$.

If the second situation occurs, we then have the reverse inequality for some point $q' \in T(p, q_j)$ near q_j for sufficiently large j so that

$$\widetilde{R}(q', q_j) \leq T(\tilde{q}', \tilde{q}_j)$$

because of Lemma 16 (3) and (4). This contradicts Lemma 5.1. Therefore, (C1) is true for some $r_2 > r_0(R)$. □

Assertion 7.5. *The condition* (C2) *is satisfied for all $q \in E(o, p\,; r)$, $r_0(R) < r < r_2$.*

Proof. Let $q \in M$ with $d(o, q) + d(p, q) < r_2$. For convenience, we set $q(t) = T(p, q)(t)$ and $\tilde{q}(t) = \widetilde{T}(p, q)(t)$, $0 \leq t \leq d(p, q)$. Let t_0 be the least upper bound of the set of all $t_1 \leq d(p, q)$ so that there exists a minimizing geodesic segment $T(\tilde{p}, \tilde{q}(t))$ with $\widetilde{T}(p, q(t)) \geq T(\tilde{p}, \tilde{q}(t))$ for all $t \in (0, t_1)$. Recall that $t_0 > 0$ because of Lemma 5.1. Suppose for indirect proof that $t_0 < d(p, q)$. If $\tilde{q}(t_0) \notin Cut(\tilde{p})$, then there exists a positive ε such that $\widetilde{T}(p, q(t_0+t)) \geq T(\tilde{p}, \tilde{q}(t_0+t))$ for all $t \in (0, \varepsilon)$ because of Lemma 5.5. This contradicts the choice of t_0. Suppose $\tilde{q}(t_0) \in Cut(\tilde{p})$. Since the minimizing geodesic segment $T(p, q(t_0))$ is unique and (C1) is satisfied, we have $\widetilde{T}(p, q(t_0)) \geq U(\tilde{p}, \tilde{q}(t_0))$. As is observed in the proof of Lemmas 6.1 and 6.2, there exists a positive ε such that $\widetilde{T}(p, q(t_0+t)) \geq U(\tilde{p}, \tilde{q}(t_0+t))$ for all $t \in (0, \varepsilon)$. This contradicts the choice of t_0. Hence, it follows that $\widetilde{T}(p, q) \geq T(\tilde{p}, \tilde{q})$. $\square$

Assertions 7.3 to 7.5 imply that $M \setminus B(o, p\,; r_0(R)) \neq \emptyset$ is false when we employ the reference surface $\widetilde{M}_{\delta(R)}$. Since $\delta(R) \to 0$ as $R \to \ell$, we conclude that $M \setminus B(o, p\,; r_0) \neq \emptyset$ is false to the original reference surface of revolution $\widetilde{M}$ $(\delta(\ell) = 0)$. This completes the proof of Theorems 2.1 and 2.3. $\square$

The following proposition has been proved in the above argument.

Proposition 7.6. *Let M and p satisfy the same assumption as in Theorem 2.1. Then, a point $q \in M$ is a cut point of p if there exists a minimizing geodesic segment $T(p, q)$ such that $\widetilde{T}(p, q) \not\geq U(\tilde{p}, \tilde{q})$.*

Proof. As was seen in the proof of Theorem 2.1, if the reference point $\tilde{q}$ is in $\mathrm{Int}(\widetilde{M}^+_{\tilde{p}})$, then there exists a minimizing geodesic T connecting p and q such that $\widetilde{T} \geq U(\tilde{p}, \tilde{q})$. Therefore, we have at least two minimizing geodesics T and $T(p, q)$ connecting p and q. This implies that $q \in Cut(p)$.

Suppose that $\theta(\tilde{q}) = 0$ or π. Then, as was seen in the proof of Assertion 7.3, there are two possibilities. One is that $\widetilde{T}(p, q) = T(\tilde{p}, \tilde{o}) \cup T(\tilde{o}, \tilde{q})$ and it is a minimizing geodesic segment in $\widetilde{M}$. Then, it follows from the curvature condition that if $q \notin Cut(p)$, then $\tilde{q} \notin Cut(\tilde{p})$. Then, $\widetilde{T}(p, q) \geq U(\tilde{p}, \tilde{q})$ is true, a contradiction. The other is that a geodesic triangle $\triangle opq$ bounds a totally geodesic 2-dimensional submanifold in M which is isometric to the comparison triangle domain $\triangle \tilde{o}\tilde{p}\tilde{q}$ in $\widetilde{M}$. In this case, for any $\delta > 0$, we regard $\widetilde{M}_\delta$ as a reference surface of $\widetilde{M}$ and M. Then, the reference point $\tilde{q} \in \widetilde{M}^+_\delta$ of q belongs to $\mathrm{Int}(\widetilde{M}^+_\delta)$ and, moreover, the boundary of the set of the reference points of all points in $\widetilde{M}$. If $q \notin Cut(p)$, then this contradicts Lemma 4.1 (1), meaning that $q \in Cut(p)$. $\square$

Remark 7.7. From the proof of Theorems we notice that the assumption $F_p(E(p)) \cap \widetilde{F}_{\tilde{p}}(Cut(\tilde{p}) \cap \mathrm{Int}(\widetilde{M}^+_{\tilde{p}})) = \emptyset$ can be replaced by the following condition: If $q \in F_p{}^{-1}(\widetilde{F}_{\tilde{p}}(Cut(\tilde{p}) \cap \mathrm{Int}(\widetilde{M}^+_{\tilde{p}})))$, then there exists a minimizing geodesic segment $T(p, q)$ with $\widetilde{T}(p, q) \geq U(\tilde{p}, \tilde{q})$.

8. Maximum perimeter and diameter

We have corollaries which are the special version of Corollary 1.3.

Corollary 8.1. *Let (M, o) and all points $p \in M$ with $p \neq o$ satisfy the same assumption in Theorem 2.1. Then evry geodesic triangle $\triangle opq$ admits its comparison triangle $\triangle \tilde{o}\tilde{p}\tilde{q}$ in $\widetilde{M}$. In particular, if $\ell < \infty$, we have*

$$d(o,p) + d(p,q) + d(o,q) \leq 2\ell$$

and the diameter of M is less than or equal to ℓ.

Proof. As was seen in the paragraph just before Lemma 3.2, we have

$$d(o,q) + d(p,q) = d(\tilde{o},\tilde{q}) + d(\tilde{p},\tilde{q}) \leq 2\ell - d(\tilde{p},\tilde{o}) = 2\ell - d(p,o).$$

Therefore, we have $d(o,p) + d(p,q) + d(o,q) \leq 2\ell$.

Let p and q be points in M such that $d(p,q)$ is the diameter of M. It is clear that if $p = o$, then $d(p,q) \leq \ell$. Suppose that $p \neq o$. If $\tilde{p}$ and $\tilde{q}$ in $\widetilde{M}$ are the reference points of p and q, respectively, we then have

$$d(\tilde{p},\tilde{q}) \leq \min\{d(\tilde{p},\tilde{o}) + d(\tilde{o},\tilde{q}), d(\tilde{p},\tilde{o}_1) + d(\tilde{o}_1,\tilde{q})\} \leq \ell,$$

where $\tilde{o}_1$ is the antipodal point of $\tilde{o}$ in $\widetilde{M}$. Therefore, we have $d(p,q) \leq \ell$. $\square$

We have the maximum diameter theorem and the maximum perimeter theorem if the assumption (2.6) is extended to the boundary of $\widetilde{M}^+_{\tilde{p}}$.

Corollary 8.2. *Let (M, o) be a complete pointed Riemannian manifold which is referred to a reference surface of revolution $(\widetilde{M}, \tilde{o})$ with $\ell < \infty$. Assume that all points $p \in M$ with $p \neq o$ satisfy*

$$F_p(E(p)) \cap \widetilde{F}_{\tilde{p}}(Cut(\tilde{p}) \cap \widetilde{M}^+_{\tilde{p}}) = \emptyset. \tag{8.1}$$

If there exists a pair of points p and q in M such that the perimeter of the geodesic triangle $\triangle opq$ is 2ℓ, then M is isometric to the warped product manifold whose warping function is the radial curvature function of $\widetilde{M}$. In particular, the same conclusion holds for M if the diameter of M is ℓ.

Proof. Suppose that the perimeter of the geodesic triangle $\triangle opq$ is 2ℓ. From Theorem 2.1, it has a comparison triangle $\triangle \tilde{o}\tilde{p}\tilde{q}$ in $\widetilde{M}^+_{\tilde{p}}$. We then have $d(\tilde{p},\tilde{q}) = d(\tilde{p},\tilde{o}_1) + d(\tilde{o}_1,\tilde{q})$, since

$$\begin{aligned}
& d(\tilde{p},\tilde{q}) + d(\tilde{p},\tilde{o}) + d(\tilde{o},\tilde{q}) \\
= \;& 2\ell \\
= \;& 2d(\tilde{o},\tilde{o}_1) \\
= \;& d(\tilde{p},\tilde{o}_1) + d(\tilde{o}_1,\tilde{q}) + d(\tilde{p},\tilde{o}) + d(\tilde{o},\tilde{q}),
\end{aligned}$$

where $\tilde{o}_1$ is the antipodal point of $\tilde{o}$ in $\widetilde{M}$. This implies that $U(\tilde{p},\tilde{q}) = T(\tilde{p},\tilde{o}_1) \cup T(\tilde{o}_1,\tilde{q})$. From the assumption (8.1) and the condition (C1), there exists a minimizing geodesic segment $T(p,q)$ in M such that

$\widetilde{T}(p,q) \geq U(\tilde{p},\tilde{q})$. Thus, we can find a point $o_1 \in T(p,q)$ whose reference point is $\tilde{o}_1$. Since $d(o,o_1) = d(\tilde{o},\tilde{o}_1) = \ell$, The farthest point theorem in [7] concludes our corollary.

Suppose that the diameter of M is ℓ. Let the distance between p and q be ℓ. If $p = o$, then the statement follows from the farthest point theorem (see [7]). Suppose $p \neq o$. As was seen in the proof of Corollary 8.1, we have

$$\ell = d(\tilde{p},\tilde{q}) \leq \min\{d(\tilde{p},\tilde{o}) + d(\tilde{o},\tilde{q}), d(\tilde{p},\tilde{o}_1) + d(\tilde{o}_1,\tilde{q})\} \leq \ell.$$

Since

$$d(\tilde{p},\tilde{o}) + d(\tilde{o},\tilde{q}) + d(\tilde{p},\tilde{o}_1) + d(\tilde{o}_1,\tilde{q}) = 2\ell,$$

we have

$$d(\tilde{p},\tilde{q}) = d(\tilde{p},\tilde{o}) + d(\tilde{o},\tilde{q}) = d(\tilde{p},\tilde{o}_1) + d(\tilde{o}_1,\tilde{q}) = \ell.$$

Thus, the perimeter of the comparison triangle $\triangle\tilde{o}\tilde{p}\tilde{q}$ in $\widetilde{M}$ of $\triangle opq$ is 2ℓ. The maximal perimeter theorem prove the maximal diameter theorem. $\square$

Remark 8.3. If the Gauss curvature of the reference surface $\widetilde{M}$ is a positive constant κ, we do not need the assumption (8.1). In fact, as was seen in the proof of Corollary 8.2, we have $U(\tilde{p},\tilde{q}) = T(\tilde{p},\tilde{o}_1) \cup T(\tilde{o}_1,\tilde{q})$ if the perimeter of $\triangle opq$ is 2ℓ. If $d(\tilde{p},\tilde{q}) < \ell = \pi/\sqrt{\kappa}$, then the minimizing geodesic segment is unique, meaning that $U(\tilde{p},\tilde{q}) = L(\tilde{p},\tilde{q})$. This implies that there exists a minimizing geodesic segment $T(p,q)$ in M such that $\widetilde{T}(p,q) \geq T(\tilde{p},\tilde{q})$ as the limit of the positional relations in $\mathrm{Int}(\widetilde{M}^+_{\tilde{p}})$. Thus, we have a point o_1 whose reference point is $\tilde{o}_1$. In particular, the diameter of M is ℓ. In the case of $d(\tilde{p},\tilde{q}) = \ell$, it is clear that the diameter of M is ℓ. Therefore, the maximum diameter theorem states that M is a sphere with constant curvature κ.

Proof of Corollary 1.4. We first prove that there exists a straight line in $\widetilde{M}$ if there is a straight line in M. Let $T(t)$, $-\infty < t < \infty$, be a straight line in M. Let t_0 be a parameter such that $d(o,T(t_0)) = d(o,T)$. We set $\widetilde{T}(t_1) = (d(o,T(t_1)),0)$ for all $t_1 \in (-\infty,t_0)$, and $\widetilde{T}(t) = \widetilde{F}_{\widetilde{T}(t_1)}{}^{-1} \circ F_{T(t_1)}(T(t))$ for any $t \in (t_1,\infty)$. Then it follows from Theorem 2.1 that $\widetilde{T}(t)$, $t \geq t_1$, is a curve in $\widetilde{M}^+_{\widetilde{T}(t_1)}$ such that $\widetilde{T} \geq T(\widetilde{T}(t_1),\widetilde{T}(t))$ for all $t \geq t_1$. The sequence of minimizing geodesic segments $S_t = T(\widetilde{T}(t_1),\widetilde{T}(t))$ connecting $\widetilde{T}(t_1)$ and $\widetilde{T}(t)$ contains a subsequence S_k converging to a ray S emanating from $\widetilde{T}(t_1)$ as $k \to \infty$. Let the ray be denoted by $S(t_1)(t)$, $t_1 \leq t$. Then, $d(\tilde{o},S(t_1)(t_0)) \leq d(o,T)$. From this fact we can find a sequence of rays $S(k)$ converging to a straight line S as $k \to -\infty$.

It is known (cf. [2, 4]) that if there is a straight line in $\widetilde{M}$, then the total curvature of $\widetilde{M}$ is nonpositive. Therefore M has no straight line. If M has at least two ends, then there is a straight line connecting distinct ends. This is impossible because the total curvature of $\widetilde{M}$ is positive. $\square$

9. Proof of Theorem 2.7

If $\tilde{p}$ and $r_2 \in (0, r_1)$ satisfy $d(\tilde{o}, \tilde{p}) > \ell - r_1^*$ and $f(r_2) = f(r(\tilde{p}))$, then the geodesic segment $\Gamma(\tilde{p})$ intersects $r^{-1}(\{r_1\})$ at most once before it meets $r^{-1}(\{r_2\})$. The terminal point of $\Gamma(\tilde{p})$ is denoted by $\Gamma(\tilde{p})_e$. Notice that $\Gamma(\tilde{p})$ converges to the meridian $\theta^{-1}(\{\pi/2\})$ as $\tilde{p} \to \tilde{o}_1$. We then have

$$\theta(\tilde{p}) < \theta(\Gamma(\tilde{p})_e) \leq \theta(\tilde{p}) + \pi.$$

Lemma 9.1 (Basic Lemma)**.** *Let $(\widetilde{M}, \tilde{o})$ be a compact surface of revolution with metric* (2.1)*. Then, there exists for an arbitrary given $r \in (0, \ell)$ a point $\tilde{p} \in \widetilde{M}$, $\tilde{p} \neq \tilde{o}_1$, such that the cut locus* $\mathrm{Cut}(\tilde{p})$ *is contained in r-ball $B(\tilde{o}, r)$ centered at $\tilde{o}$. In particular, $c_2(r) > 0$ for all $r > 0$.*

Proof. It follows from the continuity of i and $i(\tilde{o}_1) = \ell$ that there exists for an arbitrary given $r \in (0, \ell)$ a point $\tilde{p} \in \widetilde{M}$ such that

$$r + i(\tilde{p}) - d(\tilde{p}, \tilde{o}_1) > \ell.$$

In fact, if $\tilde{p}$ is sufficiently close to $\tilde{o}_1$ then $\ell - i(\tilde{p}) + d(\tilde{p}, \tilde{o}_1)$ is arbitrary small. We then have $d(\tilde{o}, \tilde{q}) < r$ for every cut point $\tilde{q} \in \mathrm{Cut}(\tilde{p})$. In fact, suppose contrary that $d(\tilde{o}, \tilde{q}) \geq r$, namely $d(\tilde{o}_1, \tilde{q}) \leq \ell - r$, then

$$\begin{aligned} d(\tilde{p}, \tilde{q}) &\leq d(\tilde{q}, \tilde{o}_1) + d(\tilde{p}, \tilde{o}_1) \\ &\leq \ell - r + d(\tilde{p}, \tilde{o}_1) < i(\tilde{p}). \end{aligned}$$

This implies that $\tilde{q} \notin \mathrm{Cut}(\tilde{p})$, a contradiction.

$\square$

From now on let (M, o) be a compact pointed Riemannian manifold which is referred to $(\widetilde{M}, \tilde{o})$ with its metric (2.1). Let $T(x, y)$ for $x, y \in M$, ($T(\tilde{x}, \tilde{y})$ for $\tilde{x}, \tilde{y} \in \widetilde{M}$, respectively) be a minimizing geodesic joining x to y, ($\tilde{x}$ to $\tilde{y}$, respectively). The following Lemmas are useful for the proof of our theorem.

Lemma 9.2 (TCT)**.** *Assume that there exists a point $p \in M$ such that $d(o, p) > \ell - c_2(i(o))$. Then, every $T(p, q) \subset M$ joining p to every point $q \in M$ admits a corresponding $T(\tilde{p}, \tilde{q}) \subset \widetilde{M}$ satisfying* (2.8). Moreover, there exists for every point $q \in M$ and for every $T(\tilde{p}, \tilde{q})$ a $T(p, q)$ in M satisfying (2.8).

Proof. It follows from the definition of $c_2(r)$ that $\mathrm{Cut}(\tilde{p}) \cap \mathrm{Int}(\widetilde{M}^+_{\tilde{p}}) \subset B(\tilde{o}, i(o))$, namely $d(\tilde{o}, \mathrm{Cut}(\tilde{p}) \cap \mathrm{Int}(\widetilde{M}^+_{\tilde{p}})) < i(o)$. Since $E(p) \subset \mathrm{Cut}(o)$ from (2.5) and since $\mathrm{Cut}(o) \subset M \setminus B(o, i(o))$, we have $d(o, E(p)) \geq i(o)$. Then (2.6) is satisfied for p. We conclude the proof by Theorem 2.3. $\square$

We say that $\widetilde{M}$ is *without conjugate points in half* if all points $\tilde{p} \in \widetilde{M}$ have no point conjugate to $\tilde{p}$ along geodesics from $\tilde{p}$ in $\mathrm{Int}(\widetilde{M}^+_{\tilde{p}})$. If $\widetilde{M}$ is without conjugate points in half, then all geodesics contained in $\mathrm{Int}(\widetilde{M}^+_{\tilde{p}})$ are minimizing, and, in particular, $c_2(r) = \ell$ for all $r > 0$. Any von Mangoldt surface is without conjugate points in half (see [21]).

We say that $\Gamma(\tilde{p})_e$ is *cross-cutting* if its turn angle is π around $\tilde{o}$, namely $\angle(\tilde{p}\,\tilde{o}\,\Gamma(\tilde{p})_e) = \theta(\Gamma(\tilde{p})_e) - \theta(\tilde{p}) = \pi$. Since any simply connected biangle domain bounded by two minimizing geodesic segments has a point conjugate to its vertexes in its interior, $\Gamma(\tilde{p})_e$ is cross-cutting for any point $\tilde{p}$ other than $\tilde{o}$ and $\tilde{o}_1$ if $\widetilde{M}$ is without conjugate points in half.

Remark 9.3. If $\widetilde{M}$ is without conjugate points in half, then Theorem 2.3 is true for all points $\tilde{p} \in \widetilde{M}$. Therefore, we do not need Lemma 9.2.

Since (M, o) is referred to $(\widetilde{M}, \tilde{o})$, there exists a unique point $o^* \in M$ such that $d(o, o^*) = \max\{d(o, x) \mid x \in M\} \leq \ell$, equality holding if and only if M is isometric to the warped product manifold $[0, \ell] \times S^{n-1}(1)$ with warping function f. Here, the uniqueness of o^* will be proved later.

Lemma 9.4. *Let $c = \min\{r_1, i(o)\}$. Assume that a farthest point o^* to o in M satisfies $d(o, o^*) > \ell - \min\{c_2(c), c_3(c)\}$. Then, M is topologically an n-sphere.*

Proof. Let $\tilde{o}^* \in \widetilde{M}$ be a reference point of o^*, namely $\tilde{o}^* = (d(o, o^*), 0)$. Since $d(\tilde{o}, \tilde{o}^*) > \ell - c_3(c)$, the endpoint $\Gamma(\tilde{o}^*)_e$ of $\Gamma(\tilde{o}^*)$ is contained in $B(\tilde{o}, c)$. Set $c' = (c + r(\Gamma(\tilde{o}^*)_e))/2$. We then have $c > c' > r(\Gamma(\tilde{o}^*)_e)$. Let $N = \{p \in M \mid d(p, o) > d(\tilde{o}, \Gamma(\tilde{o}^*)_e) = r(\Gamma(\tilde{o}^*)_e)\}$ and $N' := \{p \in M \mid d(o, p) > c'\}$. Then, we have $B(o, c) \cup N' = M$ and $N' \subset N$, since $c > c' > r(\Gamma(\tilde{o}^*)_e)$.

Obviously, there exists no critical point of the distance function to o in $B(o, c) \setminus \{o\}$ because of $i(o) \geq c$. We will prove that there exists no critical point of the distance function to o^* in N'. Then the proof of this lemma will complete.

Let $\widetilde{D}$ denote the domain bounded by $\Gamma(\tilde{o}^*)$, $T(\tilde{o}, \tilde{o}^*)$ and $T(\tilde{o}, \Gamma(\tilde{o}^*)_e)$. It follows from the Clairault relation that $r(T(\tilde{o}^*, \tilde{x})(t))$, $0 \leq t \leq d(\tilde{o}^*, \tilde{x})$, is monotone decreasing for t if $\tilde{x} \in \widetilde{D} \cap r^{-1}([r(\Gamma(\tilde{o}^*)_e), \ell])$. In particular, we have $\angle(\tilde{o}^*\,\tilde{x}\,\tilde{o}) > \pi/2$ for all $\tilde{x} \in \widetilde{D} \cap r^{-1}([r(\Gamma(\tilde{o}^*)_e), \ell])$.

Let $S(o, c') := \{p \in M \mid d(o, p) = c'\}$. We first claim that all the reference points $\tilde{q}$ of $q \in S(o, c')$ are contained in the domain $\widetilde{D}$.

If $\theta(\Gamma(\tilde{o}^*)_e) = \pi$, nothing is left to prove because of Lemma 9.2.

Suppose that $\theta(\Gamma(\tilde{o}^*)_e) < \pi$. Suppose for indirect proof that there exists a point $q \in S(o, c')$ such that $\tilde{q} \notin \widetilde{D}$. Let q_1 be a point in $T(o, o^*)$ with $d(o, q_1) = c'$. Since $c' < i(o)$, we see $S(o, c')$ is diffeomorphic to an $(n-1)$-sphere. Hence, there exists a curve $g(t)$, $0 \leq t \leq 1$, in

$S(o,c')$ connecting $g(0)=q_1$ and $g(1)=q$. The reference curve $\tilde{c}(t)=\widetilde{F}_{\tilde{o}^*}{}^{-1}(F_{o^*}(g(t)))$, $0\le t\le 1$, moves on the parallel $r^{-1}(\{c'\})=S(\tilde{o},c')$ in $\widetilde{M}^+_{\tilde{p}}$ from $\tilde{q}_1$ to $\tilde{q}$. Hence, there exists a $t_0\in(0,1)$ such that $\theta(\Gamma(\tilde{o}^*)\cap S(\tilde{o},c'))<\theta(\tilde{g}(t_0))<\theta(\Gamma(\tilde{o}^*)_e)$ and $T(\tilde{o}^*,\tilde{g}(t_0))$ contains a point $\tilde{x}$ with $r(\tilde{x})>r(\tilde{o}^*)$. In fact, since $r(\Gamma(\tilde{o}^*)_e)<c'$, we have the point $\tilde{q}_2$ where the parallel $S(\tilde{o},c')$ intersects the meridian through $\Gamma(\tilde{o}^*)_e$. Then, all points $\tilde{g}(t_0)$ lying in the subarc of $S(\tilde{o},c')$ between $\Gamma(\tilde{o}^*)\cap S(\tilde{o},c')$ and $\tilde{q}_2$ satisfy this property. Actually, since $\Gamma(\tilde{o}^*)$ is tangent to the parallel $r^{-1}(\{r(\tilde{o}^*)\})$ at $\tilde{o}^*$, we have $T(\tilde{o}^*,\tilde{g}(t_0))\not\subset r^{-1}([0,r(\tilde{o}^*)])$. Then, it follows from Lemma 9.2 that there exists a point $x\in M$ such that $d(o,x)>d(o,o^*)$, contradicting the choice of o^*. Thus all the reference points for $S(o,c')$ is contained in $\widetilde{D}$.

We secondly claim that all the reference points $\tilde{q}$ of $q\in N'$ are contained in $\widetilde{D}\cap r^{-1}((c',\ell])$.

Suppose for indirect proof that there exists a reference point $\tilde{q}\notin\widetilde{D}\cap r^{-1}((c',\ell])$ of $q\in N'$, namely $r(\tilde{q})>c'$ but $\tilde{q}\notin\widetilde{D}$. Let $\widetilde{T}(o^*,q)(t)$, $0\le t\le d(o^*,q)$, be the reference curve of a minimizing geodesic $T(o^*,q)(t)$. From the definition of o^*, there exists a $t_0>0$ such that $\widetilde{T}(o^*,q)(t)\in\widetilde{D}\cap r^{-1}((c',\ell])$ for $t\in[0,t_0]$.

There exists no $t\in[0,d(o^*,q)]$ such that $r(\widetilde{T}(o^*,q)(t))=c'$. In fact, suppose contrary, then $r(T(\tilde{o}^*,\tilde{q})(s))=c'$ for some $s\in[0,d(o^*,q)]$ because of Lemma 9.2. Recall that the strip bounded by two parallels $r^{-1}(\{r(\tilde{o}^*)\})$ and $r^{-1}(\{r(\Gamma(\tilde{o}^*)_e)\})$ is foliated by minimizing geodesic segments $R_\theta(\Gamma(\tilde{o}^*))$, $0\le\theta\le 2\pi$, where R_θ is the rotation with angle θ of $\widetilde{M}$ around $\tilde{o}$. From the fact that the reference points for $S(o,c')$ is contained in $\widetilde{D}$, it is impossible that $r(T(\tilde{o}^*,\tilde{q})(0))>c'$, $r(T(\tilde{o}^*,\tilde{q})(s))=c'$ and $r(T(\tilde{o}^*,\tilde{q})(d(o^*,q))>c'$, since, otherwise, $T(\tilde{o}^*,\tilde{q})$ intersects $R_\theta(\Gamma(\tilde{o}^*))$ twice for some θ.

Thus, it follows that there exists $t_1\ge t_0$ such that $\widetilde{T}(o^*,q)(t_1)\in\Gamma(\tilde{o}^*)$ and $r(\widetilde{T}(o^*,q)(t_1))>c'$.

Then, as was seen before, the existence of a point $\widetilde{T}(o^*,q)(t_1+\varepsilon)\notin\widetilde{D}\cap r^{-1}((c',\ell])$ for a sufficiently small ε and Lemma 9.2 implies that there exists an $x\in N'$ such that its reference point $\tilde{x}$ satisfies $r(\tilde{x})>r(\tilde{o}^*)$, a contradiction. This completes the proof of the second claim.

Let $q\in N'$. Since, as was mentioned above, $r(T(\tilde{o}^*,\tilde{q})(t))$ is monotone decreasing in $t\in[0,d(o^*,q))]$, we have $\angle(\tilde{o}^*\,\tilde{q}\,\tilde{o})>\pi/2$. From Lemma 9.2, we have $\angle(o^*\,q\,o)>\pi/2$. Thus there exists no critical point of the distance function to o^* in $N'\setminus\{o^*\}$. $\square$

Notice that, as was just seen, the reference points of N' are contained in the domain $\widetilde{D}\setminus r^{-1}((0,r(\Gamma(\tilde{o}^*)_e)))$. In particular, the farthest point o^* to o is unique, since $r(\Gamma(\tilde{o}^*)(t))$ is monotone decreasing in $t\in[0,d(\tilde{o}^*,\Gamma(\tilde{p})_e)]$.

Remark 9.5. If $\widetilde{M}$ is without conjugate points in half, then the distance function to o^* has no critical point in $N' := \{p \in M \,|\, d(o,p) > r(\Gamma(\tilde{o}^*)_e)\}$.

In fact, the Toponogov comparison theorem is true from Remark 9.3. Under the assumption, the domain $\widetilde{D}$ is bounded by the meridians $\theta^{-1}(0)$, $\theta^{-1}(\pi)$ and the minimizing geodesic $\Gamma(\tilde{o}^*)$, and, therefore, the reference points for $S(o, r(\Gamma(\tilde{o}^*)_e))$ is always contained in $\widetilde{D}$. This proves the argument in Remark 9.5.

Proof of Theorem 2.7. If $i(o) \geq r_1$, we then have

$$\frac{\min\{c_2(r_1), c_3(r_1)\}}{r_1} \geq \mu > \frac{\ell - d(o,o^*)}{r_1}.$$

Thus, we have $\min\{c_2(r_1), c_3(r_1)\} > \ell - d(o,o^*)$, meaning $d(o,o^*) > \ell - \min\{c_2(r_1), c_3(r_1)\}$. If $i(o) < r_1$, we then have

$$\frac{\min\{c_2(i(o)), c_3(i(o))\}}{i(o)} \geq \mu > \frac{\ell - d(o,o^*)}{i(o)}.$$

From the similar argument, we have $d(o,o^*) > \ell - \min\{c_2(i(o)), c_3(i(o))\}$. Therefore, Lemma 9.4 completes the proof of Theorem 2.7. $\square$

10. Examples of our constants

We first study the sphere with constant radius as a reference surface.

Example 10.1. We have $r_1 = \pi/(2\sqrt{\lambda})$ and $\mu = 1$ for the sphere with constant radius $1/\sqrt{\lambda}$.

Remark 10.2. The assumptions and the proof ideas of our theorem are compared with those of the classical diameter theorem [5], which is stated: Let M be a connected, complete Riemannian manifold with sectional curvature $K_M \geq \lambda > 0$ and diameter $\mathrm{diam}(M) > \pi/(2\sqrt{\lambda})$. Then M is homeomorphic to the n-sphere. The most important point is that one endpoint of the diameter is a critical point of the distance function to the other endpoint. From this point of view, Kondo and Ohta [10] extended the diameter sphere theorem, assuming that the base point is a critical point of the distance function to a certain point. In our theorem the base point o is not a critical point of the distance function to o^*, in general.

More generally, the following example is remarkable.

Example 10.3. Sinclair and Tanaka [19] determined the cut locus of a 2-sphere $\widetilde{M}$ of revolution satisfying that $K(t)$ is monotone and $K(t) = K(\ell - t)$ for $t \in [0, \ell/2]$. When $K(t)$, $0 \leq t \leq \ell/2$, is monotone non-decreasing, the cut locus of a point $\tilde{p}$ is a sub-arc of the parallel $r^{-1}(\{\ell - r(\tilde{p})\})$. Therefore, $c_2(r) = c_3(r) = r$ for all $r \in [0, \ell/2]$. Thus we have $\mu = 1$.

We have an estimate of μ for a von Mangoldt surface of revolution.

Example 10.4. Let $\widetilde{M}$ be a von Mangoldt surface, namely the curvature function $K(r)$ is monotone non-increasing in $r \in [0, \ell]$. Then there exists a unique $r_1 = \ell - {r_1}^*$ such that $f'(r_1) = 0$. Let $\tilde{p}$ be a point with $r(\tilde{p}) > \ell - \min\{r_1, \ell - r_1\}$. Then, $f(r(\Gamma(\tilde{o}^*)_e)) \geq f(r(\tilde{p}))$ because of the Clairaut relation. Since $K(r) \geq K(r')$ for all $0 \leq r \leq r_1 \leq r' \leq \ell$, we have $f(r) \leq f(\ell - r)$ in $r \in [0, \min\{r_1, \ell - r_1\}]$. This implies that $\ell - r(\tilde{p}) \leq r(\Gamma(\tilde{o}^*)_e)$. Therefore, we have $\mu \leq 1$.

The following example suggests us that the constant μ defined from the curvature function K and the assumption (2.10) are important to study a sphere theorem.

Example 10.5. Let $0 < a < b$. Let S_1 and S_2 be circles whose lengths are $2a$ and $2b$, respectively. Let $T = S_1 \times S_2$ and $\tilde{o} \in T$. Let o^* be the furthest point to o in T. Then, we have $i(o) = a$ and $d(o, o^*) = \sqrt{a^2 + b^2} > b$.

This example shows that for any small $\varepsilon > 0$ and large $\ell > 0$ there exists a torus T satisfying $i(o) < \varepsilon$ and $d(o, o^*) = \ell$.

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