On generalizations and refinements of triangle inequalities

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March 2015

Contents

Preface

Let $(X, \|\cdot\|)$ be a normed space. The triangle inequality

$$\|x+y\| \leq \|x\| + \|y\| \quad (x, y \in X)$$

is one of the most significant inequalities in mathematics. This inequality and the relevant topics have been studied by many authors (cf. [2], [3], [4], [6], [9], [10], [11], [12], [13], [19], [20], [21], [22], [24], [27], [29], [30], [31], [33], [36], [37], [38], [40], [42] etc.).

In this thesis, we study the triangle inequality in two directions. We first study a generalized triangle inequality of the following type. For a Hilbert space H, we recall the parallelogram law

$$\|x+y\|^2 + \|x-y\|^2 = 2\big(\|x\|^2 + \|y\|^2\big) \quad (x, y \in H).$$

This implies that the parallelogram inequality

$$\|x+y\|^2 \leq 2\big(\|x\|^2 + \|y\|^2\big) \quad (x, y \in H). \tag{1}$$

Saitoh in [38] noted the inequality (1) may be more suitable than the usual triangle inequality and used the inequality (1) to the setting of a natural sum Hilbert space for two arbitrary Hilbert spaces. This inequality actually holds in any normed space. The following inequality which is a generalization of (1)

$$\|x+y\|^q \leq 2^{q-1}\big(\|x\|^q + \|y\|^q\big) \quad (x, y \in X), \tag{2}$$

where $q \geq 1$ also holds in any normed space. Motivated by this, Belbacir et al. [6] considered that replacing the triangle inequality in the definition of a norm by (2). On the other hand, for a Hilbert space H, we recall the Euler-Lagrange type identity

$$\frac{\|x\|^2}{\mu} + \frac{\|y\|^2}{\nu} - \frac{\|ax+by\|^2}{\lambda} = \frac{\|\nu bx - \mu ay\|^2}{\lambda\mu\nu} \quad (x, y \in H),$$

where $a, b, \lambda, \mu, \nu \in \mathbb{R}$ with $\lambda = \mu a^2 + \nu b^2$. If $\lambda\mu\nu > 0$, this implies the following inequality

$$\frac{\|ax+by\|^2}{\lambda} \leq \frac{\|x\|^2}{\mu} + \frac{\|y\|^2}{\nu} \quad (x, y \in H).$$

This inequality can be considered a generalization of (1). In this direction, for a normed space X, Takahasi et al. [40] considered the following type inequality

$$\frac{\|ax+by\|^p}{\lambda} \le \frac{\|x\|^p}{\mu} + \frac{\|y\|^p}{\nu} \quad (x, y \in X), \tag{3}$$

where $p \in \mathbb{R}$ with $p > 0$, $a, b \in \mathbb{C}$ and $\lambda, \mu, \nu \in \mathbb{R}$. They gave necessary and sufficient conditions which the inequality (3) and its reverse inequality hold. In [9], Dadipour et al. characterized all $(\mu_1, \ldots, \mu_n) \in \mathbb{R}^n$ which satisfy a general case of this inequality:

$$\|x_1 + \cdots + x_n\|^p \le \frac{\|x_1\|^p}{\mu_1} + \cdots + \frac{\|x_n\|^p}{\mu_n} \quad (x_1, \ldots, x_n \in X) \tag{4}$$

and its reverse inequality. Our main aim in chapter 1 is to present a generalization of (4) by using ψ-direct sums of Banach spaces (cf. [18]). Therefore, we give another approach to characterizations of all $(\mu_1, \ldots, \mu_n) \in \mathbb{R}^n$ which satisfy (4) and its reverse inequality (cf. [9, Theorems 2.4–2.7]).

Next we study the inequalities which are sharper than the usual triangle inequality for a Banach space X. We consider the usual triangle inequality for n-elements

$$\left\|\sum_{j=1}^n x_j\right\| \le \sum_{j=1}^n \|x_j\| \quad (x_1, \cdots, x_n \in X).$$

In [19], Kato, Saito and Tamura proved the sharp triangle inequality and reverse inequality as follows: for all nonzero elements $x_1, \cdots, x_n$ in a Banach space X,

$$\begin{aligned}\left\|\sum_{j=1}^n x_j\right\| &+ \left(n - \left\|\sum_{j=1}^n \frac{x_j}{\|x_j\|}\right\|\right) \min_{1\le j\le n} \|x_j\| \\ &\le \sum_{j=1}^n \|x_j\| \\ &\le \left\|\sum_{j=1}^n x_j\right\| + \left(n - \left\|\sum_{j=1}^n \frac{x_j}{\|x_j\|}\right\|\right) \max_{1\le j\le n} \|x_j\|\end{aligned}$$

hold. In the case of two elements, above inequalities are following: for all nonzero elements $x, y \in X$,

$$\begin{aligned}\|x+y\| &+ \left(2 - \left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\|\right) \min\{\|x\|, \|y\|\} \\ &\le \|x\| + \|y\| \\ &\le \|x+y\| + \left(2 - \left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\|\right) \max\{\|x\|, \|y\|\}\end{aligned}$$

hold (cf. [14] and [20]). These sharp triangle inequalities are applied to the characterizations of geometric properties of Banach spaces like the strict convexity, the uniform convexity and so on (cf. [19] and [37]). After that, Mitani et al. [29] succeeded in the further extension of above inequalities as follows: for all nonzero elements $x_1, x_2, \ldots, x_n$ in a Banach space X with $\|x_1\| \geq \|x_2\| \geq \cdots \geq \|x_n\|, n \geq 2$,

$$\begin{aligned}
&\left\|\sum_{j=1}^{n} x_j\right\| + \sum_{k=2}^{n}\left(k - \left\|\sum_{j=1}^{k} \frac{x_j}{\|x_j\|}\right\|\right)(\|x_k\| - \|x_{k+1}\|) \\
&\quad \leq \sum_{j=1}^{n} \|x_j\| \\
&\quad \leq \left\|\sum_{j=1}^{n} x_j\right\| - \sum_{k=2}^{n}\left(k - \left\|\sum_{j=n-(k-1)}^{n} \frac{x_j}{\|x_j\|}\right\|\right)(\|x_{n-k}\| - \|x_{n-(k-1)}\|)
\end{aligned}$$

hold, where $x_0 = x_{n+1} = 0$. In the case of $n = 3$, above inequalities are following: for all nonzero elements $x, y, z \in X$ with $\|x\| \geq \|y\| \geq \|z\|$,

$$\begin{aligned}
&\|x+y+z\| + \left(3 - \left\|\frac{x}{\|x\|} + \frac{y}{\|y\|} + \frac{z}{\|z\|}\right\|\right)\|z\| \\
&\qquad + \left(2 - \left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\|\right)(\|y\| - \|z\|) \\
&\leq \|x\| + \|y\| + \|z\| \\
&\leq \|x+y+z\| + \left(3 - \left\|\frac{x}{\|x\|} + \frac{y}{\|y\|} + \frac{z}{\|z\|}\right\|\right)\|x\| \\
&\qquad - \left(2 - \left\|\frac{y}{\|y\|} + \frac{z}{\|z\|}\right\|\right)(\|x\| - \|y\|)
\end{aligned}$$

hold. They also studied equality attainedness on these inequalities in a strictly convex Banach space (cf. [19] and [27]). In this direction, Mineno, Nakamura and Ohwada [24] studied the problem that characterize all the intermediate value C which satisfies

$$0 \leq C \leq \sum_{j=1}^{n} \|x_j\| - \left\|\sum_{j=1}^{n} x_j\right\|$$

by using $x_1, x_2, \ldots, x_n$ in a Banach space X. Our main aims in Section 2 are to give the simple proof of the result of Mineno et al. and study when this norm inequality attains equality in a strictly convex Banach space.

Acknowledgements

I would like to express the deepest gratitude to my supervisor Professor Kichi-Suke Saito at Niigata University for his great guidance and useful advices. I have studied under him during a master's course from 1997 to 1999 and a doctoral course. He provided much support and instructions. I also wish to thank all my colleagues in Professor Saito's laboratory.

Moreover I would like to express his thanks to Professor Tomoyoshi Ohwada at Shizuoka University for his many instructive suggestions in Chapter 2 of this thesis.

1 On generalized triangle inequalities

1.1 Introduction

In this chapter, for a normed space $(X, \|\cdot\|)$, we consider the following generalized triangle inequality which is involved with the Euler-Lagrange type identity: for any fixed $n \in \mathbb{N}$ with $n \geq 2$ and fixed $p \in \mathbb{R}$ with $p > 0$,

$$\frac{\|a_1x_1 + \cdots + a_nx_n\|^p}{\lambda} \leq \frac{\|x_1\|^p}{\mu_1} + \cdots + \frac{\|x_n\|^p}{\mu_n} \quad (x_1, \dots, x_n \in X) \tag{5}$$

where $(a_1, \dots, a_n, \lambda, \mu_1, \dots, \mu_n) \in \mathbb{C}^n \times \mathbb{R} \times \mathbb{R}^n$. In [40], Takahasi et al. gave a necessary and sufficient condition in order that a special case of (5):

$$\frac{\|ax + by\|^p}{\lambda} \leq \frac{\|x\|^p}{\mu} + \frac{\|y\|^p}{\nu} \quad (x, y \in X)$$

and its reverse inequality hold. For simplicity, we shall write, for example,

$$\{\lambda > 0, \mu > 0, \nu > 0\}$$

for

$$\{(\lambda, \mu, \nu) \in \mathbb{R}^3 : \ \lambda > 0, \mu > 0, \nu > 0\}.$$

Theorem 1.1.1 (cf. [40, Theorem 1.1 and 4.1])**.** *Let X be a normed space. Let $p \in \mathbb{R}, p \geq 1$, $a, b \in \mathbb{C}, \lambda, \mu, \nu \in \mathbb{R}$. Put*

$$D_p^+ = \left\{(a, b, \lambda, \mu, \nu) : \frac{\|ax + by\|^p}{\lambda} \leq \frac{\|x\|^p}{\mu} + \frac{\|y\|^p}{\nu} \ (x, y \in X)\right\}.$$

If $p > 1$, then

(i) $D_p^+ \cap \{\lambda > 0, \mu > 0, \nu > 0\}$

$$= \{\lambda > 0, \mu > 0, \nu > 0, |\lambda|^{1/(p-1)} \geq |\mu|^{1/(p-1)}|a|^{p'} + |\nu|^{1/(p-1)}|b|^{p'}\}.$$

(ii) $D_p^+ \cap \{\lambda < 0, \mu < 0, \nu > 0\}$

$$= \{\lambda < 0, \mu < 0, \nu > 0, |\lambda|^{1/(p-1)} \leq |\mu|^{1/(p-1)}|a|^{p'} - |\nu|^{1/(p-1)}|b|^{p'}\}.$$

(iii) $D_p^+ \cap \{\lambda < 0, \mu > 0, \nu < 0\}$

$$= \{\lambda < 0, \mu > 0, \nu < 0, |\lambda|^{1/(p-1)} \leq -|\mu|^{1/(p-1)}|a|^{p'} + |\nu|^{1/(p-1)}|b|^{p'}\}.$$

(iv) $D_p^+ \cap \{\lambda < 0, \mu < 0, \nu < 0\} = \varnothing$.

(v) $D_p^+ \cap \{\lambda > 0, \mu > 0, \nu < 0\} = \varnothing$.

(vi) $D_p^+ \cap \{\lambda > 0, \mu < 0, \nu > 0\} = \varnothing$.

(vii) $D_p^+ \cap \{\lambda > 0, \mu < 0, \nu < 0\} = \varnothing$.

(viii) $D_p^+ \cap \{\lambda < 0, \mu > 0, \nu > 0\} = \mathbb{C} \times \mathbb{C} \times \mathbb{R}^3 \setminus \{\lambda\mu\nu = 0\}$.

where $p' = (1 - 1/p)^{-1}$.

If $p = 1$, *then*

$$\begin{aligned} D_1^+ = &\{\lambda < 0, \mu > 0, \nu > 0\} \\ &\cup \{\lambda > 0, \mu > 0, \nu > 0, \lambda \geq \mu|a|, \lambda \geq \nu|b|\} \\ &\cup \{\lambda < 0, \mu < 0, \nu > 0, \lambda \geq \mu|a|, -\nu|b| \geq \mu|a|\} \\ &\cup \{\lambda < 0, \mu > 0, \nu < 0, \lambda \geq \nu|b|, -\mu|a| \geq \nu|b|\}. \end{aligned}$$

In [9], Dadipour et al. characterized all $(\mu_1, \ldots, \mu_n) \in \mathbb{R}^n$ which satisfy a special case of (5):

$$\|x_1 + \cdots + x_n\|^p \leq \frac{\|x_1\|^p}{\mu_1} + \cdots + \frac{\|x_n\|^p}{\mu_n} \quad (x_1, \ldots, x_n \in X) \tag{6}$$

and its reverse inequality. Our aim is to present a generalization of (6) by using ψ-direct sums of Banach spaces (cf. [18]). Therefore, we give another approach to characterizations of all $(\mu_1, \ldots, \mu_n) \in \mathbb{R}^n$ which satisfy (6) and its reverse inequality (cf. [9, Theorems 2.4–2.7]).

In §1.2, we summarize basic results of absolute normalized norms by [7], [26], [28], [32], [34] and [35].

In §1.3, we present a generalization of (6) by using ψ-direct sums of Banach spaces (cf. [18]). Therefore, we give another approach to characterizations of all $(\mu_1, \ldots, \mu_n) \in \mathbb{R}^n$ which satisfy (6) and its reverse inequality.

In §1.4, we generalized a inequality (6) for infinite sequences $\{x_n\}_{n=1}^{\infty} \subset X$ by using generalized ℓ_p-spaces (cf. [25]).

1.2 Absolute normalized norms

In this section, we summarize basic results of absolute normalized norms on $\mathbb{C}^n$ by [7], [26], [28], [32], [34] and [35].

1.2.1 Absolute normalized norms on $\mathbb{C}^n$ and their characterizations

A norm $\|\cdot\|$ on $\mathbb{C}^n$ is absolute if

$$\|(z_1, z_2, \dots, z_n)\| = \|(|z_1|, |z_2|, \dots, |z_n|)\| \quad ((z_1, z_2, \dots, z_n) \in \mathbb{C}^n)$$

and normalized if

$$\|(1, 0, \dots, 0)\| = \|(0, 1, 0, \dots, 0)\| = \cdots = \|(0, \dots, 0, 1)\| = 1.$$

An ℓ_p-norm on $\mathbb{C}^n$:

$$\|(z_1, \dots, z_n)\|_p = \begin{cases} (|z_1|^p + \cdots + |z_n|^p)^{\frac{1}{p}} & (1 \le p < \infty) \\ \max\{|z_1|, \dots, |z_n|\} & (p = \infty) \end{cases}$$

is such an example. We first show some basic facts about these norms. Let AN_n be the set of all absolute normalized norms on $\mathbb{C}^n$.

Proposition 1.2.1 (cf. [35, Lemma 3.1]). *Let* $\|\cdot\| \in AN_n$. *Then*

$$\|(0, z_2, z_3, \dots, z_n)\| \le \|(z_1, z_2, \dots, z_n)\| \qquad (\mathrm{B_1})$$

$$\|(z_1, 0, z_3, \dots, z_n)\| \le \|(z_1, z_2, \dots, z_n)\| \qquad (\mathrm{B_2})$$

$$\vdots$$

$$\|(z_1, \dots, z_{n-1}, 0)\| \le \|(z_1, z_2, \dots, z_n)\|. \qquad (\mathrm{B_n})$$

In particular,

$$\|\cdot\|_\infty \le \|\cdot\| \le \|\cdot\|_1.$$

Proof. For any $(z_1, z_2, \dots, z_n) \in \mathbb{C}^n$, we have

$$\begin{aligned} \|(0, z_2, z_3, \dots, z_n)\| &= \frac{1}{2}\big(\|(z_1, z_2, z_3, \dots, z_n) + (-z_1, z_2, z_3, \dots, z_n)\|\big) \\ &\le \frac{1}{2}\big(\|(z_1, z_2, z_3, \dots, z_n)\| + \|(-z_1, z_2, z_3, \dots, z_n)\|\big) \\ &= \frac{1}{2}\big(\|(z_1, z_2, z_3, \dots, z_n)\| + \|(|-z_1|, |z_2|, |z_3|, \dots, |z_n|)\|\big) \\ &= \frac{1}{2}\big(\|(z_1, z_2, z_3, \dots, z_n)\| + \|(z_1, z_2, z_3, \dots, z_n)\|\big) \\ &= \|(z_1, z_2, z_3, \dots, z_n)\|. \end{aligned}$$

Then we have (B_1). Similarly, we have (B_2), ..., (B_n). Therefore, for any $(z_1, z_2, \ldots, z_n) \in \mathbb{C}^n$, we have

$$\begin{aligned}\|(z_1, z_2, \ldots, z_n)\|_\infty &= \max\{|z_1|, |z_2|, \ldots, |z_n|\} \\ &= \max\big\{\|(z_1, 0, \ldots, 0)\|, \|(0, z_2, 0, \ldots, 0)\|, \ldots, \|(0, \ldots, 0, z_n)\|\big\} \\ &\le \|(z_1, z_2, \ldots, z_n)\| \\ &= \|(z_1, 0, \ldots, 0) + (0, z_2, 0, \ldots, 0) + \cdots + (0, \ldots, 0, z_n)\| \\ &\le \|(z_1, 0, \ldots, 0)\| + \|(0, z_2, 0, \ldots, 0)\| + \cdots + \|(0, \ldots, 0, z_n)\| \\ &= |z_1| + |z_2| + \cdots + |z_n| \\ &= \|(z_1, z_2, \ldots, z_n)\|_1.\end{aligned}$$

This completes the proof. □

We remark that every absolute norm is monotone:

Proposition 1.2.2 (cf. [7, Proposition IV.1.1])**.** *Let* $\|\cdot\| \in AN_n$. *If* $|z_j| \le |w_j|$ *for any* $j = 1, \ldots, n$, *then*

$$\|(z_1, \ldots, z_n)\| \le \|(w_1 \ldots, w_n)\|.$$

Proof. We may suppose that $z_j, w_j \ge 0$ $(j = 1, \ldots, n)$. Since $z_1 \le w_1$, there exists $t,\ 0 \le t \le 1$ such that $z_1 = tw_1$. Then we have

$$\begin{aligned}&\|(z_1, z_2, \ldots, z_n)\| \\ &\quad = \|(tw_1, z_2, \ldots, z_n)\| \\ &\quad = \left\|\left(\frac{1+t}{2}w_1 - \frac{1-t}{2}w_1, \frac{1+t}{2}z_2 + \frac{1-t}{2}z_2, \ldots, \frac{1+t}{2}z_n + \frac{1-t}{2}z_n\right)\right\| \\ &\quad = \left\|\left(\frac{1+t}{2}w_1, \frac{1+t}{2}z_2, \ldots, \frac{1+t}{2}z_n\right) + \left(-\frac{1-t}{2}w_1, \frac{1-t}{2}z_2, \ldots, \frac{1-t}{2}z_n\right)\right\| \\ &\quad = \left\|\frac{1+t}{2}(w_1, z_2, \ldots, z_n) + \frac{1-t}{2}(-w_1, z_2, \ldots, z_n)\right\| \\ &\quad \le \frac{1+t}{2}\|(w_1, z_2, \ldots, z_n)\| + \frac{1-t}{2}\|(-w_1, z_2, \ldots, z_n)\| \\ &\quad = \frac{1+t}{2}\|(w_1, z_2, \ldots, z_n)\| + \frac{1-t}{2}\|(w_1, z_2, \ldots, z_n)\| \\ &\quad = \|(w_1, z_2, \ldots, z_n)\|.\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\|(w_1, z_2, z_3, \ldots, z_n)\| &\le \|(w_1, w_2, z_3, \ldots, z_n)\| \\
\|(w_1, w_2, z_3, \ldots, z_n)\| &\le \|(w_1, w_2, w_3, z_4, \ldots, z_n)\| \\
&\vdots \\
\|(w_1, w_2, \ldots, w_{n-1}, z_n)\| &\le \|(w_1, w_2, \ldots, w_{n-1}, w_n)\|.
\end{aligned}$$

Then we have

$$\|(z_1, z_2, \ldots, z_{n-1}, z_n)\| \le \|(w_1, w_2, \ldots, w_{n-1}, w_n)\|.$$

This completes the proof. □

Bonsal and Duncan [8] showed the following characterization of absolute normalized norms on $\mathbb{C}^2$. Let Ψ_2 denote the set of all continuous convex functions ψ on $[0,1]$ with $\psi(0) = \psi(1) = 1$ satisfying

$$\max\{1-t, t\} \le \psi(t) \le 1 \quad (t \in [0,1]).$$

Let $\|\cdot\| \in AN_2$ and let

$$\psi(t) = \|(1-t, t)\| \quad (t \in (0,1)). \tag{7}$$

Then $\psi \in \Psi_2$. Conversely, for given $\psi \in \Psi_2$ define

$$\|(z_1, z_2)\|_\psi = \begin{cases} (|z_1| + |z_2|)\psi\left(\frac{|z_2|}{|z_1|+|z_2|}\right) & ((z_1, z_2) \ne (0,0)) \\ 0 & ((z_1, z_2) = (0,0)). \end{cases}$$

Then $\|\cdot\|_\psi \in AN_2$, and $\|\cdot\|_\psi$ satisfies (7). In fact, AN_2 and Ψ_2 are in one-to-one correspondence under (7).

In [35], Saito et al. characterized absolute normalized norms on $\mathbb{C}^n$ by means of the corresponding convex function as follows. For each $n \in \mathbb{N}$ with $n \ge 2$, put

$$\Delta_n = \left\{(t_1, \ldots, t_n) \in \mathbb{R}^n : t_1, \ldots, t_n \ge 0, \sum_{j=1}^n t_j = 1\right\},$$

and let Ψ_n denote the set of all continuous convex functions ψ on Δ_n satisfying the following conditions:

$$\psi(1, 0, \ldots, 0) = \psi(0, 1, 0, \ldots, 0) = \cdots = \psi(0, \ldots, 0, 1) = 1 \tag{A$_0$}$$

$$\psi(t_1,\ldots,t_n) \geq (1-t_1)\psi\left(0, \frac{t_2}{1-t_1},\ldots,\frac{t_n}{1-t_1}\right) \tag{A$_1$}$$

$$\psi(t_1,\ldots,t_n) \geq (1-t_2)\psi\left(\frac{t_1}{1-t_2}, 0, \frac{t_3}{1-t_2},\ldots,\frac{t_n}{1-t_2}\right) \tag{A$_2$}$$

$$\vdots$$

$$\psi(t_1,\ldots,t_n) \geq (1-t_n)\psi\left(\frac{t_1}{1-t_n},\ldots,\frac{t_{n-1}}{1-t_n}, 0\right). \tag{A$_n$}$$

Then AN_n and Ψ_n are in one-to-one correspondence as follows.

Theorem 1.2.3 (cf. [35, Theorems 3.4])**.**

(i) *Let* $\|\cdot\| \in AN_n$ *and let*

$$\psi(t_1,\cdots,t_n) = \|(t_1,\ldots,t_n)\| \quad ((t_1,\ldots,t_n) \in \Delta_n). \tag{8}$$

Then $\psi \in \Psi_n$*. Conversely,*

(ii) *For given* $\psi \in \Psi_n$ *define*

$$\|(z_1,\ldots,z_n)\|_\psi = \begin{cases} (|z_1|+\cdots+|z_n|)\psi\left(\frac{|z_1|}{|z_1|+\cdots+|z_n|},\ldots,\frac{|z_n|}{|z_1|+\cdots+|z_n|}\right) & ((z_1,\ldots,z_n) \neq (0,\ldots,0)) \\ 0 & ((z_1,\ldots,z_n) = (0,\ldots,0)). \end{cases} \tag{9}$$

Then $\|\cdot\|_\psi \in AN_n$*, and* $\|\cdot\|_\psi$ *satisfies* (8).

To show this theorem, we prepare some lemmas. We remark that the case of an ℓ_p-norm on $\mathbb{C}^n$, by the Hölder inequality, $\psi = \psi_p \in \Psi_n$ is

$$\psi_p(t_1,\ldots,t_n) = \begin{cases} ({t_1}^p+\cdots+{t_n}^p)^{\frac{1}{p}} & (1 \leq p < \infty) \\ \max\{t_1,\ldots,t_n\} & (p=\infty). \end{cases}$$

Lemma 1.2.4 (cf. [35, Lemma 3.2])**.** *Let* $\psi \in \Psi_n$*. Then*

$$\frac{1}{n} \leq \psi_\infty(t_1,\cdots,t_n) \leq \psi(t_1,\cdots,t_n) \leq 1$$

for all $(t_1,\ldots,t_n) \in \Delta_n$*.*

Proof. Let $\psi \in \Psi_n$. By ($\mathrm{A_0}$) and the convexity of ψ, it is clear that $\psi(t_1, t_2, \cdots, t_n) \leq 1$. Next, from ($\mathrm{A_0}$), ($\mathrm{A_2}$), ..., ($\mathrm{A_n}$), we have

$$\begin{aligned}
\psi(t_1, \ldots, t_n) &\geq (1-t_2)\psi\left(\frac{t_1}{1-t_2}, 0, \frac{t_3}{1-t_2}, \ldots, \frac{t_n}{1-t_2}\right) \\
&\geq (1-t_2)\left(1-\frac{t_3}{1-t_2}\right)\psi\left(\frac{\frac{t_1}{1-t_2}}{1-\frac{t_3}{1-t_2}}, 0, 0, \frac{\frac{t_4}{1-t_2}}{1-\frac{t_3}{1-t_2}}, \ldots, \frac{\frac{t_n}{1-t_2}}{1-\frac{t_3}{1-t_2}}\right) \\
&= (1-t_2-t_3)\psi\left(\frac{t_1}{1-t_2-t_3}, 0, 0, \frac{t_4}{1-t_2-t_3}, \ldots, \frac{t_n}{1-t_2-t_3}\right) \\
&\geq \cdots \\
&\geq (1-t_2-t_3-\cdots-t_n)\psi\left(\frac{t_1}{1-t_2-t_3-\cdots-t_{n-1}}, 0, \ldots, 0\right) \\
&= t_1\psi(1, 0, \ldots, 0) \\
&= t_1.
\end{aligned}$$

Next, from ($\mathrm{A_0}$), ($\mathrm{A_1}$), ($\mathrm{A_3}$), ..., ($\mathrm{A_n}$), we have

$$\begin{aligned}
\psi(t_1, \ldots, t_n) &\geq (1-t_1)\psi\left(0, \frac{t_2}{1-t_1}, \frac{t_3}{1-t_1}, \ldots, \frac{t_n}{1-t_1}\right) \\
&\geq (1-t_1)\left(1-\frac{t_3}{1-t_1}\right)\psi\left(0, \frac{\frac{t_2}{1-t_1}}{1-\frac{t_3}{1-t_1}}, 0, \frac{\frac{t_4}{1-t_1}}{1-\frac{t_3}{1-t_1}}, \ldots, \frac{\frac{t_n}{1-t_1}}{1-\frac{t_3}{1-t_1}}\right) \\
&= (1-t_1-t_3)\psi\left(0, \frac{t_2}{1-t_1-t_3}, 0, \frac{t_4}{1-t_1-t_3}, \ldots, \frac{t_n}{1-t_1-t_3}\right) \\
&\geq \cdots \\
&\geq (1-t_1-t_3-\cdots-t_n)\psi\left(0, \frac{t_2}{1-t_1-t_3-\cdots-t_n}, 0, \ldots, 0\right) \\
&= t_2\psi(0, 1, 0, \ldots, 0) \\
&= t_2.
\end{aligned}$$

Similarly, we can show that for $3 \leq i \leq n-1$,

$$\psi(t_1, \ldots, t_n) \geq t_i.$$

Finally, from ($\mathrm{A_0}$), ($\mathrm{A_1}$), ..., ($\mathrm{A_{n-1}}$), we have

$$\begin{aligned}
\psi(t_1,\dots,t_n) &\geq (1-t_1)\psi\left(0,\frac{t_2}{1-t_1},\dots,\frac{t_n}{1-t_1}\right)\\
&\geq (1-t_1)\left(1-\frac{t_2}{1-t_1}\right)\psi\left(0,0,\frac{\frac{t_3}{1-t_1}}{1-\frac{t_2}{1-t_1}},\dots,\frac{\frac{t_n}{1-t_1}}{1-\frac{t_2}{1-t_1}}\right)\\
&= (1-t_1-t_2)\psi\left(0,0,\frac{t_3}{1-t_1-t_2},\dots,\frac{t_n}{1-t_1-t_2}\right)\\
&\geq \cdots\\
&\geq (1-t_1-\cdots-t_{n-1})\psi\left(0,\dots,0,\frac{t_n}{1-t_1-\cdots-t_{n-1}}\right)\\
&= t_n\psi(0,\dots,1)\\
&= t_n.
\end{aligned}$$

Then for any $1 \leq i \leq n$, we have

$$\psi(t_1,\dots,t_n) \geq t_i.$$

Thus we have

$$\psi(t_1,\dots,t_n) \geq \max\{t_1,\cdots,t_n\} \geq \frac{1}{n},$$

so,

$$\psi(t_1,\dots,t_n) \geq \psi_\infty(t_1,\dots,t_n) \geq \frac{1}{n}.$$

This completes the proof. □

Lemma 1.2.5 (cf. [35, Theorem 3.5])**.** *For any* $(p_1,\dots,p_n),(a_1,\dots,a_n) \in \mathbb{C}^n$ *such that* $0 \leq p_i \leq a_i$ $(i=1,\dots,n)$*, we have that*

$$\|(p_1,\dots,p_n)\|_\psi \leq \|(a_1,\dots,a_n)\|_\psi. \tag{10}$$

Proof. We first show that, if $0 \leq p_1 < a_1$, then

$$\|(p_1,p_2,\dots,p_n)\|_\psi \leq \|(a_1,p_2\dots,a_n)\|_\psi. \tag{11}$$

This is, we show that if $0 \leq p_1 < a_1$

$$\begin{aligned}
&(p_1+p_2+\cdots+p_n)\psi\left(\frac{p_1}{p_1+p_2+\cdots+p_n},\dots,\frac{p_n}{p_1+p_2+\cdots+p_n}\right)\\
&\quad\leq (a_1+p_2+\cdots+p_n)\\
&\qquad\times\psi\left(\frac{a_1}{a_1+p_2+\cdots+p_n},\frac{p_2}{a_1+p_2+\cdots+p_n}\dots,\frac{p_n}{a_1+p_2+\cdots+p_n}\right).
\end{aligned}$$

Take any $(t_1,\dots,t_n)\in\Delta_n$, and consider the line segment

$$\left[(1,0,\dots,0),\left(0,\frac{t_2}{1-t_1},\dots,\frac{t_n}{1-t_1}\right)\right]$$

in Δ_n. For any real number λ such that $1<\lambda\le 1/(1-t_1)$, we put

$$\begin{aligned}
(t_1',&t_2',\dots,t_n')\\
&=(1,0,\dots,0)+\lambda\left\{(t_1,t_2,\dots,t_n)-(1,0,\dots,0)\right\}\\
&=\big(1+\lambda(t_1-1),\lambda t_2,\dots,\lambda t_n\big)\\
&=\frac{(1-t_1)(\lambda-1)}{t_1}\left(0,\frac{t_2}{1-t_1},\dots,\frac{t_n}{1-t_1}\right)+\frac{1-\lambda(1-t_1)}{t_1}\,(t_1,\dots,t_n)\,.
\end{aligned}$$

By the convexity of ψ,

$$\begin{aligned}
&\psi(t_1',t_2',\dots,t_n')\\
&\quad\le\frac{(1-t_1)(\lambda-1)}{t_1}\;\psi\left(0,\frac{t_2}{1-t_1},\dots,\frac{t_n}{1-t_1}\right)+\frac{1-\lambda(1-t_1)}{t_1}\psi(t_1,\dots,t_n).
\end{aligned}$$

Then we have from (A_1)

$$\begin{aligned}
&\frac{\psi(t_1,\dots,t_n)}{1-t_1}-\frac{\psi(t_1',\dots,t_n')}{1-t_1'}\\
&\quad\ge\frac{\psi(t_1,\dots,t_n)}{1-t_1}-\frac{1}{1-t_1'}\\
&\qquad\times\left\{\frac{(1-t_1)(\lambda-1)}{t_1}\psi\left(0,\frac{t_2}{1-t_1},\dots,\frac{t_n}{1-t_1}\right)+\frac{1-\lambda(1-t_1)}{t_1}\psi(t_1,\dots,t_n)\right\}\\
&\quad=\frac{\psi(t_1,\dots,t_n)}{1-t_1}-\frac{1}{\lambda(1-t_1)}\\
&\qquad\times\left\{\frac{(1-t_1)(\lambda-1)}{t_1}\psi\left(0,\frac{t_2}{1-t_1},\dots,\frac{t_n}{1-t_1}\right)+\frac{1-\lambda(1-t_1)}{t_1}\psi(t_1,\dots,t_n)\right\}\\
&\quad=\frac{\lambda-1}{\lambda t_1(1-t_1)}\left\{\psi(t_1,\dots,t_n)-(1-t_1)\psi\left(0,\frac{t_2}{1-t_1},\cdots,\frac{t_n}{1-t_1}\right)\right\}\ge 0.
\end{aligned}$$

Thus, we have

$$\frac{\psi(t_1,t_2,\dots,t_n)}{1-t_1}\ge\frac{\psi(t_1',t_2',\dots,t_n')}{1-t_1'}. \tag{12}$$

Since $0\le p_1<a_1$, we put

$$\begin{aligned}
(t_1,\dots,t_n)&=\left(\frac{a_1}{a_1+p_2+\cdots+p_n},\frac{p_2}{a_1+p_2+\cdots+p_n},\dots,\frac{p_n}{a_1+p_2+\cdots+p_n}\right)\\
(t_1',\dots,t_n')&=\left(\frac{p_1}{p_1+p_2+\cdots+p_n},\frac{p_2}{p_1+p_2+\cdots+p_n},\dots,\frac{p_n}{p_1+p_2+\cdots+p_n}\right)\\
\lambda&=\frac{a_1+p_2+\cdots+p_n}{p_1+p_2+\cdots+p_n}>1
\end{aligned}$$

respectively in (12). Thus, we have

$$\frac{\psi\left(\frac{a_1}{a_1+p_2+\cdots+p_n},\frac{p_2}{a_1+p_2+\cdots+p_n},\ldots,\frac{p_n}{a_1+p_2+\cdots+p_n}\right)}{\frac{p_1}{a_1+p_2+\cdots+p_n}} \geq \frac{\psi\left(\frac{p_1}{p_1+p_2+\cdots+p_n},\frac{p_2}{p_1+p_2+\cdots+p_n},\ldots,\frac{p_n}{p_1+p_2+\cdots+p_n}\right)}{\frac{p_1}{p_1+p_2+\cdots+p_n}}.$$

So,

$$\begin{aligned}(a_1&+p_2+\cdots+p_n)\\ &\times\psi\left(\frac{p_1}{a_1+p_2+\cdots+p_n},\frac{p_2}{a_1+p_2+\cdots+p_n},\ldots,\frac{p_n}{a_1+p_2+\cdots+p_n}\right)\\ &\geq(p_1+p_2+\cdots+p_n)\\ &\qquad\times\psi\left(\frac{p_1}{p_1+p_2+\cdots+p_n},\frac{p_2}{p_1+p_2+\cdots+p_n},\ldots,\frac{p_n}{p_1+p_2+\cdots+p_n}\right).\end{aligned}$$

This implies (11). Similarly, we can show that for $2\leq i\leq n$,

$$\|(a_1,\ldots,a_{i-1},p_i,p_{i+1},\ldots,p_n)\|_\psi\leq\|(a_1,\ldots,a_{i-1},a_i,p_{i+1},\ldots,p_n)\|_\psi.$$

Therefor, we have (10). This completes the proof. □

Proof of Theorem 1.2.3.

(i) A property of the continuation is clear. We check a property of the convexity. For any $(t_1,\ldots,t_n),(t'_1,\ldots,t'_n)\in\Delta_n$ and any $\nu\in[0,1]$, we have

$$\begin{aligned}\psi\big((1-\nu)&(t_1,\ldots,t_n)+\nu(t'_1,\ldots,t'_n)\big)\\ &=\psi\big((1-\nu)t_1+\nu t'_1,\ldots,(1-\nu)t_n+\nu t'_n\big)\\ &=\big\|\big((1-\nu)t_1+\nu t'_1,\ldots,(1-\nu)t_n+\nu t'_n\big)\big\|\\ &=\|(1-\nu)(t_1,\ldots,t_n)+\nu(t'_1,\ldots,t'_n)\|\\ &\leq(1-\nu)\|(t_1,\ldots,t_n)\|+\nu\|(t'_1,\ldots.t'_n)\|\\ &=(1-\nu)\psi(t_1,\ldots,t_n)+\nu\psi(t'_1,\ldots,t'_n).\end{aligned}$$

Then ψ is a convex function on Δ_n. Next, we check properties of (A_0), (A_1), ...,

($\mathrm{A_n}$). ($\mathrm{A_0}$) is clear. By ($\mathrm{B_1}$), we have

$$\begin{aligned}
&\psi(t_1,\ldots,t_n)\\
&\quad= \|(t_1,t_2,\ldots,t_n)\|\\
&\quad\geq \|(0,t_2,\ldots,t_n)\|\\
&\quad= (t_2+\cdots+t_n)\left\|\left(0,\frac{t_2}{t_2+\cdots+t_n},\ldots,\frac{t_n}{t_2+\cdots+t_n}\right)\right\|\\
&\quad= (1-t_1)\left\|\left(0,\frac{t_2}{1-t_1},\ldots,\frac{t_n}{1-t_1}\right)\right\|\\
&\quad= (1-t_1)\psi\left(0,\frac{t_2}{1-t_1},\ldots,\frac{t_n}{1-t_1}\right).
\end{aligned}$$

Thus we have ($\mathrm{A_1}$). We similarly have ($\mathrm{A_2}$), ..., ($\mathrm{A_n}$).

(ii) By Lemma 1.2.4, all properties of an absolute normalized norm are clear except the triangle inequality

$$\|(z_1,\ldots,z_n)+(w_1,\ldots,w_n)\|_\psi \leq \|(z_1,\ldots,z_n)\|_\psi + \|(w_1,\ldots,w_n)\|_\psi \tag{13}$$

for any $(z_1,\ldots,z_n),(w_1,\ldots,w_n)\in\mathbb{C}^n$. From Lemma 1.2.5 and the convexity of ψ, for any $(z_1,\ldots,z_n),(w_1,\ldots,w_n)\in\mathbb{C}^n$, we have

$$\begin{aligned}
&\|(z_1,\ldots,z_n)+(w_1,\ldots,w_n)\|_\psi\\
&\quad= \|(z_1+w_1,\ldots,z_n+w_n)\|_\psi\\
&\quad= \|(|z_1+w_1|,\ldots,|z_n+w_n|)\|_\psi\\
&\quad\leq \|(|z_1|+|w_1|,\ldots,|z_n|+|w_n|)\|_\psi\\
&\quad= (|z_1|+|w_1|+\cdots+|z_n|+|w_n|)\\
&\qquad\times\psi\left(\frac{|z_1|+|w_1|}{|z_1|+|w_1|+\cdots+|z_n|+|w_n|},\ldots,\frac{|z_n|+|w_n|}{|z_1|+|w_1|+\cdots+|z_n|+|w_n|}\right)\\
&\quad= (|z_1|+\cdots+|z_n|+|w_1|+\cdots+|w_n|)\\
&\qquad\times\psi\left(\frac{|z_1|}{|z_1|+\cdots+|w_n|}+\frac{|w_1|}{|z_1|+\cdots+|w_n|},\ldots\right.\\
&\qquad\qquad\left.\ldots,\frac{|z_n|}{|z_1|+\cdots+|w_n|}+\frac{|w_n|}{|z_1|+\cdots+|w_n|}\right)
\end{aligned}$$

$$
\begin{aligned}
&= (|z_1| + \cdots + |z_n| + |w_1| + \cdots + |w_n|) \\
&\quad \times \psi\left(\left(\frac{|z_1|}{|z_1| + \cdots + |w_n|}, \ldots, \frac{|z_n|}{|z_1| + \cdots + |w_n|}\right)\right. \\
&\qquad\qquad\qquad \left. + \left(\frac{|w_1|}{|z_1| + \cdots + |w_n|}, \ldots, \frac{|w_n|}{|z_1| + \cdots + |w_n|}\right)\right) \\
&= (|z_1| + \cdots + |z_n| + |w_1| + \cdots + |x_n|) \\
&\quad \times \psi\left(\frac{|z_1| + \cdots + |z_n|}{|z_1| + \cdots + |z_n| + |w_1| + \cdots + |w_n|}\right. \\
&\qquad\qquad \left(\frac{|z_1|}{|z_n| + \cdots + |z_n|}, \ldots, \frac{|z_n|}{|z_1| + \cdots + |z_n|}\right) \\
&\qquad\qquad\qquad + \frac{|w_1| + \cdots + |w_n|}{|z_1| + \cdots + |z_n| + |w_1| + \cdots + |w_n|} \\
&\qquad\qquad\qquad\qquad \left.\left(\frac{|w_1|}{|w_1| + \cdots + |w_n|}, \ldots, \frac{|w_n|}{|w_1| + \cdots + |w_n|}\right)\right) \\
&\leq (|z_1| + \cdots + |z_n|)\psi\left(\frac{|z_1|}{|z_1| + \cdots + |z_n|}, \ldots, \frac{|z_n|}{|z_1| + \cdots + |z_n|}\right) \\
&\qquad + (|w_1| + \cdots + |w_n|)\psi\left(\frac{|w_1|}{|w_1| + \cdots + |w_n|}, \ldots, \frac{|w_n|}{|w_1| + \cdots + |w_n|}\right) \\
&= \|(z_1, \ldots, z_n)\|_\psi + \|(w_1, \ldots, w_n)\|_\psi.
\end{aligned}
$$

This completes the proof. □

The inequality (13) is called the generalized Minkowski inequality.

1.2.2 The dual space of $\mathbb{C}^n$ with an absolute normalized norm

In this subsection, we consider the dual space of $\mathbb{C}^n$ with absolute normalized norm (cf. [28]). Let $\psi \in \Psi_n$. For any $(z_1, z_2, \ldots, z_n) \in \mathbb{C}^n$, the dual norm $\|\cdot\|_\psi^*$ of $\|\cdot\|_\psi$ is defined by

$$
\|(z_1, z_2, \ldots, z_n)\|_\psi^* = \sup\left\{\left|\sum_{j=1}^n z_j w_j\right| : \|(w_1, w_2, \ldots, w_n)\|_\psi = 1\right\}. \tag{14}
$$

Following properties hold.

Theorem 1.2.6 (cf. [28, Section 3]). *For $\|\cdot\|_\psi \in AN_n$,*

(i) $\|\cdot\|_\psi^* \in AN_n$.

(ii) *A corresponding convex function of* $\|\cdot\|_\psi^*$ *in* Ψ_n *is given by*

$$\psi^*(s_1,\ldots,s_n) = \sup_{(t_1,\ldots,t_n)\in\Delta_n} \frac{\sum_{j=1}^n t_j s_j}{\psi(t_1,\ldots,t_n)}$$

for $(s_1,\ldots,s_n)\in\Delta_n$.

(iii) $\|\cdot\|_\psi^* = \|\cdot\|_{\psi^*}$.

Proof.

(i) All properties of an absolute normalized norm are clear except the normalization.

$$\begin{aligned}
&\|(1,0,\ldots,0)\|_\psi^* \\
&\quad= \sup\big\{|y_1| : \|(y_1,\ldots,y_n)\|_\psi = 1\big\} \\
&\quad= \sup\big\{y_1 : \|(y_1,\ldots,y_n)\|_\psi = 1, y_1,\ldots,y_n \geq 0, (y_1,\ldots,y_n)\neq(0,\ldots,0)\big\} \\
&\quad= \sup\left\{\frac{s_1}{\psi(s_1,\ldots,s_n)} : (s_1,\ldots,s_n)\in\Delta_n\right\} \\
&\quad= \frac{1-0-\cdots-0}{\psi(0,\ldots,0)} \\
&\quad= 1,
\end{aligned}$$

where

$$s_1 = \frac{y_1}{y_1+\cdots+y_n}, \ldots, s_n = \frac{y_n}{y_1+\cdots+y_n}.$$

Similarly, we have the others.

(ii) For all $(s_1,\ldots,s_n)\in\Delta_n$.

$$\begin{aligned}
&\psi^*(s_1,\ldots,s_n) \\
&\quad= \|(s_1,\ldots,s_n)\|_\psi^* \\
&\quad= \sup\big\{|s_1y_1+\cdots+s_ny_n| : \|(y_1,\ldots,y_n)\|_\psi = 1\big\} \\
&\quad= \sup\big\{|s_1y_1+\cdots+s_ny_n| : \|(y_1,\ldots,y_n)\|_\psi = 1, \\
&\qquad\qquad y_1,\ldots,y_n \geq 0, (y_1,\ldots,y_n)\neq(0,\ldots,0)\big\} \\
&\quad= \psi^*(s_1,\ldots,s_n) \\
&\quad= \sup_{(t_1,\ldots,t_n)\in\Delta_n} \frac{\sum_{j=1}^n t_j s_j}{\psi(t_1,\ldots,t_n)},
\end{aligned}$$

where

$$t_1 = \frac{y_1}{y_1+\cdots+y_n}, \ldots, t_n = \frac{y_n}{y_1+\cdots+y_n}.$$

(iii) If $(x_1, \ldots, x_n) \neq (0, \ldots, 0)$, then

$$
\begin{aligned}
&\|(x_1, \ldots, x_n)\|_{\psi^*} \\
&\quad = (|x_1| + \cdots + |x_n|)\psi^* \left(\frac{|x_1|}{|x_1| + \cdots + |x_n|}, \ldots, \frac{|x_n|}{|x_1| + \cdots + |x_n|} \right) \\
&\quad = (|x_1| + \cdots + |x_n|) \sup_{(t_1, \ldots, t_n) \in \Delta_n} \frac{t_1 \frac{|x_1|}{|x_1|+\cdots+|x_n|} + \cdots + t_n \frac{|x_n|}{|x_1|+\cdots+|x_n|}}{\psi(t_1, \ldots, t_n)} \\
&\quad = \sup_{(t_1, \ldots, t_n) \in \Delta_n} \frac{t_1|x_1| + \cdots + t_n|x_n|}{\psi(t_1, \ldots, t_n)} \\
&\quad = \sup\big\{|x_1|y_1 + \cdots + |x_n|y_n : \|(y_1, \ldots, y_n)\|_\psi = 1, \\
&\qquad\qquad\qquad y_1, \ldots, y_n \geq 0, (y_1, \ldots, y_n) \neq (0, \ldots, 0)\big\} \\
&\quad = \sup\big\{|x_1 y_1 + \cdots + x_n y_n| : \|(y_1, \ldots, y_n)\|_\psi = 1\big\} \\
&\quad = \|(x_1, \ldots, x_n)\|_\psi^*,
\end{aligned}
$$

where

$$y_1 = \frac{t_1}{\psi(t_1, \ldots, t_n)}, \ldots, y_n = \frac{t_n}{\psi(t_1, \ldots, t_n)}.$$

If $(x_1, \ldots, x_n) = (0, \ldots, 0)$, it is clear. This completes the proof. □

The case of an ℓ_p-norm on $\mathbb{C}^n$, its dual norm $\|\cdot\|_p^* = \|\cdot\|_{\psi_p}^*$ is

$$\|(z_1, \ldots, z_n)\|_p^* = \begin{cases} (|z_1|^q + \cdots + |z_n|^q)^{\frac{1}{q}} & (1 < p < \infty) \\ \max\{|z_1|, \ldots, |z_n|\} & (p = 1), \end{cases} \tag{15}$$

where $1/p + 1/q = 1$. Moreover, we have the generalized Hölder inequality.

Proposition 1.2.7 (cf. [7, Exercise IV.1.14])**.** *Let $\psi \in \Psi_n$. Then we have*

$$\sum_{j=1}^n |z_j w_j| \leq \|(z_1, \ldots, z_n)\|_\psi^* \ \|(w_1, \ldots, w_n)\|_\psi \tag{16}$$

for any $(z_1, \ldots, z_n), (w_1, \ldots, w_n) \in \mathbb{C}^n$.

Proof. For (14), we have

$$\left| \sum_{j=1}^n z_j y_j \right| \leq \|(z_1, \ldots, z_n)\|_\psi^*$$

for any $(y_1, \ldots, y_n) \in \mathbb{C}^n$ such that $\|(y_1, \ldots, y_n)\|_\psi = 1$. Put $y_i = w_i / \|(w_1, \ldots, w_n)\|_\psi$ for all $i = 1, \ldots, n$, where $(w_1, \ldots, w_n) \neq (0, \ldots, 0) \in \mathbb{C}^n$. Then we have

$$\left| \sum_{j=1}^{n} z_j w_j \right| \leq \|(z_1, \ldots, z_n)\|_\psi^* \ \|(w_1, \ldots, w_n)\|_\psi.$$

Put z_i, w_i for $|z_i|, |w_i|$ respectively, we have (16). This complete the proof. □

1.2.3 The set $\widetilde{\Psi}_n$ of all continuous concave functions on Δ_n

In this subsection, we define the set $\widetilde{\Psi}_n$ of all continuous concave functions on Δ_n satisfying

$$\tilde{\psi}(1, 0, \ldots, 0) = \tilde{\psi}(0, 1, 0, \ldots, 0) = \cdots = \tilde{\psi}(0, \ldots, 0, 1) = 1$$

for $\tilde{\psi} \in \widetilde{\Psi}_n$ (cf. [32]). Just as $\psi \in \Psi_n$ (9), for any $\tilde{\psi} \in \widetilde{\Psi}_n$, we define a mapping $\|\cdot\|_{\tilde{\psi}}$ on $\mathbb{C}^n$ as

$$\|(z_1, \ldots, z_n)\|_{\tilde{\psi}} = \begin{cases} (|z_1| + \cdots + |z_n|)\tilde{\psi}\left(\frac{|z_1|}{|z_1|+\cdots+|z_n|}, \ldots, \frac{|z_n|}{|z_1|+\cdots+|z_n|}\right) & ((z_1, \ldots, z_n) \neq (0, \ldots, 0)) \\ 0 & ((z_1, \ldots, z_n) = (0, \ldots, 0)). \end{cases} \tag{17}$$

This mapping is not a norm, however we have the generalized inverse Minkowski inequality.

Proposition 1.2.8 (cf. [32, Proposition 6]). *Let $\tilde{\psi} \in \widetilde{\Psi}_n$. Then we have*

$$\|(|z_1|, \ldots, |z_n|) + (|w_1|, \ldots, |w_n|)\|_{\tilde{\psi}} \geq \|(|z_1|, \ldots, |z_n|)\|_{\tilde{\psi}} + \|(|w_1|, \ldots, |w_n|)\|_{\tilde{\psi}} \tag{18}$$

for any $(z_1, \ldots, z_n), (w_1, \ldots, w_n) \in \mathbb{C}^n$.

Proof. From the concavity of $\tilde{\psi}$, for any $(z_1, \ldots, z_n), (z_1, \ldots, y_n) \in \mathbb{C}^n$, we have

$$\begin{aligned} &\|(|z_1|, \ldots, |z_n|) + (|w_1|, \ldots, |w_n|)\|_{\tilde{\psi}} \\ &\quad = \|(|z_1| + |w_1|, \ldots, |z_n| + |w_n|)\|_{\tilde{\psi}} \\ &\quad = (|z_1| + |w_1| + \cdots + |z_n| + |w_n|) \\ &\qquad \times \tilde{\psi}\left(\frac{|z_1| + |w_1|}{|z_1| + |w_1| + \cdots + |z_n| + |w_n|}, \ldots, \frac{|z_n| + |w_n|}{|z_1| + |w_1| + \cdots + |z_n| + |w_n|}\right) \end{aligned}$$

$$
\begin{aligned}
&= (|z_1| + \cdots + |z_n| + |w_1| + \cdots + |w_n|) \\
&\quad \times \tilde{\psi}\Bigg(\frac{|z_1|}{|z_1| + \cdots + |w_n|} + \frac{|w_1|}{|z_1| + \cdots + |w_n|}, \dots \\
&\qquad\qquad \dots, \frac{|z_n|}{|z_1| + \cdots + |w_n|} + \frac{|w_n|}{|z_1| + \cdots + |w_n|}\Bigg) \\
&= (|z_1| + \cdots + |z_n| + |w_1| + \cdots + |w_n|) \\
&\quad \times \tilde{\psi}\Bigg(\bigg(\frac{|z_1|}{|z_1| + \cdots + |w_n|}, \dots, \frac{|z_n|}{|z_1| + \cdots + |w_n|}\bigg) \\
&\qquad\qquad + \bigg(\frac{|w_1|}{|z_1| + \cdots + |w_n|}, \dots, \frac{|w_n|}{|z_1| + \cdots + |w_n|}\bigg)\Bigg) \\
&= (|z_1| + \cdots + |z_n| + |w_1| + \cdots + |x_n|) \\
&\quad \times \tilde{\psi}\Bigg(\frac{|z_1| + \cdots + |z_n|}{|z_1| + \cdots + |z_n| + |w_1| + \cdots + |w_n|} \\
&\qquad\qquad \bigg(\frac{|z_1|}{|z_n| + \cdots + |z_n|}, \dots, \frac{|z_n|}{|z_1| + \cdots + |z_n|}\bigg) \\
&\qquad\qquad + \frac{|w_1| + \cdots + |w_n|}{|z_1| + \cdots + |z_n| + |w_1| + \cdots + |w_n|} \\
&\qquad\qquad\qquad \bigg(\frac{|w_1|}{|w_1| + \cdots + |w_n|}, \dots, \frac{|w_n|}{|w_1| + \cdots + |w_n|}\bigg)\Bigg) \\
&\geq (|z_1| + \cdots + |z_n|)\tilde{\psi}\left(\frac{|z_1|}{|z_1| + \cdots + |z_n|}, \dots, \frac{|z_n|}{|z_1| + \cdots + |z_n|}\right) \\
&\qquad + (|w_1| + \cdots + |w_n|)\tilde{\psi}\left(\frac{|w_1|}{|w_1| + \cdots + |w_n|}, \dots, \frac{|w_n|}{|w_1| + \cdots + |w_n|}\right) \\
&= \|(|z_1|, \dots, |z_n|)\|_{\tilde{\psi}} + \|(|w_1|, \dots, |w_n|)\|_{\tilde{\psi}}.
\end{aligned}
$$

This completes the proof. □

Especially, for all $p \in \mathbb{R}$ with $0 < p \leq 1$,

$$\tilde{\psi_p}(t_1, \dots, t_n) = (t_1^p + \cdots + t_n^p)^{\frac{1}{p}}$$

is an element of $\widetilde{\Psi}_n$ by the inverse Hölder inequality, and $\|(z_1, \dots, z_n)\|_{\tilde{\psi}_p} = (|z_1|^p + \cdots + |z_n|^p)^{\frac{1}{p}}$.

This mapping $\| \cdot \|_{\tilde{\psi}_p}$ is monotone since the following proposition holds.

Proposition 1.2.9. *For any $(p_1, \dots, p_n), (a_1, \dots, a_n) \in \mathbb{C}^n$ such that $0 \leq p_i \leq a_i$ $(i = 1, \dots, n)$, we have that*

$$\|(p_1, \dots, p_n)\|_{\tilde{\psi}} \leq \|(a_1, \dots, a_n)\|_{\tilde{\psi}}. \tag{19}$$

Proof. We first show that, if $0 \le p_1 < a_1$, then

$$\|(p_1, p_2, \ldots, p_n)\|_{\tilde{\psi}} \le \|(a_1, p_2 \ldots, p_n)\|_{\tilde{\psi}}. \tag{20}$$

This is, we show that if $0 \le p_1 < a_1$

$$(p_1 + p_2 + \cdots + p_n)\tilde{\psi}\left(\frac{p_1}{p_1 + p_2 + \cdots + p_n}, \ldots, \frac{p_n}{p_1 + p_2 + \cdots + p_n}\right)$$
$$\le (a_1 + p_2 + \cdots + p_n)\tilde{\psi}\left(\frac{a_1}{a_1 + p_2 + \cdots + p_n}, \ldots, \frac{p_n}{a_1 + p_2 + \cdots + p_n}\right).$$

Take any $(s_1, \ldots, s_n) \in \Delta_n$ such that $s_1 + \cdots + s_n = 1$, and consider the line segment

$$\left[(1, 0, \ldots, 0), \left(0, \frac{s_2}{1 - s_1}, \ldots, \frac{s_n}{1 - s_1}\right)\right]$$

in Δ_n. For any real number λ such that $1 < \lambda \le 1/(1 - s_1)$, we put

$$(s'_1, s'_2, \ldots, s'_n) = (1, 0, \ldots, 0) + \lambda\{(s_1, s_2, \ldots, s_n) - (1, 0, \ldots, 0)\}.$$

Then we have

$$(s_1, s_2, \ldots, s_n) = \frac{1}{\lambda}(s'_1, s'_2, \ldots, s'_n) + \left(1 - \frac{1}{\lambda}\right)(1, 0, \ldots, 0).$$

By the concavity of $\tilde{\psi}$,

$$\begin{aligned}
\tilde{\psi}(s_1, s_2, \ldots, s_n) &\ge \frac{1}{\lambda}\tilde{\psi}(s'_1, s'_2, \ldots, s'_n) + \left(1 - \frac{1}{\lambda}\right)\tilde{\psi}(1, 0, \ldots, 0) \\
&\ge \frac{1}{\lambda}\tilde{\psi}(s'_1, s'_2, \ldots, s'_n) \\
&= \frac{1 - s_1}{1 - s'_1}\tilde{\psi}(s'_1, s'_2, \ldots, s'_n).
\end{aligned}$$

Thus, we have

$$\frac{\tilde{\psi}(s_1, s_2, \ldots, s_n)}{1 - s_1} \ge \frac{\tilde{\psi}(s'_1, s'_2, \ldots, s'_n)}{1 - s'_1}. \tag{21}$$

Since $0 \le p_1 < a_1$, we put

$$\begin{aligned}
&(s_1, s_2, \ldots, s_n) \\
&\quad = \left(\frac{a_1}{a_1 + p_2 + \cdots + p_n}, \frac{p_2}{a_1 + p_2 + \cdots + p_n}, \ldots, \frac{p_n}{a_1 + p_2 + \cdots + p_n}\right), \\
&(s'_1, s'_2, \ldots, s'_n) \\
&\quad = \left(\frac{p_1}{p_1 + p_2 + \cdots + p_n}, \frac{p_2}{p_1 + p_2 + \cdots + p_n}, \ldots, \frac{p_n}{p_1 + p_2 + \cdots + p_n}\right), \\
&\lambda = \frac{a_1 + p_2 + \cdots + p_n}{p_1 + p_2 + \cdots + p_n} > 1
\end{aligned}$$

respectively in (21). Thus, we have

$$\frac{\tilde{\psi}\left(\frac{a_1}{a_1+p_2+\cdots+p_n}, \frac{p_2}{a_1+p_2+\cdots+p_n}, \ldots, \frac{p_n}{a_1+p_2+\cdots+p_n}\right)}{1-\frac{a_1}{a_1+p_2+\cdots+p_n}}$$
$$\geq \frac{\tilde{\psi}\left(\frac{p_1}{p_1+p_2+\cdots+p_n}, \frac{p_2}{p_1+p_2+\cdots+p_n}, \ldots, \frac{p_n}{p_1+p_2+\cdots+p_n}\right)}{1-\frac{p_1}{p_1+p_2+\cdots+p_n}}.$$

This implies (20). Similarly, we can show that for $2 \leq i \leq n$,

$$\|(a_1, \ldots, a_{i-1}, p_i, p_{i+1}, \ldots, p_n)\|_{\tilde{\psi}} \leq \|(a_1, \ldots, a_{i-1}, a_i, p_{i+1}, \ldots, p_n)\|_{\tilde{\psi}}.$$

Therefor, we have (19). This completes the proof. □

Let $\tilde{\psi} \in \widetilde{\Psi}_n$. Denote

$$\tilde{\psi}_*(s_1, \ldots, s_n) = \inf_{(t_1,\ldots,t_n)\in\Delta_n} \frac{\sum_{j=1}^n t_j s_j}{\tilde{\psi}(t_1, \ldots, t_n)}. \tag{22}$$

The corresponding map $\|\cdot\|_{\tilde{\psi}_*}$ is defined by (17). Then the following generalized inverse Hölder inequality holds.

Proposition 1.2.10 (cf. [32, Proposition 9])**.** *Let $\tilde{\psi} \in \widetilde{\Psi}_n$. Then we have*

$$\sum_{j=1}^n |z_j w_j| \geq \|(z_1, \ldots, z_n)\|_{\tilde{\psi}_*} \|(w_1, \ldots, w_n)\|_{\tilde{\psi}} \tag{23}$$

for any $(z_1, \ldots, z_n), (w_1, \ldots, w_n) \in \mathbb{C}^n$.

Proof. By (22), we get that

$$\sum_{j=1}^n s_j t_j \geq \tilde{\psi}_*(s_1, \ldots, s_n)\ \tilde{\psi}(t_1, \ldots, t_n).$$

Put

$$s_j = \frac{|z_j|}{|z_1| + \cdots + |z_n|},\ t_j = \frac{|w_j|}{|w_1| + \cdots + |w_n|}$$

for $j = 1, \ldots, n$, where $(z_1, \ldots, z_n), (w_1, \ldots, w_n) \in \mathbb{C}^n$, then we have (23). This completes the proof. □

1.3 Characterizations of generalized triangle inequalities

In this section, by using ψ-dirct sum of Banach space, we show generalizations of a following obvious fact: let X be a normed space and $a_1, a_2 \in \mathbb{C}$, then

$$\|a_1x_1 + a_2x_2\| \leq \|x_1\| + \|x_2\| \quad (x_1, x_2 \in X)$$

if and only if

$$\max\{|a_1|, |a_2|\} \leq 1.$$

1.3.1 Generalizations of preceding results

Let $(X, \|\cdot\|)$ be a normed space. For any $n \in \mathbb{N}$ with $n \geq 2$ and $\psi \in \Psi_n$, we can consider a product space X^n with the norm

$$\|(x_1, x_2, \ldots, x_n)\|_\psi = \|(\|x_1\|, \|x_2\|, \ldots, \|x_n\|)\|_\psi \quad (x_1, \ldots, x_n \in X). \tag{24}$$

We call $(X^n, \|\cdot\|_\psi)$ ψ-direct sum of X (cf. [18]), and denote it by $\ell_\psi^n(X)$. We first prove the following result.

Theorem 1.3.1. *Let X be a normed space and $\psi \in \Psi_n$. Let $(a_1, \ldots, a_n) \in \mathbb{C}^n$. Then*

$$\left\|\sum_{j=1}^n a_jx_j\right\| \leq \|(x_1, \ldots, x_n)\|_\psi \quad (x_1, \ldots, x_n \in X) \tag{25}$$

if and only if

$$\|(a_1, \ldots, a_n)\|_{\psi^*} \leq 1.$$

Proof. Assume that $\|(a_1, \ldots, a_n)\|_{\psi^*} \leq 1$. Let $x_1, \ldots, x_n \in X$. From the generalized Hölder inequality (16), we have

$$\begin{aligned}
\left\|\sum_{j=1}^n a_jx_j\right\| &\leq \sum_{j=1}^n |a_j|\|x_j\| \\
&\leq \|(|a_1|, \ldots, |a_n|)\|_{\psi^*} \ \|(\|x_1\|, \ldots, \|x_n\|)\|_\psi \\
&= \|(a_1, \ldots, a_n)\|_{\psi^*} \ \|(x_1, \ldots, x_n)\|_\psi \\
&\leq \|(x_1, \ldots, x_n)\|_\psi.
\end{aligned}$$

Hence $(a_1, \ldots, a_n)$ satisfies (25).

Conversely, assume that $(a_1, \ldots, a_n)$ satisfies (25). Take any $e \in X$ with $\|e\| = 1$, and any $(\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n$ such that $\|(\alpha_1, \ldots, \alpha_n)\|_\psi = 1$, we have

$$\begin{aligned}\left|\sum_{j=1}^n a_j \alpha_j\right| &= \left\|\sum_{j=1}^n a_j \alpha_j e\right\| \\ &\leq \|(\alpha_1 e, \ldots, \alpha_n e)\|_\psi \\ &= \|(\|\alpha_1 e\|, \ldots, \|\alpha_n e\|)\|_\psi \\ &= \|(|\alpha_1|, \ldots, |\alpha_n|)\|_\psi \\ &= \|(\alpha_1, \ldots, \alpha_n)\|_\psi \\ &= 1.\end{aligned}$$

Thus

$$\sup\left\{\left|\sum_{j=1}^n a_j \alpha_j\right| : \|(\alpha_1, \ldots, \alpha_n)\|_\psi = 1\right\} \leq 1,$$

and so we have $\|(a_1, \ldots, a_n)\|_{\psi^*} \leq 1$. This completes the proof. □

In the above theorem, $(a_1, \ldots, a_n)$ is an element in the unit ball of $(\ell_\psi^n)^* = \ell_{\psi^*}^n$, where $(\ell_\psi^n)^*$ is a dual space of ℓ_ψ^n.

From Theorem 1.3.1, we have the following corollary by putting $\psi = \psi_p$ and using (15).

Corollary 1.3.2. *Let X be a normed space and $p \in \mathbb{R}$ with $p > 1$. Let $(a_1, \ldots, a_n) \in \mathbb{C}^n$. Then*

$$\left\|\sum_{j=1}^n a_j x_j\right\|^p \leq \|x_1\|^p + \cdots + \|x_n\|^p \quad (x_1, \ldots, x_n \in X)$$

if and only if

$$(|a_1|^q + \cdots + |a_n|^q)^{\frac{1}{q}} \leq 1,$$

where $1/p + 1/q = 1$.

For $\tilde{\psi} \in \widetilde{\Psi}_n$, we define $\|\cdot\|_{\tilde{\psi}}$ as in (24), and consider $(X^n, \|\cdot\|_{\tilde{\psi}})$ which is denoted by $\ell_{\tilde{\psi}}^n(X)$. Note that $\ell_{\tilde{\psi}}^n(X)$ is not a normed space. In this case, we have the following result.

Theorem 1.3.3. *Let X be a normed space and $\tilde{\psi} \in \widetilde{\Psi}_n$. Let $(a_1, \ldots, a_n) \in \mathbb{C}^n$. Then*

$$\left\|\sum_{j=1}^n a_j x_j\right\| \leq \|(x_1, \ldots, x_n)\|_{\tilde{\psi}} \quad (x_1, \ldots, x_n \in X) \tag{26}$$

if and only if

$$\max\{|a_1|,\ldots,|a_n|\} \le 1.$$

Proof. Assume that $\max\{|a_1|,\ldots,|a_n|\} \le 1$. Let $x_1,\ldots,x_n \in X$. From the generalized inverse Minkowski inequality (18), we have

$$\begin{aligned}\left\|\sum_{j=1}^n a_j x_j\right\| &\le \sum_{j=1}^n |a_j|\|x_j\| \\ &\le \sum_{j=1}^n \|x_j\| \\ &= \|(\|x_1\|,0,\ldots,0)\|_{\tilde\psi} + \cdots + \|(0,\ldots,0,\|x_n\|)\|_{\tilde\psi} \\ &\le \|(\|x_1\|,0,\ldots,0) + \cdots + (0,\ldots,0,\|x_n\|)\|_{\tilde\psi} \\ &= \|(\|x_1\|,\ldots,\|x_n\|)\|_{\tilde\psi} \\ &= \|(x_1,\ldots,x_n)\|_{\tilde\psi}.\end{aligned}$$

Hence (26) holds.

Conversely, assume that $(a_1,\ldots,a_n)$ satisfies (26). Take any $e \in X$ with $\|e\| = 1$. From (26), $|a_i| = \|a_i e\| \le \|(0,\ldots,0,\overset{(i)}{e},0,\cdots 0)\|_{\tilde\psi} = 1$ for all $i = 1,\ldots,n$. So we have $\max\{|a_1|,\ldots,|a_n|\} \le 1$. This completes the proof. □

From Theorem 1.3.3, we have the following corollary by putting $\tilde\psi = \tilde\psi_p$, where $0 < p \le 1$.

Corollary 1.3.4. *Let X be a normed space and $p \in \mathbb{R}$ with $0 < p \le 1$. Let $(a_1,\ldots,a_n) \in \mathbb{C}^n$. Then*

$$\left\|\sum_{j=1}^n a_j x_j\right\|^p \le \|x_1\|^p + \cdots + \|x_n\|^p \quad (x_1,\ldots,x_n \in X)$$

if and only if

$$\max\{|a_1|,\ldots,|a_n|\} \le 1.$$

1.3.2 Applications to inequalities involved with the Euler-Lagrange type identity

In this subsection, we give another proof of characterizations of $(\mu_1,\ldots,\mu_n) \in \mathbb{R}^n$ which satisfies

$$\|x_1 + \cdots + x_n\|^p \le \frac{\|x_1\|^p}{\mu_1} + \cdots + \frac{\|x_n\|^p}{\mu_n} \quad (x_1,\ldots,x_n \in X), \tag{27}$$

where $p \in \mathbb{R}$ with $p > 0$, proved by [9, Theorems 2.4 and 2.5]. We put

$$F(p) = \left\{ (\mu_1, \ldots, \mu_n) \in \mathbb{R}^n : \left\| \sum_{j=1}^{n} x_j \right\|^p \le \sum_{j=1}^{n} \frac{\|x_j\|^p}{\mu_j} \ \ (x_1, \ldots, x_n \in X) \right\},$$

and also for each $k = 0, 1, \ldots, n$, we place in correspondence $F(p\ ;\ k)$ as the subset of $F(p)$ consisting of all n-tuples $(\mu_1, \ldots, \mu_n) \in \mathbb{R}^n$ for which inequality (27) holds and exactly k numbers of $\mu_1, \ldots, \mu_n$ are negative. We note that

$$F(p) = \bigcup_{k=0}^{n} F(p\ ;\ k).$$

Theorem 1.3.5 (cf. [9, Theorems 2.4 and 2.5]). *Let X be a normed space and $p \in \mathbb{R}$ with $p > 0$. Then the following assertions hold*:

(i) $F(p\ ;\ 0)$

$$= \begin{cases} \left\{ (\mu_1, \ldots, \mu_n) \in \mathbb{R}^n : \mu_1, \ldots, \mu_n > 0,\ \sum_{j=1}^{n} \mu_j^{\frac{1}{p-1}} \le 1 \right\} & (p > 1) \\ (0, 1] \times \cdots \times (0, 1] & (0 < p \le 1); \end{cases}$$

(ii) $F(p\ ;\ k) = \varnothing$ *for all* $k = 1, \ldots, n$;

(iii) $F(p) = F(p\ ;\ 0)$.

Proof. (i) If $(\mu_1, \ldots, \mu_n)$ satisfies (27), then

$$\begin{aligned} \left\| \mu_1^{\frac{1}{p}} x_1 + \cdots + \mu_n^{\frac{1}{p}} x_n \right\|^p &\le \frac{\|\mu_1^{\frac{1}{p}} x_1\|^p}{\mu_1} + \cdots + \frac{\|\mu_n^{\frac{1}{p}} x_n\|^p}{\mu_n} \\ &= \|x_1\|^p + \cdots + \|x_n\|^p \quad (x_1, \ldots, x_n \in X). \end{aligned} \tag{28}$$

In the case where $p > 1$, from Corollary 1.3.2, we have

$$\begin{aligned} (28) \ &\Leftrightarrow\ \left(\left| \mu_1^{\frac{1}{p}} \right|^q + \cdots + \left| \mu_n^{\frac{1}{p}} \right|^q \right)^{\frac{1}{q}} \le 1 \\ &\Leftrightarrow\ \sum_{j=1}^{n} \mu_j^{\frac{1}{p-1}} \le 1, \end{aligned}$$

where $1/p + 1/q = 1$. In the case where $0 < p \le 1$, from Corollary 1.3.4, we have

$$\begin{aligned} (28) \ &\Leftrightarrow\ \max\left(\left| \mu_1^{\frac{1}{p}} \right|^p, \ldots, \left| \mu_n^{\frac{1}{p}} \right|^p \right) \le 1 \\ &\Leftrightarrow\ (\mu_1, \ldots, \mu_n) \in (0, 1] \times \cdots \times (0, 1]. \end{aligned}$$

(ii) Suppose $\mu_i < 0$. In (27), take any $e \in X$ with $\|e\| = 1$, and put $x_j = \delta_{ij} e$ for all $j = 1, \dots, n$, where δ_{ij} is a Kronecker's delta. Hence $1 \leq 1/\mu_i$. This is a contradiction, and so $F(p\ ;\ k) = \varnothing$.

(iii) is clear. This completes the proof. □

Similarly, we give another proof of characterizations of $(\mu_1, \dots, \mu_n) \in \mathbb{R}^n$ which holds a reverse inequality of (27):

$$\|x_1 + \cdots + x_n\|^p \geq \frac{\|x_1\|^p}{\mu_1} + \cdots + \frac{\|x_n\|^p}{\mu_n} \quad (x_1, \dots, x_n \in X)\,, \tag{29}$$

where $p \in \mathbb{R}$ with $p > 0$, proved by [9, Theorems 2.6 and 2.7]. We put

$$G(p) = \left\{ (\mu_1, \dots, \mu_n) \in \mathbb{R}^n : \left\| \sum_{j=1}^n x_j \right\|^p \geq \sum_{j=1}^n \frac{\|x_j\|^p}{\mu_j} \quad (x_1, \dots, x_n \in X) \right\},$$

and also for each $k = 0, 1, \dots, n$, we place in correspondence $G(p\ ;\ k)$ as the subset of $G(p)$ consisting of all n-tuples $(\mu_1, \dots, \mu_n) \in \mathbb{R}^n$ for which inequality (29) holds and exactly k numbers of $\mu_1, \dots, \mu_n$ are negative. We note that

$$G(p) = \bigcup_{k=0}^n G(p\ ;\ k).$$

Theorem 1.3.6 (cf. [9, Theorems 2.6 and 2.7]). *Let X is a normed space and $p \in \mathbb{R}$ with $p > 0$. Then the following assertions hold*:

(i) $G(p\ ;\ k) = \varnothing$ *for all* $k = 0, 1, \dots, n-2$;

(ii) $G(p\ ;\ n-1)$

$$= \begin{cases} \left\{ (\mu_1, \dots, \mu_n) \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \mu_i > 0,\ \mu_j < 0\ (j \neq i) \\ \mu_i^{\frac{1}{p-1}} \geq 1 + \sum_{j=1,\, j \neq i}^n |\mu_j|^{\frac{1}{p-1}} \\ \textit{for some } i \in \{1, \dots, n\} \end{array} \right\} & (p > 1) \\ \left\{ (\mu_1, \dots, \mu_n) \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \mu_i > 0,\ \mu_j < 0\ (j \neq i) \\ \mu_i \geq \max_{j \in \{1, \dots, n\} \setminus \{i\}} (1, |\mu_j|) \\ \textit{for some } i \in \{1, \dots, n\} \end{array} \right\} & (0 < p \leq 1); \end{cases}$$

(iii) $G(p\ ;\ n) = (-\infty, 0) \times \cdots \times (-\infty, 0)$;

(iv) $G(p) = G(p\ ;\ n-1) \cup G(p\ ;\ n)$.

Proof. (i) In (29), put $x_j = 0$ for all $j \in \{1, \ldots, n\}$ such that $\mu_j < 0$. Next, for all $i \in \{1, \ldots, n\}$ such that $\mu_i > 0$, select x_i such that $x_i \neq 0$ and $x_1 + \cdots + x_n = 0$. Hence the left-hand side of (29) vanishes, and the right-hand side of (29) is strictly positive. This is a contradiction, and so $G(p\ ;\ k) = \varnothing$.

(ii) If $(\mu_1, \ldots, \mu_n)$ satisfies (29), there exists $i \in \{1, \ldots, n\}$ such that $\mu_i > 0$. Thus (29) is equivalent to

$$\sum_{\substack{j=1\\ j\neq i}}^{n} \frac{\|x_j\|^p}{-\frac{\mu_j}{\mu_i}} + \frac{\|x_1 + \cdots + x_n\|^p}{\frac{1}{\mu_i}} \geq \|x_i\|^p \quad (x_1, \ldots, x_n \in X). \tag{30}$$

In (30), put $y_j = -x_j$ for all $j \in \{1, \ldots, n\} \setminus \{i\}$. Next, put $y_i = x_1 + \cdots + x_n$. Hence it is equivalent to

$$\sum_{\substack{j=1\\ j\neq i}}^{n} \frac{\|y_j\|^p}{-\frac{\mu_j}{\mu_i}} + \frac{\|y_i\|^p}{\frac{1}{\mu_i}} \geq \|y_1 + \cdots + y_n\|^p \quad (y_1, \ldots, y_n \in X). \tag{31}$$

From Theorem 1.3.5 (i), in the case where $p > 1$, we have

$$\begin{aligned}(31) \quad &\Leftrightarrow \quad \sum_{\substack{j=1\\ j\neq i}}^{n} \left|-\frac{\mu_j}{\mu_i}\right|^{\frac{1}{p-1}} + \left|\frac{1}{\mu_i}\right|^{\frac{1}{p-1}} \leq 1 \\ &\Leftrightarrow \quad \mu_i^{\frac{1}{p-1}} \geq 1 + \sum_{\substack{j=1\\ j\neq i}}^{n} |\mu_j|^{\frac{1}{p-1}},\end{aligned}$$

and in the case $0 < p \leq 1$, we have

$$\begin{aligned}(31) \quad &\Leftrightarrow \quad \frac{1}{\mu_i} \in (0,1] \text{ and } -\frac{\mu_j}{\mu_i} \in (0,1]\ (j \in \{1, \ldots, n\} \setminus \{i\}) \\ &\Leftrightarrow \quad \mu_i \geq \max_{j \in \{1,\ldots,n\}\setminus\{i\}} (1, |\mu_j|).\end{aligned}$$

(iii) and (iv) are clear. This completes the proof. □

At the end of this subsection, we give another proof of characterizations of $(a_1, \ldots, a_n, \lambda, \mu_1, \ldots, \mu_n) \in \mathbb{C}^n \times \mathbb{R} \times \mathbb{R}^n$ which holds

$$\frac{\|a_1 x_1 + \cdots + a_n x_n\|^p}{\lambda} \leq \frac{\|x_1\|^p}{\mu_1} + \cdots + \frac{\|x_n\|^p}{\mu_n} \quad (x_1, \ldots, x_n \in X),$$

where $p \in \mathbb{R}$ with $p > 0$.

Theorem 1.3.7 (cf. [2, Theorems 2]). *Let X be a normed space and $p, q \in \mathbb{R}$ with $p > 1, 1/p + 1/q = 1$. Let $(a_1, \ldots, a_n, \lambda, \mu_1, \ldots, \mu_n) \in \mathbb{C}^n \times \mathbb{R} \times \mathbb{R}^n$. Then*

$$\frac{\|a_1x_1 + \cdots + a_nx_n\|^p}{\lambda} \leq \frac{\|x_1\|^p}{\mu_1} + \cdots + \frac{\|x_n\|^p}{\mu_n} \quad (x_1, \ldots, x_n \in X) \tag{32}$$

holds if:

(i) $\lambda > 0, \mu_j > 0$ *for all* $j \in \{1, \ldots, n\}$ *and*

$$\lambda^{\frac{1}{p-1}} \geq \sum_{j=1}^{n} \mu_j^{\frac{1}{p-1}} |a_j|^q,$$

(ii) $\lambda < 0, \mu_i < 0$ *for some* $i \in \{1, \ldots, n\}$, $\mu_j > 0$ *for all* $j \in \{1, \ldots, n\} \setminus \{i\}$ *and*

$$(-\mu_i)^{\frac{1}{p-1}} |a_i|^q \geq (-\lambda)^{\frac{1}{p-1}} + \sum_{\substack{j=1 \\ j \neq i}}^{n} \mu_j^{\frac{1}{p-1}} |a_j|^q,$$

(iii) $\lambda < 0, \mu_j > 0$ *for all* $j \in \{1, \ldots, n\}$ *and any* $a_1, \ldots, a_n \in \mathbb{C}$.

Proof. (i) Remark that

$$\lambda^{\frac{1}{p-1}} \geq \mu_1^{\frac{1}{p-1}} |a_1|^q + \cdots + \mu_n^{\frac{1}{p-1}} |a_n|^q$$

$$\Leftrightarrow \left(\left| a_1 \left(\frac{\mu_1}{\lambda} \right)^{\frac{1}{p}} \right|^q + \cdots + \left| a_n \left(\frac{\mu_n}{\lambda} \right)^{\frac{1}{p}} \right|^q \right)^{\frac{1}{q}} \leq 1. \tag{33}$$

From Corollary 1.3.2, we have

$$(33) \quad \Leftrightarrow \quad \left\| \sum_{j=1}^{n} a_j \left(\frac{\mu_j}{\lambda} \right)^{\frac{1}{p}} \mu_j^{-\frac{1}{p}} x_j \right\|^p \leq \sum_{j=1}^{n} \left\| \mu_j^{-\frac{1}{p}} x_j \right\|^p \quad (x_1, \ldots, x_n \in X),$$

hence we have (32).

(ii) The case where $a_i \neq 0$, remark that

$$(-\mu_i)^{\frac{1}{p-1}} |a_i|^q \geq (-\lambda)^{\frac{1}{p-1}} + \sum_{\substack{j=1 \\ j \neq i}}^{n} \mu_j^{\frac{1}{p-1}} |a_j|^q$$

$$\Leftrightarrow (-\mu_i)^{\frac{1}{p-1}} \geq (-\lambda)^{\frac{1}{p-1}} \left| \frac{1}{a_i} \right|^q + \sum_{\substack{j=1 \\ j \neq i}}^{n} \mu_j^{\frac{1}{p-1}} \left| \frac{a_j}{a_i} \right|^q. \tag{34}$$

From (i), we have

$$(34) \Leftrightarrow \frac{\left\| \frac{1}{a_i}x_i + \sum_{\substack{j=1\\ j\neq i}}^n \frac{a_j}{a_i}x_j \right\|^p}{-\mu_i} \le \frac{\|x_i\|^p}{-\lambda} + \sum_{\substack{j=1\\ j\neq i}}^n \frac{\|x_j\|^p}{\mu_j} \quad (x_1, \ldots, x_n \in X). \qquad (35)$$

In (35), put $y_j = -x_j$ for any $j = 2, 3, \ldots, n$. Next, put $y_1 = \frac{1}{a_1}x_1 + \sum_{\substack{j=1\\ j\neq i}}^n \frac{a_j}{a_i}x_j$. Hence we have (32).

The case where $a_i = 0$, on one hand there exist no $(a_1, \ldots, a_n, \lambda, \mu_1, \ldots, \mu_n)$ such that

$$0 \ge (-\lambda)^{\frac{1}{p-1}} + \sum_{\substack{j=1\\ j\neq i}}^n \mu_j^{\frac{1}{p-1}} |a_j|^q.$$

On the other hand, in (32), put $x_j = 0$ for $j \in \{1, \ldots, n\} \setminus \{i\}$, then we have $0 \le \|x_i\|^p/\mu_i$ $(x_i \in X)$, however there exist no $\mu_i > 0$ such that this inequality holds. That is not a contradiction.

(iii) is clear. This completes the proof. □

By using Corollary 1.3.4, we have the following theorem:

Theorem 1.3.8. *Let X be a normed space and $p \in \mathbb{R}$ with $0 < p \le 1$. Let $(a_1, \ldots, a_n, \lambda, \mu_1, \ldots, \mu_n) \in \mathbb{C}^n \times \mathbb{R} \times \mathbb{R}^n$. Then*

$$\frac{\|a_1x_1 + \cdots + a_nx_n\|^p}{\lambda} \le \frac{\|x_1\|^p}{\mu_1} + \cdots + \frac{\|x_n\|^p}{\mu_n} \quad (x_1, \ldots, x_n \in X)$$

holds if:

(i) $\lambda > 0, \mu_j > 0$ *for all* $j \in \{1, \ldots, n\}$ *and*

$$\lambda \ge \max\{\mu_1|a_1|^p, \ldots, \mu_n|a_1|^p\},$$

(ii) $\lambda < 0, \mu_i < 0$ *for some* $i \in \{1, \ldots, n\}$, $\mu_j > 0$ *for all* $j \in \{1, \ldots, n\} \setminus \{i\}$ *and*

$$-\mu_i|a_i|^p \ge \max_{j\in\{1,\ldots,n\}\setminus\{i\}} \{\mu_j|a_j|^p, -\lambda\},$$

(iii) $\lambda < 0, \mu_j > 0$ *for all* $j \in \{1, \ldots, n\}$ *and any* $a_1, \ldots, a_n \in \mathbb{C}$.

By using these theorems, we have characterizations of $(a_1, \dots, a_n, \lambda, \mu_1, \dots, \mu_n) \in \mathbb{C}^n \times \mathbb{R} \times \mathbb{R}^n$ which holds

$$\frac{\|a_1x_1 + \cdots + a_nx_n\|^p}{\lambda} \geq \frac{\|x_1\|^p}{\mu_1} + \cdots + \frac{\|x_n\|^p}{\mu_n} \qquad (x_1, \dots, x_n \in X),$$

where $p \in \mathbb{R}$ with $p > 0$.

Theorem 1.3.9. *Let X be a normed space and $p, q \in \mathbb{R}$ with $p > 1, 1/p + 1/q = 1$. Let $(a_1, \dots, a_n, \lambda, \mu_1, \dots, \mu_n) \in \mathbb{C}^n \times \mathbb{R} \times \mathbb{R}^n$. Then*

$$\frac{\|a_1x_1 + \cdots + a_nx_n\|^p}{\lambda} \geq \frac{\|x_1\|^p}{\mu_1} + \cdots + \frac{\|x_n\|^p}{\mu_n} \qquad (x_1, \dots, x_n \in X)$$

holds if:

(i) $\lambda < 0, \mu_j < 0$ *for all* $j \in \{1, \dots, n\}$ *and*

$$(-\lambda)^{\frac{1}{p-1}} \geq \sum_{j=1}^{n} (-\mu_j)^{\frac{1}{p-1}} |a_j|^q,$$

(ii) $\lambda > 0, \mu_i > 0$ *for some* $i \in \{1, \dots, n\}$, $\mu_j < 0$ *for all* $j \in \{1, \dots, n\} \setminus \{i\}$ *and*

$$\mu_i^{\frac{1}{p-1}} |a_i|^q \geq \lambda^{\frac{1}{p-1}} + \sum_{\substack{j=1 \\ j \neq i}}^{n} (-\mu_j)^{\frac{1}{p-1}} |a_j|^q,$$

(iii) $\lambda > 0, \mu_j < 0$ *for all* $j \in \{1, \dots, n\}$ *and any* $a_1, \dots, a_n \in \mathbb{C}$.

Theorem 1.3.10. *Let X be a normed space and $p \in \mathbb{R}$ with $0 < p \leq 1$. Let $(a_1, \dots, a_n, \lambda, \mu_1, \dots, \mu_n) \in \mathbb{C}^n \times \mathbb{R} \times \mathbb{R}^n$. Then*

$$\frac{\|a_1x_1 + \cdots + a_nx_n\|^p}{\lambda} \geq \frac{\|x_1\|^p}{\mu_1} + \cdots + \frac{\|x_n\|^p}{\mu_n} \qquad (x_1, \dots, x_n \in X)$$

holds if:

(i) $-\lambda < 0, \mu_j < 0$ *for all* $j \in \{1, \dots, n\}$ *and*

$$\lambda \geq \max\{-\mu_1|a_1|^p, \dots, -\mu_n|a_1|^p\},$$

(ii) $\lambda > 0, \mu_i > 0$ *for some* $i \in \{1, \dots, n\}$, $\mu_j < 0$ *for all* $j \in \{1, \dots, n\} \setminus \{i\}$ *and*

$$\mu_i|a_i|^p \geq \max_{j \in \{1, \dots, n\} \setminus \{i\}} \{-\mu_j|a_j|^p, \lambda\},$$

(iii) $\lambda > 0, \mu_j < 0$ *for all* $j \in \{1, \dots, n\}$ *and any* $a_1, \dots, a_n \in \mathbb{C}$.

1.3.3 An application to the parallelogram inequality

In this subsection, we consider another aspect of classical triangle inequality $\|x + y\| \le \|x\| + \|y\|$. For a Hilbert space H, we recall the parallelogram law

$$\|x+y\|^2 + \|x-y\|^2 = 2\big(\|x\|^2 + \|y\|^2\big) \quad (x, y \in H).$$

This implies that the parallelogram inequality

$$\|x+y\|^2 \le 2\big(\|x\|^2 + \|y\|^2\big) \quad (x, y \in H).$$

Saitoh in [38] noted the parallelogram inequality may be more suitable than the classical triangle inequality. Motivated by this, Belbachir et al. introduced the notion of q-norm: let X be a vector space over $\mathbb{K}(= \mathbb{R}$ or $\mathbb{C})$. For $1 \le q < \infty$, a mapping $\|\cdot\|$ from X into $\mathbb{R}^+(= \{a \in \mathbb{R} : a \ge 0\})$ is called a q-norm if it satisfies the following conditions:

(i) $\|x\| = 0 \Leftrightarrow x = 0$,

(ii) $\|\alpha x\| = |\alpha|\|x\| \quad (x \in X, \alpha \in \mathbb{K})$,

(iii) $\|x+y\|^q \le 2^{q-1}\big(\|x\|^q + \|y\|^q\big) \quad (x, y \in X)$.

They proved that for all q with $1 \le q < \infty$, every norm is a q-norm, and conversely, every q-norm is a norm in the usual sense (cf. [6]).

Saito et al. in [36] generalized the notion of q-norm, that is, the notion of ψ-norm by considering function $\psi \in \Psi_2$: let X be a vector space over $\mathbb{K}(= \mathbb{R}$ or $\mathbb{C})$ and $\psi \in \Psi_2$. A mapping $\|\cdot\|$ from X into $\mathbb{R}^+(= \{a \in \mathbb{R} : a \ge 0\})$ is called ψ-norm if it satisfies the following conditions:

(i) $\|x\| = 0 \Leftrightarrow x = 0$,

(ii) $\|\alpha x\| = |\alpha|\|x\| \quad (x \in X, \alpha \in \mathbb{K})$,

(iii) $\|x+y\| \le \frac{1}{\min_{0\le t\le 1}\psi(t)}\big\|(\|x\|, \|y\|)\big\|_\psi \quad (x, y \in X)$.

Note that the function $\psi(t) = \psi_q(t) = \{(1-t)^q + t^q\}^{1/q}$ take the minimum at $t = 1/2$ and

$$\psi_q\left(\frac{1}{2}\right) = \left\{\left(\frac{1}{2}\right)^q + \left(\frac{1}{2}\right)^q\right\}^{\frac{1}{q}} = 2^{\frac{1}{q}-1},$$

then the condition (iii) of a ψ-norm implies

$$\|x+y\| \leq \frac{1}{\psi_q(\frac{1}{2})}\big\|(\|x\|,\|y\|)\big\|_{\psi_q} = 2^{1-\frac{1}{q}}\big(\|x\|^q+\|y\|^q\big).$$

Then we have $\|x+y\|^q \leq 2^{q-1}(\|x\|^q+\|y\|^q)$ and so $\|\cdot\|$ becomes a q-norm. They proved that for all $\psi \in \Psi_2$, every norm is a ψ-norm, and conversely, every ψ-norm is a norm in the usual sense (cf. [36]).

By using Theorem 1.3.1, we can proof that the constant $1/\min_{0\leq t\leq 1}\psi(t)$ of the condition (iii) in a ψ-norm is the best constant of the condition.

Theorem 1.3.11. *Let X be a normed space and $\psi \in \Psi_2$. If there exist a positive constant C satisfying*

$$\|x+y\| \leq C\big\|(\|x\|,\|y\|)\big\|_{\psi} \quad (x,y\in X). \tag{36}$$

Then

$$C \geq \frac{1}{\min_{0\leq t\leq 1}\psi(t)}.$$

Proof. Remark that

$$(36) \Leftrightarrow \left\|\frac{1}{C}x+\frac{1}{C}y\right\| \leq \big\|(\|x\|,\|y\|)\big\|_{\psi} \quad (x,y\in X). \tag{37}$$

From Theorem 1.3.1, we have

$$(37) \Leftrightarrow \left\|\left(\frac{1}{C},\frac{1}{C}\right)\right\|_{\psi^*} \leq 1.$$

Then

$$C \geq \|(1,1)\|_{\psi^*} = 2\psi^*\left(\frac{1}{2}\right) = \sup_{0\leq t\leq 1}\frac{1}{\psi(t)} = \frac{1}{\min_{0\leq t\leq 1}\psi(t)}.$$

This completes the proof. □

1.4 A generalization of ℓ_p-spaces

In this section, our aim is to present generalizations of Theorem 1.3.1 and Theorem 1.3.3 for infinite sequences $\{x_n\}_{n=1}^{\infty} \subset X$ by using generalized ℓ_p-spaces.

1.4.1 ℓ_ψ-spaces

In this subsection, we summarize basic results of ℓ_ψ-spaces which is a generalization of ℓ_p-spaces by [25].

Let ℓ_0 denote the set of all infinite sequences of complex numbers with only finitely many non-zero elements. A norm $\|\cdot\|$ on ℓ_0 is called absolute if $\|\{z_n\}_{n=1}^\infty\| = \|\{|z_n|\}_{n=1}^\infty\|$ for all $\{z_n\}_{n=1}^\infty \in \ell_0$, and normalized if $\|e_n\| = 1$ for all $n = 1, 2, \ldots$, where $e_n = (0, \ldots, 0, \overset{(n)}{1}, 0, \ldots) \in \ell_0$. We remark that every absolute normalized norm is monotone: if $|z_i| \leq |w_i|$ for every $i = 1, 2, \cdots$, then $\|\{z_n\}_{n=1}^\infty\| \leq \|\{w_n\}_{n=1}^\infty\|$, where $\{z_n\}_{n=1}^\infty, \{w_n\}_{n=1}^\infty \in \ell_0$.

Let AN_∞ is the family of all absolute normalized norms on ℓ_0, and put

$$\Delta_\infty = \left\{ t = \{t_n\}_{n=1}^\infty \in \ell_0 : t_n \geq 0, \sum_{n=1}^\infty t_n = 1 \right\}.$$

For every $\|\cdot\| \in AN_\infty$, we define the function on Δ_∞ such that

$$\psi(t) = \|t\| \quad (t = \{t_n\}_{n=1}^\infty \in \Delta_\infty), \tag{38}$$

then ψ is a continuous convex function on Δ_∞ satisfying the following conditions:

$$\psi(e_n) = 1 \tag{A$'_0$}$$

$$\psi(t) \geq (1 - t_n)\psi\left(\frac{t_1}{1-t_n}, \ldots, \frac{t_{n-1}}{1-t_n}, 0, \frac{t_{n+1}}{1-t_n}, \ldots\right) \tag{A$'_\mathrm{n}$}$$

for all $n = 1, 2, \ldots$ and every $t = \{t_n\}_{n=1}^\infty \in \Delta_\infty$ with $t_n \neq 1$, where $e_n = (0, \ldots, 0, \overset{(n)}{1}, 0, \ldots) \in \ell_0$.

Conversely, we define the set Ψ_∞ of all continuous convex functions on Δ_∞ satisfying the conditions (A$'_\mathrm{n}$) for all $n = 0, 1, 2, \ldots$. For any $\psi \in \Psi_\infty$, we define the mapping on ℓ_0:

$$\|\{z_n\}_{n=1}^\infty\|_\psi = \begin{cases} \left(\sum_{j=1}^\infty |z_j|\right) \psi\left(\frac{|z_1|}{\sum_{j=1}^\infty |z_j|}, \ldots, \frac{|z_n|}{\sum_{j=1}^\infty |z_j|}, \ldots\right) & (\{z_n\}_{n=1}^\infty \neq 0) \\ 0 & (\{z_n\}_{n=1}^\infty = 0), \end{cases}$$

then $\|\cdot\|_\psi \in AN_\infty$ and it satisfies (38).

In fact, AN_∞ and Ψ_∞ are in a one-to-one correspondence under the equation (38).

Using this, we introduce the ℓ_ψ-spaces. Let ℓ_∞ be the Banach space of all bounded infinite sequences of complex numbers. For $\{z_n\}_{n=1}^\infty \in \ell_\infty$ and $\psi \in \Psi_\infty$, by Proposition 1.2.2, $\{\|(z_1, \ldots, z_n, 0, 0, \ldots)\|_\psi\}_{n=1}^\infty$ is an increasing sequence. Thus we define the space ℓ_ψ by

$$\ell_\psi = \left\{ \{z_n\}_{n=1}^\infty \in \ell_\infty : \lim_{n\to\infty} \|(z_1, \ldots, z_n, 0, 0, \ldots)\|_\psi < \infty \right\}. \tag{39}$$

Proposition 1.4.1 (cf. [25, Proposition 2.4]). *ℓ_ψ is a Banach space with the norm*

$$\|\{z_n\}_{n=1}^\infty\|_\psi = \lim_{n\to\infty} \|(z_1, \ldots, z_n, 0, 0, \ldots)\|_\psi.$$

Proof. Let $\{y_k\}_{k=1}^\infty \subset \ell_\psi$ be any Cauchy sequence of ℓ_ψ. We put $y_k = \{z_n^{(k)}\}_{n=1}^\infty$ for every $k \in \mathbb{N}$. By Proposition 1.2.2, for each $n \in \mathbb{N}$, for any $k, l \in \mathbb{N}$, we have

$$\begin{aligned} |z_n^{(k)} - z_n^{(l)}| &= \|(0, \ldots, 0, z_n^{(k)} - z_n^{(l)}, 0, \ldots)\|_\psi \\ &\leq \|(z_1^{(k)} - z_1^{(l)}, \ldots, z_n^{(k)} - z_n^{(l)}, 0, \ldots)\|_\psi, \end{aligned}$$

then $\{z_n^{(k)}\}_{k=1}^\infty \subset \mathbb{C}$ is a Cauchy sequence in $\mathbb{C}$. Thus there exists $z_n \in \mathbb{C}$ such that $z_n = \lim_{k\to\infty} z_n^{(k)}$ for each $n \in \mathbb{N}$. By Proposition 1.2.2, we have

$$\begin{aligned} &\|(z_1^{(k)} - z_1, \ldots, z_n^{(k)} - z_n, 0, 0, \ldots)\|_\psi \\ &\qquad = \lim_{m\to\infty} \|(z_1^{(k)} - z_1^{(m)}, \ldots, z_n^{(k)} - z_n^{(m)}, 0, 0, \ldots)\|_\psi \\ &\qquad = \liminf_{m\to\infty} \|(z_1^{(k)} - z_1^{(m)}, \ldots, z_n^{(k)} - z_n^{(m)}, 0, 0, \ldots)\|_\psi \\ &\qquad \leq \liminf_{m\to\infty} \|y_k - y_m\|_\psi. \end{aligned}$$

As $n \to \infty$, we have $y = \{z_n\}_{n=1}^\infty \in \ell_\psi$ and $\|y_k - y\|_\psi \to 0$. This completes the proof. □

Next, we consider the dual space of ℓ_ψ. Let $\psi \in \Psi_\infty$. For any $\{z_n\}_{n=1}^\infty \in \ell_0$, the dual norm of $\|\cdot\|_\psi$ is defined by following:

$$\|\{z_n\}\|_\psi^* = \sup\left\{ \left| \sum_{n=1}^\infty z_n w_n \right| : w = \{w_n\}_{n=1}^\infty \in \ell_0, \|w\|_\psi = 1 \right\}.$$

Then $\|\cdot\|_\psi^* \in AN_\infty$ and the corresponding convex function in Ψ_∞ is given by

$$\psi^*(s) = \sup_{t\in\Delta_\infty} \frac{\sum_{n=1}^\infty s_n t_n}{\psi(t)} \quad (s = \{s_n\}_{n=1}^\infty \in \Delta_\infty),$$

and $\|\cdot\|_\psi^* = \|\cdot\|_{\psi^*}$. Then

$$\ell_{\psi^*} = \left\{ \{w_n\}_{n=1}^\infty \in \ell_\infty : \lim_{n\to\infty} \|(w_1, \ldots, w_n, 0, 0, \ldots)\|_{\psi^*} < \infty \right\}$$

is also a Banach space with the norm

$$\|\{w_n\}_{n=1}^\infty\|_{\psi^*} = \lim_{n\to\infty} \|(w_1, \ldots, w_n, 0, 0, \ldots)\|_{\psi^*}.$$

Moreover we have the Generalized Hölder inequality:

$$\sum_{n=1}^\infty |z_n w_n| \le \|\{z_n\}_{n=1}^\infty\|_\psi \ \|\{w_n\}_{n=1}^\infty\|_{\psi^*} \tag{40}$$

for any $\{z_n\}_{n=1}^\infty \in \ell_\psi$ and any $\{w_n\}_{n=1}^\infty \in \ell_{\psi^*}$.

Now we note the ℓ_p-norm which is a good example of absolute normalized norms. For any $\{x_n\}_{n=1}^\infty \in \ell_0$, it is

$$\|\{z_n\}_{n=1}^\infty\|_p = \begin{cases} (\sum_{n=1}^\infty |z_n|^p)^{\frac{1}{p}} & (1 \le p < \infty) \\ \max_{1\le n<\infty} |z_n| & (p = \infty), \end{cases}$$

and also for every $\|\cdot\| \in AN_\infty$, we have $\|\cdot\|_\infty \le \|\cdot\| \le \|\cdot\|_1$. In this case, $\psi = \psi_p \in \Psi_\infty$ is

$$\psi_p(t) = \begin{cases} (\sum_{n=1}^\infty t_n^{\,p})^{\frac{1}{p}} & (1 \le p < \infty) \\ \max_{1\le n<\infty} t_n & (p = \infty) \end{cases}$$

for any $t = \{t_n\}_{n=1}^\infty \in \Delta_\infty$. For any $\{z_n\}_{n=1}^\infty \in \ell_\psi$, a norm $\|\cdot\|_p = \|\cdot\|_{\psi_p}$ is

$$\|\{z_n\}_{n=1}^\infty\|_p = \begin{cases} (\sum_{n=1}^\infty |z_n|^p)^{\frac{1}{p}} & (1 \le p < \infty) \\ \sup_{1\le n<\infty} |z_n| & (p = \infty), \end{cases}$$

and a dual norm $\|\cdot\|_p^* = \|\cdot\|_{\psi_p}^*$ is

$$\|\{z_n\}_{n=1}^\infty\|_p^* = \begin{cases} (\sum_{n=1}^\infty |z_n|^q)^{\frac{1}{q}} & (1 < p < \infty) \\ \sup_{1\le n<\infty} |z_n| & (p = 1), \end{cases} \tag{41}$$

where $1/p + 1/q = 1$. Thus ℓ_ψ is a generalization of ℓ_p.

Let $(X, \|\cdot\|)$ be a Banach space. For any $\psi \in \Psi_\infty$, we define the ψ-direct sums of X to be the space

$$\ell_\psi(X) = \big\{ \{x_n\}_{n=1}^\infty \subset X : \{\|x_n\|\}_{n=1}^\infty \in \ell_\psi \big\},$$

where ℓ_ψ is (39). Then it is a Banach space with the norm $\|\{x_n\}_{n=1}^\infty\|_\psi = \|\{\|x_n\|\}_{n=1}^\infty\|_\psi$ (cf. [41]). We first prove the following result.

Theorem 1.4.2. *Let X be a Banach space, $\psi \in \Psi_\infty$ and $\{a_n\}_{n=1}^\infty \in \ell_\infty$. Then following conditions are equivalent*:

(i) *for all $\{x_n\}_{n=1}^\infty \in \ell_\psi(X)$, $\sum_{n=1}^\infty a_n x_n$ converges in X and satisfies*

$$\left\| \sum_{n=1}^{\infty} a_n x_n \right\| \leq \|\{x_n\}_{n=1}^\infty\|_\psi;$$

(ii) *$\{a_n\}_{n=1}^\infty \in \ell_{\psi^*}$ and satisfies*

$$\|\{a_n\}_{n=1}^\infty\|_{\psi^*} \leq 1.$$

Proof. If $\{a_n\}_{n=1}^\infty$ satisfies (ii), we remark that $\|(a_1, \ldots, a_n)\|_{\psi^*} \leq 1$ for all $n \in \mathbb{N}$. From the Generalized Hölder inequality (40), we have

$$\sum_{j=1}^{n} \|a_j x_j\| \leq \|(x_1, \ldots, x_n)\|_\psi$$

for all $x_1, \ldots, x_n \in X$. Then we have (i).

Conversely, assume that $\{a_n\}_{n=1}^\infty \in \ell_\infty$ satisfies (i). For all fixed $n \in \mathbb{N}$, put $x_{n+1} = x_{n+2} = \cdots = 0$, then we have

$$\left\| \sum_{j=1}^{n} a_j x_j \right\| \leq \|(x_1, \ldots, x_n)\|_\psi.$$

From Theorem 1.3.1, we have $\|(a_1, \ldots, a_n)\|_{\psi^*} \leq 1$ for all $n \in \mathbb{N}$. Hence $\{a_n\}_{n=1}^\infty$ holds (ii). □

In this theorem, $\|\{a_n\}_{n=1}^\infty\|_{\psi^*} \leq 1$ is an element in the unit ball of $(\ell_\psi(X))^* = \ell_{\psi^*}(X)$, where $(\ell_\psi(X))^*$ is a dual space of $\ell_\psi(X)$.

From this theorem, we have a following corollary by putting $\psi = \psi_p$ and using (41).

Corollary 1.4.3. *Let X be a Banach space, $n \in \mathbb{N}$ with $n \geq 2$, $p \in \mathbb{R}$ with $p > 1$, and $\{a_n\}_{n=1}^\infty \in \ell_\infty$. Then following conditions are equivalent*:

(i) *for all $\{x_n\}_{n=1}^\infty \in \ell_p(X)$, $\sum_{n=1}^\infty a_n x_n$ converges in X and satisfies*

$$\left\| \sum_{n=1}^{\infty} a_n x_n \right\|^p \leq \|\{x_n\}_{n=1}^\infty\|^p;$$

(ii) *$\{a_n\}_{n=1}^\infty \in \ell_q$ and satisfies*

$$\|\{a_n\}_{n=1}^\infty\|_q \leq 1,$$

where $1/p + 1/q = 1$.

1.4.2 $\ell_{\tilde{\psi}}$-spaces

We define the set $\widetilde{\Psi}_\infty$ of all continuous concave functions on Δ_∞ satisfying the following conditions: if $\tilde{\psi} \in \widetilde{\Psi}_\infty$, then $\tilde{\psi}(e_n) = 1$ for all $n = 1, 2, \ldots$, where $e_n = (0, \ldots, 0, \overset{(n)}{1}, 0, \ldots) \in \ell_0$. For any $\tilde{\psi} \in \widetilde{\Psi}_\infty$, we define a mapping $\| \cdot \|_{\tilde{\psi}}$ on ℓ_0 such that

$$\|\{z_n\}_{n=1}^\infty\|_{\tilde{\psi}} = \begin{cases} (\sum_{j=1}^\infty |z_j|)\tilde{\psi}\left(\frac{|z_1|}{\sum_{j=1}^\infty |z_j|}, \ldots, \frac{|z_n|}{\sum_{j=1}^\infty |z_j|}, \ldots\right) & (\{z_n\}_{n=1}^\infty \neq 0) \\ 0 & (\{z_n\}_{n=1}^\infty = 0). \end{cases}$$

Using this, we introduce the $\ell_{\tilde{\psi}}$-spaces. Let ℓ_∞ is the Banach space of all bounded infinite sequences of complex numbers. For any $\tilde{\psi} \in \widetilde{\Psi}_\infty$, we define the space $\ell_{\tilde{\psi}}$ by

$$\ell_{\tilde{\psi}} = \left\{\{z_n\}_{n=1}^\infty \in \ell_\infty : \lim_{n\to\infty} \|(z_1, \ldots, z_n, 0, 0, \ldots)\|_{\tilde{\psi}} < \infty\right\}.$$

For any $\{z_n\}_{n=1}^\infty \in \ell_{\tilde{\psi}}$, we define the mapping

$$\|\{z_n\}_{n=1}^\infty\|_{\tilde{\psi}} = \lim_{n\to\infty} \|(z_1, \ldots, z_n, 0, 0, \ldots)\|_{\tilde{\psi}}.$$

This mapping is not a norm, however, we have the generalized inverse Minkowski inequality:

$$\|\{|z_n| + |w_n|\}_{n=1}^\infty\|_{\tilde{\psi}} \geq \|\{|z_n|\}_{n=1}^\infty\|_{\tilde{\psi}} + \|\{|w_n|\}_{n=1}^\infty\|_{\tilde{\psi}} \tag{42}$$

for any $\{z_n\}_{n=1}^\infty, \{w_n\}_{n=1}^\infty \in \ell_{\tilde{\psi}}$. For all $p \in \mathbb{R}$ with $0 < p \leq 1$,

$$\tilde{\psi}_p(t) = \left(\sum_{n=1}^\infty t_n^{\,p}\right)^{\frac{1}{p}}$$

is a element of $\widetilde{\Psi}_\infty$ and $\|\{z_n\}_{n=1}^\infty\|_{\tilde{\psi}_p} = \|\{z_n\}_{n=1}^\infty\|_p = (\sum_{n=1}^\infty |z_n|^p)^{\frac{1}{p}}$.

Let $(X, \| \cdot \|)$ be a Banach space. For any $\tilde{\psi} \in \widetilde{\Psi}_\infty$, we define the $\tilde{\psi}$-direct sums of X to be the space

$$\ell_{\tilde{\psi}}(X) = \left\{\{x_n\}_{n=1}^\infty \subset X : \{\|x_n\|\}_{n=1}^\infty \in \ell_{\tilde{\psi}}\right\},$$

with the mapping $\|\{x_n\}_{n=1}^\infty\|_{\tilde{\psi}} = \|\{\|x_n\|\}_{n=1}^\infty\|_{\tilde{\psi}}$. We have a following result.

Theorem 1.4.4. *Let X be a Banach space, $\tilde{\psi} \in \widetilde{\Psi}_\infty$ and $\{a_n\}_{n=1}^\infty \in \ell_\infty$. Then following conditions are equivalent*:

(i) *for all* $\{x_n\}_{n=1}^\infty \in \ell_{\tilde\psi}(X)$*,* $\sum_{n=1}^\infty a_n x_n$ *converges in* X *and satisfies*

$$\left\|\sum_{n=1}^\infty a_n x_n\right\| \le \|\{x_n\}_{n=1}^\infty\|_{\tilde\psi};$$

(ii) $\sup_{1\le n<\infty} |a_n| \le 1$.

Proof. If $\{a_n\}_{n=1}^\infty$ satisfies (ii), we remark that $\max\{|a_1|,\dots,|a_n|\} \le 1$ for all $n \in \mathbb{N}$. As in the proof of Theorem 1.3.3, from the generalized inverse Minkowski inequality (42), we have

$$\sum_{j=1}^n \|a_j x_j\| \le \|(x_1,\dots,x_n)\|_{\tilde\psi}$$

for all $x_1,\dots,x_n \in X$. Then we have (i).

Conversely, assume that $\{a_n\}_{n=1}^\infty \in \ell_\infty$ satisfies (i). For all fixed $n \in \mathbb{N}$, put $x_{n+1} = x_{n+2} = \cdots = 0$, then we have

$$\left\|\sum_{j=1}^n a_j x_j\right\| \le \|(x_1,\dots,x_n)\|_{\tilde\psi}.$$

From Theorem 1.3.3, we have $\max\{|a_1|,\dots,|a_n|\} \le 1$ for all $n \in \mathbb{N}$. Hence $\{a_n\}_{n=1}^\infty$ holds (ii). □

From this theorem, we have a following corollary by putting $\tilde\psi = \tilde\psi_p$, where $0 < p \le 1$.

Corollary 1.4.5. *Let* X *be a Banach space,* $n \in \mathbb{N}$ *with* $n \ge 2$*,* $p \in \mathbb{R}$ *with* $0 < p \le 1$ *and* $\{a_n\}_{n=1}^\infty \in \ell_\infty$*. Then following conditions are equivalent:*

(i) *for all* $\{x_n\}_{n=1}^\infty \in \ell_p(X)$*,* $\sum_{n=1}^\infty a_n x_n$ *converges in* X *and satisfies*

$$\left\|\sum_{n=1}^\infty a_n x_n\right\|^p \le \|\{x_n\}_{n=1}^\infty\|^p;$$

(ii) $\sup_{1\le n<\infty} |a_n| \le 1$.

2 On sharp triangle inequalities

2.1 Introduction

In this chapter, we consider the inequalities which are sharper than the usual triangle inequality for a Banach space X. In [14], Hudizik and Landes remarked a sharp triangle inequality for two elements. In [19], Kato, Saito and Tamura proved the sharp triangle inequality and reverse inequality as follows: for all nonzero elements $x_1, \cdots, x_n \in X$,

$$\begin{aligned} &\left\|\sum_{j=1}^{n} x_j\right\| + \left(n - \left\|\sum_{j=1}^{n} \frac{x_j}{\|x_j\|}\right\|\right) \min_{1\le j\le n} \|x_j\| \\ &\quad \le \sum_{j=1}^{n} \|x_j\| \qquad (43) \\ &\quad \le \left\|\sum_{j=1}^{n} x_j\right\| + \left(n - \left\|\sum_{j=1}^{n} \frac{x_j}{\|x_j\|}\right\|\right) \max_{1\le j\le n} \|x_j\| \end{aligned}$$

hold. In [29], Mitani et al. succeeded in the further extensions of above inequalities as follows: for all nonzero elements $x_1, x_2, \ldots, x_n \in X, n \ge 2$,

$$\begin{aligned} &\left\|\sum_{j=1}^{n} x_j\right\| + \sum_{k=2}^{n}\left(k - \left\|\sum_{j=1}^{k} \frac{x_j^*}{\|x_j^*\|}\right\|\right)(\|x_k^*\| - \|x_{k+1}^*\|) \\ &\quad \le \sum_{j=1}^{n} \|x_j\| \qquad (44) \\ &\quad \le \left\|\sum_{j=1}^{n} x_j\right\| - \sum_{k=2}^{n}\left(k - \left\|\sum_{j=n-(k-1)}^{n} \frac{x_j^*}{\|x_j^*\|}\right\|\right)(\|x_{n-k}^*\| - \|x_{n-(k-1)}^*\|) \end{aligned}$$

hold, where $x_1^*, x_2^*, \ldots, x_n^*$ are the rearrangement of $x_1, x_2, \ldots, x_n$ satisfying $\|x_1^*\| \ge \|x_2^*\| \ge \cdots \ge \|x_n^*\|$, and $x_0^* = x_{n+1}^* = 0$. Moreover they studied equality attainedness on these inequalities in a strictly convex Banach space (cf. [19] and [27]). In [24], Mineno, Nakamura and Ohwada studied the problem that characterize all the intermediate value C which satisfies

$$0 \le C \le \sum_{j=1}^{n} \|x_j\| - \left\|\sum_{j=1}^{n} x_j\right\|$$

by using $x_1, x_2, \dots, x_n$ in a Banach space X. Inequalities (43) and (44) give partial solutions of this problem: for all $x_1, x_2, \dots, x_n \in X$,

$$0 \leq \left(n - \left\| \sum_{j=1}^{n} \frac{x_j}{\|x_j\|} \right\| \right) \min_{1 \leq j \leq n} \|x_j\| \leq \sum_{j=1}^{n} \|x_j\| - \left\| \sum_{j=1}^{n} x_j \right\|$$

and

$$0 \leq \sum_{k=2}^{n} \left(k - \left\| \sum_{j=1}^{k} \frac{x_j^*}{\|x_j^*\|} \right\| \right) (\|x_k^*\| - \|x_{k+1}^*\|) \leq \sum_{j=1}^{n} \|x_j\| - \left\| \sum_{j=1}^{n} x_j \right\|$$

hold. Our main aims in this chapter are to give the simple proof of the result of Mineno et al. and study when this norm inequality attains equality in a strictly convex Banach space.

In §2.2, we summarize basic results of sharp triangle inequalities and their reverse inequalities in Banach spaces.

In §2.3, we present the simple proof of the result of Mineno et al. and study when this norm inequality attains equality in strictly convex Banach spaces.

In §2.4, by using sharp triangle inequalities, we characterize some geometrical properties of Banach spaces.

2.2 Sharp triangle inequalities in Banach spaces

2.2.1 Sharp triangle inequalities and their reverse inequalities

In this section, we summarize basic results of sharp triangle inequalities and their reverse inequalities in Banach spaces. In [19], Kato et al. proved the sharp triangle inequality and reverse inequalities for an arbitrary number of finitely many nonzero elements $x_1, x_2, \dots, x_n \in X$ as follows:

Theorem 2.2.1 (cf. [19, Theorem 1]). *For all nonzero elements $x_1, x_2, \dots, x_n$ in a Banach space X,*

$$\left\| \sum_{j=1}^{n} x_j \right\| + \left(n - \left\| \sum_{j=1}^{n} \frac{x_j}{\|x_j\|} \right\| \right) \min_{1 \leq j \leq n} \|x_j\|$$

$$\leq \sum_{j=1}^{n} \|x_j\| \tag{45}$$

$$\leq \left\| \sum_{j=1}^{n} x_j \right\| + \left(n - \left\| \sum_{j=1}^{n} \frac{x_j}{\|x_j\|} \right\| \right) \max_{1 \leq j \leq n} \|x_j\| \tag{46}$$

hold.

In the case $n = 2$, we have the following inequalities which will be used to prove the general n elements case.

Theorem 2.2.2 (cf. [14] and [20]). *For all nonzero elements x, y in a Banach space X,*

$$\|x+y\| + \left(2 - \left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\|\right) \min\{\|x\|, \|y\|\}$$
$$\leq \|x\| + \|y\| \tag{47}$$
$$\leq \|x+y\| + \left(2 - \left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\|\right) \max\{\|x\|, \|y\|\} \tag{48}$$

hold.

Proof. If $\|x\| = \|y\|$, both inequalities (47) and (48) hold with equality. Therefore we may assume this is not the case. Let us see the first inequality. Let $\|x\| > \|y\|$. Then we have

$$\begin{aligned}
\left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\| &= \left\|\frac{x}{\|y\|} + \frac{y}{\|y\|} - \frac{x}{\|y\|} + \frac{x}{\|x\|}\right\| \\
&= \left\|\frac{x}{\|y\|} + \frac{y}{\|y\|} - \left(\frac{1}{\|y\|} - \frac{1}{\|x\|}\right) x\right\| \\
&\geq \left\|\frac{x}{\|y\|} + \frac{y}{\|y\|}\right\| - \left(\frac{1}{\|y\|} - \frac{1}{\|x\|}\right) \|x\| \\
&= \frac{\|x+y\|}{\|y\|} - \left(\frac{1}{\|y\|} - \frac{1}{\|x\|}\right) \|x\| - \left(\frac{1}{\|y\|} - \frac{1}{\|y\|}\right) \|y\| \\
&= \frac{\|x+y\|}{\|y\|} - \left(\frac{\|x\| + \|y\|}{\|y\|} - 2\right),
\end{aligned}$$

for which it follows that

$$\frac{\|x\| + \|y\|}{\|y\|} \geq \frac{\|x+y\|}{\|y\|} + \left(2 - \left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\|\right).$$

Hence we obtain

$$\|x\| + \|y\| \geq \|x+y\| + \left(2 - \left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\|\right) \|y\|,$$

or the inequality (47). For the second inequality as in the above proof, let $\|x\| > \|y\|$. Then we have

$$\begin{aligned}\left\|\frac{x}{\|x\|}+\frac{y}{\|y\|}\right\| &= \left\|\frac{x}{\|x\|}+\frac{y}{\|x\|}-\frac{y}{\|x\|}+\frac{y}{\|y\|}\right\| \\ &= \left\|\frac{x}{\|x\|}+\frac{y}{\|x\|}+\left(\frac{1}{\|y\|}-\frac{1}{\|x\|}\right)y\right\| \\ &\leq \left\|\frac{x}{\|x\|}+\frac{y}{\|y\|}\right\|+\left(\frac{1}{\|y\|}-\frac{1}{\|x\|}\right)\|y\| \\ &= \frac{\|x+y\|}{\|x\|}+\left(\frac{1}{\|y\|}-\frac{1}{\|x\|}\right)\|y\|+\left(\frac{1}{\|x\|}-\frac{1}{\|x\|}\right)\|x\| \\ &= \frac{\|x+y\|}{\|x\|}+\left(2-\frac{\|x\|+\|y\|}{\|x\|}\right),\end{aligned}$$

for which it follows that

$$\frac{\|x\|+\|y\|}{\|x\|} \leq \frac{\|x+y\|}{\|x\|}+\left(2-\left\|\frac{x}{\|x\|}+\frac{y}{\|y\|}\right\|\right).$$

Hence we obtain

$$\|x\|+\|y\| \leq \|x+y\|+\left(2-\left\|\frac{x}{\|x\|}+\frac{y}{\|y\|}\right\|\right)\|x\|,$$

or the inequality (48). Thus we have the conclusion. □

Proof of Theorem 2.2.1. If $\|x_1\| = \|x_2\| = \cdots = \|x_n\|$, both inequalities (45) and (46) hold with equality. Therefore we may assume this is not the case. Let us see the first inequality. Let $\|x_{j_0}\| = \min\{\|x_j\| : 1 \leq j \leq n\}$ and $J_0 = \{j : \|x_j\| = \|x_{j_0}\|, 1 \leq j \leq n\}$. Then for any nonzero $x_1, \ldots, x_n \in X$ we have

$$\begin{aligned}\left\|\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}\right\| &= \left\|\sum_{j\in J_0}\frac{x_j}{\|x_j\|}+\sum_{j\in J_0^c}\frac{x_j}{\|x_j\|}\right\| \\ &= \left\|\sum_{j=1}^{n}\frac{x_j}{\|x_{j_0}\|}-\sum_{j\in J_0^c}\frac{x_j}{\|x_{j_0}\|}+\sum_{j\in J_0^c}\frac{x_j}{\|x_j\|}\right\| \\ &= \left\|\sum_{j=1}^{n}\frac{x_j}{\|x_{j_0}\|}-\sum_{j\in J_0^c}\left(\frac{1}{\|x_{j_0}\|}-\frac{1}{\|x_j\|}\right)x_j\right\| \\ &\geq \left\|\sum_{j=1}^{n}\frac{x_j}{\|x_{j_0}\|}\right\|-\sum_{j\in J_0^c}\left(\frac{1}{\|x_{j_0}\|}-\frac{1}{\|x_j\|}\right)\|x_j\|\end{aligned}$$

$$= \left\| \sum_{j=1}^{n} \frac{x_j}{\|x_{j_0}\|} \right\| - \sum_{j=1}^{n} \left(\frac{1}{\|x_{j_0}\|} - \frac{1}{\|x_j\|} \right) \|x_j\|$$
$$= \left\| \sum_{j=1}^{n} \frac{x_j}{\|x_{j_0}\|} \right\| - \left(\sum_{j=1}^{n} \frac{\|x_j\|}{\|x_{j_0}\|} - n \right),$$

for which it follows that

$$\sum_{j=1}^{n} \frac{\|x_j\|}{\|x_{j_0}\|} \geq \frac{\| \sum_{j=1}^{n} x_j \|}{\|x_{j_0}\|} + \left(n - \left\| \sum_{j=1}^{n} \frac{x_j}{\|x_j\|} \right\| \right).$$

Hence we obtain

$$\sum_{j=1}^{n} \|x_j\| \geq \left\| \sum_{j=1}^{n} x_j \right\| + \left(n - \left\| \sum_{j=1}^{n} \frac{x_j}{\|x_j\|} \right\| \right) \|x_{j_0}\|,$$

or the inequality (45).

For the second inequality let $\|x_{j_1}\| = \max\{\|x_j\| : 1 \leq j \leq n\}$ and $J_1 = \{j : \|x_j\| = \|x_{j_1}\|, 1 \leq j \leq n\}$. Then we have

$$\begin{aligned}
\left\| \sum_{j=1}^{n} \frac{x_j}{\|x_j\|} \right\| &= \left\| \sum_{j \in J_1} \frac{x_j}{\|x_j\|} + \sum_{j \in J_1^c} \frac{x_j}{\|x_j\|} \right\| \\
&= \left\| \sum_{j=1}^{n} \frac{x_j}{\|x_{j_1}\|} - \sum_{j \in J_1^c} \frac{x_j}{\|x_{j_1}\|} + \sum_{j \in J_1^c} \frac{x_j}{\|x_j\|} \right\| \\
&= \left\| \sum_{j=1}^{n} \frac{x_j}{\|x_{j_1}\|} + \sum_{j \in J_1^c} \left(\frac{1}{\|x_j\|} - \frac{1}{\|x_{j_1}\|} \right) x_j \right\| \\
&\leq \left\| \sum_{j=1}^{n} \frac{x_j}{\|x_{j_1}\|} \right\| + \sum_{j \in J_1^c} \left(\frac{1}{\|x_j\|} - \frac{1}{\|x_{j_1}\|} \right) \|x_j\| \\
&= \left\| \sum_{j=1}^{n} \frac{x_j}{\|x_{j_1}\|} \right\| + \sum_{j=1}^{n} \left(\frac{1}{\|x_j\|} - \frac{1}{\|x_{j_1}\|} \right) \|x_j\| \\
&= \left\| \sum_{j=1}^{n} \frac{x_j}{\|x_{j_1}\|} \right\| + n - \sum_{j=1}^{n} \frac{\|x_j\|}{\|x_{j_1}\|},
\end{aligned}$$

and hence

$$\sum_{j=1}^{n} \|x_j\| \leq \left\| \sum_{j=1}^{n} x_j \right\| + \left(n - \left\| \sum_{j=1}^{n} \frac{x_j}{\|x_j\|} \right\| \right) \|x_{j_1}\|.$$

Thus we have the conclusion. □

2.2.2 Sharper triangle inequalities

In [29], Mitani et al. succeeded in the further extension of above inequalities.

Theorem 2.2.3 (cf. [29, Theorem 1]). *For all nonzero elements $x_1, x_2, \ldots, x_n$ in a Banach space $X, n \geq 2$,*

$$\left\|\sum_{j=1}^{n} x_j\right\| + \sum_{k=2}^{n}\left(k - \left\|\sum_{j=1}^{k} \frac{x_j^*}{\|x_j^*\|}\right\|\right)(\|x_k^*\| - \|x_{k+1}^*\|)$$

$$\leq \sum_{j=1}^{n} \|x_j\| \tag{49}$$

$$\leq \left\|\sum_{j=1}^{n} x_j\right\| - \sum_{k=2}^{n}\left(k - \left\|\sum_{j=n-(k-1)}^{n} \frac{x_j^*}{\|x_j^*\|}\right\|\right)(\|x_{n-k}^*\| - \|x_{n-(k-1)}^*\|) \tag{50}$$

hold, where $x_1^*, x_2^*, \ldots, x_n^*$ are the rearrangement of $x_1, x_2, \ldots, x_n$ satisfying $\|x_1^*\| \geq \|x_2^*\| \geq \cdots \geq \|x_n^*\|$, and $x_0^* = x_{n+1}^* = 0$.

As the case $n = 2$ the above theorem includes Theorem 2.2.2. To see explicitly that the Theorem 2.2.3 refines Theorem 2.2.1, we reformulate Theorem 2.2.3 as follows: for all nonzero elements $x_1, x_2, \ldots, x_n$ in a Banach space X, $n \geq 3$,

$$\left\|\sum_{j=1}^{n} x_j\right\| + \left(n - \left\|\sum_{j=1}^{n} \frac{x_j^*}{\|x_j^*\|}\right\|\right)\|x_n^*\|$$

$$+ \sum_{k=2}^{n-1}\left(k - \left\|\sum_{j=1}^{k} \frac{x_j^*}{\|x_j^*\|}\right\|\right)(\|x_k^*\| - \|x_{k+1}^*\|)$$

$$\leq \sum_{j=1}^{n} \|x_j\|$$

$$\leq \left\|\sum_{j=1}^{n} x_j\right\| + \left(n - \left\|\sum_{j=1}^{n} \frac{x_j^*}{\|x_j^*\|}\right\|\right)\|x_1^*\|$$

$$- \sum_{k=2}^{n-1}\left(k - \left\|\sum_{j=n-(k-1)}^{n} \frac{x_j^*}{\|x_j^*\|}\right\|\right)(\|x_{n-k}^*\| - \|x_{n-(k-1)}^*\|)$$

hold, where $x_1^*, x_2^*, \ldots, x_n^*$ are the rearrangement of $x_1, x_2, \ldots, x_n$ satisfying $\|x_1^*\| \geq \|x_2^*\| \geq \cdots \geq \|x_n^*\|$.

In the case of $n = 3$ we have the following inequalities: for all nonzero elements x, y, z in a Banach space X with $\|x\| \geq \|y\| \geq \|z\|$,

$$
\begin{aligned}
\|x+y+z\| &+ \left(3 - \left\|\frac{x}{\|x\|} + \frac{y}{\|y\|} + \frac{z}{\|z\|}\right\|\right)\|z\| \\
&\quad + \left(2 - \left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\|\right)(\|y\| - \|z\|) \\
&\leq \|x\| + \|y\| + \|z\| \\
&\leq \|x+y+z\| + \left(3 - \left\|\frac{x}{\|x\|} + \frac{y}{\|y\|} + \frac{z}{\|z\|}\right\|\right)\|x\| \\
&\quad - \left(2 - \left\|\frac{y}{\|y\|} + \frac{z}{\|z\|}\right\|\right)(\|x\| - \|y\|)
\end{aligned}
$$

hold.

In [27], Mitani and Saito gave a simple proof of these inequalities.

Proof of Theorem 2.2.3. To prove this theorem we shall prove that for all nonzero elements $x_1, x_2, \ldots, x_n$ in a Banach space X with $\|x_1\| \geq \|x_2\| \geq \cdots \geq \|x_n\|, n \geq 2$,

$$
\begin{aligned}
&\left\|\sum_{j=1}^{n} x_j\right\| + \sum_{k=2}^{n}\left(k - \left\|\sum_{j=1}^{k} \frac{x_j}{\|x_j\|}\right\|\right)(\|x_k\| - \|x_{k+1}\|) \\
&\qquad \leq \sum_{j=1}^{n} \|x_j\| \qquad (51) \\
&\qquad \leq \left\|\sum_{j=1}^{n} x_j\right\| - \sum_{k=2}^{n}\left(k - \left\|\sum_{j=n-(k-1)}^{n} \frac{x_j}{\|x_j\|}\right\|\right)(\|x_{n-k}\| - \|x_{n-(k-1)}\|) \qquad (52)
\end{aligned}
$$

hold, where $x_0 = x_{n+1} = 0$. According to Theorem 2.2.1, inequalities (51) and (52) hold for the case $n = 2$. Therefore let $n \geq 3$. We first prove the case $\|x_1\| > \|x_2\| > \cdots > \|x_n\|$. We prove inequality (51) by the induction. Assume that the inequality holds true for all $n-1$ elements in X. Let $x_1, x_2, \ldots, x_n$ be any n elements in X with $\|x_1\| > \|x_2\| > \cdots > \|x_n\| > 0$. Let

$$u_j = (\|x_j\| - \|x_n\|)\frac{x_j}{\|x_j\|},$$

for all positive j with $1 \leq j \leq n$. Then

$$\sum_{j=1}^{n} x_j = \|x_n\| \sum_{j=1}^{n} \frac{x_j}{\|x_j\|} + \sum_{j=1}^{n-1} u_j \qquad (53)$$

and $\|u_1\| > \|u_2\| > \cdots > \|u_{n-1}\| > 0$. By assumption,

$$\left\|\sum_{j=1}^{n-1} u_j\right\| \leq \sum_{j=1}^{n-1}\|u_j\| - \sum_{k=2}^{n-1}\left(k - \left\|\sum_{j=1}^{k}\frac{u_j}{\|u_j\|}\right\|\right)(\|u_k\| - \|u_{k+1}\|) \tag{54}$$

holds, where $u_n = 0$. Since $\|u_k\| - \|u_{k+1}\| = \|x_k\| - \|x_{k+1}\|$, from (53) and (54),

$$\begin{aligned}
\left\|\sum_{j=1}^{n} x_j\right\| &= \left\|\|x_n\|\sum_{j=1}^{n}\frac{x_j}{\|x_j\|} + \sum_{j=1}^{n-1} u_j\right\| \\
&\leq \|x_n\|\left\|\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}\right\| + \left\|\sum_{j=1}^{n-1} u_j\right\| \\
&\leq \|x_n\|\left\|\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}\right\| + \sum_{j=1}^{n-1}\|u_j\| \\
&\qquad - \sum_{k=2}^{n-1}\left(k - \left\|\sum_{j=1}^{k}\frac{u_j}{\|u_j\|}\right\|\right)(\|u_k\| - \|u_{k+1}\|) \\
&= \|x_n\|\left\|\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}\right\| + \sum_{j=1}^{n-1}(\|x_j\| - \|x_n\|) \\
&\qquad - \sum_{k=2}^{n-1}\left(k - \left\|\sum_{j=1}^{k}\frac{x_j}{\|x_j\|}\right\|\right)(\|x_k\| - \|x_{k+1}\|) \\
&= \sum_{j=1}^{n}\|x_j\| - \sum_{k=2}^{n}\left(k - \left\|\sum_{j=1}^{k}\frac{x_j}{\|x_j\|}\right\|\right)(\|x_k\| - \|x_{k+1}\|)
\end{aligned}$$

and hence (51). Thus (51) holds true for all finite elements in X.

Next we show inequality (52). Let

$$v_j = (\|x_1\| - \|x_{n-j+1}\|)\frac{x_{n-j+1}}{\|x_{n-j+1}\|}$$

for all positive j with $1 \leq j \leq n-1$. Then

$$\sum_{j=1}^{n} x_j = \|x_1\|\sum_{j=1}^{n}\frac{x_j}{\|x_j\|} - \sum_{j=1}^{n-1} v_j$$

and $\|v_1\| > \|v_2\| > \cdots > \|v_{n-1}\| > 0$. Applying Inequality (51) to $v_1, v_2, \ldots, v_{n-1}$,

$$\begin{aligned}
\left\|\sum_{j=1}^{n} x_j\right\| &\geq \|x_1\| \left\|\sum_{j=1}^{n} \frac{x_j}{\|x_j\|}\right\| - \left\|\sum_{j=1}^{n-1} v_j\right\| \\
&\geq \|x_1\| \left\|\sum_{j=1}^{n} \frac{x_j}{\|x_j\|}\right\| - \sum_{j=1}^{n-1} \|v_j\| \\
&\qquad + \sum_{k=2}^{n-1} \left(k - \left\|\sum_{j=1}^{k} \frac{v_j}{\|v_j\|}\right\|\right) (\|v_k\| - \|v_{k+1}\|) \\
&= \|x_1\| \left\|\sum_{j=1}^{n} \frac{x_j}{\|x_j\|}\right\| - \sum_{j=1}^{n-1} (\|x_1\| - \|x_{n-j+1}\|) \\
&\qquad + \sum_{k=2}^{n-1} \left(k - \left\|\sum_{j=1}^{k} \frac{x_{n-j+1}}{\|x_{n-j+1}\|}\right\|\right) (\|x_{n-k+1}\| - \|x_{n-k}\|) \\
&= \sum_{j=1}^{n} \|x_j\| - \sum_{k=2}^{n} \left(k - \left\|\sum_{j=n-(k-1)}^{n} \frac{x_j}{\|x_j\|}\right\|\right) (\|x_{n-k+1}\| - \|x_{n-k}\|)
\end{aligned}$$

hold, where $v_n = 0$. Thus we obtain (52).

Now we show the case of $\|x_1\| \geq \|x_2\| \geq \cdots \geq \|x_n\| > 0$. For all positive numbers m with $m > n$, let

$$x_{k,m} = \left(1 - \frac{k}{m}\right) x_k, \quad (k = 1, 2, \ldots, n).$$

Then $\|x_{1,m}\| > \|x_{2,m}\| > \cdots > \|x_{n,m}\| > 0$. Applying inequalities (51) and (52) to $x_{1,m}, x_{2,m}, \ldots, x_{n,m}$,

$$\begin{aligned}
&\left\|\sum_{j=1}^{n} x_{j,m}\right\| + \sum_{k=2}^{n} \left(k - \left\|\sum_{j=1}^{k} \frac{x_{j,m}}{\|x_{j,m}\|}\right\|\right) (\|x_{k,m}\| - \|x_{k+1,m}\|) \\
&\quad \leq \sum_{j=1}^{n} \|x_{j,m}\| \\
&\quad \leq \left\|\sum_{j=1}^{n} x_{j,m}\right\| - \sum_{k=2}^{n} \left(k - \left\|\sum_{j=n-(k-1)}^{n} \frac{x_{j,m}}{\|x_{j,m}\|}\right\|\right) (\|x_{n-k,m}\| - \|x_{n-(k-1),m}\|)
\end{aligned}$$

hold, where $x_{0,m} = x_{n+1,m} = 0$ for all positive numbers m with $m > n$. As $m \to +\infty$, we have Inequalities (51) and (52) for the case of $\|x_1\| \geq \|x_2\| \geq \cdots \geq \|x_n\| > 0$. This completes the proof. □

2.3 Characterization of the intermediate values of the triangle inequality

2.3.1 A simple proof of the preceding result

In [24], Mineno et al. characterised the intermediate values of the triangle inequality in normed spaces as follows. For positive integer $n \geq 2$, let $M_n([0,1])$ be the set of all $n \times n$ matrices whose all elements belong to the interval $[0,1]$, and L_n denote the set of all lower triangular matrices of $M_n([0,1])$, i.e.,

$$L_n = \Big\{ a = (a_{ij}) \in M_n([0,1]) : a_{ij} = 0,\ i < j \Big\}.$$

Let $1 \leq m \leq n$. For each $a = (a_{ij})$ in L_n, we set $\ell^a_{mj}(m) = a_{mj}, 1 \leq j \leq m$, and if $2 \leq n$, then, for each m with $2 \leq m \leq n$, we put

$$\ell^a_{ij}(m) = a_{ij} \prod_{k=i+1}^{m} (1 - a_{kj}), \quad 1 \leq i \leq m-1,\ 1 \leq j \leq m.$$

Lemma 2.3.1 (cf. [24, Lemma 3.1])**.** *Keep the notation as above. Let $a = (a_{ij}), b = (b_{ij})$ in L_n. Then the following statements hold.*

(i) *Let $1 \leq m \leq n$. For each i with $1 \leq i \leq m$,*

$$0 \leq \ell^a_{ij}(m) \leq a_{ij} \leq 1 \quad (1 \leq j \leq n),$$

and $n \times n$ matrix $\big(\ell^a_{ij}(n)\big)$ belongs to L_n.

(ii) *Let $n \geq 3$. For each m with $3 \leq m \leq n$,*

$$\ell^a_{ij}(m) = \begin{cases} \ell^a_{ij}(m-1)(1 - a_{mj}) & (2 \leq i \leq m-1, 1 \leq j \leq m-1) \\ a_{mj} & (i = m, 1 \leq j \leq m). \end{cases}$$

(iii) *For each i with $1 \leq i \leq n$,*

$$\big|\ell^a_{ij}(m) - \ell^b_{ij}(m)\big| \leq \sum_{k=j}^{m} |a_{ik} - b_{ik}| \quad (1 \leq j \leq m).$$

Proof. We only prove (iii). It is clear in case of $n = 1$ and 2. Let $n \geq 3$. It is also clear in case of $i = 1$ and m. By using induction on m, we see that

$$\left| \prod_{k=i+1}^{m} (1 - a_{kj}) - \prod_{k=i+1}^{m} (1 - b_{kj}) \right| \leq \sum_{k=i+1}^{m} |a_{kj} - b_{kj}|.$$

Thus we have

$$\begin{aligned}
&\left|\ell_{ij}^a(m) - \ell_{ij}^b(m)\right| \\
&\quad = \left| a_{ij} \prod_{k=i+1}^{m} (1-a_{kj}) - b_{ij} \prod_{k=i+1}^{m} (1-b_{kj}) \right| \\
&\quad \le \left| a_{ij} \prod_{k=i+1}^{m} (1-a_{kj}) - a_{ij} \prod_{k=i+1}^{m} (1-b_{kj}) \right| \\
&\qquad\qquad + \left| a_{ij} \prod_{k=i+1}^{m} (1-b_{kj}) - b_{ij} \prod_{k=i+1}^{m} (1-b_{kj}) \right| \\
&\quad \le \left| \prod_{k=i+1}^{m} (1-a_{kj}) - \prod_{k=i+1}^{m} (1-b_{kj}) \right| + |a_{ij} - b_{ij}| \\
&\quad \le \sum_{k=i+1}^{m} |a_{kj} - b_{kj}| + |a_{ij} - b_{ij}| \\
&\quad \le \sum_{k=i}^{m} |a_{ij} - b_{ij}|.
\end{aligned}$$

This completes the proof. □

Theorem 2.3.2 (cf. [24, Lemma 3.2])**.** *Let $n \ge 2$. With the above notation, take any $a = (a_{ij})$ in L_n. For any elements $x_1, x_2, \cdots, x_n$ in a normed space X, the following inequalities hold*:

$$0 \le \sum_{i=1}^{n} \left(\sum_{j=1}^{i} \|\ell_{ij}^a(n) x_j\| - \left\| \sum_{j=1}^{i} \ell_{ij}^a(n) x_j \right\| \right) \le \sum_{j=1}^{n} \|x_j\| - \left\| \sum_{j=1}^{n} x_j \right\|. \tag{55}$$

In [24], the above theorem was established for $\ell_{1j}^a(m) = a_{1j},\ 1 \le j \le m$, which values do not influence (55). We made a modification in order to consider the equality attainability in (55) in a particular space.

Take any $a = (a_{ij}) \in L_n$ and fix it. Considering $\left(\ell_{ij}^a(n)\right)$ as the matrix acting on a Banach space $\underbrace{X \oplus X \oplus \cdots \oplus X}_{n \text{ times}}$, we have

$$\begin{pmatrix} \ell_{11}^a(n) & & & \\ \ell_{21}^a(n) & \ell_{22}^a(n) & \text{\Large 0} & \\ \vdots & \vdots & \ddots & \\ \ell_{n1}^a(n) & \dots & \ell_{nn-1}^a(n) & \ell_{nn}^a(n) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \ell_{11}^a(n) x_1 \\ \sum_{j=1}^{2} \ell_{2j}^a(n) x_j \\ \vdots \\ \sum_{j=1}^{n} \ell_{nj}^a(n) x_j \end{pmatrix},$$

where $x_1, x_2, \ldots, x_n \in X$. For each entries, we have the triangle inequalities

$$\left\|\sum_{j=1}^{i} \ell_{ij}^a(n)x_j\right\| \leq \sum_{j=1}^{i} \|\ell_{ij}^a(n)x_j\| \quad (1 \leq i \leq n). \tag{56}$$

Theorem 2.3.2 means that the sum of all differences of (56) is less than the difference of the triangle inequality

$$\left\|\sum_{j=1}^{n} x_j\right\| \leq \sum_{j=1}^{n} \|x_j\|.$$

To present a simple proof of Theorem 2.3.2, we would need the following lemma.

Lemma 2.3.3. *Let $n \geq 2$ and $a = (a_{ij}) \in L_n$. Take m with $1 \leq m \leq n$. For each j with $m \leq j \leq n$, the following identity holds*:

$$\sum_{i=j}^{n} \ell_{im}^a(n) + \prod_{i=j}^{n}(1 - a_{im}) = 1.$$

Proof. We shall prove this lemma by using induction on j. If $j = n$, then

$$(1 - a_{nm}) + \ell_{nm}^a(n) = (1 - a_{nm}) + a_{nm} = 1.$$

Next let $j = n - 1$. Then

$$\begin{aligned}
&\prod_{i=n-1}^{n} (1 - a_{i\,m}) + \sum_{i=n-1}^{n} \ell_{i\,m}^a(n) \\
&\quad = (1 - a_{n-1\,m})(1 - a_{n\,m}) + \ell_{n-1\,m}^a(n) + \ell_{n\,m}^a(n) \\
&\quad = (1 - a_{n-1\,m})(1 - a_{n\,m}) + a_{n-1\,m}(1 - a_{n\,m}) + a_{n\,m} \\
&\quad = (1 - a_{n-1\,m})(1 - a_{n\,m}) + \{1 - (1 - a_{n-1\,m})\}(1 - a_{n\,m}) + a_{n\,m} \\
&\quad = (1 - a_{n\,m}) + a_{n\,m} \\
&\quad = 1.
\end{aligned}$$

Now if we assume that, for each j with $m + 1 \leq j \leq n$,

$$\prod_{i=j}^{n}(1 - a_{i\,m}) + \sum_{i=j}^{n} \ell_{i\,m}^a(n) = 1,$$

then we have

$$\begin{aligned}
&\prod_{i=j-1}^{n}(1-a_{i\,m})+\sum_{i=j-1}^{n}\ell^a_{i\,m}(n)\\
&\quad=\prod_{i=j-1}^{n}(1-a_{i\,m})+\ell^a_{j-1\,m}(n)+\sum_{i=j}^{n}\ell^a_{j\,m}(n)\\
&\quad=\prod_{i=j-1}^{n}(1-a_{i\,m})+a_{j-1\,m}\prod_{i=j}^{n}(1-a_{i\,m})+\sum_{i=j}^{n}\ell^a_{j\,m}(n)\\
&\quad=\prod_{i=j-1}^{n}(1-a_{i\,m})+\{1-(1-a_{j-1\,m})\}\prod_{i=j}^{n}(1-a_{i\,m})+\sum_{i=j}^{n}\ell^a_{j\,m}(n)\\
&\quad=\prod_{i=j}^{n}(1-a_{i\,m})+\sum_{i=j}^{n}\ell^a_{j\,m}(n)\\
&\quad=1.
\end{aligned}$$

Thus, by induction on j, we obtain the lemma. □

Proof of Theorem 2.3.2. For each $x_1,\dots,x_n\in X$ and $a=(a_{ij})\in L_n$, the following equations hold:

$$\begin{aligned}
\sum_{j=1}^{n}x_j&=\sum_{j=1}^{n}\left(1-\sum_{i=j}^{n}\ell^a_{ij}(n)\right)x_j+\sum_{j=1}^{n}\sum_{i=j}^{n}\ell^a_{ij}(n)x_j\\
&=\sum_{j=1}^{n}\left(1-\sum_{i=j}^{n}\ell^a_{ij}(n)\right)x_j+\sum_{i=1}^{n}\sum_{j=1}^{i}\ell^a_{ij}(n)x_j.
\end{aligned}$$

By Lemma 2.3.3, we see that

$$0\le\prod_{i=j}^{n}(1-a_{ij})=1-\sum_{i=j}^{n}\ell^a_{ij}(n).$$

Moreover, $0\le\ell^a_{ij}(n)\le1$ for all $i,j\in\{1,\dots,n\}$. Thus, applying the triangle inequality, we have

$$\begin{aligned}
\left\|\sum_{j=1}^{n}x_j\right\|&=\left\|\sum_{j=1}^{n}\left(1-\sum_{i=j}^{n}\ell^a_{ij}(n)\right)x_j+\sum_{i=1}^{n}\sum_{j=1}^{i}\ell^a_{ij}(n)x_j\right\|\\
&\le\sum_{j=1}^{n}\left\|\left(1-\sum_{i=j}^{n}\ell^a_{ij}(n)\right)x_j\right\|+\sum_{i=1}^{n}\left\|\sum_{j=1}^{i}\ell^a_{ij}(n)x_j\right\| \qquad (57)\\
&=\sum_{j=1}^{n}\left(1-\sum_{i=j}^{n}\ell^a_{ij}(n)\right)\|x_j\|+\sum_{i=1}^{n}\left\|\sum_{j=1}^{i}\ell^a_{ij}(n)x_j\right\|
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{n}\left(\|x_j\| - \sum_{i=j}^{n}\left\|\ell_{ij}^a(n)x_j\right\|\right) + \sum_{i=1}^{n}\left\|\sum_{j=1}^{i}\ell_{ij}^a(n)x_j\right\| \\
&= \sum_{j=1}^{n}\|x_j\| - \sum_{j=1}^{n}\sum_{i=j}^{n}\left\|\ell_{ij}^a(n)x_j\right\| + \sum_{i=1}^{n}\left\|\sum_{j=1}^{i}\ell_{ij}^a(n)x_j\right\| \\
&= \sum_{j=1}^{n}\|x_j\| - \sum_{i=1}^{n}\sum_{j=1}^{i}\left\|\ell_{ij}^a(n)x_j\right\| + \sum_{i=1}^{n}\left\|\sum_{j=1}^{i}\ell_{ij}^a(n)x_j\right\| \\
&= \sum_{j=1}^{n}\|x_j\| - \sum_{i=1}^{n}\left(\sum_{j=1}^{i}\left\|\ell_{ij}^a(n)x_j\right\| - \left\|\sum_{j=1}^{i}\ell_{ij}^a(n)x_j\right\|\right).
\end{aligned}
$$

This completes the proof. □

Let $x_1, x_2, \cdots, x_n \in X$. For each $a \in L_n$, if we put

$$
f(a) = \sum_{i=1}^{n}\left(\sum_{j=1}^{i}\left\|\ell_{ij}^a(n)x_j\right\| - \left\|\sum_{j=1}^{i}\ell_{ij}^a(n)x_j\right\|\right),
$$

then we see that

$$
f(a_0) = 0, \quad f(a_1) = \sum_{j=1}^{n}\|x_j\| - \left\|\sum_{j=1}^{n}x_j\right\|,
$$

where $a_0, a_1 \in L_n$ with

$$
a_0 = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}, \quad a_1 = \begin{pmatrix} a_{11} & 0 & \cdots & \cdots & 0 \\ a_{21} & a_{22} & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & 0 & \vdots \\ a_{n-11} & a_{n-12} & \cdots & a_{n-1n-1} & 0 \\ 1 & 1 & \cdots & 1 & 1 \end{pmatrix}.
$$

Moreover, by Lemma 2.3.1, we see that, for each $a = (a_{ij}), b = (b_{ij}) \in L_n$,

$$
\begin{aligned}
|f(a) - f(b)| &\leq 2\sum_{i=2}^{n}\sum_{j=1}^{i}\left|\ell_{ij}^a(n) - \ell_{ij}^b(n)\right| \cdot \|x_j\| \\
&\leq 2\max_{1\leq j\leq n}\|x_j\| \cdot \sum_{i=2}^{n}\sum_{j=1}^{i}\sum_{k=i}^{n}|a_{ki} - b_{ki}|.
\end{aligned}
$$

Thus, considering f to a function on $\prod_{i=1}^{n(n+1)/2}[0,\ 1]$, f is continuous on $\prod_{i=1}^{n(n+1)/2}[0,\ 1]$. Therefor, we obtain the following.

Corollary 2.3.4 (cf. [24, Corollary 3.3]). *Let $n \geq 2$ and $x_1, x_2, \cdots, x_n$ in a Banach space X. For each ω with $0 \leq \omega \leq \sum_{j=1}^n \|x_j\| - \left\|\sum_{j=1}^n x_j\right\|$, there exists $a \in L_n$ such that*

$$\omega = \sum_{i=1}^{n} \left(\sum_{j=1}^{i} \|\ell_{ij}^a(n)x_j\| - \left\| \sum_{j=1}^{i} \ell_{ij}^a(n)x_j \right\| \right).$$

Of course inequality (55) contains (45) and (49).

Theorem 2.3.5 (cf. [24, Corollary 3.4]). *Let $n \geq 2$. For all elements $x_1, x_2, \cdots, x_n$ in a Banach space X, the following inequalities hold*

$$0 \leq \left(n - \left\| \sum_{j=1}^{n} \frac{x_j}{\|x_j\|} \right\| \right) \min_{1 \leq j \leq n} \|x_j\| \leq \sum_{j=1}^{n} \|x_j\| - \left\| \sum_{j=1}^{n} x_j \right\|.$$

Proof. We may assume that $\|x_n\| = \min_{1 \leq j \leq n} \|x_j\|$. If we take $a \in L_n$ as

$$a = \begin{pmatrix} 0 & & & \\ \vdots & \ddots & \text{\Large 0} & \\ 0 & \cdots & 0 & \\ \frac{\|x_n\|}{\|x_1\|} & \cdots & \frac{\|x_n\|}{\|x_{n-1}\|} & \frac{\|x_n\|}{\|x_n\|} \end{pmatrix},$$

then it is clear that

$$\ell_{ij}^a(n) = 0 \quad (1 \leq i \leq n-1), \qquad \ell_{nj}^a(n) = \frac{\|x_n\|}{\|x_j\|} \quad (1 \leq j \leq n).$$

In this case, we see that

$$\begin{aligned} f(a) &= \sum_{i=1}^{n} \left(\sum_{j=1}^{i} \|\ell_{ij}^a(n)x_j\| - \left\| \sum_{j=1}^{i} \ell_{ij}^a(n)x_j \right\| \right) \\ &= \sum_{j=1}^{n} \|\ell_{nj}^a(n)x_j\| - \left\| \sum_{j=1}^{n} \ell_{nj}^a(n)x_j \right\| \\ &= \sum_{j=1}^{n} \left\| \frac{\|x_n\|}{\|x_j\|} x_j \right\| - \left\| \sum_{j=1}^{n} \frac{\|x_n\|}{\|x_j\|} x_j \right\| \\ &= \left(n - \left\| \sum_{j=1}^{n} \frac{x_j}{\|x_j\|} \right\| \right) \min_{1 \leq j \leq n} \|x_j\|. \end{aligned}$$

Thus, by Theorem 2.3.2, we have

$$0 \leq \left(n - \left\| \sum_{j=1}^{n} \frac{x_j}{\|x_j\|} \right\| \right) \min_{1 \leq j \leq n} \|x_j\| \leq \sum_{j=1}^{n} \|x_j\| - \left\| \sum_{j=1}^{n} x_j \right\|.$$

This completes the proof. □

Theorem 2.3.6 (cf. [24, Corollary 3.5]). *Let $n \geq 2$. For all elements $x_1, x_2, \cdots, x_n$ in a Banach space X, the following inequalities hold*

$$0 \leq \sum_{i=2}^{n} \left(i - \left\| \sum_{j=1}^{i} \frac{x_j^*}{\|x_j^*\|} \right\| \right) (\|x_i^*\| - \|x_{i+1}^*\|) \leq \sum_{j=1}^{n} \|x_j\| - \left\| \sum_{j=1}^{n} x_j \right\|,$$

where $x_1^*, x_2^*, \ldots, x_n^*$ are the rearrangement of $x_1, x_2, \ldots, x_n$ which satisfies $\|x_1^*\| \geq \|x_2^*\| \geq \cdots \geq \|x_n^*\|$ and $x_{n+1}^* = 0$.

Proof. Let us first show in the case $\|x_1^*\| > \|x_2^*\| > \cdots > \|x_n^*\|$. If we take $a = (a_{ij}) \in L_n$ as

$$a_{ij} = \frac{\|x_i^*\| - \|x_{i+1}^*\|}{\|x_j^*\| - \|x_{i+1}^*\|} \qquad (1 \leq j \leq i \leq n),$$

then, for all $j \leq i$ with $2 \leq i \leq n$, we have $\ell_{ij}^a(n) = (\|x_i^*\| - \|x_{i+1}^*\|)/\|x_j^*\|$. Indeed, when $2 \leq j \leq n-1$, for all i with $1 \leq j \leq i$, we have

$$\begin{aligned}
\ell_{ij}^a(n) &= a_{ij} \prod_{k=i+1}^{n} (1 - a_{kj}) \\
&= \frac{\|x_i^*\| - \|x_{i+1}^*\|}{\|x_j^*\| - \|x_{i+1}^*\|} \cdot \prod_{k=i+1}^{n} \left(1 - \frac{\|x_k^*\| - \|x_{k+1}^*\|}{\|x_j^*\| - \|x_{k+1}^*\|} \right) \\
&= \frac{\|x_i^*\| - \|x_{i+1}^*\|}{\|x_j^*\| - \|x_{i+1}^*\|} \cdot \prod_{k=i+1}^{n} \frac{\|x_j^*\| - \|x_k^*\|}{\|x_j^*\| - \|x_{k+1}^*\|} \\
&= \frac{\|x_i^*\| - \|x_{i+1}^*\|}{\|x_j^*\| - \|x_{i+1}^*\|} \cdot \frac{\|x_j^*\| - \|x_{i+1}^*\|}{\|x_j^*\| - \|x_{i+2}^*\|} \cdot \frac{\|x_j^*\| - \|x_{i+2}^*\|}{\|x_j^*\| - \|x_{i+3}^*\|} \cdots \frac{\|x_j^*\| - \|x_n^*\|}{\|x_j^*\| - \|x_{n+1}^*\|} \\
&= \frac{\|x_i^*\| - \|x_{i+1}^*\|}{\|x_j^*\|}.
\end{aligned}$$

Moreover, when $j = n$, we see that $\ell_{in}^a(n) = \|x_n^*\|/\|x_j^*\| = (\|x_n^*\| - \|x_{n+1}^*\|)/\|x_j^*\|$. In this case, we see that

$$\begin{aligned}
f(a) &= \sum_{i=1}^{n} \left(\sum_{j=1}^{i} \|\ell_{ij}^a(n) x_j^*\| - \left\| \sum_{j=1}^{i} \ell_{ij}^a(n) x_j^* \right\| \right) \\
&= \sum_{j=2}^{n} \left(\sum_{j=1}^{i} \left\| \frac{\|x_i^*\| - \|x_{i+1}^*\|}{\|x_j^*\|} x_j^* \right\| - \left\| \sum_{j=1}^{i} \frac{\|x_i^*\| - \|x_{i+1}^*\|}{\|x_j^*\|} x_j^* \right\| \right) \\
&= \sum_{i=2}^{n} \left\{ \sum_{j=1}^{i} \left\| \frac{x_j^*}{\|x_j^*\|} \right\| (\|x_i^*\| - \|x_{i+1}^*\|) - \left\| \sum_{j=1}^{i} \frac{x_j^*}{\|x_j^*\|} \right\| (\|x_i^*\| - \|x_{i+1}^*\|) \right\}
\end{aligned}$$

$$= \sum_{i=2}^{n} \left\{ \sum_{j=1}^{i} (\|x_i^*\| - \|x_{i+1}^*\|) - \left\| \sum_{j=1}^{i} \frac{x_j^*}{\|x_j^*\|} \right\| (\|x_i^*\| - \|x_{i+1}^*\|) \right\}$$

$$= \sum_{i=1}^{n} \left(i - \left\| \sum_{j=1}^{i} \frac{x_j^*}{\|x_j^*\|} \right\| \right) (\|x_i^*\| - \|x_{i+1}^*\|).$$

Applying Theorem 2.3.2, we have

$$0 \le \sum_{i=2}^{n} \left(i - \left\| \sum_{j=1}^{i} \frac{x_j^*}{\|x_j^*\|} \right\| \right) (\|x_i^*\| - \|x_{i+1}^*\|) \le \sum_{j=1}^{n} \|x_j^*\| - \left\| \sum_{j=1}^{n} x_j^* \right\|.$$

We next prove a general case by using a technique in the proof of Theorem 2.2.3. Let $\|x_1^*\| \ge \|x_2^*\| \ge \cdots \ge \|x_n^*\|$. For each fixed integer m with $m > n$, we set

$$x_{i,m}^* = \left(1 - \frac{i}{m}\right) x_i^* \quad (1 \le i \le n).$$

Then we see that $\|x_{1,m}^*\| > \|x_{2,m}^*\| > \cdots > \|x_{n,m}^*\| > 0$, and so we have

$$0 \le \sum_{i=2}^{n} \left(i - \left\| \sum_{j=1}^{i} \frac{x_{j,m}^*}{\|x_{j,m}^*\|} \right\| \right) (\|x_{i,m}^*\| - \|x_{i+1,m}^*\|) \le \sum_{j=1}^{n} \|x_{j,m}^*\| - \left\| \sum_{j=1}^{n} x_{j,m}^* \right\|,$$

where $x_{n+1,m}^* = 0$. Since $x_{i,m}^* \to x_i^*$ $(m \to \infty)$, we have the conclusion. This completes the proof. □

2.3.2 Equality attainedness in a strictly convex Banach space

In this subsection, we consider equality attainedness for (55) in a strictly convex Banach space. For a Banach space X, let S_X be a unit sphere in X: $S_X = \{x \in X : \|x\| = 1\}$. A Banach space X is *strictly convex* if for all $x, y \in S_X$ with $x \ne y$, $\|x+y\| < 2$ holds. For the case of the usual triangle inequality, equality attainedness is following.

Lemma 2.3.7 (cf. [1, Problem 11.1]). *Let $(X, \|\cdot\|)$ be a strictly convex Banach space. For $x_1, \ldots, x_n \in X$, the following assertions are equivalent:*

(i) $\left\| \sum_{j=1}^{n} x_j \right\| = \sum_{j=1}^{n} \|x_j\|$;

(ii) $\|x_i\| x_j = \|x_j\| x_i$ *for all* $i, j \in \{1, \cdots, n\}$.

In [19] and [27], Kato et al. consider equality attainedness for the inequality (45) and (49). For each m with $1 \leq m \leq n$, we put $I_m = \{1, 2, \dots, m\}$. For $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{R}^n$ and $1 \leq m \leq n$, we define

$$I_m^+(\alpha) = \{k \in I_m : \alpha_k > 0\},$$
$$I_m^-(\alpha) = \{k \in I_m : \alpha_k < 0\}.$$

$|I_m^+(\alpha)|$ and $|I_m^-(\alpha)|$ are the cardinal numbers of $I_m^+(\alpha)$ and $I_m^-(\alpha)$ respectively.

Theorem 2.3.8 (cf. [27, Theorem 3.7]). *Let X be a strictly convex Banach space and $x_1, x_2, \dots, x_n$ nonzero elements in X with $\|x_1\| > \|x_2\| > \cdots > \|x_n\|$. Then the equality*

$$\left\| \sum_{j=1}^n x_j \right\| + \sum_{k=2}^n \left(k - \left\| \sum_{j=1}^k \frac{x_j}{\|x_j\|} \right\| \right) (\|x_k\| - \|x_{k+1}\|) = \sum_{j=1}^n \|x_j\|$$

holds if and only if there exists $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{R}^n$ with $1 = \alpha_1 > |\alpha_2| > |\alpha_3| > \cdots > |\alpha_n|$ such that

$$x_m = \alpha_m x_1,$$
$$|I_m^+(\alpha)| \geq |I_m^-(\alpha)|$$

for every m with $1 \geq m \geq n$.

For two or three elements cases are following.

Theorem 2.3.9 (cf. [27, Theorem 3.4]). *Let X be a strictly convex Banach space and x, y nonzero elements in X with $\|x\| > \|y\|$. Then the equality*

$$\|x + y\| + \left(2 - \left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\| \right) \|y\| = \|x\| + \|y\|$$

holds if and only if there exists a real number α with $0 < \alpha < 1$ satisfying $y = \pm \alpha x$.

Theorem 2.3.10 (cf. [27, Theorem 3.5]). *Let X be a strictly convex Banach space and x, y, z nonzero elements in X with $\|x\| > \|y\| > \|z\|$. Then the equality*

$$\|x + y + z\| + \left(3 - \left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} + \frac{z}{\|z\|} \right\| \right) \|z\| + \left(2 - \left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\| \right) (\|y\| - \|z\|) = \|x\| + \|y\| + \|z\|$$

holds if and only if there exist α, β with $0 < \beta < \alpha < 1$ satisfying one of the following conditions:

(a) $y = \alpha x, z = \pm\beta x$,

(b) $y = -\alpha x, z = \beta x$.

Now we consider equality attainedness for (55). For $a = (a_{ij}) \in L_n$ and $x_1, \ldots, x_n \in X$, put

$$J = \left\{ j \in \{1, \ldots, n\} : \sum_{i=j}^{n} \ell_{ij}^a(n) \neq 1 \right\},$$
$$I = \left\{ i \in \{1, \ldots, n\} : \sum_{j=1}^{i} \ell_{ij}^a(n) x_j \neq 0 \right\}.$$

Lemma 2.3.11. *Take $a = (a_{ij}) \in L_n$. Then*

(i) *$J = \{1, \ldots, n\}$ if and only if $a_{ij} \in [0, 1)$ for all $i, j \in \{1, \ldots, n\}$; and*

(ii) *$J = \varnothing$ if and only if, for each $j \in \{1, \ldots, n\}$, there exists i with $j \leq i \leq n$ such that $a_{ij} = 1$*

Proposition 2.3.12. *Let $n \geq 2$ and $a = (a_{ij})$ in L_n. For $x_1, \ldots, x_n$ in a strictly convex Banach space X, the equality*

$$\sum_{i=1}^{n} \left(\sum_{j=1}^{i} \|\ell_{ij}^a(n) x_j\| - \left\| \sum_{j=1}^{i} \ell_{ij}^a(n) x_j \right\| \right) = \sum_{j=1}^{n} \|x_j\| - \left\| \sum_{j=1}^{n} x_j \right\| \tag{58}$$

holds if and only if either $I = \varnothing$ and $J = \varnothing$ or, for each $g, h, k, m \in \{1, \ldots, n\}$, the following assertions hold:

(i) *if $J \neq \varnothing$, then, for each $g, h, k \in J$ and $m \in \{1, \ldots, n\}$,*

$$\|x_h\| x_g = \|x_g\| x_h, \tag{59}$$

$$\left\| \sum_{j=1}^{m} \ell_{mj}^a(n) x_j \right\| x_k = \|x_k\| \sum_{j=1}^{m} \ell_{mj}^a(n) x_j; \tag{60}$$

(ii) *if $J = \varnothing$, then, for each $k, m \in \{1, \ldots, n\}$,*

$$\left\| \sum_{j=1}^{m} \ell_{mj}^a(n) x_j \right\| \sum_{j=1}^{k} \ell_{kj}^a(n) x_j = \left\| \sum_{j=1}^{k} \ell_{kj}^a(n) x_j \right\| \sum_{j=1}^{m} \ell_{mj}^a(n) x_j. \tag{61}$$

Proof. Let $I = \varnothing$ and $J = \varnothing$. In this case, for each $i, j \in \{1, \dots, n\}$,

$$\sum_{j=1}^{i} \ell_{ij}^a(n) x_j = 0 \quad \text{and} \quad \sum_{i=j}^{n} \ell_{ij}^a(n) = 1.$$

Hence

$$0 = \sum_{i=1}^{n} \sum_{j=1}^{i} \ell_{ij}^a(n) x_j = \sum_{j=1}^{n} \sum_{i=j}^{n} \ell_{ij}^a(n) x_j = \sum_{j=1}^{n} x_j \sum_{i=j}^{n} \ell_{ij}^a(n) = \sum_{j=1}^{n} x_j.$$

Therefore we have

$$\begin{aligned}
\sum_{i=1}^{n} \left(\sum_{j=1}^{i} \|\ell_{ij}^a(n) x_j\| - \left\| \sum_{j=1}^{i} \ell_{ij}^a(n) x_j \right\| \right) &= \sum_{i=1}^{n} \sum_{j=1}^{i} \ell_{ij}^a(n) \|x_j\| \\
&= \sum_{j=1}^{n} \sum_{i=j}^{n} \ell_{ij}^a(n) \|x_j\| \\
&= \sum_{j=1}^{n} \|x_j\| \sum_{i=j}^{n} \ell_{ij}^a(n) \\
&= \sum_{j=1}^{n} \|x_j\| \\
&= \sum_{j=1}^{n} \|x_j\| - \left\| \sum_{j=1}^{n} x_j \right\|.
\end{aligned}$$

Next let $J \neq \varnothing$. If (58) holds, then the equality (57) also holds. Thus, by Lemma 2.3.7, the following equalities hold for any $g, h, k, m \in \{1, \dots, n\}$:

$$\begin{aligned}
&\left\| \left(1 - \sum_{i=h}^{n} \ell_{ih}^a(n) \right) x_h \right\| \left(1 - \sum_{i=g}^{n} \ell_{ig}^a(n) \right) x_g \\
&\qquad = \left\| \left(1 - \sum_{i=g}^{n} \ell_{ig}^a(n) \right) x_g \right\| \left(1 - \sum_{i=h}^{n} \ell_{ih}^a(n) \right) x_h, \\
&\left\| \sum_{j=1}^{m} \ell_{mj}^a(n) x_j \right\| \left(1 - \sum_{i=k}^{n} \ell_{ik}^a(n) \right) x_k \\
&\qquad = \left\| \left(1 - \sum_{i=k}^{n} \ell_{ik}^a(n) \right) x_k \right\| \sum_{j=1}^{m} \ell_{mj}^a(n) x_j.
\end{aligned}$$

By Lemma 2.3.3, we see that $0 < 1 - \sum_{i=j}^{n} \ell_{ij}^a(n)$ for all $j \in J$. Thus the above equalities equivalent to $\|x_h\| x_g = \|x_g\| x_h, g, h \in J$, and

$$\left\| \sum_{j=1}^{m} \ell_{mj}^a(n) x_j \right\| x_k = \|x_k\| \sum_{j=1}^{m} \ell_{mj}^a(n) x_j, \qquad k \in J,\ m \in \{1, \dots, n\},$$

respectively. The converse is clear. Finally let $J = \varnothing$ and $I \neq \varnothing$. In this case if (58) holds, then, for each $k, m \in I$, we have

$$\left\|\sum_{j=1}^{m} \ell_{mj}^{a}(n)x_j\right\| \sum_{j=1}^{k} \ell_{kj}^{a}(n)x_j = \left\|\sum_{j=1}^{k} \ell_{kj}^{a}(n)x_j\right\| \sum_{j=1}^{m} \ell_{mj}^{a}(n)x_j.$$

The converse is clear, and the proof is completed. □

Next, we consider equality attainedness of (58) in more detail.

Theorem 2.3.13. *Let $n \geq 2$ and $a = (a_{ij})$ in L_n. Assume that $J \neq \varnothing$ and put $j_0 = \min_{j \in J} j$. For nonzero elements $x_1, \ldots, x_n$ in a strictly convex Banach space X, the equality (58) holds if and only if there exist positive real numbers $\alpha_j, j \in J$, such that $x_j = \alpha_j x_{j_0}$, and, for each $i \in \{1, \ldots, n\}$, there are real numbers β_i such that*

$$0 \leq \sum_{j \in J_i} \alpha_j \ell_{ij}^{a}(n) + \beta_i \quad \textit{and} \quad \sum_{j \in J_i^c} \ell_{ij}^{a}(n)x_j = \beta_i x_{j_0},$$

where $J_i = \{j \in J : j \leq i\}$ and J_i^c is a complement of J_i in $\{1, \ldots, n\}$.

Proof. ($\Rightarrow$) Assume that (58) holds. By Proposition 2.3.12 (i), if we put $\alpha_j = \|x_j\|/\|x_{j_0}\|, j \in J$, then $\alpha_j > 0$ and $x_j = \alpha_j x_{j_0}, j \in J$. Moreover, for each $i \in \{1, \ldots, n\}$,

$$\sum_{j=1}^{i} \ell_{ij}^{a}(n)x_j = \left\|\sum_{j=1}^{i} \ell_{ij}^{a}(n)x_j\right\| \frac{x_{j_0}}{\|x_{j_0}\|}.$$

Since

$$\begin{aligned}\sum_{j=1}^{i} \ell_{ij}^{a}(n)x_j &= \sum_{j \in J_i} \ell_{ij}^{a}(n)x_j + \sum_{j \in J_i^c} \ell_{ij}^{a}(n)x_j \\ &= \sum_{j \in J_i} \alpha_j \ell_{ij}^{a}(n)x_{j_0} + \sum_{j \in J_i^c} \ell_{ij}^{a}(n)x_j, \qquad i \in \{1, \ldots, n\},\end{aligned}$$

we have

$$\sum_{j \in J_i^c} \ell_{ij}^{a}(n)x_j = \left(\frac{1}{\|x_{j_0}\|}\left\|\sum_{j=1}^{i} \ell_{ij}^{a}(n)x_j\right\| - \sum_{j \in J_i} \alpha_j \ell_{ij}^{a}(n)\right) x_{j_0}.$$

Put

$$\beta_i = \frac{1}{\|x_{j_0}\|}\left\|\sum_{j=1}^{i} \ell_{ij}^{a}(n)x_j\right\| - \sum_{j \in J_i} \alpha_j \ell_{ij}^{a}(n).$$

Then we see that $0 \leq \sum_{j\in J_i} \alpha_j \ell_{ij}^a(n) + \beta_i$ and

$$\sum_{j\in J_i^c} \ell_{ij}^a(n) x_j = \beta_i x_{j_0}, \qquad i \in \{1,\ldots,n\}. \tag{62}$$

($\Leftarrow$) Assume that there exist positive real numbers α_j and real numbers β_i such that $x_j = \alpha_j x_{j_0}, j \in J$, $0 \leq \sum_{j\in J_i} \alpha_j \ell_{ij}^a(n) + \beta_i$ and (62) holds. Since $\|x_j\| = \alpha_j \|x_{j_0}\|$ for all $j \in J$, it is clear that (59) is valid. So, by Proposition 2.3.12 (i), we need only to prove (60). Since

$$\sum_{j=1}^{i} \ell_{ij}^a(n) x_j = \left(\sum_{j\in J_i} \alpha_j \ell_{ij}^a(n) + \beta_i \right) x_{j_0}, \qquad i \in \{1,\ldots,n\},$$

we have

$$\left\| \sum_{j=1}^{i} \ell_{ij}^a(n) x_j \right\| = \left(\sum_{j\in J_i} \alpha_j \ell_{ij}^a(n) + \beta_i \right) \|x_{j_0}\|, \qquad i \in \{1,\ldots,n\}.$$

Thus

$$\sum_{j=1}^{i} \ell_{ij}^a(n) x_j = \frac{1}{\|x_{j_0}\|} \left\| \sum_{j=1}^{i} \ell_{ij}^a(n) x_j \right\| x_{j_0}, \qquad i \in \{1,\ldots,n\},$$

and the proof is completed. □

By Lemma 2.3.11, for $a = (a_{ij}) \in L_n$, recall that $a_{ij} \in [0,1), i,j \in \{1,\ldots,n\}$, is equivalent to $J = \{1,\ldots,n\}$. Thus we immediately have the following corollary.

Corollary 2.3.14. *Let $n \geq 2$ and $a = (a_{ij})$ in L_n. For nonzero elements $x_1,\ldots,x_n$ in a strictly convex Banach space X, if $a_{ij} \in [0,1)$ for all $i,j \in \{1,\ldots,n\}$, then the equality* (58) *holds if and only if there exist positive real numbers $\alpha_j, j \in \{1,\ldots,n\}$, such that $x_j = \alpha_j x_1, j \in \{1,\ldots,n\}$.*

Next, we consider the case $J = \varnothing$ for $a = (a_{ij}) \in L_n$. In this case we note that $a_{nn} = 1$. Take $a = (a_{ij}) \in L_n$ with $a_{nj} = 1$ for all $j \in \{1,\ldots,n\}$. Then we see that

$$(\ell_{ij}^a(n)) = \begin{pmatrix} 0 & \ldots & \ldots & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & 0 & 0 \\ 1 & \ldots & 1 & 1 \end{pmatrix}.$$

Thus it is clear that, for each $x_1,\ldots,x_n$ in a normed space X, (58) always holds. Thus we are interested in the case $a_{nj} \neq 1$ for some $j \in \{1,\ldots,n-1\}$. However, generally it is complicated. We give the following special case.

Theorem 2.3.15. *Let $n \geq 2$ and $a = (a_{ij})$ in L_n. For nonzero elements $x_1, \ldots, x_n$ in a strictly convex Banach space X, if $a_{ij} \in [0, 1)$ for all $i > j$ and $a_{ii} = 1$ for all $i \in \{1, \ldots, n\}$, then the equality (58) holds if and only if there exist real numbers $\alpha_j, j \in \{1, \ldots, n\}$, such that $x_j = \alpha_j x_1$ and $\sum_{j=1}^{i} \alpha_j \ell_{ij}^a(n) \geq 0, i \in \{1, \ldots, n\}$.*

Proof. ($\Rightarrow$) Since $a_{ii} = 1$ for all $i \in \{1, \ldots, n\}$, by Lemma 2.3.3, $\sum_{i=k}^{n} \ell_{ik}^a(n) = 1, k \in \{1, \ldots, n\}$, and we note that $\ell_{11}^a(n) > 0$. If (58) holds, then, by Proposition 2.3.12, for all $i \in \{1, \ldots, n\}$,

$$\left\| \sum_{j=1}^{i} \ell_{ij}^a(n) x_j \right\| \ell_{11}^a(n) x_1 = \| \ell_{11}^a(n) x_1 \| \sum_{j=1}^{i} \ell_{ij}^a(n) x_j. \tag{63}$$

Let $i = 2$ and assume that $\sum_{j=1}^{2} \ell_{2j}^a(n) x_j = 0$. Since $\ell_{22}^a(n) > 0$, if we put $\alpha_2 = -\ell_{21}^a(n)/\ell_{22}^a(n)$, then $x_2 = \alpha_2 x_1$ and $\sum_{j=1}^{2} \alpha_j \ell_{2j}^a(n) = 0$. Next assume that $\sum_{j=1}^{2} \ell_{2j}^a(n) x_j \neq 0$. By (63), we have

$$x_2 = \frac{1}{\ell_{22}^a(n)} \left(\frac{1}{\|x_1\|} \left\| \sum_{j=1}^{2} \ell_{2j}^a(n) x_j \right\| - \alpha_1 \ell_{21}^a(n) \right) x_1 \overset{\text{def}}{=} \alpha_2 x_1.$$

We see that

$$\left| \sum_{j=1}^{2} \alpha_j \ell_{2j}^a(n) \right| x_1 = \left\| \sum_{j=1}^{2} \ell_{2j}^a(n) x_j \right\| \frac{x_1}{\|x_1\|} = \sum_{j=1}^{2} \ell_{2j}^a(n) x_j = \left(\sum_{j=1}^{2} \alpha_j \ell_{2j}^a(n) \right) x_1.$$

Thus $\sum_{j=1}^{2} \alpha_j \ell_{2j}^a(n) > 0$, and hence in any case we have

$$\sum_{j=1}^{2} \alpha_j \ell_{2j}^a(n) \geq 0.$$

Now we assume that, for each $i = m$ with $2 \leq m \leq n-1$, (58) holds. Then there exist real numbers $\alpha_j, j \in \{1, \ldots, m\}$, such that $x_j = \alpha_j x_1, j \in \{1, \ldots, m\}$, and $\sum_{j=1}^{i} \alpha_j \ell_{ij}^a(n) \geq 0, i \in \{1, \ldots, m\}$.

If $\sum_{j=1}^{m+1} \ell_{m+1\,j}^a(n) x_j = 0$, then we see that $x_{m+1} = \alpha_{m+1} x_1$ and $\sum_{j=1}^{m+1} \alpha_j \ell_{m+1\,j}^a(n) = 0$ for $\alpha_{m+1} = -\sum_{j=1}^{m} \alpha_j \ell_{m+1\,j}^a(n)/\ell_{m+1\,m+1}^a(n)$. In the case that $\sum_{j=1}^{m+1} \ell_{m+1\,j}^a(n) x_j \neq 0$, if we put

$$\alpha_{m+1} = \frac{1}{\ell_{m+1\,m+1}^a(n)} \left(\frac{1}{\|x_1\|} \left\| \sum_{j=1}^{m+1} \ell_{m+1\,j}^a(n) x_j \right\| - \alpha_1 \ell_{m+1\,1}^a(n) \right),$$

then we see that $x_{m+1} = \alpha_{m+1}x_1$ and, by (63),

$$\begin{aligned}\left|\sum_{j=1}^{m+1} \alpha_j \ell^a_{m+1\,j}(n)\right| x_1 &= \left\|\sum_{j=1}^{m+1} \ell^a_{m+1\,j}(n)x_j\right\| \frac{x_1}{\|x_1\|} \\ &= \sum_{j=1}^{m+1} \ell^a_{m+1\,j}(n)x_j \\ &= \left(\sum_{j=1}^{m+1} \alpha_j \ell^a_{m+1\,j}(n)\right) x_1.\end{aligned}$$

Thus $\sum_{j=1}^{m+1} \alpha_j \ell^a_{m+1\,j}(n) > 0$, and hence we have

$$\sum_{j=1}^{m+1} \alpha_j \ell^a_{m+1\,j}(n) \geq 0.$$

By using induction on i, we have desired result.
($\Leftarrow$) Suppose that there exist real numbers $\alpha_j, j \in \{1, \ldots, n\}$, such that $x_j = \alpha_j x_1, j \in \{1, \ldots, n\}$, and $\sum_{j=1}^{i} \alpha_j \ell^a_{ij}(n) \geq 0, i \in \{1, \ldots, n\}$, where $\alpha_1 = 1$. If $m \notin I$, then, for each $k \in \{1, \ldots, n\}$,

$$\left\|\sum_{j=1}^{m} \ell^a_{mj}(n)x_j\right\| x_k = 0 = \|x_k\| \sum_{j=1}^{m} \ell^a_{mj}(n)x_j.$$

On the other hand, since

$$\left\|\sum_{j=1}^{i} \ell^a_{ij}(n)x_j\right\| = \|x_1\| \sum_{j=1}^{i} \alpha_j \ell^a_{ij}(n)$$

for each $i \in I$, we have

$$\sum_{j=1}^{i} \ell^a_{ij}(n)x_j = \left\|\sum_{j=1}^{i} \ell^a_{ij}(n)x_j\right\| \frac{x_1}{\|x_1\|}.$$

Therefore, by Proposition 2.3.12 (ii), the proof is completed. □

As a corollary of Theorem 2.3.15, we can get [27, Theorem 3.7].

Corollary 2.3.16 (cf. [27, Theorem 3.7]). *For all nonzero elements $x_1, \ldots, x_n$ in a strictly convex Banach space X with $\|x_1\| > \|x_2\| > \ldots > \|x_n\|$, the equality*

$$\sum_{i=1}^{n} \left(i - \left\|\sum_{j=1}^{i} \frac{x_j}{\|x_j\|}\right\|\right) (\|x_i\| - \|x_{i+1}\|) = \sum_{j=1}^{n} \|x_j\| - \left\|\sum_{j=1}^{n} x_j\right\| \tag{64}$$

holds if and only if there exist real numbers $\alpha_j, j \in \{1, \ldots, n\}$*, with* $1 = \alpha_1 > |\alpha_2| > \ldots > |\alpha_n|$ *such that* $x_j = \alpha_j x_1, j \in \{1, \ldots, n\}$*, and* $|I_m^+(\alpha)| \geq |I_m^-(\alpha)|$ *for every* m *with* $1 \leq m \leq n$*, where* $|I_m^+(\alpha)|$ *and* $|I_m^-(\alpha)|$ *are the cardinal numbers of* $I_m^+(\alpha) = \{j \in \{1, \ldots, m\} : \alpha_j > 0\}$ *and* $I_m^-(\alpha) = \{j \in \{1, \ldots, m\} : \alpha_j < 0\}$*, respectively.*

Proof. As in the proof of [24, Corollary 3.5] or the proof of Theorem 2.3.6, if we put

$$a_{ij} = \frac{\|x_i\| - \|x_{i+1}\|}{\|x_j\| - \|x_{i+1}\|}, \quad i, j \in \{1, \ldots, n\},$$

then $a = (a_{ij}) \in L_n$ and

$$\ell_{ij}^a(n) = \frac{\|x_i\| - \|x_{i+1}\|}{\|x_j\|}, \quad i, j \in \{1, \ldots, n\}.$$

In this case, (63) is equivalent to (64). Thus, by (61), if (64) holds, then there exists $\alpha_j \in \mathbb{R}$ such that

$$x_j = \alpha_j x_1 \quad \text{and} \quad \sum_{j=1}^{m} \alpha_j \ell_{mj}^a(n) \geq 0, \quad j, m \in \{1, \ldots, n\}.$$

Since

$$\ell_{mj}^a(n) = \frac{\|x_m\| - \|x_{m+1}\|}{\|x_j\|} = \frac{|\alpha_m| - |\alpha_{m+1}|}{|\alpha_j|},$$

we see that

$$0 \leq \sum_{j=1}^{m} \alpha_j \ell_{mj}^a(n) = (|\alpha_m| - |\alpha_{m+1}|) \sum_{j=1}^{m} \frac{\alpha_j}{|\alpha_j|}.$$

Thus we have

$$\begin{aligned} 0 &\leq \sum_{j=1}^{m} \frac{\alpha_j}{|\alpha_j|} \\ &= \sum_{j \in I_m^+(\alpha)}^{m} \frac{\alpha_j}{|\alpha_j|} + \sum_{j \in I_m^-(\alpha)}^{m} \frac{\alpha_j}{|\alpha_j|} \\ &= \sum_{j \in I_m^+(\alpha)}^{m} 1 + \sum_{j \in I_m^-(\alpha)}^{m} (-1) \\ &= |I_m^+(\alpha)| - |I_m^-(\alpha)|, \end{aligned}$$

$m \in \{1, \ldots, n\}$. The converse is clear, and the proof is completed. □

2.4 Applications of sharp triangle inequalities

By using sharp triangle inequalities, we can characterize some geometrical properties of Banach spaces. In this section, for a Banach space X, S_X is a unit sphere in X: $S_X = \{x \in X : \|x\| = 1\}$, and B_X is a unit ball in X: $B_X = \{x \in X : \|x\| \leq 1\}$.

2.4.1 The strict convexity and the uniform convexity

We first consider the strict convexity. A Banach space X is *strictly convex* if for all $x, y \in S_X$ with $x \neq y$, $\|x + y\| < 2$ holds (cf. [23]).

Theorem 2.4.1 (cf. [37, Proposition 8])**.** *Let X be a Banach space. The following conditions are equivalent*:

(i) *X is strictly convex.*

(ii) *Let $x, y \in X$ such that $\|x + y\| = \|x\| + \|y\|$ and $x, y \neq 0$. Then there exists $\alpha > 0$ such that $x = \alpha y$.*

Proof. Assume (i). Suppose that $\|x + y\| = \|x\| + \|y\|$ and $x, y \neq 0$. By the shape triangle inequality (47):

$$\|x + y\| + \left(2 - \left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\|\right) \min\{\|x\|, \|y\|\} \leq \|x\| + \|y\|,$$

we have

$$\left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\| = 2.$$

Since X is strictly convex, we have $x/\|x\| = y/\|y\|$. Thus $x = \alpha y$ and $\alpha > 0$, we have (ii). The opposite implication is clear. □

Next we consider the uniform convexity. A Banach space X is *uniformly convex* if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $x, y \in S_X$ and $\|x - y\| \geq \varepsilon$ imply $\|x + y\| < 2(1 - \delta)$ (cf. [23]).

Theorem 2.4.2 (cf. [37, Proposition 9])**.** *Let X be a Banach space. The following conditions are equivalent*:

(i) *X is uniformly convex.*

(ii) *For all $\{x_n\}, \{y_n\}$ in S_X such that $\|x_n + y_n\| \to 2$, we have $\|x_n - y_n\| \to 0$.*

(iii) *For all* $\{x_n\}, \{y_n\}$ *in* B_X *such that* $\|x_n + y_n\| \to 2$*, we have* $\|x_n - y_n\| \to 0$*.*

Proof. We only show the implication (ii) $\Rightarrow$ (iii). Take $\{x_n\}, \{y_n\}$ in B_X such that $\|x_n+y_n\| \to 2$. Since $\|x_n+y_n\| \le \|x_n\|+\|y_n\| \le 2$, we have $\|x_n\| \to 1$ and $\|y_n\| \to 1$. Hence we may suppose that $x_n \neq 0$ and $y_n \neq 0$ for all n. Then, since

$$\|x_n + y_n\| + \left(2 - \left\|\frac{x_n}{\|x_n\|} + \frac{y_n}{\|y_n\|}\right\|\right) \min\{\|x_n\|, \|y_n\|\} \le \|x_n\| + \|y_n\| \le 2,$$

we have

$$\left\|\frac{x_n}{\|x_n\|} + \frac{y_n}{\|y_n\|}\right\| \to 2.$$

By the assumption,

$$\left\|\frac{x_n}{\|x_n\|} - \frac{y_n}{\|y_n\|}\right\| \to 0.$$

Since

$$\|x_n - y_n\| \le \left\|x_n - \frac{x_n}{\|x_n\|}\right\| + \left\|\frac{x_n}{\|x_n\|} - \frac{y_n}{\|y_n\|}\right\| + \left\|\frac{y_n}{\|y_n\|} - y_n\right\|,$$

we have $\|x_n - y_n\| \to 0$ as $n \to \infty$. □

2.4.2 The uniform non-squareness

Next we consider the uniform non-squareness. A Banach space X is *uniformly non-square* if there exist $\varepsilon > 0$ such that for all $x, y \in S_X$, we have $\min\{\|x+y\|, \|x-y\|\} \le 2(1-\varepsilon)$ (cf. [17] and [5]).

Theorem 2.4.3 (cf. [37, Proposition 10])**.** *Let* X *be a Banach space. The following conditions are equivalent*:

(i) X *is uniformly non-square.*

(ii) *There exists* $\varepsilon > 0$ *such that for all* $x, y \in B_X$*, we have* $\min\{\|x+y\|, \|x-y\|\} \le 2(1-\varepsilon)$*.*

Proof. Assume (i). Then there exists $\varepsilon > 0$ such that for all $x, y \in S_X$, we have $\min\{\|x + y\|, \|x - y\|\} \le 2(1 - \varepsilon)$. Take any $x, y \in B_X$. If $\min\{\|x\|, \|y\|\} \le 1/2$, then

$$\|x \pm y\| \le \|x\| + \|y\| \le 1 + \frac{1}{2} = \frac{3}{2} = 2\left(1 - \frac{1}{4}\right).$$

If $\min\{\|x\|, \|y\|\} \geq 1/2$, then

$$\min\left\{\left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\|, \left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\|\right\} \leq 2(1-\varepsilon).$$

Let

$$\left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\| \leq \left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\|.$$

Then by the shape triangle inequality,

$$\begin{aligned}\|x+y\| &\leq \|x\| + \|y\| - \left(2 - \left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\|\right)\min\{\|x\|, \|y\|\} \\ &\leq 2 - \frac{2\varepsilon}{2} = 2\left(1 - \frac{\varepsilon}{2}\right).\end{aligned}$$

Similarly, if

$$\left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\| \geq \left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\|,$$

then

$$\begin{aligned}\|x-y\| &\leq \|x\| + \|y\| - \left(2 - \left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\|\right)\min\{\|x\|, \|y\|\} \\ &\leq 2 - \frac{2\varepsilon}{2} = 2\left(1 - \frac{\varepsilon}{2}\right).\end{aligned}$$

Hence $\min\{\|x+y\|, \|x-y\|\} \leq 2(1-\varepsilon/2)$. Put $\varepsilon_0 = \min\{\varepsilon/2, 1/4\}$, then we have (ii). The opposite implication is clear. □

In [19], Kato et al. mentioned a direct application of (45) of Theorem 2.2.1 to a uniformly non-ℓ_1^n Banach space. A Banach space X is called *uniformly non-*ℓ_1^n provided there exists ε $(0 < \varepsilon < 1)$ such that for any $x_1, \ldots, x_n \in S_X$, there exists $\theta = (\theta_j)$ of n signs ± 1 for which

$$\left\|\sum_{j=1}^{n} \theta_j x_j\right\| \leq n(1-\varepsilon). \tag{65}$$

When $n = 2$, X is uniformly non-square (cf. [17] and [5]).

Theorem 2.4.4 (cf. [19, Corollary 4]). *For a Banach space X the following conditions are equivalent*:

(i) *X is uniformly non-ℓ_1^n.*

(ii) *There exists* ε $(0 < \varepsilon < 1)$ *such that for any* $x_1, \ldots, x_n \in B_X$*, there exists* $\theta = (\theta_j)$ *of* n *signs* ± 1 *for which* (65) *holds true.*

Proof. Assume (i). Then there exists ε $(0 < \varepsilon < 1)$ such that for any $x_1, \ldots, x_n \in S_X$, there exists $\theta = (\theta_j)$ of n signs ± 1 for which (65) is valid. Take $x_1, \ldots, x_n \in B_X$. If $\|x_{j_0}\| := \min\{\|x_1\|, \ldots, \|x_n\|\} \leq 1/2$, we have

$$\left\| \sum_{j=1}^{n} \theta_j x_j \right\| \leq \sum_{j \neq j_0} \|x_j\| + \|x_{j_0}\| \leq (n-1) + \frac{1}{2} \leq n \left(1 - \frac{1}{2n}\right).$$

Let $\|x_{j_0}\| \geq 1/2$. According to our assumption there exists n signs (θ_j) for which (65) is valid for $x_1/\|x_1\|, \ldots, x_n/\|x_n\|$. Therefore by the first inequality of Theorem 2.2.1,

$$\begin{aligned} \left\| \sum_{j=1}^{n} \theta_j x_j \right\| &\leq \sum_{j=1}^{n} \|x_j\| - \left(n - \left\| \sum_{j=1}^{n} \theta_j \frac{x_j}{\|x_j\|} \right\| \right) \|x_{j_0}\| \\ &\leq 2 - \frac{n\varepsilon}{2} \\ &= n \left(1 - \frac{\varepsilon}{2}\right). \end{aligned}$$

Consequently by letting $\varepsilon_0 = \min\{\varepsilon/2, 1/(2n)\}$ we have the conclusion. □

References

[1] Y. A. Abramovich and C. D. Aliprantis, *Problems in operator theory. Graduate Studies in Mathematics*, 51. AMS, Providence, RI, (2002).

[2] S. Abramovich, S. Ivelić and J. Pečarić, *Extension of the Euler-Lagrange identity by superquadratic power functions*, International Journal of Pure and Applied Math., **74** (2012), no. 2, 209–220.

[3] A. H. Ansari and M. S. Moslehian, *More on reverse triangle inequality in inner products spaces*, Int. J. Math. Math. Sci. (2005), no.18, 2883–2893.

[4] A. H. Ansari and M. S. Moslehian, *Refinements of reverse triangle inequalities in inner product spaces*, J. Inequal. Pure Appl. Math., **6**(3) (2005), article 64, 12pp.

[5] B. Beauzamy, *Introduction to Banach Spaces and Their Geometry, 2nd ed.*, North-Holland, Amsterdam-New York-Oxford, 1985.

[6] H. Belbachir, M. Mirzavaziri and S. Moslehian, *q-norms are really norms*, The Australian Journal of Mathematical Analysis and Applications, vol. 3, no. 1 (2006), article 2, 1–3.

[7] R. Bhatia, *Matrix Analysis*, Springer-Verlag, Berlin/New York, 1997.

[8] F. F. Bonsall, and J. Duncan, *Numerical Ranges* II, Lecture Note Series, Vol. 10, London Math. Soc., London, 1973.

[9] F. Dadipour, M. S. Moslehian, J. M. Rassias and S.-E. Takahasi, *Characterizations of a generalized triangle inequality in normed spaces*, Nonlinear Analysis, **75** (2012), 735–741.

[10] S. S. Dragomir, *Reverses of the triangle inequality in Banach spaces*, JIPAM. J. Inequal. Pure Appl. Math. **6**(5) (2005), Art. 129, pp. 46.

[11] S. S. Dragomir, *Generalizations of the Pećarić-Rajić inequality in normed linear spaces*, Math. Inequal. Appl. **12** no.1 (2009), 53–65.

[12] M. Fujii, M. Kato, K.-S. Saito and T. Tamura, *Sharp mean triangle inequality*, Math. Inequal. Appl. **13** (2010), 743–752.

[13] C.-Y. Hsu, S.-Y. Shaw and H.-J. Wong, *Refinements of generalized triangle inequalities*, J. Math. Anal. Appl. **344** (2008), 17–31.

[14] H. Hudzik and T. R. Landes, *Characteristic of convexity of Köthe function spaces*, Math. Ann., **294** (1992), 117–124.

[15] T. Izumida, K.-I. Mitani and K.-S. Saito, *On generalized triangle inequalities in normed spaces*, RIMS Kokyuroku, **1893** (2014), 20–27.

[16] T. Izumida, K.-I. Mitani and K.-S. Saito, *Another approach to characterizations of generalized triangle inequalities in normed spaces*, Cent. Eur. J. Math., **12**(11) (2014), 1615–1623.

[17] C. James, *Uniformly non-square Banach spaces*, Ann. of Math., **80** (1964), 542–550.

[18] M. Kato, K.-S. Saito and T. Tamura, *On ψ-direct sums of Banach spaces and convexity*, J. Aust. Math. Soc., **75** (2003), 413–422.

[19] M. Kato, K.-S. Saito and T. Tamura, *Sharp triangle inequality and its reverse in Banach spaces*, Math. Inequal. Appl., **10**(2) (2007), 451–460.

[20] L. Maligranda, *Simple norm inequalities*, Amer. Math. Monthly, **113** (2006), 256–260.

[21] L. Maligranda, *Some remarks on the triangle inequality for norms*, Banach J. Math. Anal., **2**(2) (2008), 31–41.

[22] M. S. Martirosyan and S. V. Samarchyan, *Inversion of the triangle inequality in $\mathbb{R}^n$*, Izv. Nats. Akad. Nauk Armenii Mat. **38** (2003), no.4, 65–72.

[23] R. E. Megginson, *An Introduction to Banach Space Theory*, Springer, New York, 1998

[24] K. Mineno, Y. Nakamura and T. Ohwada, *Characterization of the intermediate values of the triangle inequality*, Math. Inequal. Appl. **15** (2012), no. 4, 1019–1035.

[25] K.-I. Mitani and K.-S. Saito, *On generalized ℓ_p-spaces*, Hiroshima Math. J., **37** (2007), 1–12.

[26] K.-I. Mitani and K.-S. Saito, *Dual of two dimensional Lorentz sequence spaces*, Nonlinear Analysis, **71** (2009), 5238–5247.

[27] K.-I. Mitani and K.-S. Saito, *On sharp triangle inequalities in Banach spaces* II, J. Inequal. Appl., 2010, Art. ID 323609, 17pp.

[28] K.-I. Mitani, S. Oshiro and K.-S. Saito, *Smoothness of ψ-direct sums of Banach spaces*, Math. Ineq. Appl., **8** (2005), 147–157.

[29] K.-I. Mitani, K.-S. Saito, M. Kato and T. Tamura, *On sharp triangle Inequalities in Banach spaces,* J. Math. Anal. Appl. **10** (2007), 451–460.

[30] H. Mizuguchi, K.-S. Saito and R. Tanaka, *On the calculation of Dunkl-Williams constant of normed linear spaces,* Cent. Eur. J. Math., **11**(7) (2013), 1212–1227.

[31] M. S. Moslehian, F. Dadipour, R. Rajić and A. Marić *A glimpse at the Dunkl-Williams inequality,* Banach J. Math. Anal. **5** no. 2 (2011), 138–151.

[32] L. Nikolova, L.-E. Persson and S. Varošanec, *The Beckenbach-Dresher inequality in the ψ-direct sums of spaces and related results*, J. Ineq. Appl., **2012**, 2012:7, 14pp.

[33] T. Ohwada, *On a continuous mapping and sharp triangle inequalities*, Inequalities and Applications '10, International Series of Numerical Mathematics, Springer, Vol. 161 (2011), 125–136.

[34] K.-S. Saito, M. Kato and Y. Takahashi, *Von Neumann-Jordan constant of absolute normalized norms on* $\mathbb{C}^2$, J. Math. Anal. Appl., **244** (2000), 515–532.

[35] K.-S. Saito, M. Kato and Y. Takahashi, *Absolute norms on* $\mathbb{C}^n$, J. Math. Anal. Appl., **252** (2000), 879–905.

[36] K.-S. Saito, R. An, H. Mizuguchi and K.-I. Mitani, *Another aspect of triangle inequality*, ISRN Math. Anal., 2011, Art. ID 514184, 5pp.

[37] K.-S. Saito and K.-I. Mitani, *On sharp triangle inequalities in Banach spaces and their applications*, Banach and Function Spaces III, Yokohama Publ., Yokohama, (2009), 295–304.

[38] S. Saitoh, *Generalizations of triangle inequality*, J. Inequal. Pure Appl. Math. **4** (2003), No. 3, Article 62, 5 pp.

[39] H. Sano, T. Izumida, K.-I. Mitani, T. Ohwada and K.-S. Saito, *Characterization of the intermediate values of the triangle inequality II*, Cent. Eur. J. Math., **12**(5) (2014), 778–786.

[40] S.-E. Takahasi, J.M. Rassias, S. Saitoh and Y. Takahashi, *Refined generalizations of the triangle inequality on Banach spaces*, Math. Ineq. Appl., **13** (2010), 733–741.

[41] T. Zachariades, *On ℓ_p-spaces and ψ-direct sums of Banach space*, Rocky Mountain J. M., **41** (2011), 971–997.

[42] L. Zhang, T. Ohwada and M. Chō, *Reverses of the triangle inequality in inner product spaces,* Math. Ineq. Appl., **17** (2014), 539–555.