Metric preservers on groups and gyrogroups

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CONTENTS

1. **Introduction**

Let (X_1, d_1) and (X_1, d_2) be two metric spaces. A map $T : X_1 \to X_2$ is called an isometry if it preserves the metric, that is, $d_2(Ta, Tb) = d_1(a, b)$ for every pair of points a and b in X_1. For a normed space $(N, \|\cdot\|)$, the metric d induced by $\|\cdot\|$ is defined by $d(a, b) = \|a - b\|$ for every pair of points a and b in N. For two normed spaces $(N_1, \|\cdot\|_1)$ and $(N_2, \|\cdot\|_2)$, a map $T : N_1 \to N_2$ is an isometry if $\|Ta - Tb\|_2 = \|a - b\|$ for every pair of points a and b in N_1.

The celebrated Mazur-Ulam Theorem [14] states that every bijective isometry T between two real normed spaces $(N_1, \|\cdot\|_1)$ and $(N_2, \|\cdot\|_2)$ is affine, that is, $T((a + b)/2) = (Ta + Tb)/2$ for every pair of points a and b in N_1. In particular, if $T(0) = 0$, then T is real linear. In other words, surjective isometries between noremed spaces are real linear isomorphisms followed by translations. This theorem asserts that a bijecion between two real normed spaces which preserves the metric structure also preserves the algebraic structure automatically. On the other hand, surjective isometries between two complex normed spaces are not necessarily complex linear followed by translations.

It is an interesting problem whether the mappings between spaces which preserve particular objects or properties preserve other objects or properties. The study of isometries has a long history. It dates back at least to the Banach-Stone theorem of 1930's. Let X be a compact Hausdorff space and $C(X)$ the algebra of all complex-valued continuous functions on X. The algebra $C(X)$ is a commutative Banach algebra

equipped with the supremum norm. The Banach-Stone theorem states that $C(X)$ and $C(Y)$ are isometrically complex linear isomorphic to each other if and only if X and Y are homeomorphic. It implies that $C(X)$ and $C(Y)$ are isometrically isomorphic as Banach spaces if and only if $C(X)$ and $C(Y)$ are isometrically isomorphic as Banach algebras. In [10], Kadison describes the structure of all surjective complex linear isometries between two unital C^*-algebras. It follows that two C^*-algebras are isometrically isomorphic as Banach spaces if and only if they are isometrically isomorphic as Jordan*-algebras. It is a non-commutative generalization of the Banach-Stone theorem. There is vast literature of isometries on various linear spaces.

In this paper, we study the algebraic structures of isometries on some structures which need not be linear spaces. In 2003, Väisälä gave a simple proof of the Mazur-Ulam theorem based on the idea of Vogt [19] and reflections in points. Hatori, Hirasawa, Miura and Molnár [6] studied algebraic properties of surjective isometries on groups and proved a Mazur-Ulam theorem on metric groups applying the idea of Väisälä. Applying this theorem, Hatori and Molnár gave a complete description of surjective isometries (with respect to the metric induced by the operator norm) from unitary groups on Hilbert spaces onto itself. By the result, surjective isometries are group automorphisms or group anti-automorphisms followed by left multiplications. They [5, 8] also studied isometries on the unitary groups of unital C^*-algebras. In

section 2, we study isometries on the special orthogonal group based on [2].

In special relativity, the set of all relativistically admissible velocities coincides to $\mathbb{R}_c^3 = \{\boldsymbol{u} \in \mathbb{R}^3 : \|\boldsymbol{u}\| < c\}$, where c is the speed of light in vacuum. The Einstein velocity addition $\oplus_E$ in $\mathbb{R}_c^3$ is given by the equation

$$\boldsymbol{u} \oplus_E \boldsymbol{v} = \frac{1}{1 + \frac{\langle \boldsymbol{u}, \boldsymbol{v} \rangle}{c^2}} \left\{ \boldsymbol{u} + \frac{1}{\gamma_u} \boldsymbol{v} + \frac{1}{c^2} \frac{\gamma_u}{1 + \gamma_u} \langle \boldsymbol{u}, \boldsymbol{v} \rangle \boldsymbol{u} \right\}$$

for all $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}$, where $\langle \cdot, \cdot \rangle$ is the Euclidean inner product and γ_u is the Lorentz factor given by $\gamma_u = (1 - \|\boldsymbol{u}\|^2/c^2)^{-\frac{1}{2}}$. The Einstein velocity addition $\oplus_E$ is not associative in $\mathbb{R}_c^3$. Hence, $(\mathbb{R}_c^3, \oplus_E)$ is not a group. Along with the study of the Einstein's velocity addition law, it turned out that $(\mathbb{R}_c^3, \oplus_E)$ has a structure which is called the gyrogroup. The gyrogroup is a generalization of the group which is not necessarily associative. Some gyrogroups equipped with their own gyrometrics. In the Einstein gyrogroup $(\mathbb{R}_c^3, \oplus_E)$, its gyrometric is given by $\|\boldsymbol{u} \oplus_E (-\boldsymbol{v})\|$ for any pair $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}_c^3$, where $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^3$. In section 3, we study gyrometric preserving maps on some gyrogroups based on [1].

2. **Isometries of the special orthogonal group**

In this section, let n be a positive integer. Denote by $M_n(\mathbb{R})$ the real algebra of all $n \times n$ real matrices with the identity matrix I_n, $O(n)$ the group of the all orthogonal matrices and $SO(n)$ the group of the all special orthogonal matrices in $M_n(\mathbb{R})$. Let $\mathbb{R}^n_+\downarrow$ denote the set of all nonzero vectors $(x_1, \ldots, x_n) \in \mathbb{R}$ satisfying $x_1 \geq \cdots \geq x_n \geq 0$. For any $\mathfrak{c} = (c_1, \ldots, c_n) \in \mathbb{R}^n_+\downarrow$, we define the $\mathfrak{c}$-spectral norm of $A \in M_n(\mathbb{R})$ by

$$\|A\|_\mathfrak{c} = \sum_{i=1}^n c_i \sigma_i(A),$$

where $\sigma_1(A) \geq \cdots \geq \sigma_n(A)$ are the singular values of A. Recall that the singular values of A is the eigenvalues of the square roots of A^*A, where A^* is the conjugate transpose of A. In the following of the section, we assume that $c_1 = 1$ for $\mathfrak{c} = (c_1, \ldots, c_n) \in \mathbb{R}^n_+\downarrow$. Note that $\mathfrak{c}$-spectral norm is a generalization of the operator norm [when $c_1 = 1$, $c_i = 0$ $(i \neq 1)$] and the Ky Fan k-norm [when $c_i = 1$ $(i \leq k)$, $c_i = 0$ $(k < i)$].

In the paper [7], Hatori and Molnár gave a complete description of surjective isometries (with respect to the metric induced by the operator norm) from unitary groups on Hilbert spaces onto itself. By the result, we can verify that a surjective isometry (with respect to the metric induced by the operator norm) ϕ on a unitary group $U(H)$ on a Hilbert space H with the identity map I is only of the form $\phi(\cdot) = \phi(I)\phi_0(\cdot)$, where ϕ_0 is a group automorphism or a group anti-automorphism. Furthermore, ϕ_0 can be extended to an algebra isomorphism or anti-isomorphism on $B(H)$, where $B(H)$ is the algebra of all

bounded linear transformations of H. In this section, we give a complete description of all isometries on $SO(n)$ with respect to the metric induced by the $\mathfrak{c}$-spectral norm. We show that there are isometries on $SO(4)$ which have exceptional forms.

Let $K_n(\mathbb{R})$ be the real linear space of all $n \times n$ skew-symmetric matrices of real entries. Note that $\exp K_n(\mathbb{R}) = SO(n)$. For $A \in K_4(\mathbb{R})$, the skew-symmetric matrix A^+ is obtained from A by interchanging its $(1, 4)$ and $(2, 3)$ entries, and interchanges the $(4, 1)$ and $(3, 2)$ entries accordingly.

2.1. The main result of section 2.

The following theorem is the main result of this section.

Theorem 2.1.1. *Let T be a map from $SO(n)$ into itself and $\mathfrak{c} \in \mathbb{R}_+^n \downarrow$. Then the following* (S-i) *and* (S-ii) *are equivalent.*

(S-i) *T is an isometry with respect to the metric induced by $\|\cdot\|_{\mathfrak{c}}$, that is, $\|T(X) - T(Y)\|_{\mathfrak{c}} = \|X - Y\|_{\mathfrak{c}}$ for every pair $X, Y \in SO(n)$.*

(S-ii) *There exists $O \in O(n)$ such that T is of one of the following form:*

(S-a): *$T(X) = T(I_n)OXO^{-1}$ for every $X \in SO(n)$,*

(S-b): *$T(X) = T(I_n)OX^{-1}O^{-1}$ for every $X \in SO(n)$,*

(S-c): *$n = 4$ and $T(\exp(A)) = T(I_4)O(\exp(A^+))O^{-1}$ for every $A \in K_4(\mathbb{R})$,*

(S-d): *$n = 4$ and $T(\exp(A)) = T(I_4)O(\exp(A^+))^{-1}O^{-1}$ for every $A \in K_4(\mathbb{R})$.*

In particular, if $T(I_n) = I_n$, then T is a group automorphism on $SO(n)$ for (S-a); *T is a group anti-automorphism on $SO(n)$ for* (S-b); *T preserves the Jordan products $T(XYX) = T(X)T(Y)T(X)$ for every pair of X and Y in $SO(4)$, while T is neither group automorphism nor group anti-automorphism on $SO(4)$ for* (S-c) *and* (S-d).

Note that for $n = 4$, $\exp(A) = \exp(B)$ if and only if $\exp(A^+) = \exp(B^+)$ for $A, B \in K_4(\mathbb{R})$. This theorem is proved in later subsection 2.6. Note also that T needs not be surjective in (S-i).

2.2. **Preparations of the proof.**

In subsection 2.3, we will show that any ismetry (with respect to the metric induced by the $\mathfrak{c}$-spectral norm) on $SO(n)$ is of one of the form (S-a), (S-b), (S-c) or (S-d) of Theorem 2.1.1. In this subsection, we exhibit necessary definitions and results which are applied in subsection 2.3.

Any isometry (with respect to the metric induced by $\| \cdot \|_{\mathfrak{c}}$) from $SO(n)$ into itself is surjective as $SO(n)$ is compact. In general, we see the following.

Lemma 2.2.1. [3, Excerecise 2.4.1] *Let (X, d) be a compact metric space. Suppose that T is an isometry from (X, d) into itself. Then T is surjective.*

Proof. Clearly, $T^m(X)$ is a non-empty compact closed subset of X, where T^m denotes the m-times composition of T for a positive integer

m. For any positive integer m, $T^{m+1}(X) \subseteq T^m(x)$ holds. Thus, we have $\bigcap_{m \in \mathbb{N}} T^m(X)$ is non-empty and compact. Assume now that T is not surjective. From this assumption, there exists an element $x_0 \in X \setminus \bigcap_{m \in \mathbb{N}} T^m(X)$. Since $\bigcap_{m \in \mathbb{N}} T^m(X)$ is compact, we have

$$d(x_0, \bigcap_{m \in \mathbb{N}} T^m(X)) > 0.$$

Since X is a compact metric space, the sequence $\{T^m(x_0)\}_{m \in \mathbb{N}}$ have a convergent subsequence $\{T^{i_k}(x_0)\}_{k \in \mathbb{N}}$ and denote

$$(1) \qquad\qquad y_0 = \lim_{k \to \infty} T^{i_k}(x_0).$$

We claim that

$$(2) \qquad\qquad y_0 \in \bigcap_{k \in \mathbb{N}} T^{i_k}(X).$$

If $y_0 \notin \bigcap_{k \in \mathbb{N}} T^{i_k}(X)$, then there exists k_0 such that $y_0 \notin T^{i_{k_0}}(X)$. Thus, $r = d(y_0, T^{i_{k_0}}(X)) > 0$. Hence $d(y_0, T^{i_k}(X)) \geq r$ for all $k \geq k_0$ because $\{T^{i_k}(X)\}_{k \in \mathbb{N}}$ is a decreasing sequence. It is contradictory to (1) and we have (2). On the other hand, we have

$$
\begin{aligned}
d(y_0, \bigcap_{m \in \mathbb{N}} T^m(X)) \quad &= d(\lim_{k \to \infty} T^{i_k}(x_0), \bigcap_{m \in \mathbb{N}} T^m(X)) \\
&= \lim_{k \to \infty} d(T^{i_k}(x_0), \bigcap_{m \in \mathbb{N}} T^m(X)) \\
&= \lim_{k \to \infty} d(x_0, \bigcap_{m \in \mathbb{N}} T^m(X)) \\
&= d(x_0, \bigcap_{m \in \mathbb{N}} T^m(X)) > 0
\end{aligned}
$$

because T is an isometry. It is contradictory to (2). The proof is complete. $\qquad\square$

Following Definitions 2.2.2, 2.2.3, Proposition 2.2.4 and Lemma 2.2.5 are described in the paper [7]. Proposition 2.2.4 and Lemma 2.2.5 are applied to prove Lemma 2.3.2. Definitions 2.2.2, 2.2.3 and Proposition 2.2.4 are studied more generally in the paper [6].

Definition 2.2.2. Let (X, d) be a metric space, where X is a nonempty subset of a group G with the property that $yx^{-1}y \in X$ for every pair $x, y \in X$. Let $a, b \in X$. We say that $B(a, b)$ holds for (X, d) if the following are fulfilled:

(B1): $d(bx^{-1}b, by^{-1}b) = d(x, y)$ for every $x, y \in X$.

(B2): There exists a positive real number $K > 1$ such that

$$d(bx^{-1}b, x) \geq Kd(x, b)$$

for all $x \in L_{a,b}$, where

$$L_{a,b} = \{x \in X : d(a, x) = d(ba^{-1}b, x) = d(a, b)\}.$$

Definition 2.2.3. Let (X, d) be a metric space, where X is a nonempty subset of a group G with the property that $yx^{-1}y \in X$ for every pair $x, y \in X$. Let $a, b \in X$. We say that $C_1(a, b)$ holds for (X, d) if the following are fulfilled:

(C1): $ax^{-1}b, bx^{-1}a \in X$ for any $x \in X$.

(C2): $d(ax^{-1}b, ay^{-1}b) = d(x, y)$ for any pair $x, y \in X$.

Proposition 2.2.4. [7] *Let (X_i, d_i) be a metric space, where X_i is a nonempty subset of a group G_i with the property that $yx^{-1}y \in X_i$ for every pair $x, y \in X_i$, for 1=1,2. Let $\phi : X_1 \to X_2$ be a surjective*

isometry. Pick $a, b \in X_1$. Suppose that $B(a, b)$ holds for (X_1, d_1) and $C_1(\phi(a), \phi(ba^{-1}b))$ holds for (X_2, d_2). Then we have

$$\phi(ba^{-1}b) = \phi(b)(\phi(a))^{-1}\phi(b).$$

Lemma 2.2.5. [7] *For i=1,2, let G_i be a group and X_i a nonempty subset of G_i such that $yx^{-1}y \in X_i$ for every pair $x, y \in X_i$. Suppose that $\phi : X_1 \to X_2$ is a map, m is a positive integer, and $\{a_k\}_{k=0}^{2^m}$ is a finite sequence in X_1 such that we have*

$$a_{k+1}a_k^{-1}a_{k+1} = a_{k+2}$$

and

$$\phi(a_{k+1}a_k^{-1}a_{k+1}) = \phi(a_{k+1})(\phi(a_k))^{-1}\phi(a_{k+1})$$

for all $0 \le k \le 2^m - 2$. Then we have that

$$a_{2^m-1}a_0^{-1}a_{2^m-1} = a_{2^m}$$

and

$$\phi(a_{2^m-1}a_0^{-1}a_{2^m-1}) = \phi(a_{2^m-1})(\phi(a_0))^{-1}\phi(a_{2^m-1}).$$

In the paper [12], Li and Tsing studied isometries (with respect to the metric induced by $\|\cdot\|_{\mathfrak{c}}$) on the space of the symmetric matrices and the space of the skew-symmetric matrices. Theorem 2.2.6 is a part of Theorem 4.1 in [12].

Theorem 2.2.6. *Let S be a linear map from $K_n(\mathbb{R})$ into itself and $\mathfrak{c} \in \mathbb{R}_+^n\!\downarrow$. Then the following (K-i) and (K-ii) are equivalent*

(K-i) *S is an isometry with respect to the metric induced by $\|\cdot\|_{\mathfrak{c}}$, that is, $\|S(A) - S(B)\|_{\mathfrak{c}} = \|A - B\|_{\mathfrak{c}}$ for every pair $A, B \in K_n(\mathbb{R})$.*

(K-ii) *There exists $O \in O(n)$ such that S is of one of the following form:*

    (K-a): $S(A) = OAO^{-1}$ *for every* $A \in K_n(\mathbb{R})$,

    (K-b): $S(A) = -OAO^{-1}$ *for every* $A \in K_n(\mathbb{R})$,

    (K-c): $n = 4$ *and* $S(A) = OA^+O^{-1}$ *for every* $A \in K_4(\mathbb{R})$,

    (K-d): $n = 4$ *and* $S(A) = -OA^+O^{-1}$ *for every* $A \in K_4(\mathbb{R})$.

2.3. Necessary conditions for isometries.

Lemma 2.3.1. *Let $\mathfrak{c} \in \mathbb{R}_+^n\downarrow$. Suppose that $T : SO(n) \to SO(n)$ is an isometry with respect to the metric d induced by the norm $\| \cdot \|_{\mathfrak{c}}$. Then*

$$T(YX^{-1}Y) = T(Y)(T(X))^{-1}T(Y)$$

for every pair $X, Y \in SO(n)$ that satisfy $\|X - Y\|_{\mathfrak{c}} < \frac{1}{2}$.

Proof. First, we note that T is surjective by Lemma 2.2.1 since $SO(n)$ is compact. Clearly, the conditions $C_1(T(Y), T(YX^{-1}Y))$ and (B1) of $B(X, Y)$ are satisfied. It remains to check (B2). Let $X, Y \in SO(n)$ such that $d(X, Y) < \frac{1}{2}$ and setting $K = 2 - 2d(X, Y) > 1$. We assert that the inequality

$$d(YZ^{-1}Y, Z) \geq Kd(Z, Y)$$

holds for any $Z \in L_{X,Y}$, where

$$L_{X,Y} = \{Z \in SO(n) : d(X, Z) = d(YX^{-1}Y, Z) = d(X, Y)\}.$$

To prove this, let $Z \in L_{X,Y}$. Then

$$d(Z, Y) \leq d(Z, X) + d(X, Y) = 2d(X, Y)$$

and thus

$$2 - d(Z, Y) \geq 2 - 2d(X, Y) = K.$$

We check

$$d(Z, Y) = \|Z - Y\|_{\mathfrak{c}} = \|Y^{-1}Z - I_n\|_{\mathfrak{c}}$$

and

$$d(YZ^{-1}Y, Z) = \|YZ^{-1}Y - Z\|_{\mathfrak{c}} = \|YZ^{-1}YZ^{-1} - I_n\|_{\mathfrak{c}}$$
$$= \|(YZ^{-1} + I_n)(YZ^{-1} - I_n)\|_{\mathfrak{c}}.$$

From the assumption that $c_1 = 1$, we have

$$2\|YZ^{-1} - I_n\|_{\mathfrak{c}} - \|(YZ^{-1} + I_n)(YZ^{-1} - I_n)\|_{\mathfrak{c}}$$
$$\leq \|\big(2I_n - (YZ^{-1} + I_n)\big)(YZ^{-1} - I_n)\|_{\mathfrak{c}}$$
$$\leq \|YZ^{-1} - I_n\|_{\mathfrak{c}}^2.$$

Thus,

$$Kd(Z, Y) \leq \big(2 - d(Z, Y)\big)d(Z, Y)$$
$$= 2\|YZ^{-1} - I_n\|_{\mathfrak{c}} - \|YZ^{-1} - I_n\|_{\mathfrak{c}}^2$$
$$\leq \|(YZ^{-1} + I_n)(YZ^{-1} - I_n)\|_{\mathfrak{c}}$$
$$= d(YZ^{-1}Y, Z)$$

This gives us that the condition (B2) holds. Applying Proposition 2.2.4 we have

$$T(YX^{-1}Y) = T(Y)(T(X))^{-1}T(Y)$$

for all $X, Y \in SO(n)$ with $d(X, Y) < \frac{1}{2}$. $\qquad \square$

The following lemma asserts that any isometry (with respect to the metric induced by $\|\cdot\|_{\mathfrak{c}}$) on $SO(n)$ preserves the inverted Jordan product.

Lemma 2.3.2. *Let* $\mathfrak{c} \in \mathbb{R}^n_+\downarrow$. *Suppose that* $T : SO(n) \to SO(n)$ *is an isometry with respect to the metric induced by the norm* $\|\cdot\|_{\mathfrak{c}}$. *Then*

$$(3) \qquad T(YX^{-1}Y) = T(Y)(T(X))^{-1}T(Y)$$

for every pair $X, Y \in SO(n)$.

Proof. Pick $X, Y \in SO(n)$. Since $X^{-1}Y \in SO(n)$, there exists $W \in K_n(\mathbb{R})$ such that $\exp(W) = X^{-1}Y$. Let m be a positive integer such that $\exp\left(\frac{\|W\|_{\mathfrak{c}}}{2^m}\right) < \frac{3}{2}$. Then

$$\|\exp\frac{W}{2^m} - I_n\|_{\mathfrak{c}} \leq \exp\frac{\|W\|_{\mathfrak{c}}}{2^m} - 1 < \frac{1}{2}$$

by the assumption $c_1 = 1$. Let

$$A_k = X\exp\frac{kW}{2^m}$$

for each integer $0 \leq k \leq 2^{m+1}$. Then $A_0 = X$, $A_{2^m} = Y$, $A_{2^{m+1}} = YX^{-1}Y$. We have

$$\begin{aligned}
A_{k+1}(A_k)^{-1}A_{k+1} &= (X\exp\frac{(k+1)W}{2^m})(X\exp\frac{kW}{2^m})^{-1}(X\exp\frac{(k+1)W}{2^m}) \\
&= (X\exp\frac{(k+1)W}{2^m})(\frac{kW}{2^m})^{-1}X^{-1}(X\exp\frac{(k+1)W}{2^m}) \\
&= X\exp\frac{(k+2)W}{2^m} \\
&= A_{k+2}
\end{aligned}$$

for any $0 \leq k \leq 2^{m+1} - 2$. We also have

$$
\begin{aligned}
\|A_{k+1} - A_k\| &= \|X \exp \frac{(k+1)W}{2^m} - X \exp \frac{kW}{2^m}\|_{\mathfrak{c}} \\
&= \|(X \exp \frac{kW}{2^m})(\exp \frac{W}{2^m} - I_n)\|_{\mathfrak{c}} \\
&= \|\exp \frac{W}{2^m} - I_n\|_{\mathfrak{c}} < \frac{1}{2}
\end{aligned}
$$

since $X \exp \frac{kW}{2^m} \in SO(n)$ for any $0 \leq k \leq 2^{m+1} - 1$. By Lemma 2.3.1, it follows that

$$
T(A_{k+1}(A_k)^{-1}A_{k+1}) = T(A_{k+1})(T(A_k))^{-1}T(A_{k+1})
$$

for every $0 \leq k \leq 2^{m+1} - 2$. Applying Lemma 2.2.5, we deduce that

$$
\begin{aligned}
T(YX^{-1}Y) &= T(A_{2^m}(A_0)^{-1}A_{2^m}) \\
&= T(A_{2^m})(T(A_0))^{-1}T(A_{2^m}) \\
&= T(Y)(T(X))^{-1}T(Y).
\end{aligned}
$$

So we have (3). $\square$

Corollary 2.3.3. *Let $\mathfrak{c} \in \mathbb{R}_+^n \downarrow$. Suppose that $T_0 : SO(n) \to SO(n)$ is an isometry with respect to the metric induced by the norm $\| \cdot \|_{\mathfrak{c}}$ and $T_0(I_n) = I_n$. Then*

$$
(4) \qquad\qquad T_0(YXY) = T_0(Y)T_0(X)T_0(Y)
$$

for every pair $X, Y \in SO(n)$.

Proof. By Lemma 2.3.2, $T_0(YX^{-1}Y) = T_0(Y)(T_0(X))^{-1}T_0(Y)$ for any pair $X, Y \in SO(n)$. In particular,

$$
\begin{aligned}
T_0(X^{-1}) &= T_0(I_n X^{-1} I_n) \\
&= T_0(I_n)(T_0(X))^{-1}T_0(I_n) = (T_0(X))^{-1}
\end{aligned}
$$

for any $X \in SO(n)$. Thus

$$T_0(YXY) = T_0(Y)T_0(X)T_0(Y)$$

for any pair $X, Y \in SO(n)$. $\square$

Lemma 2.3.4. *Let $A \in K_n(\mathbb{R})$ and $\mathfrak{c} \in \mathbb{R}_+^n \downarrow$. Suppose that $T_0 : SO(n) \to SO(n)$ is an isometry with respect to the metric induced by the norm $\| \cdot \|_{\mathfrak{c}}$ and $T_0(I_n) = I_n$. Let $S_A(t) = T_0(\exp(tA))$ for $t \in \mathbb{R}$. Then $S_A : \mathbb{R} \to SO(n)$ is a one-parameter group.*

Proof. As $T_0(I_n) = I_n$, for every $X \in SO(n)$ and for any non-negative integer m, we have

$$T_0(X^m) = (T_0(X))^m$$

by the equation (4). Moreover,

$$T_0(X^k) = (T_0(X))^k$$

for any integer k since $T_0(X^{-1}) = (T_0(X))^{-1}$. We show that $S_A(t+t') = S_A(t)S_A(t')$ holds for any pair $t, t' \in \mathbb{R}$. First, let $r = \frac{k}{m}$ and $r' = \frac{k'}{m'}$ be rational numbers with integers k, k' and natural numbers m, m'. We compute

$$S_A(r + r') = T_0(\exp(\frac{km' + k'm}{mm'}A)) = (T_0(\exp(\frac{1}{mm'}A)))^{km'+k'm}$$
$$= (T_0(\exp(\frac{1}{mm'})))^{km'}(T_0(\exp(\frac{1}{mm'})))^{k'm} = S_A(r)S_A(r').$$

So we have $S_A(r + r') = S_A(r)S_A(r')$ for any rational numbers r, r'. Thus $S_A(t + t') = S_A(t)S_A(t')$ holds for any pair $t, t' \in \mathbb{R}$ because T_0 is continuous. $\square$

The following proposition gives us the necessary condition for the isometries on $SO(n)$.

Proposition 2.3.5. *Let $\mathfrak{c} \in \mathbb{R}_+^n\downarrow$. Suppose that $T : SO(n) \to SO(n)$ is an isometry with respect to $\|\cdot\|_{\mathfrak{c}}$. Then there exists $O \in O(n)$ such that T is of one of the following form:*

(S-a): $T(X) = T(I_n)OXO^{-1}$ *for every $X \in SO(n)$,*

(S-b): $T(X) = T(I_n)OX^{-1}O^{-1}$ *for every $X \in SO(n)$,*

(S-c): $n = 4$ *and* $T(\exp(A)) = T(I_4)O(\exp(A^+))O^{-1}$ *for every $A \in K_4(\mathbb{R})$,*

(S-d): $n = 4$ *and* $T(\exp(A)) = T(I_4)O(\exp(A^+))^{-1}O^{-1}$ *for every $A \in K_4(\mathbb{R})$.*

Proof. First, we note that T is surjective by Lemma 2.2.1. Put $T_0(\cdot) = (T(I_n))^{-1}T(\cdot)$. Then T_0 is also a surjective isometry on $SO(n)$. Lemma 2.3.4 shows that $S_A : \mathbb{R} \to SO(n)$ is a one-parameter group for any $A \in K_n(\mathbb{R})$, where $S_A(t) = T_0(\exp(tA))$ for any $t \in \mathbb{R}$. It follows that there exists a unique element $f(A) \in K_n(\mathbb{R})$ such that $S_A(t) = \exp(tf(A))$ for all $t \in \mathbb{R}$. We constitute the map $f : K_n(\mathbb{R}) \to K_n(\mathbb{R})$.

We assert that f is surjective. Since $T_0^{-1} : SO(n) \to SO(n)$ is surjective isometry, in the same way as above, there is a map $g : K_n(\mathbb{R}) \to K_n(\mathbb{R})$ such that $T_0^{-1}(\exp(tA)) = \exp(tg(A))$ for every $t \in \mathbb{R}$ and $A \in K_n(\mathbb{R})$. We have $\exp(tA) = T_0(\exp(tg(A))) = \exp(tf(g(A)))$ for all $t \in \mathbb{R}$ and $A \in K_n(\mathbb{R})$. It follows that $f(g(A)) = A$ for any $A \in K_n(\mathbb{R})$, so f is surjective.

We next show that f is a real-linear isometry. It is easy to check $f(0) = 0$. As T_0 is an isometry, we have

$$\|A - B\|_{\mathfrak{c}} = \lim_{t \to 0} \left\| \frac{\exp(tA) - \exp(tB)}{t} \right\|_{\mathfrak{c}}$$
$$= \lim_{t \to 0} \left\| \frac{T_0(\exp(tA)) - T_0(\exp(tB))}{t} \right\|_{\mathfrak{c}}$$
$$= \lim_{t \to 0} \left\| \frac{\exp(tf(A)) - \exp(tf(B))}{t} \right\|_{\mathfrak{c}}$$
$$= \|f(A) - f(B)\|_{\mathfrak{c}}$$

for any pair $A, B \in K_n(\mathbb{R})$. We observe that $f : K_n(\mathbb{R}) \to K_n(\mathbb{R})$ is a surjective isometry. Then f is a real-linear isometry by the celebrated Mazur-Ulam theorem.

By Theorem 2.2.6 there exists $O \in O(n)$ such that f is of one of the form (K-a), (K-b), (K-c) or (K-d). Suppose that f is of the form of (K-a), then

$$T_0(X) = \exp(f(A)) = \exp(OAO^{-1}) = O\exp(A)O^{-1} = OXO^{-1}$$

for any $X \in SO(n)$, where $X = \exp(A)$ for $A \in K_n(\mathbb{R})$. So we have (S-a). In the same way, (S-b), (S-c) and (S-d) obtained by (K-b), (K-c) and (K-d) respectively. $\square$

2.4. The B-C-H formula of Fujii and Suzuki.

If a map T on $SO(n)$ is of the form (S-a) or (S-b) in Theorem 2.1.1, then T is a surjective isometry since $\mathfrak{c}$-spectral norm is a unitarily invariant norm. We will show in subsection 2.5 that if T is of the form (S-c) or (S-d), then T is also an isometry. For the proof of this,

we make use of the Baker-Cambell-Hausdorff (B-C-H) formula of Fujii

and Suzuki [4] for $SO(4)$ (Theorem 2.4.2). A special emphasis is on the

range of $\sin^{-1}\rho$, which is not stated clearly in [4], that $0 \leq \sin^{-1}\rho \leq \pi$

depending not only on ρ itself but also the value

$$\cos|\boldsymbol{x}|\cos|\boldsymbol{y}| - \frac{\sin|\boldsymbol{x}|\sin|\boldsymbol{y}|}{|\boldsymbol{x}||\boldsymbol{y}|}\langle \boldsymbol{x}, \boldsymbol{y}\rangle.$$

To prove Theorem 2.4.2, Fujii and Suzuki applied B-C-H formula for

$SU(2)$, the special unitary group of the degree 2 (Theorem 2.4.1).

Let $\{\sigma_1, \sigma_2, \sigma_3\}$ be Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and

$$H_0(2; \mathbb{C}) = \{x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3 : x_1, x_2, x_3 \in \mathbb{R}\}.$$

It is well known that

$$\sigma_i^2 = E_2 \quad (i = 1, 2, 3),$$

$$\sigma_1\sigma_2 = i\sigma_3, \quad \sigma_2\sigma_3 = i\sigma_1, \quad \sigma_3\sigma_1 = i\sigma_2,$$

$$\sigma_i\sigma_j = -\sigma_j\sigma_i \quad (i, j = 1, 2, 3, \ i \neq j)$$

and $iH_0(2; \mathbb{C})$ is the Lie algebra of the 2-dimensional special unitary

group $SU(2)$. For any $X = x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3 \in H_0(2; \mathbb{C})$, we denote

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

For any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^3$, we put

$$\alpha(\boldsymbol{x}, \boldsymbol{y}) = \frac{\sin^{-1}\rho}{\rho} \frac{\sin|\boldsymbol{x}|\cos|\boldsymbol{y}|}{|\boldsymbol{x}|},$$

$$\beta(\boldsymbol{x}, \boldsymbol{y}) = \frac{\sin^{-1}\rho}{\rho} \frac{\cos|\boldsymbol{x}|\sin|\boldsymbol{y}|}{|\boldsymbol{y}|},$$

$$\gamma(\boldsymbol{x}, \boldsymbol{y}) = \frac{\sin^{-1}\rho}{\rho} \frac{\sin|\boldsymbol{x}|\sin|\boldsymbol{y}|}{|\boldsymbol{x}||\boldsymbol{y}|}$$

with

$$\rho \equiv \rho(\boldsymbol{x}, \boldsymbol{y}) = \left\{ \sin^2 |\boldsymbol{x}| \cos^2 |\boldsymbol{y}| + \sin^2 |\boldsymbol{y}| - \frac{\sin^2 |\boldsymbol{x}| \sin^2 |\boldsymbol{y}|}{|\boldsymbol{x}|^2 |\boldsymbol{y}|^2} \langle \boldsymbol{x}, \boldsymbol{y} \rangle^2 \right.$$
$$\left. + \frac{2 \sin |\boldsymbol{x}| \cos |\boldsymbol{x}| \sin |\boldsymbol{y}| \cos |\boldsymbol{y}|}{|\boldsymbol{x}| |\boldsymbol{y}|} \langle \boldsymbol{x}, \boldsymbol{y} \rangle \right\}^{\frac{1}{2}}$$

and

(5)
$$0 \leq \sin^{-1} \rho \leq \pi,$$
$$\cos(\sin^{-1} \rho) = \cos |\boldsymbol{x}| \cos |\boldsymbol{y}| - \frac{\sin |\boldsymbol{x}| \sin |\boldsymbol{y}|}{|\boldsymbol{x}| |\boldsymbol{y}|} \langle \boldsymbol{x}, \boldsymbol{y} \rangle.$$

The following is the B-C-H formula for $SU(2)$. In the paper [4], it is not stated clearly the condition (5). We restate it with a proof.

Theorem 2.4.1 ([4]). *Let* $X, Y \in H_0(2; \mathbb{C})$. *Then*

$$\exp(iX) \exp(iY) = \exp(iZ_0),$$

where $Z_0 = \alpha(\boldsymbol{x}, \boldsymbol{y}) X + \beta(\boldsymbol{x}, \boldsymbol{y}) Y + \frac{i}{2} \gamma(\boldsymbol{x}, \boldsymbol{y}) (XY - YX)$.

Proof. It is well known that

$$\exp(i(a_1 \sigma_1 + a_2 \sigma_2 + a_3 \sigma_3)) = \cos r I_2 + \frac{\sin r}{r} i(a_1 \sigma_1 + a_2 \sigma_2 + a_3 \sigma_3),$$

where $r = \sqrt{a_1^2 + a_2^2 + a_3^2}$, for any triple $a_1, a_2, a_3 \in \mathbb{R}$. Pick $X, Y \in H_0(2; \mathbb{C})$. By an elementary calculation, we have

$$XY = \langle \boldsymbol{x}, \boldsymbol{y} \rangle I_2 + \frac{1}{2}(XY - YX),$$

where $\langle\cdot,\cdot\rangle$ is the Euclidean inner product on $\mathbb{R}^3$. Thus, we have

$$
\begin{aligned}
\exp(iX)\exp(iY) &= \left\{\cos|\boldsymbol{x}|I_2 + \frac{\sin|\boldsymbol{x}|}{|\boldsymbol{x}|}iX\right\}\left\{\cos|\boldsymbol{y}|I_2 + \frac{\sin|\boldsymbol{y}|}{|\boldsymbol{y}|}iY\right\} \\
&= \left\{\cos|\boldsymbol{x}|\cos|\boldsymbol{y}| - \frac{\sin|\boldsymbol{x}|\sin|\boldsymbol{y}|}{|\boldsymbol{x}||\boldsymbol{y}|}\langle\boldsymbol{x},\boldsymbol{y}\rangle\right\}I_2 \\
&\quad + i\frac{\sin|\boldsymbol{x}|\cos|\boldsymbol{y}|}{|\boldsymbol{x}|}X + i\frac{\cos|\boldsymbol{x}|\sin|\boldsymbol{y}|}{|\boldsymbol{y}|}Y \\
&\quad - \frac{\sin|\boldsymbol{x}|\sin|\boldsymbol{y}|}{|\boldsymbol{x}||\boldsymbol{y}|}\frac{1}{2}(XY - YX).
\end{aligned}
$$

First, there is $Z \in H_0(2;\mathbb{C})$ such that $\exp(iX)\exp(iY) = \exp(iZ)$ because $\exp(iH_0(2;\mathbb{C})) = SU(2)$. Thus, we have

$$
\exp(iZ) = \cos|\boldsymbol{z}|I_2 + \frac{\sin|\boldsymbol{z}|}{|\boldsymbol{z}|}iZ.
$$

Since $\{I_2, \sigma_1, \sigma_2, \sigma_3\}$ is linearly independent, we have

$$
\cos|\boldsymbol{z}| = \cos|\boldsymbol{x}|\cos|\boldsymbol{y}| - \frac{\sin|\boldsymbol{x}|\sin|\boldsymbol{y}|}{|\boldsymbol{x}||\boldsymbol{y}|}\langle\boldsymbol{x},\boldsymbol{y}\rangle
$$

and

$$
\begin{aligned}
\frac{\sin|\boldsymbol{z}|}{|\boldsymbol{z}|}Z &= \frac{\sin|\boldsymbol{x}|\cos|\boldsymbol{y}|}{|\boldsymbol{x}|}X + \frac{\cos|\boldsymbol{x}|\sin|\boldsymbol{y}|}{|\boldsymbol{y}|}Y \\
&\quad + \frac{\sin|\boldsymbol{x}|\sin|\boldsymbol{y}|}{|\boldsymbol{x}||\boldsymbol{y}|}\frac{i}{2}(XY - YX).
\end{aligned}
$$

Hence

$$
\left|\cos|\boldsymbol{x}|\cos|\boldsymbol{y}| - \frac{\sin|\boldsymbol{x}|\sin|\boldsymbol{y}|}{|\boldsymbol{x}||\boldsymbol{y}|}\langle\boldsymbol{x},\boldsymbol{y}\rangle\right| \le 1.
$$

Moreover,

$$\sin^2|\boldsymbol{z}| = 1 - \cos^2|\boldsymbol{z}|$$

$$= 1 - \left\{ \cos|\boldsymbol{x}|\cos|\boldsymbol{y}| - \frac{\sin|\boldsymbol{x}|\sin|\boldsymbol{y}|}{|\boldsymbol{x}||\boldsymbol{y}|}\langle \boldsymbol{x}, \boldsymbol{y}\rangle \right\}^2$$

$$= \sin^2|\boldsymbol{x}|\cos^2|\boldsymbol{y}| + \sin^2|\boldsymbol{y}| - \frac{\sin^2|\boldsymbol{x}|\sin^2|\boldsymbol{y}|}{|\boldsymbol{x}|^2|\boldsymbol{y}|^2}\langle \boldsymbol{x}, \boldsymbol{y}\rangle^2$$

$$+ \frac{2\sin|\boldsymbol{x}|\cos|\boldsymbol{x}|\sin|\boldsymbol{y}|\cos|\boldsymbol{y}|}{|\boldsymbol{x}||\boldsymbol{y}|}\langle \boldsymbol{x}, \boldsymbol{y}\rangle$$

$$= \rho^2$$

Therefore, we can choose $0 \le r \le \pi$ such satisfies

$$\cos r = \cos|\boldsymbol{x}|\cos|\boldsymbol{y}| - \frac{\sin|\boldsymbol{x}|\sin|\boldsymbol{y}|}{|\boldsymbol{x}||\boldsymbol{y}|}\langle \boldsymbol{x}, \boldsymbol{y}\rangle$$

$$\sin r = \rho,$$

and denote $\sin^{-1}\rho = r$. Put

$$Z_0 = \alpha X + \beta Y + \gamma\frac{i}{2}(XY - YX).$$

Then

$$Z_0 = \frac{\sin^{-1}\rho}{\rho}\left\{ \frac{\sin|\boldsymbol{x}|\cos|\boldsymbol{y}|}{|\boldsymbol{x}|}X + \frac{\cos|\boldsymbol{x}|\sin|\boldsymbol{y}|}{|\boldsymbol{y}|}Y \right.$$

$$\left. + \frac{\sin|\boldsymbol{x}|\sin|\boldsymbol{y}|}{|\boldsymbol{x}||\boldsymbol{y}|}\frac{i}{2}(XY - YX) \right\}$$

$$= \frac{\sin^{-1}\rho}{\rho}\frac{\sin|\boldsymbol{z}|}{|\boldsymbol{z}|}Z.$$

It follows that

$$|\boldsymbol{z}_0| = \frac{\sin^{-1}\rho}{\rho}\frac{|\sin|\boldsymbol{z}||}{|\boldsymbol{z}|}|\boldsymbol{z}| = \sin^{-1}\rho = r$$

since $\rho = |\sin|\boldsymbol{z}||$. Thus, we have

$$\sin|\boldsymbol{z}_0| = \sin r = \rho,$$

$$\cos|\boldsymbol{z}_0| = \cos r = \cos|\boldsymbol{x}|\cos|\boldsymbol{y}| - \frac{\sin|\boldsymbol{x}|\sin|\boldsymbol{y}|}{|\boldsymbol{x}||\boldsymbol{y}|}\langle \boldsymbol{x}, \boldsymbol{y}\rangle$$

and hence

$$\frac{\sin|\boldsymbol{z}|}{|\boldsymbol{z}|}Z = \frac{\sin|\boldsymbol{z_0}|}{|\boldsymbol{z_0}|}Z_0.$$

It follows that $\exp(iZ_0) = \exp(iZ)$. As desired, we have

$$\exp(iZ_0) = \exp(iX)\exp(iY).$$

$\square$

We note that $a_{ii} = 0$ and $a_{ij} = -a_{ji}$ for any $1 \leq i,j \leq 4$ for $A = (a_{ij}) \in K_4(\mathbb{R})$. Put

$$\varphi_1(A) = \frac{a_{12} + a_{34}}{2}, \ \varphi_2(A) = \frac{a_{13} - a_{24}}{2}, \ \varphi_3(A) = \frac{a_{14} + a_{23}}{2},$$

$$\psi_1(A) = \frac{a_{12} - a_{34}}{2}, \ \psi_2(A) = -\frac{a_{13} + a_{24}}{2}, \ \psi_3(A) = \frac{a_{14} - a_{23}}{2}.$$

for any $A = (a_{ij}) \in K_4(\mathbb{R})$. For any $A = (a_{ij}) \in K_4(\mathbb{R})$, define the element of $H_0(2;\mathbb{C})$

$$\Phi(A) = \varphi_1(A)\sigma_1 + \varphi_2(A)\sigma_2 + \varphi_3(A)\sigma_3,$$

$$\Psi(A) = \psi_1(A)\sigma_1 + \psi_2(A)\sigma_2 + \psi_3(A)\sigma_3$$

and

$$\overrightarrow{\Phi}(A) = \begin{pmatrix} \varphi_1(A) \\ \varphi_2(A) \\ \varphi_3(A) \end{pmatrix}, \ \overrightarrow{\Psi}(A) = \begin{pmatrix} \psi_1(A) \\ \psi_2(A) \\ \psi_3(A). \end{pmatrix} \in \mathbb{R}^3.$$

Put

$$\alpha_1(A,B) = \alpha(\overrightarrow{\Phi}(A), \overrightarrow{\Phi}(B)), \ \alpha_2(A,B) = \alpha(\overrightarrow{\Psi}(A), \overrightarrow{\Psi}(B)),$$

$$\beta_1(A,B) = \beta(\overrightarrow{\Phi}(A), \overrightarrow{\Phi}(B)), \ \beta_2(A,B) = \beta(\overrightarrow{\Psi}(A), \overrightarrow{\Psi}(B)),$$

$$\gamma_1(A,B) = \gamma(\overrightarrow{\Phi}(A), \overrightarrow{\Phi}(B)), \ \gamma_2(A,B) = \gamma(\overrightarrow{\Psi}(A), \overrightarrow{\Psi}(B)),$$

for any pair $A, B \in K_4(\mathbb{R})$. Let R be the unitary matrix which is called the magic matrix by Makhlin

$$R = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & -i \\ 0 & -i & -1 & 0 \\ 0 & -i & 1 & 0 \\ 1 & 0 & 0 & i \end{pmatrix}.$$

Put

$$
\begin{aligned}
fs(A, B) = \Big\{ & \alpha_1(A, B)\Phi(A) + \beta_1(A, B)\Phi(B) \\
& + \frac{i}{2}\gamma_1(A, B)\left(\Phi(A)\Phi(B) - \Phi(B)\Phi(A)\right) \Big\} \otimes E_2 \\
& + E_2 \otimes \Big\{ \alpha_2(A, B)\Psi(A) + \beta_2(A, B)\Psi(B) \\
& + \frac{i}{2}\gamma_2(A, B)\left(\Psi(A)\Psi(B) - \Psi(B)\Psi(A)\right) \Big\}
\end{aligned}
$$

and define

$$
(6) \qquad\qquad BCH(A, B) = iR^* fs(A, B)R.
$$

The following is the B-C-H formula for $SO(4)$. It was proved by applying Theorem 2.4.1.

Theorem 2.4.2 ([4]). *Let $A, B \in K_4(\mathbb{R})$. Then*

$$
\exp(A)\exp(B) = \exp(BCH(A, B)).
$$

2.5. **Exceptional forms of isometries on $SO(4)$.**

In this subsection, we prove that the map $\exp(A) \mapsto \exp(A^+)$ is an isometry (with respect to the metric induced by $\|\cdot\|_{\mathfrak{c}}$) on $SO(4)$ (Theorem 2.5.3).

The following proposition is easily proved by an elementary calculation. We can apply to prove Lemma 2.5.2.

Proposition 2.5.1.

$$\det\left\{\begin{pmatrix} a & \overline{z} & \overline{w} & 0 \\ z & b & 0 & \overline{w} \\ w & 0 & -b & \overline{z} \\ 0 & w & z & -a \end{pmatrix} - \lambda I_4\right\}$$
$$=\lambda^4 - \{a^2 + b^2 + 2(|z|^2 + |w|^2)\}\lambda^2 + (ab + |w|^2 - |z|^2)^2$$

for every $a, b \in \mathbb{R}$, $z, w, \lambda \in \mathbb{C}$.

Proof.

$$\begin{vmatrix} a-\lambda & \overline{z} & \overline{w} & 0 \\ z & b-\lambda & 0 & \overline{w} \\ w & 0 & -b-\lambda & \overline{z} \\ 0 & w & z & -a-\lambda \end{vmatrix}$$

$$=(a-\lambda)\begin{vmatrix} b-\lambda & 0 & \overline{w} \\ 0 & -b-\lambda & \overline{z} \\ w & z & -a-\lambda \end{vmatrix}$$

$$-z\begin{vmatrix} \overline{z} & \overline{w} & 0 \\ 0 & -b-\lambda & \overline{z} \\ w & z & -a-\lambda \end{vmatrix} + w\begin{vmatrix} \overline{z} & \overline{w} & 0 \\ b-\lambda & 0 & \overline{w} \\ w & z & -a-\lambda \end{vmatrix}$$

$$=(a-\lambda)\{(b-\lambda)(-b-\lambda)(-a-\lambda) - (b-\lambda)|z|^2 - (b-\lambda)|w|^2\}$$

$$\quad - z\{(-a-\lambda)(-b-\lambda)\overline{z} + |w^2\overline{z}| - \overline{z}|z|^2\}$$

$$\quad + w\{\overline{w}|w|^2 - \overline{w}|z|^2 - (-a-\lambda)(b-\lambda)\overline{w}\}$$

$$=(a^2 - \lambda^2)(b^2 - \lambda^2) - \{ab - (a + b\lambda + \lambda^2)\}|z|^2$$

$$\quad + \{ab + (a-b)\lambda - \lambda^2\}|w|^2 - \{ab + (a + b\lambda + \lambda^2)\}|z|^2$$

$$\quad + \{ab - (a-b)\lambda - \lambda^2\}|w|^2 + |z|^4 - 2|z|^2|w|^2 + |w|^4$$

$$=\{\lambda^4 - (a^2 + b^2)\lambda^2 + a^2b^2\}$$

$$\quad - 2ab|z|^2 - 2|z|^2\lambda^2 + 2ab|w|^2 - 2|w|^2\lambda^2 + (|z|^2 - |w|^2)^2$$

$$=\lambda^4 - \{a^2 + b^2 + 2(|z|^2 + |w|^2)\}\lambda^2 + (ab + |w|^2 - |z|^2)^2$$

$\square$

For any matrix X, let P_X denote the characteristic polynomial of X.

Lemma 2.5.2.

$$P_{BCH(A,B)} = P_{BCH(A^+,B^+)}$$

holds for every pair of $A, B \in K_4(\mathbb{R})$.

Proof. To begin the proof, we describe the form of $P_{fs(A,B)}$ for $A, B \in K_4(\mathbb{R})$. Pick $A, B \in K_4(\mathbb{R})$. It is easy to check

$$\Phi(A)\Phi(B) - \Phi(B)\Phi(A) = 2i\{\varphi_2(A)\varphi_3(B) - \varphi_3(A)\varphi_2(B)\}\sigma_1$$
$$+ 2i\{\varphi_3(A)\varphi_1(B) - \varphi_1(A)\varphi_3(B)\}\sigma_2$$
$$+ 2i\{\varphi_1(A)\varphi_2(B) - \varphi_2(A)\varphi_1(B)\}\sigma_3$$

and hence

$$\alpha_1(A, B)\Phi(A) + \beta_1(A, B)\Phi(B)$$
$$+ \frac{i}{2}\gamma_1(A, B)\left(\Phi(A)\Phi(B) - \Phi(B)\Phi(A)\right)$$
$$= \sum_{j=1}^{3} X_j(A, B)\sigma_j,$$

where

$$X_1(A, B) = \alpha_1(A, B)\varphi_1(A) + \beta_1(A, B)\varphi_1(B)$$
$$- \gamma_1(A, B)(\varphi_2(A)\varphi_3(B) - \varphi_3(A)\varphi_2(B)),$$

$$X_2(A, B) = \alpha_1(A, B)\varphi_2(A) + \beta_1(A, B)\varphi_2(B)$$
$$- \gamma_1(A, B)(\varphi_3(A)\varphi_1(B) - \varphi_1(A)\varphi_3(B)),$$

$$X_3(A, B) = \alpha_1(A, B)\varphi_3(A) + \beta_1(A, B)\varphi_3(B)$$
$$- \gamma_1(A, B)(\varphi_1(A)\varphi_2(B) - \varphi_2(A)\varphi_1(B)).$$

In the same way,

$$\alpha_2(A, B)\Psi(A) + \beta_2(A, B)\Psi(B)$$

$$+ \frac{i}{2}\gamma_2(A, B)\left(\Psi(A)\Phi(B) - \Psi(B)\Psi(A)\right)$$

$$= \sum_{j=1}^{3} Y_j(A, B)\sigma_j,$$

where

$$Y_1(A, B) = \alpha_2(A, B)\psi_1(A) + \beta_2(A, B)\psi_1(B)$$

$$- \gamma_2(A, B)(\psi_2(A)\psi_3(B) - \psi_3(A)\psi_2(B)),$$

$$Y_2(A, B) = \alpha_2(A, B)\psi_2(A) + \beta_2(A, B)\psi_2(B)$$

$$- \gamma_2(A, B)(\psi_3(A)\psi_1(B) - \psi_1(A)\psi_3(B)),$$

$$Y_3(A, B) = \alpha_2(A, B)\psi_3(A) + \beta_2(A, B)\psi_3(B)$$

$$- \gamma_2(A, B)(\psi_1(A)\psi_2(B) - \psi_2(A)\psi_1(B)).$$

Then

$$fs(A, B)$$

$$= (\sum_{j=1}^{3} X_j(A, B)\sigma_j) \otimes I_2 + I_2 \otimes \sum_{j=1}^{3} Y_j(A, B)\sigma_j$$

$$= \begin{pmatrix} X_3(A, B) & X_1(A, B) - iX_2(A, B) \\ X_1(A, B) + iX_2(A, B) & -X_3(A, B) \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$+ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} Y_3(A, B) & Y_1(A, B) - iY_2(A, B) \\ Y_1(A, B) + iY_2(A, B) & -Y_3(A, B) \end{pmatrix}$$

$$= \begin{pmatrix} X_3(A,B) + Y_3(A,B) & Y_1(A,B) - iY_2(A,B) & X_1(A,B) - iX_2(A,B) & 0 \\ Y_1(A,B) + iY_2(A,B) & X_3(A,B) - Y_3(A,B) & 0 & X_1(A,B) - iX_2(A,B) \\ X_1(A,B) + iX_2(A,B) & 0 & -X_3(A,B) + Y_3(A,B) & Y_1(A,B) - iY_2(A,B) \\ 0 & X_1(A,B) + iX_2(A,B) & Y_1(A,B) + iY_2(A,B) & -X_3(A,B) - Y_3(A,B) \end{pmatrix}.$$

By applying proposition 2.5.1, we obtain

$$P_{fs(A,B)}(\lambda) = \lambda^4 - 2\left\{\sum_{j=1}^{3} X_j(A, B)^2 + \sum_{j=1}^{3} Y_j(A, B)^2\right\}\lambda^2$$

$$+ \left\{\sum_{j=1}^{3} X_j(A, B)^2 - \sum_{j=1}^{3} Y_j(A, B)^2\right\}^2.$$

We assert that

$$\sum_{j=1}^{3} X_j(A, B)^2 = \sum_{j=1}^{3} X_j(A^+, B^+)^2,$$

$$\sum_{j=1}^{3} Y_j(A, B)^2 = \sum_{j=1}^{3} Y_j(A^+, B^+)^2$$

for any pair $A, B \in K_4(\mathbb{R})$. By definition, it is apparent that

$$\overrightarrow{\Phi}(C^+) = \begin{pmatrix} \varphi_1(C^+) \\ \varphi_2(C^+) \\ \varphi_3(C^+) \end{pmatrix} = \begin{pmatrix} \varphi_1(C) \\ \varphi_2(C) \\ \varphi_3(C) \end{pmatrix} = \overrightarrow{\Phi}(C),$$

for $C \in K_4(\mathbb{R})$ and hence

$$\alpha_1(A^+, B^+) = \alpha_1(A, B), \ \beta_1(A^+, B^+) = \beta_1(A, B), \ \gamma_1(A^+, B^+) = \gamma_1(A, B).$$

It follows that $X_i(A^+, B^+) = X_i(A, B)$ for $i = 1, 2, 3$. Hence

$$\sum_{i=1}^{3} X_i(A^+, B^+) = \sum_{i=1}^{3} X_i(A, B).$$

In the same way, we have

$$\overrightarrow{\Psi}(C^+) = \begin{pmatrix} \psi_1(C^+) \\ \psi_2(C^+) \\ \psi_3(C^+) \end{pmatrix} = \begin{pmatrix} \psi_1(C) \\ \psi_2(C) \\ -\psi_3(C) \end{pmatrix},$$

and $|\overrightarrow{\Psi}(C^+)| = |\overrightarrow{\Psi}(C)|$ for $C \in K_4(\mathbb{R})$. Hence

$$\alpha_2(A^+, B^+) = \alpha_2(A, B), \ \beta_2(A^+, B^+) = \beta_2(A, B), \ \gamma_2(A^+, B^+) = \gamma_2(A, B).$$

Moreover, by an elementary calculation, we see that

$$\sum_{j=1}^{3} Y_j(A, B)^2 = \alpha_2(A, B)^2 \sum_{j=1}^{3} \psi_j(A)^2 + \beta_2(A, B)^2 \sum_{j=1}^{3} \psi_j(B)^2$$

$$+ \gamma_2(A, B)^2 \left\{ \left(\psi_2(A)\psi_3(B) - \psi_3(A)\psi_2(B) \right)^2 \right.$$

$$+ \left(\psi_3(A)\psi_1(B) - \psi_1(A)\psi_3(B) \right)^2 + \left. \left(\psi_1(A)\psi_2(B) - \psi_2(A)\psi_1(B) \right)^2 \right\}$$

$$+ 2\alpha_2(A, B)\beta_2(A, B) \left(\psi_1(A)\psi_1(B) + \psi_2(A)\psi_2(B) + \psi_3(A)\psi_3(B) \right).$$

We have

$$\sum_{i=1}^{3} Y_i(A^+, B^+)^2 = \sum_{i=1}^{3} Y_i(A, B)^2.$$

Thus, we obtain $P_{fs(A^+,B^+)} = P_{fs(A,B)}$. It follows that $P_{BCH(A^+,B^+)} = P_{BCH(A,B)}$ since (6). $\quad\square$

Theorem 2.5.3. *For every pair of $A, B \in K_4(\mathbb{R})$*

$$\| \exp(A) - \exp(B) \|_{\mathfrak{c}} = \| \exp(A^+) - \exp(B^+) \|_{\mathfrak{c}}.$$

Proof. Pick $A, B \in K_4(\mathbb{R})$. By Lemma 2.5.2, we have $P_{BCH(A^+,-B^+)} = P_{BCH(A,-B)}$. We see that the eigenvalues of $\exp(BCH(A^+, B^+)) - I_4$ agree with the eigenvalues of $\exp(BCH(A, B)) - I_4$ by applying the spectral mapping theorem. Since $\exp(BCH(A^+, B^+))$ and $\exp(BCH(A, B))$ are special orthogonal matrices, these are also normal matrices. Hence, $\exp(BCH(A^+, B^+)) - I_4$ and $\exp(BCH(A, B)) - I_4$ are also normal matrices. For any normal matrix, the singular values coincide with the absolute values of the eigenvalues. It follows that

$$\|\exp(A)\exp(-B) - I_4\|_{\mathfrak{c}} = \|\exp(A^+)\exp(-B^+) - I_4\|_{\mathfrak{c}}$$

Since $\|\cdot\|$ is a unitarily invariant norm, we have

$$\|\exp(A) - \exp(B)\|_{\mathfrak{c}} = \|\exp(A^+) - \exp(B^+)\|_{\mathfrak{c}}$$

as desired. $\quad\square$

2.6. **The proof of the main result of section 2.**

Proof of Theorem 2.1.1. (S-i)$\Rightarrow$(S-ii): By Proposition 2.3.5.

(S-ii)$\Rightarrow$(S-i): It is clear that T is an isometry if T is of the form (S-a) or (S-b) because $\|\cdot\|_{\mathfrak{c}}$ is a unitarily invariant norm. Suppose that T is of the form (S-c). Note that Theorem 2.5.3 implies that $\exp(A^+) = \exp(B^+)$ if and only if $\exp(A) = \exp(B)$ for $A, B \in K_4(\mathbb{R})$. Hence, T is well defined map on $SO(n)$. Since $\|\cdot\|_{\mathfrak{c}}$ is a unitarily invariant norm, by Theorem 2.5.3, we have that

$$\|T(\exp(A) - T(\exp(B)))\|_{\mathfrak{c}} = \|O\exp(A^+)O^{-1} - O\exp(B^+)O^{-1}\|_{\mathfrak{c}}$$
$$= \|\exp(A^+) - \exp(B^+)\|_{\mathfrak{c}}$$
$$= \|\exp(A) - \exp(B)\|_{\mathfrak{c}}$$

as desired. If T is of the form (S-d), in a way similar to the case of (S-c), we have

$$\|T(\exp(A) - T(\exp(B)))\|_{\mathfrak{c}} = \|O\exp(-A^+)O^{-1} - O\exp(-B^+)O^{-1}\|_{\mathfrak{c}}$$
$$= \|\exp(-A^+) - \exp(-B^+)\|_{\mathfrak{c}}$$
$$= \|\exp(-A) - \exp(-B)\|_{\mathfrak{c}}$$
$$= \|\exp(A) - \exp(B)\|_{\mathfrak{c}}.$$

The proof is complete. $\qquad\square$

3. Gyrometric preserving maps

In Newtonian mechanics, the set of all velocities coincides with 3-dimensional Euclidean space $\mathbb{R}^3$ and can treat as an inner product space. On the other hand, in special relativity, the magnitude of a velocity must not exceed the speed of light in vacuum c. The set of all Einstein velocities coincides to $\mathbb{R}_c^3 = \{\boldsymbol{u} \in \mathbb{R}^3 : \|\boldsymbol{u}\| < c\}$ and the Einstein velocity addition $\oplus_E$ in $\mathbb{R}_c^3$ is given by the equation

$$\boldsymbol{u} \oplus_E \boldsymbol{v} = \frac{1}{1 + \frac{\langle \boldsymbol{u}, \boldsymbol{v} \rangle}{c^2}} \left\{ \boldsymbol{u} + \frac{1}{\gamma_u} \boldsymbol{v} + \frac{1}{c^2} \frac{\gamma_u}{1 + \gamma_u} \langle \boldsymbol{u}, \boldsymbol{v} \rangle \boldsymbol{u} \right\}$$

for all $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}$, where $\langle \cdot, \cdot \rangle$ is the Euclidean inner product and γ_u is the Lorentz factor given by

$$\gamma_u = \sqrt{\frac{1}{1 - \frac{\|\boldsymbol{u}\|^2}{c^2}}}.$$

The Einstein velocity addition $\oplus_E$ is non-commutative and non-associative on $\mathbb{R}_c^3$ and hence $(\mathbb{R}_c^3, \oplus_E)$ does not have a group structure. Thus, it is not appropriate to treat the set of all Einstein velocities as a linear space. However, $(\mathbb{R}_c^3, \oplus_E)$ has a gyrocommutative gyrogroup structure and is called the Einstein gyrogroup. The (gyrocommutative) gyrogroup is a generalization of the (commutative) group. Some gyrocommutative gyrogroups can be treated as a gyrovector space with a scalar multiplication. The gyrovector space is a generalization of the positive definite real inner product space. The gyrovector space has several linds of structures. Especially, we consider the gyrometric in this section.

In this section, we give a complete description of all gyrometric preserving self-maps on the models of the gyrovector space, the Einstein gyrovector spaces, the Möbius gyrovector spaces and the PV (Proper Velocity) gyrovector spaces. We can show that the gyrometric preserving self-maps on these models preserve their gyrovector space structures.

In the following of the section, $\mathbb{V}$ denotes a real inner product space with the vector addition $+$ and a positive definite inner product $\langle \cdot, \cdot \rangle$. We say that an inner product $\langle \cdot, \cdot \rangle$ is positive definite if the following holds; $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = 0$ for all $\boldsymbol{u} \in \mathbb{V}$ implies $\boldsymbol{v} = \boldsymbol{0}$. We denote by $\| \cdot \|$ the norm on $\mathbb{V}$ induced by $\langle \cdot, \cdot \rangle$ and $\boldsymbol{B}$ denotes the open unit ball of $\mathbb{V}$; $\boldsymbol{B} = \{ \boldsymbol{u} \in \mathbb{V} : \| \boldsymbol{u} \| < 1 \}$.

3.1. Gyrogroups.

In the book [16], Ungar studied gyrocommutative gyrogroups. In this subsection, we recall the definition of (gyrocommutative) gyrogroups and some examples based on [16].

Definition 3.1.1. A groupoid $(S, +)$ is a nonempty set, S, with a binary operation, $+ : S \times S \to S$. An automorphism ϕ of a groupoid $(S, +)$ is a bijective self-map of S, $\phi : S \to S$, which preserves its groupoid operation, that is, $\phi(a + b) = \phi(a) + \phi(b)$ for all $a, b \in S$. $\mathrm{Aut}(S, +)$ is the set of all automorphism of a groupoid $(S, +)$.

A gyrogroup is defined as follows in [16].

Definition 3.1.2. A groupoid $(G, \oplus)$ is a gyrogroup if it satisfies the following axioms.

(G1): There is an element, $\mathbf{0} \in G$, called a left identity, satisfying

$$\mathbf{0} \oplus \boldsymbol{a} = \mathbf{0},$$

for all $\boldsymbol{a} \in G$;

(G2): There is an element $\mathbf{0}$ satisfying axiom (G1) such that for each $\boldsymbol{a} \in G$ there is an element $\ominus \boldsymbol{a}$, called a left inverse of $\boldsymbol{a}$, satisfying

$$\ominus \boldsymbol{a} \oplus \boldsymbol{a} = \mathbf{0};$$

(G3): For any triple $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in G$ there exists a unique element $\mathrm{gyr}[\boldsymbol{a}, \boldsymbol{b}]\boldsymbol{c} \in G$ such that the binary operation obeys the left gyroassociative law

$$\boldsymbol{a} \oplus (\boldsymbol{b} \oplus \boldsymbol{c}) = (\boldsymbol{a} \oplus \boldsymbol{b}) \oplus \mathrm{gyr}[\boldsymbol{a}, \boldsymbol{b}]\boldsymbol{c};$$

(G4): The map $\mathrm{gyr}[\boldsymbol{a}, \boldsymbol{b}] : G \to G$ given by $\boldsymbol{c} \mapsto \mathrm{gyr}[\boldsymbol{a}, \boldsymbol{b}]\boldsymbol{c}$ is an automorphism of the groupoid $(G, \oplus)$,

$$\mathrm{gyr}[\boldsymbol{a}, \boldsymbol{b}] \in \mathrm{Aut}(G, \oplus).$$

The automorphism $\mathrm{gyr}[\boldsymbol{a}, \boldsymbol{b}]$ of G is called gyroautomorphism of G generated by $\boldsymbol{a}, \boldsymbol{b} \in G$. The operator $\mathrm{gyr} : G \times G \to \mathrm{Aut}(G, \oplus)$ is called gyrator of G;

(G5): The gyroautomorphism $\mathrm{gyr}[a, b]$ generated by any $\boldsymbol{a}, \boldsymbol{b} \in G$ possesses the left loop property

$$\mathrm{gyr}[\boldsymbol{a} \oplus \boldsymbol{b}, \boldsymbol{b}] = \mathrm{gyr}[\boldsymbol{a}, \boldsymbol{b}]$$

for any $\boldsymbol{a}, \boldsymbol{b} \in G$.

As in group theory, we use the notation

$$\boldsymbol{a} \ominus \boldsymbol{b} = \boldsymbol{a} \oplus (\ominus \boldsymbol{b})$$

in gyrogroup theory as well.

Definition 3.1.3. [16] A gyrogroup $(G, \oplus)$ is gyrocommutative if its binary operation obey the gyrocommutative law

(G6): $\boldsymbol{a} \oplus \boldsymbol{b} = \mathrm{gyr}[\boldsymbol{a}, \boldsymbol{b}](\boldsymbol{b} \oplus \boldsymbol{a})$.

for all $\boldsymbol{a}, \boldsymbol{b} \in G$.

By definition, it is easy to see that a (commutative) group is a (gyro-commutative) gyrogroup which all of gyroautomorphisms are the identity map on G. The following examples are studied in [16] and an object of our study in this section.

Example 3.1.4. Let $s > 0$ and $\mathbb{V}_s$ be the s-ball of $\mathbb{V}$. Einstein addition $\oplus_E$ is the binary operation in $\mathbb{V}_s$ given by the equation

$$\begin{aligned}
\boldsymbol{u} \oplus_E \boldsymbol{v} &= \frac{1}{1 + \frac{\langle \boldsymbol{u}, \boldsymbol{v} \rangle}{s^2}} \left\{ \left(1 + \frac{1}{s^2} \frac{\gamma_u \langle \boldsymbol{u}, \boldsymbol{v} \rangle}{1 + \gamma_u} \right) \boldsymbol{u} + \frac{1}{\gamma_u} \boldsymbol{v} \right\} \\
&= \frac{1}{1 + \frac{\langle \boldsymbol{u}, \boldsymbol{v} \rangle}{s^2}} \left\{ \left(1 + \frac{1}{s^2} \frac{\langle \boldsymbol{u}, \boldsymbol{v} \rangle}{1 + \alpha_u} \right) \boldsymbol{u} + \alpha_u \boldsymbol{v} \right\}
\end{aligned}$$

where γ_u is the gamma factor $\gamma_u = (1 - \frac{\|\boldsymbol{u}\|^2}{s^2})^{-\frac{1}{2}}$ in the s-ball $\mathbb{V}_s$ and $\alpha_u = \gamma_u^{-1}$. $(\mathbb{V}_s, \oplus_E)$ is a gyrocommutative gyrogroup and called the Einstein gyrogroup. The identity of $(\mathbb{V}_s, \oplus_E)$ is the zero vector of $\mathbb{V}$ and $\ominus_E \boldsymbol{u} = -\boldsymbol{u}$ for any $\boldsymbol{u} \in \mathbb{V}_s$.

Example 3.1.5. Möbius addition $\oplus_M$ is the binary operation in the open unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ given by the equation

$$a \oplus_M b = \frac{a + b}{1 + \overline{a}b}$$

$(\mathbb{D}, \oplus_M)$ is a gyrocommutative gyrogroup. The identity of $(\mathbb{D}, \oplus_M)$ is 0 and the inverse element of $a \in (\mathbb{D}, \oplus_M)$ is $-a$.

Let us identify the complex plane $\mathbb{C}$ with the Euclidean plane $\mathbb{R}^2$ in the usual sense, we have a natural extension of Example 3.1.5 as the following.

Example 3.1.6. Let $s > 0$ and $\mathbb{V}_s$ be the s-ball of $\mathbb{V}$. Möbius addition $\oplus_M$ is the binary operation in $\mathbb{V}_s$ given by the equation

$$\boldsymbol{u} \oplus_M \boldsymbol{v} = \frac{(1 + \frac{2}{s^2}\langle \boldsymbol{u}, \boldsymbol{v}\rangle + \frac{1}{s^2}\|\boldsymbol{v}\|^2)\boldsymbol{u} + (1 - \frac{1}{s^2}\|\boldsymbol{u}\|^2)\boldsymbol{v}}{1 + \frac{2}{s^2}\langle \boldsymbol{u}, \boldsymbol{v}\rangle + \frac{1}{s^4}\|\boldsymbol{u}\|^2\|\boldsymbol{v}\|^2}$$

$(\mathbb{V}_s, \oplus_M)$ is a gyrocommutative gyrogroup and called the Möbius gyrogroup. The identity of $(\mathbb{V}_s, \oplus_M)$ is the zero vector of $\mathbb{V}$ and $\ominus_M \boldsymbol{u} = -\boldsymbol{u}$ for any $\boldsymbol{u} \in \mathbb{V}_s$.

Example 3.1.7. Let $s > 0$ and $\mathbb{V}$ be a real inner product space. PV (Proper Velocity) addition $\oplus_P$ is the binary operation in $\mathbb{V}$ given by the equation

$$\boldsymbol{u} \oplus_P \boldsymbol{v} = \left\{ \frac{\beta_u}{1 + \beta_u} \frac{\langle \boldsymbol{u}, \boldsymbol{v}\rangle}{s^2} + \frac{1}{\beta_v} \right\} \boldsymbol{u} + \boldsymbol{v}$$
$$= \left\{ \frac{1}{1 + \delta_u} \frac{\langle \boldsymbol{u}, \boldsymbol{v}\rangle}{s^2} + \delta_v \right\} \boldsymbol{u} + \boldsymbol{v}$$

where β_u, called the beta factor, is given by the equation $\beta_u = (1 + \frac{\|\boldsymbol{u}\|^2}{s^2})^{-\frac{1}{2}}$ and $\delta_u = \beta_u^{-1}$. $(\mathbb{V}, \oplus_P)$ is a gyrocommutative gyrogroup and

called the PV (Proper Velocity) gyrogroup. The identity of $(\mathbb{V}, \oplus_P)$ is the zero vector of $\mathbb{V}$ and $\ominus_P \boldsymbol{u} = -\boldsymbol{u}$.

3.2. **Gyrovector spaces and gyrometrics.**

Ungar also studied the gyrovector space in his book [16]. A gyrovector space is defined as follows in [16]

Definition 3.2.1. Let G be a subset of a real inner product space $\mathbb{V}$ ($\mathbb{V}$ is called the carrier of G). A real inner product gyrovector space (gyrovector space, in short) $(G, \oplus, \otimes)$ is a gyrocommutative gyrogroup $(G, \oplus)$ with a scalar multiplication $\otimes : \mathbb{R} \times G \to G$ that satisfy the following axioms:

(V0): $\langle \mathrm{gyr}[\boldsymbol{u}, \boldsymbol{v}]\boldsymbol{a}, \mathrm{gyr}[\boldsymbol{u}, \boldsymbol{v}]\boldsymbol{b} \rangle = \langle \boldsymbol{a}, \boldsymbol{b} \rangle$ for all $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{a}, \boldsymbol{b} \in G$;

(V1): $1 \otimes \boldsymbol{a} = \boldsymbol{a}$ for all $\boldsymbol{a} \in G$;

(V2): $(r_1 + r_2) \otimes \boldsymbol{a} = (r_1 \otimes \boldsymbol{a}) \oplus (r_2 \otimes \boldsymbol{a})$ for all $\boldsymbol{a} \in G$, $r_1, r_2 \in \mathbb{R}$;

(V3): $(r_1 r_2) \otimes \boldsymbol{a} = r_1 \otimes (r_2 \otimes \boldsymbol{a})$ for all $\boldsymbol{a} \in G\backslash$, $r_1, r_2 \in \mathbb{R}$;

(V4): $\dfrac{|r| \otimes \boldsymbol{a}}{\|r \otimes \boldsymbol{a}\|} = \dfrac{\boldsymbol{a}}{\|\boldsymbol{a}\|}$ for all $\boldsymbol{a} \in G \setminus \{\boldsymbol{0}\}$, $r \in \mathbb{R} \setminus \{0\}$;

(V5): $\mathrm{gyr}[\boldsymbol{u}, \boldsymbol{v}](r \otimes \boldsymbol{a}) = r \otimes \mathrm{gyr}[\boldsymbol{u}, \boldsymbol{v}]\boldsymbol{a}$ for all $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{a} \in G$, $r \in \mathbb{R}$;

(V6): $\mathrm{gyr}[r_1 \otimes \boldsymbol{v}, r_2 \otimes \boldsymbol{v}] = id_G$ for all $\boldsymbol{v} \in G$, $r_1, r_2 \in \mathbb{R}$;

(VV): $\|G\| = \{\pm\|\boldsymbol{a}\| \in \mathbb{R} : \boldsymbol{a} \in G\}$ is an one-dimensional real vector space with vector addition $\oplus$ and scalar multiplication $\otimes$;

(V7): $\|r \otimes \boldsymbol{a}\| = |r| \otimes \|\boldsymbol{a}\|$ for all $\boldsymbol{a} \in G$, $r \in \mathbb{R}$;

(V8): $\|\boldsymbol{a} \oplus \boldsymbol{b}\| \le \|\boldsymbol{a}\| \oplus \|\boldsymbol{b}\|$ for all $\boldsymbol{a}, \boldsymbol{b} \in G$.

A bijective self-map on a gyrovector space is called an automorphism if the map preserves its structure as follows.

Definition 3.2.2. [16] An automorphism τ of a gyrovector space $(G, \oplus, \otimes)$ is a bijective self-map of G, $\tau : G \to G$ which preserves its structure, that is,

(a): $\tau(\boldsymbol{a} \oplus \boldsymbol{b}) = \tau\boldsymbol{a} \oplus \tau\boldsymbol{b}$ for any $\boldsymbol{a}, \boldsymbol{b} \in G$,

(b): $\tau(r \otimes \boldsymbol{a}) = r \otimes \tau\boldsymbol{a}$ for any $r \in \mathbb{R}$, $\boldsymbol{a} \in G$,

(c): $\langle \tau\boldsymbol{a}, \tau\boldsymbol{b} \rangle = \langle \boldsymbol{a}, \boldsymbol{b} \rangle$ for any $\boldsymbol{a}, \boldsymbol{b} \in G$.

Denote $\mathrm{Aut}(G, \oplus, \otimes)$ the set of all automorphism of the gyrovector space $(G, \oplus, \otimes)$.

Gyrovector spaces have the structure which is called the gyrometric. The gyrometric of a gyrovector space is defined in [16] as follows.

Definition 3.2.3. [16] Let $(G, \oplus, \otimes)$ be a gyrovector space. Its gyrometric ϱ is given by the function $\varrho : G \times G \to \mathbb{R}$,

$$\varrho(\boldsymbol{a}, \boldsymbol{b}) = \| \ominus \boldsymbol{a} \oplus \boldsymbol{b} \| = \| \boldsymbol{b} \ominus \boldsymbol{a} \|.$$

For any gyrovector space, the gyrometric is invariant under the automorphisms and the left gyrotranslations as follows.

Theorem 3.2.4. [16] *Suppose that ϱ is the gyrometric on a gyrovector space $(G, \oplus, \otimes)$. We have*

$$\varrho(\boldsymbol{a} \oplus \boldsymbol{b}, \boldsymbol{a} \oplus \boldsymbol{c}) = \varrho(\boldsymbol{b}, \boldsymbol{c}),$$

$$\varrho(\tau\boldsymbol{b}, \tau\boldsymbol{c}) = \varrho(\boldsymbol{b}, \boldsymbol{c})$$

for any $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in G$, $\tau \in \mathrm{Aut}(G, \oplus, \otimes)$.

A real inner product space $(\mathbb{V}, +, \cdot)$ is a gyrovector space and its gyrometric is the metric induced by its norm. The Einstein gyrogroups, the Möbius gyrogroups and the PV gyrogroups admit the scalar multiplications are turning themselves into gyrovector spaces as following examples.

Example 3.2.5. [16] The Einstein gyrogroup $(\mathbb{V}_s, \oplus_E)$ is a gyrovector space $(\mathbb{V}_s, \oplus_E, \otimes_E)$ with the scalar multiplication $\otimes_E$ on $(\mathbb{V}_s, \oplus_E)$ defined by

$$r \otimes_E \boldsymbol{v} = s \tanh(r \tanh^{-1} \frac{\|\boldsymbol{v}\|}{s}) \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|},$$

where $r \in \mathbb{R}$, $\boldsymbol{v} \in \mathbb{V}_s \setminus \{\boldsymbol{0}\}$; and $r \otimes_E \boldsymbol{0} = \boldsymbol{0}$. The gyrometric $\varrho_E(\boldsymbol{u}, \boldsymbol{v}) = \| - \boldsymbol{u} \oplus_E \boldsymbol{v}\|$ on the Einstein gyrogroup is called Einstein gyrometric. Let $d_E(\boldsymbol{u}, \boldsymbol{v}) = \tanh^{-1} \frac{\varrho_E(\boldsymbol{u}, \boldsymbol{v})}{s}$ then d_E is the metric on $\mathbb{V}_s$.

Example 3.2.6. [16]. The Möbius gyrogroup $(\mathbb{V}_s, \oplus_M)$ is a gyrovector space $(\mathbb{V}_s, \oplus_M, \otimes_M)$ with the scalar multiplication $\otimes_M$ on $(\mathbb{V}_s, \oplus_M)$ defined by

$$r \otimes_M \boldsymbol{v} = s \tanh(r \tanh^{-1} \frac{\|\boldsymbol{v}\|}{s}) \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|},$$

where $r \in \mathbb{R}$, $\boldsymbol{v} \in \mathbb{V}_s \setminus \{\boldsymbol{0}\}$; and $r \otimes_M \boldsymbol{0} = \boldsymbol{0}$. The gyrometric $\varrho_M(\boldsymbol{u}, \boldsymbol{v}) = \| - \boldsymbol{u} \oplus_M \boldsymbol{v}\|$ on the Möbius gyrogroup is called the Möbius gyrometric. Let $d_M(\boldsymbol{u}, \boldsymbol{v}) = \tanh^{-1} \frac{\varrho_M(\boldsymbol{u}, \boldsymbol{v})}{s}$. Then $(\mathbb{V}_s, d_M)$ is the metric space and we call d_M the Möbius metric. In the special case when we consider the Möbius gyrogroup on the complex open unit disc $(\mathbb{D}, \oplus_M)$, Möbius gyrometric reduces to

$$\varrho_M(a, b) = | - a \oplus_M b| = \left| \frac{a - b}{1 - \overline{a}b} \right|.$$

The Möbius gyrometric on $\mathbb{D}$ is known as the pesudo-hyperbolic metric and Möbius metric d_M on $\mathbb{D}$ is also known as the Poincaré metric.

Example 3.2.7. [16] The PV gyrogroup $(\mathbb{V}, \oplus_P)$ is a gyrovector space $(\mathbb{V}, \oplus_P, \otimes_P)$ with the scalar multiplication $\otimes_P$ on $(\mathbb{V}, \oplus_P)$ defined by

$$r \otimes_P \boldsymbol{v} = s \sinh(r \sinh^{-1} \frac{\|\boldsymbol{v}\|}{s}) \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|},$$

where $r \in \mathbb{R}$, $\boldsymbol{v} \in \mathbb{V} \setminus \{\boldsymbol{0}\}$; and $r \otimes_P \boldsymbol{0} = \boldsymbol{0}$. ϱ_P denotes the gyrometric on the PV gyrogroup; $\varrho_P(\boldsymbol{u}, \boldsymbol{v}) = \| - \boldsymbol{u} \oplus_P \boldsymbol{v}\|$.

The following of the section, we consider the Einstein gyrogroups, the Möbius gyrogroups and the PV gyrogroups with the assumption $s = 1$ for simplicity. Indeed, $\mathbb{V}_s = \boldsymbol{B}$ if $s = 1$.

The gyrometrics ϱ_E, ϱ_M and ϱ_P can be represented as in the equations of the following proposition.

Proposition 3.2.8. *For any $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{B}$,*

$$(7) \qquad \varrho_E(\boldsymbol{u}, \boldsymbol{v}) = \left\{ 1 - \frac{(1 - \|\boldsymbol{u}\|^2)(1 - \|\boldsymbol{v}\|^2)}{(1 - \langle \boldsymbol{u}, \boldsymbol{v} \rangle)^2} \right\}^{\frac{1}{2}},$$

$$(8) \qquad \varrho_M(\boldsymbol{u}, \boldsymbol{v}) = \left\{ 1 - \frac{(1 - \|\boldsymbol{u}\|^2)(1 - \|\boldsymbol{v}\|^2)}{1 + \|\boldsymbol{u}\|^2 \|\boldsymbol{v}\|^2 - 2\langle \boldsymbol{u}, \boldsymbol{v} \rangle} \right\}^{\frac{1}{2}}.$$

For any $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{V}$,

$$(9) \quad \varrho_P(\boldsymbol{u}, \boldsymbol{v}) = (\langle \boldsymbol{u}, \boldsymbol{v} \rangle^2 - 2\delta_u \delta_v \langle \boldsymbol{u}, \boldsymbol{v} \rangle + \|\boldsymbol{u}\|^2 + \|\boldsymbol{v}\|^2 + \|\boldsymbol{u}\|^2 \|\boldsymbol{v}\|^2)^{\frac{1}{2}}.$$

Proof. Put $a = \|\boldsymbol{u}\|$, $b = \|\boldsymbol{v}\|$ and $x = \langle \boldsymbol{u}, \boldsymbol{v} \rangle$. By elementary calculation, we have

$$
\begin{aligned}
\| -\boldsymbol{u} \oplus_E \boldsymbol{v} \|^2 &= \left\| \frac{1}{1 + \langle -\boldsymbol{u}, \boldsymbol{v} \rangle} \left\{ \left(1 + \frac{\langle -\boldsymbol{u}, \boldsymbol{v} \rangle}{1 + \alpha_{-u}} \right)(-\boldsymbol{u}) + \alpha_{-u} \boldsymbol{v} \right\} \right\|^2 \\
&= \left\| \frac{1}{1 - x} \left\{ \left(-1 + \frac{x}{1 + \alpha_u} \right) \boldsymbol{u} + \alpha_u \boldsymbol{v} \right\} \right\|^2 \\
&= \frac{\left(1 - \frac{x}{1 + \alpha_u} \right)^2 a^2 + (1 - a^2) b^2 - 2\alpha_u \left(1 - \frac{x}{1 + \alpha_u} \right) x}{(1 - x)^2} \\
&= \frac{x^2 - 2x + a^2 + (1 - a^2) b^2}{(1 - x)^2} \\
&= \frac{(1 - x)^2 - (1 - a^2)(1 - b^2)}{(1 - x)^2} \\
&= 1 - \frac{(1 - a^2)(1 - b^2)}{(1 - x)^2},
\end{aligned}
$$

$$
\begin{aligned}
\| -\boldsymbol{u} \oplus_M \boldsymbol{v} \|^2 &= \left\| \frac{(1 + 2\langle -\boldsymbol{u}, \boldsymbol{v} \rangle + \|\boldsymbol{v}\|^2)(-\boldsymbol{u}) + (1 - \|\boldsymbol{u}\|^2) \boldsymbol{v}}{1 + 2\langle -\boldsymbol{u}, \boldsymbol{v} \rangle + \|\boldsymbol{u}\|^2 \|\boldsymbol{v}\|^2} \right\|^2 \\
&= \left\| \frac{-(1 + b^2 - 2x) \boldsymbol{u} + (1 - a^2) \boldsymbol{v}}{1 + a^2 b^2 - 2x} \right\|^2 \\
&= \frac{(1 + b^2 - 2x)^2 a^2 + (1 - a^2)^2 b^2 - 2(1 + b^2 - 2x)(1 - a^2) x}{(1 + a^2 b^2 - 2x)^2} \\
&= \frac{4x^2 - 2(1 + a^2)(1 + b^2) x + (a^2 + b^2)(1 + a^2 b^2)}{(1 + a^2 b^2 - 2x)^2} \\
&= \frac{(1 + a^2 b^2 - 2x)^2 - (1 - a^2)(1 - b^2)(1 + a^2 b^2 - 2x)}{(1 + a^2 b^2 - 2x)^2} \\
&= 1 - \frac{(1 - a^2)(1 - b^2)}{1 + a^2 b^2 - 2x},
\end{aligned}
$$

$$
\begin{aligned}
\| -\boldsymbol{u} \oplus_P \boldsymbol{v} \|^2 &= \left\| \left\{ \frac{1}{1 + \delta_u} \langle -\boldsymbol{u}, \boldsymbol{v} \rangle + \delta_v \right\}(-\boldsymbol{u}) + \boldsymbol{v} \right\| \\
&= \left\{ \frac{-x}{1 + \delta_u} + \delta_v \right\}^2 a^2 + 2 \left\{ \frac{-x}{1 + \delta_u} + \delta_v \right\}(-x) + b^2 \\
&= \frac{-(1 - \delta_u) + 2}{1 + \delta_u} x^2 + \{ 2\delta_v (1 - \delta_u) - 2\delta_v \} x + a^2 \delta_v^2 + b^2 \\
&= x^2 - 2\delta_u \delta_v x + a^2 + b^2 + a^2 b^2.
\end{aligned}
$$

$\square$

The Bergman metric β on the open unit ball in $\mathbb{C}^n$ is given by

$$\beta(\boldsymbol{z}, \boldsymbol{w}) = \frac{1}{2} \log \frac{1 + \varphi(\boldsymbol{z}, \boldsymbol{w})}{1 - \varphi(\boldsymbol{z}, \boldsymbol{w})},$$

where

$$\varphi(\boldsymbol{z}, \boldsymbol{w}) = \left\{ 1 - \frac{(1 - \|\boldsymbol{z}\|^2)(1 - \|\boldsymbol{w}\|^2)}{\|1 - 2\langle \boldsymbol{z}, \boldsymbol{w}\rangle\|^2} \right\}^{\frac{1}{2}},$$

for any points $\boldsymbol{z}, \boldsymbol{w}$ of the open unit ball in $\mathbb{C}^n$ [20, Lemma 1.2, Proposition 1.20]. Moreover, φ is also a metric on the open unit ball in $\mathbb{C}^n$ [20, Corollary 1.22] and called the pesudo-hyperbolic metric. By Proposition 3.2.8, if $\mathbb{V} = \mathbb{R}^n$, then Einstein gyrometric ϱ_E on $(\boldsymbol{B}, \oplus_E, \otimes)$ is the restriction of φ and hence d_E is the restriction of β.

3.3. **The main results of section 3.**

The following Theorems 3.3.1, 3.3.2 and 3.3.3 are the main results in this section.

Theorem 3.3.1. *Let T be a self-map on the Einstein gyrovector space* $(\boldsymbol{B}, \oplus_E, \otimes_E)$. *Then the following conditions* (E-1), (E-2) *and* (E-3) *are equivalent.*

  (E-1) *T satisfies the following conditions* (E-a), (E-b) *and* (E-c)*:*

  (E-a): *$T(\boldsymbol{0}) = \boldsymbol{0}$,*

  (E-b): *T is a surjection,*

  (E-c): *T is an Einstein gyrometric preserving map,*

    that is, $\varrho_E(T\boldsymbol{u}, T\boldsymbol{v}) = \varrho_E(\boldsymbol{u}, \boldsymbol{v})$ for all $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{B}$.

(E-2) *There exists a surjective inner product preserving linear map*
$O : \mathbb{V} \to \mathbb{V}$ *such that* $T = O|_{\boldsymbol{B}}$.

(E-3) $T \in \mathrm{Aut}(\boldsymbol{B}, \oplus_E, \otimes_E)$.

In particular, if $\dim \mathbb{V} < \infty$, *then the conditions* (E-a) *and* (E-c) *together imply the condition* (E-b).

Theorem 3.3.2. *Let* T *be a self-map on the Möbius gyrovector space*
$(\boldsymbol{B}, \oplus_M, \otimes_M)$. *Then the following conditions* (M-1), (M-2) *and* (M-3)
are equivalent.

(M-1) T *satisfies the following conditions* (M-a), (M-b) *and* (M-c)*:*

(M-a): $T(\boldsymbol{0}) = \boldsymbol{0}$,

(M-b): T *is a surjection,*

(M-c): T *is a Möbius gyrometric preserving map,*

that is, $\varrho_M(T\boldsymbol{u}, T\boldsymbol{v}) = \varrho_M(\boldsymbol{u}, \boldsymbol{v})$ *for all* $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{B}$.

(M-2) *There exists a surjective inner product preserving linear map*
$O : \mathbb{V} \to \mathbb{V}$ *such that* $T = O|_{\boldsymbol{B}}$.

(M-3) $T \in \mathrm{Aut}(\boldsymbol{B}, \oplus_M, \otimes_M)$.

In particular, if $\dim \mathbb{V} < \infty$, *then the conditions* (M-a) *and* (M-c)
together imply the condition (M-b).

Theorem 3.3.3. *Let* T *be a self-map on the PV gyrovector space*
$(\mathbb{V}, \oplus_P, \otimes)$. *Then the following conditions* (PV-1), (PV-2) *and* (PV-3) *are equivalent.*

(PV-1) T *satisfies the following conditions* (PV-a), (PV-b) *and* (PV-c)*:*

(PV-a): $T(\boldsymbol{0}) = \boldsymbol{0}$,

(PV-b): *T is a surjection,*

(PV-c): *T is a PV gyrometric preserving map,*

that is, $\varrho_P(T\boldsymbol{u}, T\boldsymbol{v}) = \varrho_P(\boldsymbol{u}, \boldsymbol{v})$ for all $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{V}$.

(PV-2) *T is a surjective inner product preserving linear map on $\mathbb{V}$.*

(PV-3) $T \in \mathrm{Aut}(\mathbb{V}, \oplus_P, \otimes_P)$.

In particular, if $\dim \mathbb{V} < \infty$, then the conditions (PV-a) *and* (PV-c) *together imply the condition* (PV-b).

3.4. **Lemmas.**

In this subsection, we give necessary lemmas to prove the main results in this section. For any $a > 0$, put $\pi_a = \{\boldsymbol{u} \in \mathbb{V} : \|\boldsymbol{u}\| = a\}$.

The following Lemmas 3.4.1, 3.4.2 and 3.4.3 state that The Einstein gyrometric preserving maps, the Möbius gyrometric preserving maps and the PV gyrometric preserving maps preserve the inner products, respectively.

Lemma 3.4.1. *Let T be an Einstein gyrometric preserving self-map on the Einstein gyrovector space $(\boldsymbol{B}, \oplus_E, \otimes_E)$. Suppose that $T(\boldsymbol{0}) = \boldsymbol{0}$. Then T is an inner product preserving map.*

Proof. We first note that $T(\pi_a) \subseteq \pi_a$ for any $0 \leq a < 1$ since $\|T\boldsymbol{u}\| = \varrho_E(T\boldsymbol{0}, T\boldsymbol{u}) = \varrho_E(\boldsymbol{0}, \boldsymbol{u}) = \|\boldsymbol{u}\|$ for all $\boldsymbol{u} \in \boldsymbol{B}$. Let $0 \leq a, b < 1$ be arbitrary. Put

$$f[a, b](x) = \left\{ 1 - \frac{(1 - a^2)(1 - b^2)}{(1 - x)^2} \right\}^{\frac{1}{2}}.$$

Then the function $f[a,b] : [-ab, ab] \to \mathbb{R}$ is a monotone decreasing function because $x \leq ab < 1$ for any $x \in [-ab, ab]$. Thus $f[a,b]$ is injective. Let $\boldsymbol{u} \in \pi_a, \boldsymbol{v} \in \pi_b$. Note that $-ab \leq \langle \boldsymbol{u}, \boldsymbol{v} \rangle \leq ab$. We have $\varrho_E(\boldsymbol{u}, \boldsymbol{v}) = f[a,b](\langle \boldsymbol{u}, \boldsymbol{v} \rangle)$ by the equation (7). We also have $\varrho_E(T\boldsymbol{u}, T\boldsymbol{v}) = f[a,b](\langle T\boldsymbol{u}, T\boldsymbol{v} \rangle)$ because $T\boldsymbol{u} \in \pi_a$, $T\boldsymbol{v} \in \pi_b$. Hence $f[a,b](\langle T\boldsymbol{u}, T\boldsymbol{v} \rangle) = f[a,b](\langle \boldsymbol{u}, \boldsymbol{v} \rangle)$ as $\varrho_E(T\boldsymbol{u}, T\boldsymbol{v}) = \varrho_E(\boldsymbol{u}, \boldsymbol{v})$. It implies that $\langle T\boldsymbol{u}, T\boldsymbol{v} \rangle = \langle \boldsymbol{u}, \boldsymbol{v} \rangle$ because f is injective. $\qquad \square$

Lemma 3.4.2. *Let T be a Möbius gyrometric preserving self-map on the Möbius gyrovector space $(\boldsymbol{B}, \oplus_M, \otimes_M)$. Suppose that $T(\boldsymbol{0}) = \boldsymbol{0}$. Then T is an inner product preserving map.*

Proof. Note $T(\pi_a) \subseteq \pi_a$ for any $0 \leq a < 1$ since $\|T\boldsymbol{u}\| = \varrho_M(T\boldsymbol{0}, T\boldsymbol{u}) = \varrho_M(\boldsymbol{0}, \boldsymbol{u}) = \|\boldsymbol{u}\|$ for all $\boldsymbol{u} \in \boldsymbol{B}$. Let $0 \leq a, b < 1$ be arbitrary. Put

$$g[a,b](x) = \left\{ 1 - \frac{(1-a^2)(1-b^2)}{1 + a^2 b^2 - 2x} \right\}^{\frac{1}{2}}.$$

Then the function $g[a,b] : [-ab, ab] \to \mathbb{R}$ is a monotone decreasing function. Thus $g[a,b]$ is injective. Let $\boldsymbol{u} \in \pi_a, \boldsymbol{v} \in \pi_b$. Note that $-ab \leq \langle \boldsymbol{u}, \boldsymbol{v} \rangle \leq ab$. We have $\varrho_M(\boldsymbol{u}, \boldsymbol{v}) = g[a,b](\langle \boldsymbol{u}, \boldsymbol{v} \rangle)$ by the equation (8). We also have $\varrho_M(T\boldsymbol{u}, T\boldsymbol{v}) = g[a,b](\langle T\boldsymbol{u}, T\boldsymbol{v} \rangle)$ because $T\boldsymbol{u} \in \pi_a$, $T\boldsymbol{v} \in \pi_b$. Hence $g[a,b](\langle T\boldsymbol{u}, T\boldsymbol{v} \rangle) = g[a,b](\langle \boldsymbol{u}, \boldsymbol{v} \rangle)$ as $\varrho_M(T\boldsymbol{u}, T\boldsymbol{v}) = \varrho_M(\boldsymbol{u}, \boldsymbol{v})$. It implies that $\langle T\boldsymbol{u}, T\boldsymbol{v} \rangle = \langle \boldsymbol{u}, \boldsymbol{v} \rangle$. $\qquad \square$

Lemma 3.4.3. *Let T be a PV gyrometric preserving self-map on the PV gyrovector space $(\mathbb{V}, \oplus_P, \otimes_P)$. Suppose that $T(\boldsymbol{0}) = \boldsymbol{0}$. Then T is an inner product preserving map.*

Proof. Note $T(\pi_a) \subseteq \pi_a$ for any $0 \le a$ since $\|T\boldsymbol{u}\| = \varrho_P(\boldsymbol{0}, T\boldsymbol{u}) = \varrho_P(T\boldsymbol{0}, T\boldsymbol{u}) = \varrho_P(\boldsymbol{0}, \boldsymbol{u}) = \|\boldsymbol{u}\|$ for any $\boldsymbol{u} \in \mathbb{V}$. Let $a, b \ge 0$ be arbitrary. Put

$$h[a, b](x) = (x^2 - 2\delta_u \delta_v x + a^2 + b^2 + a^2 b^2)^{\frac{1}{2}}.$$

We show that $h[a, b] : [-ab, ab] \to \mathbb{R}$ is a monotone decreasing function. We have

$$(h^2[a, b])'(x) = \frac{dh^2[a, b](x)}{dx} = 2x - 2\delta_u \delta_v.$$

Thus $(h^2[a, b])'(x) < 0$ for any $x \in [-ab, ab]$ because $x \le ab < \delta_u \delta_v$. It implies that $h^2[a, b]$ is a monotone decreasing function and hence $h[a, b]$ is also monotone decreasing. Therefore $h[a, b]$ is injective. Let $\boldsymbol{u} \in \pi_a, \boldsymbol{v} \in \pi_b$. Note that $-ab \le \langle \boldsymbol{u}, \boldsymbol{v} \rangle \le ab$. We have $\varrho_P(\boldsymbol{u}, \boldsymbol{v}) = h[a, b](\langle \boldsymbol{u}, \boldsymbol{v} \rangle)$ by the equation (9). We also have $\varrho_P(T\boldsymbol{u}, T\boldsymbol{v}) = h[a, b](\langle T\boldsymbol{u}, T\boldsymbol{v} \rangle)$ because $T\boldsymbol{u} \in \pi_a$ and $T\boldsymbol{v} \in \pi_b$. Hence $h[a, b](\langle T\boldsymbol{u}, T\boldsymbol{v} \rangle) = h[a, b](\langle \boldsymbol{u}, \boldsymbol{v} \rangle)$ as $\varrho_P(T\boldsymbol{u}, T\boldsymbol{v}) = \varrho_P(\boldsymbol{u}, \boldsymbol{v})$. It implies that $\langle T\boldsymbol{u}, T\boldsymbol{v} \rangle = \langle \boldsymbol{u}, \boldsymbol{v} \rangle$. $\square$

The following lemma shows that an inner product preserving map on $\boldsymbol{B}$ is extendible to the whole space.

Lemma 3.4.4. *Let $\boldsymbol{B}$ be the open unit ball of a real inner product space $\mathbb{V}$. Suppose that $T : \boldsymbol{B} \to \boldsymbol{B}$ is an inner product preserving map. Then T can be extended to an inner product preserving map $S : \mathbb{V} \to \mathbb{V}$ defined by*

$$S(\boldsymbol{w}) = 2\|\boldsymbol{w}\| T\left(\frac{\boldsymbol{w}}{2\|\boldsymbol{w}\|}\right)$$

for any $\boldsymbol{w} \in \mathbb{V} \setminus \{\boldsymbol{0}\}$; and $S(\boldsymbol{0}) = \boldsymbol{0}$. Moreover, S is a linear operator if T is surjective.

Proof. First we show that $T = S|_{\boldsymbol{B}}$. Let $\boldsymbol{u} \in \boldsymbol{B} \setminus \{\boldsymbol{0}\}$, $r > 0$ which satisfy $r\boldsymbol{u} \in \boldsymbol{B}$. We have

$$\langle T(r\boldsymbol{u}), T\boldsymbol{u} \rangle = \langle r\boldsymbol{u}, \boldsymbol{u} \rangle = r\|\boldsymbol{u}\|^2 = \|r\boldsymbol{u}\|\|\boldsymbol{u}\| = \|T(r\boldsymbol{u})\|\|T\boldsymbol{u}\|.$$

It follows that $T(r\boldsymbol{u})$ and $T\boldsymbol{u}$ are parallel by the Cauchy-Schwarz inequality. Hence there exists a real number s such that $T(r\boldsymbol{u}) = sT\boldsymbol{u}$. We have $s = t$ as

$$r\|\boldsymbol{u}\|^2 = \langle r\boldsymbol{u}, \boldsymbol{u} \rangle = \langle T(r\boldsymbol{u}), T\boldsymbol{u} \rangle = \langle sT\boldsymbol{u}, T\boldsymbol{u} \rangle = s\|T\boldsymbol{u}\|^2 = s\|\boldsymbol{u}\|^2.$$

Therefore, $T(r\boldsymbol{u}) = rT(\boldsymbol{u})$ for any $\boldsymbol{u} \in \boldsymbol{B} \setminus \{\boldsymbol{0}\}$ and $r > 0$ which satisfy $r\boldsymbol{u} \in \boldsymbol{B}$. In particular,

$$S(\boldsymbol{u}) = 2\|\boldsymbol{u}\|T\left(\frac{\boldsymbol{u}}{2\|\boldsymbol{u}\|}\right) = T(\boldsymbol{u})$$

for any $\boldsymbol{u} \in \boldsymbol{B} \setminus \{\boldsymbol{0}\}$.

Next we show that S is an inner product preserving map. It is clear that $\langle S\boldsymbol{w}, S\boldsymbol{z} \rangle = 0 = \langle \boldsymbol{w}, \boldsymbol{z} \rangle$ if $\boldsymbol{w} = 0$ or $\boldsymbol{z} = 0$. For any pair $\boldsymbol{w}, \boldsymbol{z} \in \mathbb{V} \setminus \{\boldsymbol{0}\}$, we have

$$\begin{aligned}
\langle S\boldsymbol{w}, S\boldsymbol{z} \rangle &= \left\langle 2\|\boldsymbol{w}\|T\left(\frac{\boldsymbol{w}}{2\|\boldsymbol{w}\|}\right), 2\|\boldsymbol{z}\|T\left(\frac{\boldsymbol{z}}{2\|\boldsymbol{z}\|}\right) \right\rangle \\
&= 2\|\boldsymbol{w}\|2\|\boldsymbol{z}\|\left\langle T\left(\frac{\boldsymbol{w}}{2\|\boldsymbol{w}\|}\right), T\left(\frac{\boldsymbol{z}}{2\|\boldsymbol{z}\|}\right) \right\rangle \\
&= 2\|\boldsymbol{w}\|2\|\boldsymbol{z}\|\left\langle \frac{\boldsymbol{w}}{2\|\boldsymbol{w}\|}, \frac{\boldsymbol{z}}{2\|\boldsymbol{z}\|} \right\rangle \\
&= \langle \boldsymbol{w}, \boldsymbol{z} \rangle.
\end{aligned}$$

We can prove that $S(t\boldsymbol{v}) = tS(\boldsymbol{v})$ for any $t > 0$ and $\boldsymbol{v} \in \mathbb{V}$ in a way similar to the case where $T(r\boldsymbol{u}) = rT(\boldsymbol{u})$ for any $r > 0$ and $\boldsymbol{u} \in \mathbb{V}$ such that $r\boldsymbol{u} \in \boldsymbol{B}$.

Finally, we show that S is a linear map if T is surjective. Suppose that T is surjective. T^{-1} is also an inner product preserving map and hence $T^{-1}(r\boldsymbol{u}) = rT^{-1}(\boldsymbol{u})$ for any $r > 0$ and $\boldsymbol{u} \in \mathbb{V}$ such that $r\boldsymbol{u} \in \boldsymbol{B}$. Therefore, for any $\boldsymbol{y} \in \mathbb{V}$, we have

$$\boldsymbol{y} = TT^{-1}(\boldsymbol{y})$$

$$= 2\|\boldsymbol{y}\|T(T^{-1}(\frac{\boldsymbol{y}}{2\|\boldsymbol{y}\|}))$$

$$= 2\|\boldsymbol{y}\|S(T^{-1}(\frac{\boldsymbol{y}}{2\|\boldsymbol{y}\|}))$$

$$= S(2\|\boldsymbol{y}\|T^{-1}(\frac{\boldsymbol{y}}{2\|\boldsymbol{y}\|})).$$

Thus we have that S is surjective. Hence S is a surjective isometry from a normed space onto itself. The Mazur-Ulam Theorem asserts that S is a real linear map since $S(\boldsymbol{0}) = \boldsymbol{0}$. $\square$

3.5. **The proofs of the main results of section 3.**

Proof of Theorem 3.3.1. (E-2)$\Rightarrow$(E-3): Suppose that $T = O|_{\boldsymbol{B}}$ for a surjective inner product preserving linear operator $O : \mathbb{V} \to \mathbb{V}$. For any $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{B}$, $r \in \mathbb{R}$, we have

$$\langle T\boldsymbol{u}, T\boldsymbol{v} \rangle = \langle O\boldsymbol{u}, O\boldsymbol{v} \rangle = \langle \boldsymbol{u}, \boldsymbol{v} \rangle,$$

$$r \otimes_E T(\boldsymbol{u}) = \tanh(r \tanh^{-1}\|T\boldsymbol{u}\|)\frac{T\boldsymbol{u}}{\|T\boldsymbol{u}\|}$$

$$= \tanh(r \tanh^{-1}\|O\boldsymbol{u}\|)\frac{O\boldsymbol{u}}{\|O\boldsymbol{u}\|}$$

$$= O\left(\tanh(r \tanh^{-1}\|\boldsymbol{u}\|)\frac{\boldsymbol{u}}{\|\boldsymbol{u}\|}\right)$$

$$= T\left(\tanh(r \tanh^{-1}\|\boldsymbol{u}\|)\frac{\boldsymbol{u}}{\|\boldsymbol{u}\|}\right)$$

$$= T(r \otimes_E \boldsymbol{u}),$$

$$T(\boldsymbol{u}) \oplus_E T(\boldsymbol{v}) = \frac{1}{1 + \langle T\boldsymbol{u}, T\boldsymbol{v} \rangle} \left\{ \left(1 + \frac{\langle T\boldsymbol{u}, T\boldsymbol{v} \rangle}{1 + \alpha_{Tu}} \right) T\boldsymbol{u} + \alpha_{Tu} T\boldsymbol{v} \right\}$$

$$= \frac{1}{1 + \langle \boldsymbol{u}, \boldsymbol{v} \rangle} \left\{ \left(1 + \frac{\langle \boldsymbol{u}, \boldsymbol{v} \rangle}{1 + \alpha_u} \right) O\boldsymbol{u} + \alpha_u O\boldsymbol{v} \right\}$$

$$= O \left(\frac{1}{1 + \langle \boldsymbol{u}, \boldsymbol{v} \rangle} \left\{ \left(1 + \frac{\langle \boldsymbol{u}, \boldsymbol{v} \rangle}{1 + \alpha_u} \right) \boldsymbol{u} + \alpha_u \boldsymbol{v} \right\} \right)$$

$$= T(\boldsymbol{u} \oplus_E \boldsymbol{v}).$$

Finally, $O(\boldsymbol{B}) = \boldsymbol{B}$ since O is surjective and satisfies $\|O\boldsymbol{u}\| = \|\boldsymbol{u}\|$ for any $\boldsymbol{u} \in \mathbb{V}$. It follows that T is surjctive.

(E-3)$\Rightarrow$(E-1): Suppose that $T \in \mathrm{Aut}(\boldsymbol{B}, \oplus_E, \otimes_E)$. Clearly, T is surjective. T is a gyrometric preserving map by Theorem 3.2.4. $T(\boldsymbol{0}) = \boldsymbol{0}$ because T is an inner product preserving map.

(E-1)$\Rightarrow$(E-2): Suppose that T satisfies the condition (E-1). Then Lemma 3.4.1 asserts that T is a surjective inner product preserving map. Furthermore, Lemma 3.4.4 asserts that T can be extended to a surjective inner product preserving linear operator $O : \mathbb{V} \to \mathbb{V}$.

Finally, suppose that $\dim \mathbb{V} < \infty$. Assume that the conditions (E-a) and (E-c) are satisfied. Clearly, $T(\pi_a) \subset \pi_a$ for any $0 \le a < 1$ since $\|T(\boldsymbol{u})\| = \|\boldsymbol{u}\|$ for all $\boldsymbol{u} \in \boldsymbol{B}$. As $\dim \mathbb{V} < \infty$, π_a is compact for all $0 \le a < 1$. Lemma 2.2.1 asserts that $T(\pi_a) = \pi_a$ for all $0 \le a < 1$ and hence $T(\boldsymbol{B}) = \boldsymbol{B}$. $\square$

Proof of Theorem 3.3.2. (M-2)$\Rightarrow$(M-3): Suppose that $T = O|_{\boldsymbol{B}}$ for a surjective inner product preserving linear operator $O : \mathbb{V} \to \mathbb{V}$. For any $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{B}$, $r \in \mathbb{R}$, we have

$$\langle T\boldsymbol{u}, T\boldsymbol{v} \rangle = \langle O\boldsymbol{u}, O\boldsymbol{v} \rangle = \langle \boldsymbol{u}, \boldsymbol{v} \rangle,$$

$$r \otimes_M T(\boldsymbol{u}) = \tanh(r \tanh^{-1} \|T\boldsymbol{u}\|) \frac{T\boldsymbol{u}}{\|T\boldsymbol{u}\|}$$

$$= \tanh(r \tanh^{-1} \|O\boldsymbol{u}\|) \frac{O\boldsymbol{u}}{\|O\boldsymbol{u}\|}$$

$$= O\left(\tanh(r \tanh^{-1} \|\boldsymbol{u}\|) \frac{\boldsymbol{u}}{\|\boldsymbol{u}\|}\right)$$

$$= T\left(\tanh(r \tanh^{-1} \|\boldsymbol{u}\|) \frac{\boldsymbol{u}}{\|\boldsymbol{u}\|}\right)$$

$$= T(r \otimes_M \boldsymbol{u}),$$

$$T(\boldsymbol{u}) \oplus_M T(\boldsymbol{v}) = \frac{(1 + 2\langle T\boldsymbol{u}, T\boldsymbol{v}\rangle + \|T\boldsymbol{v}\|^2)T\boldsymbol{u} + (1 - \|T\boldsymbol{u}\|^2)T\boldsymbol{v}}{1 + 2\langle T\boldsymbol{u}, T\boldsymbol{v}\rangle + \|T\boldsymbol{u}\|^2\|T\boldsymbol{v}\|^2}$$

$$= \frac{(1 + 2\langle \boldsymbol{u}, \boldsymbol{v}\rangle + \|\boldsymbol{v}\|^2)O\boldsymbol{u} + (1 - \|\boldsymbol{u}\|^2)O\boldsymbol{v}}{1 + 2\langle \boldsymbol{u}, \boldsymbol{v}\rangle + \|\boldsymbol{u}\|^2\|\boldsymbol{v}\|^2}$$

$$= O\left(\frac{(1 + 2\langle \boldsymbol{u}, \boldsymbol{v}\rangle + \|\boldsymbol{v}\|^2)\boldsymbol{u} + (1 - \|\boldsymbol{u}\|^2)\boldsymbol{v}}{1 + 2\langle \boldsymbol{u}, \boldsymbol{v}\rangle + \|\boldsymbol{u}\|^2\|\boldsymbol{v}\|^2}\right)$$

$$= T(\boldsymbol{u} \oplus_M \boldsymbol{v}).$$

Finally, $O(\boldsymbol{B}) = \boldsymbol{B}$ since O is a surjective and satisfies $\|T\boldsymbol{u}\| = \|\boldsymbol{u}\|$ for all $\boldsymbol{u} \in \mathbb{V}$.

(M-3)$\Rightarrow$(M-1): Suppose that $T \in \mathrm{Aut}(\boldsymbol{B}, \oplus_M, \otimes_M)$. Clearly, T is surjective. T is a gyrometric preserving map by Theorem 3.2.4. $T(\boldsymbol{0}) = \boldsymbol{0}$ because T is an inner product preserving map.

(M-1)$\Rightarrow$(M-2): Suppose that T satisfies the condition (M-1). Then Lemma 3.4.2 asserts that T is a surjective inner product preserving map. Furthermore, Lemma 3.4.4 asserts that T can be extended to a surjective inner product preserving linear operator $O : \mathbb{V} \to \mathbb{V}$.

Finally, suppose that $\dim \mathbb{V} < \infty$. Assume that the conditions (M-a) and (M-c) are satisfied. Clearly, $T(\pi_a) \subset \pi_a$ for any $0 \le a < 1$ since $\|T(\boldsymbol{u})\| = \|\boldsymbol{u}\|$ for all $\boldsymbol{u} \in \boldsymbol{B}$. As $\dim \mathbb{V} < \infty$, π_a is compact for all $0 \le a < 1$. Lemma 2.2.1 asserts that $T(\pi_a) = \pi_a$ for all $0 \le a < 1$,

hence $T(\boldsymbol{B}) = \boldsymbol{B}$. $\square$

Proof of Theorem 3.3.3. (PV-2)$\Rightarrow$(PV-3): Suppose that T is a surjective inner product preserving linear operator $T : \mathbb{V} \to \mathbb{V}$. For any $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{V}$, $r \in \mathbb{R}$, we have

$$\langle T\boldsymbol{u}, T\boldsymbol{v} \rangle = \langle \boldsymbol{u}, \boldsymbol{v} \rangle,$$

$$r \otimes_P T(\boldsymbol{u}) = \sinh\left(r \sinh^{-1} \|T\boldsymbol{u}\|\right) \frac{T\boldsymbol{u}}{\|T\boldsymbol{u}\|}$$

$$= \sinh\left(r \sinh^{-1} \|\boldsymbol{u}\|\right) \frac{T\boldsymbol{u}}{\|\boldsymbol{u}\|}$$

$$= T\left(\sinh\left(r \sinh^{-1} \|\boldsymbol{u}\|\right) \frac{\boldsymbol{u}}{\|\boldsymbol{u}\|}\right)$$

$$= T(r \otimes_P \boldsymbol{u}),$$

$$T(\boldsymbol{u}) \oplus_P T(\boldsymbol{v}) = \left(\frac{1}{1 + \delta_{Tu}} \langle T\boldsymbol{u}, T\boldsymbol{v} \rangle\right) T\boldsymbol{u} + T\boldsymbol{v}$$

$$= \left(\frac{1}{1 + \delta_u} \langle \boldsymbol{u}, \boldsymbol{v} \rangle\right) T\boldsymbol{u} + T\boldsymbol{v}$$

$$= T\left(\left(\frac{1}{1 + \delta_u} \langle \boldsymbol{u}, \boldsymbol{v} \rangle\right) \boldsymbol{u} + \boldsymbol{v}\right)$$

$$= T(\boldsymbol{u} \oplus_P \boldsymbol{v}).$$

(PV-3)$\Rightarrow$(PV-1): Suppose that $T \in \mathrm{Aut}(\boldsymbol{B}, \oplus_P, \otimes_P)$. Clearly, T is surjective. T is a gyrometric preserving map by Theorem3.2.4. $T(\boldsymbol{0}) = \boldsymbol{0}$ because T is an inner product preserving map.

(PV-1)$\Rightarrow$(PV-2): Suppose that T satisfies the condition (PV-1). Then Lemma 3.4.3 asserts that T is an inner product preserving map. Moreover, T is surjective and hence the Mazur-Ulam theorem asserts that T is a linear operator. Indeed, $T : \mathbb{V} \to \mathbb{V}$ is a surjective inner product preserving linear operator.

Finally, suppose that $\dim \mathbb{V} < \infty$. Assume that the conditions (PV-a) and (PV-c) are satisfied. Clearly, $T(\pi_a) \subset \pi_a$ for any $0 \le a$ since $\|T(\boldsymbol{u})\| = \|\boldsymbol{u}\|$ for all $\boldsymbol{u} \in \mathbb{V}$. As $\dim \mathbb{V} < \infty$, then π_a is compact for all $0 \le a$. Lemma 2.2.1 asserts that $T(\pi_a) = \pi_a$ for all $0 \le a$ and hence $T(\mathbb{V}) = \mathbb{V}$. $\square$

3.6. Gyrometric preserving maps on the Einstein gyrovector space, the Möbius gyorovector space and the PV gyrovector space.

In subsection 3.3, we have the representation of the surjective gyrometric preserving maps under the hypothesis $T(\boldsymbol{0}) = \boldsymbol{0}$. In general, a gyrometric preserving map does not necessarily fix the point $\boldsymbol{0}$. However, the general forms of the surjective gyrometric preserving self-maps is obtained as corollaries of our main results in this section.

Any gyrogroup $(G, \oplus)$ satisfies $\boldsymbol{a} \oplus (\ominus \boldsymbol{a} \oplus \boldsymbol{b}) = \boldsymbol{b}$ for any $\boldsymbol{a}, \boldsymbol{b} \in G$ (the left cancellation law [16])). Let T be a self-map on a gyrovector space $(G, \oplus, \otimes)$. Put $T_0(\cdot) = \ominus T(\boldsymbol{0}) \oplus T(\cdot)$. Then $T(\cdot) = T(\boldsymbol{0}) \oplus T_0(\boldsymbol{0})$ as the left cancellation law. Moreover, Theorem 3.2.4 shows that $\varrho(T_0(\boldsymbol{a}), T_0(\boldsymbol{b})) = \varrho(T(\boldsymbol{a}), T(\boldsymbol{b}))$ for any pair $\boldsymbol{a}, \boldsymbol{b} \in G$. Thus, T_0 is a gyrometric preserving map if and only if so is T. Also, T_0 is surjective if and only if so is T. Needless to say, $T_0(\boldsymbol{0}) = \boldsymbol{0}$. Applying Theorems 3.3.1, 3.3.2 and 3.3.3 to T_0 we obtain Corollaries 3.6.1, 3.6.2 and 3.6.3, respectively. These corollaries give us complete descriptions

of all surjective gyrometric preserving self-maps on our models without the assumption $T(\mathbf{0}) = \mathbf{0}$.

Corollary 3.6.1. *Let T be a self-map on an Einstein gyrogroup $(\boldsymbol{B}, \oplus_E)$. Then the following conditions are equivalent.*

(E-A) T is a surjective Einstein gyrometric preserving map, that is, $\varrho_E(T\boldsymbol{u}, T\boldsymbol{v}) = \varrho_E(\boldsymbol{u}, \boldsymbol{v})$ for $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{B}$

(E-B) There exists a surjective inner product preserving linear operator $O : \mathbb{V} \to \mathbb{V}$ such that $T(\boldsymbol{u}) = T(\mathbf{0}) \oplus_E O\boldsymbol{u}$ for any $\boldsymbol{u} \in \boldsymbol{B}$.

(E-C) T is a surjective isometry with respect to the metric d_E, that is, $d_E(T\boldsymbol{u}, T\boldsymbol{v}) = d_E(\boldsymbol{u}, \boldsymbol{v})$ for $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{B}$

Proof. (E-A)$\Leftrightarrow$(E-C): It is obvious since $d_E = \tanh^{-1} \circ \varrho_E$.

(E-A)$\Leftrightarrow$(E-B): Let T be a self-map on $(\boldsymbol{B}, \oplus_E)$. Put $T_0(\cdot) = -T(\mathbf{0}) \oplus_E T(\cdot)$.

First, we assume that T satisfies the condition (E-A). Then we have $T_0(\mathbf{0}) = \mathbf{0}$ and T_0 is a surjective Einstein gyrometric preserving map. Theorem 3.3.1 shows that T_0 is the restriction of some surjective inner product preserving linear operator $O : \mathbb{V} \to \mathbb{V}$. It follows that $T(\boldsymbol{u}) = T(\mathbf{0}) \oplus_E O\boldsymbol{u}$ for any $\boldsymbol{u} \in \boldsymbol{B}$.

Conversely, let $O : \mathbb{V} \to \mathbb{V}$ be a surjective inner product preserving linear operator and $T(\boldsymbol{u}) = T(\mathbf{0}) \oplus_E O\boldsymbol{u}$ for any $\boldsymbol{u} \in \boldsymbol{B}$. Then we have $T_0(\boldsymbol{u}) = O(\boldsymbol{u})$ for all $\boldsymbol{u} \in \boldsymbol{B}$. Theorem 3.3.1 asserts that T_0 is a surjective Einstein gyrometric preserving map and hence T is. $\qquad\square$

Corollary 3.6.2. *Let T be a self-map on the Möbius gyrogroup $(\boldsymbol{B}, \oplus_M)$. Then the following conditions are equivalent.*

(M-A) *T is a surjective Möbius gyrometric preserving map,*

that is, $\varrho_M(T\boldsymbol{u}, T\boldsymbol{v}) = \varrho_M(\boldsymbol{u}, \boldsymbol{v})$ for $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{B}$

(M-B) *There exists a surjective inner product preserving linear operator $O : \mathbb{V} \to \mathbb{V}$ such that $T(\boldsymbol{u}) = T(\boldsymbol{0}) \oplus_M O\boldsymbol{u}$ for any $\boldsymbol{u} \in \boldsymbol{B}$.*

(M-C) *T is a surjective isometry with respect to the Möbius metric d_M, that is, $d_M(T\boldsymbol{u}, T\boldsymbol{v}) = d_M(\boldsymbol{u}, \boldsymbol{v})$ for $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{B}$*

Proof. (M-A)$\Leftrightarrow$(M-C): It is obvious since $d_M = \tanh^{-1}\circ\varrho_M$.

(M-A)$\Leftrightarrow$(M-B): Let T be a self-map on $(\boldsymbol{B}, \oplus_M)$. Put $T_0(\cdot) = -T(\boldsymbol{0}) \oplus_M T(\cdot)$.

First, we assume that T satisfies the condition (M-A). Then we have $T_0(\boldsymbol{0}) = \boldsymbol{0}$ and T_0 is a surjective Möbius gyrometric preserving map. Theorem 3.3.2 asserts that T_0 is the restriction of some surjective inner product preserving linear operator $O : \mathbb{V} \to \mathbb{V}$. It follows that $T(\boldsymbol{u}) = T(\boldsymbol{0}) \oplus_E O\boldsymbol{u}$ for any $\boldsymbol{u} \in \boldsymbol{B}$.

Conversely, let $O : \mathbb{V} \to \mathbb{V}$ be a surjective inner product preserving linear operator and $T(\boldsymbol{u}) = T(\boldsymbol{0}) \oplus_M O\boldsymbol{u}$ for any $\boldsymbol{u} \in \boldsymbol{B}$. Then we have $T_0(\boldsymbol{u}) = O(\boldsymbol{u})$ for any $\boldsymbol{u} \in \boldsymbol{B}$. Theorem 3.3.2 shows that T_0 is a surjective Möbius gyrometric preserving map and hence T is. $\square$

Corollary 3.6.3. *Let T be a self-map on the PV gyrovector space $(\mathbb{V}, \oplus_P)$. Then the following conditions are equivalent.*

(P-A) *T is a surjective gyrometric preserving map on $(\mathbb{V}, \oplus_P)$, that is, $\varrho_P(T\boldsymbol{u}, T\boldsymbol{v}) = \varrho_P(\boldsymbol{u}, \boldsymbol{v})$ for all $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{V}$.*

(P-B) *There exists a surjective inner product preserving linear operator $O : \mathbb{V} \to \mathbb{V}$ such that $T(\boldsymbol{u}) = T(\boldsymbol{0}) \oplus_P O\boldsymbol{u}$ for any $\boldsymbol{u} \in \mathbb{V}$.*

Proof. Let T be a self-map on $(\boldsymbol{B}, \oplus_P)$. Put $T_0(\cdot) = -T(\boldsymbol{0}) \oplus_P T(\cdot)$.

First, we assume that T satisfies the condition (P-A). Then we have $T_0(\boldsymbol{0}) = \boldsymbol{0}$ and T_0 is a surjective PV gyrometric preserving map. Theorem 3.3.3 shows that T_0 is a surjective inner product preserving linear operator on $\mathbb{V}$. Since $T(\cdot) = T(\boldsymbol{0}) \oplus_P T_0(\cdot)$, T satisfies the condition (P-B).

Conversely, let $O : \mathbb{V} \to \mathbb{V}$ be a surjective inner product preserving linear operator and $T(\boldsymbol{u}) = T(\boldsymbol{0}) \oplus_P O\boldsymbol{u}$ for any $\boldsymbol{u} \in \mathbb{V}$. Then we have $T_0(\boldsymbol{u}) = O(\boldsymbol{u})$ for any $\boldsymbol{u} \in \mathbb{V}$. Theorem 3.3.3 asserts that T_0 is surjective and preserves the gyrometric on the PV gyrogroup and hence so is T. $\qquad\square$

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