

**Geometrical Constants and  
Norm Inequalities in Banach Spaces**

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# Introduction

There are many important norm inequalities in mathematics such as triangle inequality, Clarkson's inequalities [12], Dunkl-Williams inequality [16] and so on. These inequalities are very useful to study geometrical structure of Banach spaces. For example, the notion of strict convexity is based on the occurrence of equality in the triangle inequality. In 1936, Clarkson [12] showed that  $L_p$ -spaces with  $1 < p < \infty$  are uniformly convex by using Clarkson's inequalities.

We have several geometrical constants of Banach spaces, for example, von Neumann-Jordan constant [13], James constant [19], Dunkl-Williams constant [25] and so on. Some of them are induced by norm inequalities. The von Neumann-Jordan constant and Dunkl-Williams constant measure how much the space is close (or far) to be a Hilbert space. The James constant represents non-squareness of the unit ball. These constants play an important role in the description of geometrical properties of Banach spaces, and have been investigated in many papers. In particular, many authors have calculated and estimated constants for Banach spaces.

Our aim in this thesis is to present recent results on geometrical constants and norm inequalities. In Chapter 1, we study modified von Neuman-Jordan constant and Zbăganu constant of  $\mathbb{R}^2$  with absolute normalized norms. In Chapter 2, we investigate how to calculate the Dunkl-Williams constant of Banach spaces. In Chapter 3, we describe some remarks on Clarkson's inequalities and triangle inequality.

In Chapter 1, we consider some geometrical constants of  $\mathbb{R}^2$  with absolute normalized norms. The notion of the von Neumann-Jordan constant (hereafter referred to as NJ constant) of Banach spaces was introduced by Clarkson in [13] and it has been studied by several authors ([26, 28, 37, 63, 66] and so on). The NJ constant  $C_{NJ}(X)$  of a Banach space  $X$  is defined as

$$C_{NJ}(X) = \sup \left\{ \frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} \mid x, y \in X, (x, y) \neq (0, 0) \right\}.$$

Some similar constants have been defined and studied:

$$C'_{NJ}(X) = \sup \left\{ \frac{\|x+y\|^2 + \|x-y\|^2}{4} \mid x, y \in S_X \right\},$$
$$C_Z(X) = \sup \left\{ \frac{\|x+y\|\|x-y\|}{\|x\|^2 + \|y\|^2} \mid x, y \in X, (x, y) \neq (0, 0) \right\},$$

where  $S_X$  is the unit sphere of  $X$ . The constant  $C'_{NJ}(X)$ , called the modified von Neumann-Jordan constant (shortly, modified NJ constant) was introduced by Gao in [18]. The constant  $C_Z(X)$  was introduced by Zbăganu [70]. It has been shown that  $C'_{NJ}(X)$  and  $C_Z(X)$  do not necessarily coincide with  $C_{NJ}(X)$  (cf. [3, 4, 20, 21, 41]).

In 2000, Saito, Kato and Takahashi [58] calculated and estimated the NJ constant of  $\mathbb{C}^2$  with absolute normalized norms. The results obtained in [58] also hold on  $\mathbb{R}^2$ . We investigate the conditions of absolute normalized norms on  $\mathbb{R}^2$  that the modified NJ constant coincides with NJ constant or that the Zbăganu constant coincides with NJ constant. Further we calculate modified NJ constant and Zbăganu constant for some Banach spaces.

In Chapter 2, we study how to calculate the Dunkl-Williams constant. In 1964, Dunkl and Williams [16] showed that for any nonzero elements  $x, y$  in a Banach space  $X$ ,

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq \frac{4\|x - y\|}{\|x\| + \|y\|}.$$

This inequality is called the Dunkl-Williams inequality and have been studied in many papers ([6, 31, 42, 43, 54, 60, 61] and so on). In the paper [16], the authors proved that the constant 4 can be replaced by 2 if  $X$  is a Hilbert space. A bit later, Kirk and Smiley [34] completed this result by showing that the inequality with 2 in place of 4 in fact characterizes Hilbert spaces.

In [25], Jiménez-Melado et al. introduced the notion of the Dunkl-Williams constant of Banach spaces. The Dunkl-Williams constant  $DW(X)$  of a Banach space  $X$  is the smallest constant which can replace the 4 in the Dunkl-Williams inequality, that is,

$$DW(X) = \sup \left\{ \frac{\|x\| + \|y\|}{\|x - y\|} \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \mid x, y \in X, x, y \neq 0, x \neq y \right\}.$$

With this notion, our previous comments can be written as:  $2 \leq DW(X) \leq 4$  for any Banach space  $X$ , and  $X$  is a Hilbert space if and only if  $DW(X) = 2$ . On the other hand, from Baronti and Papini [7] we have that  $X$  is uniformly non-square if and only if  $DW(X) < 4$ . However, it is very difficult to calculate the Dunkl-Williams constant and so, except for Hilbert spaces, there exists no uniformly non-square Banach space in which the constant has been calculated.

We introduce some notations related to Birkhoff orthogonality. Then we present a characterization of the Dunkl-Williams constant. We define the frame of the unit

ball of a Banach space which is related to the norming functionals for elements of the unit sphere. Then we improve the characterization. Thereafter, as an application, we calculate the Dunkl-Williams constant of the Day-James space  $\ell_2\text{-}\ell_\infty$ .

In Chapter 3, we consider the generalized Clarkson's inequalities in the real and complex cases. In 1935, Jordan and von Neumann [26] characterized Hilbert spaces as Banach spaces satisfying the parallelogram law. In the next year 1936, as a generalization of the parallelogram law, Clarkson [12] proved famous norm inequalities for  $L_p$  so called Clarkson's inequalities. To study the generalizations of Clarkson's inequalities, for two numbers  $p, q$  with  $0 < p, q \leq \infty$ , the generalized Clarkson's inequality in the complex case have been considered:

$$(|z + w|^q + |z - w|^q)^{\frac{1}{q}} \leq C(|z|^p + |w|^p)^{\frac{1}{p}}$$

for all  $z, w \in \mathbb{C}$ . The best constant in this inequality is denoted by  $C_{p,q}(\mathbb{C})$ . Clarkson [12] proved that  $C_{p,p'}(\mathbb{C}) = 2^{1/p'}$  for  $1 \leq p \leq 2$ . Later on the best constant  $C_{p,q}(\mathbb{C})$  for the remaining pairs of  $p$  and  $q$  such that  $0 < p, q \leq \infty$  were found by Koskela [40], Maligranda and Persson [44]. In 1997, Kuriyama et al. [35] obtained an elementary proof of the generalized Clarkson's inequality in the complex case.

Moreover, we can consider the generalized Clarkson's inequality in the real case:

$$(|a + b|^q + |a - b|^q)^{\frac{1}{q}} \leq C(|a|^p + |b|^p)^{\frac{1}{p}}$$

for all  $a, b \in \mathbb{R}$ . As in the complex case, the best constant is denoted by  $C_{p,q}(\mathbb{R})$ . In 2007, Maligranda and Sabourova [45] computed the best constant  $C_{p,q}(\mathbb{R})$  for all  $0 < p, q \leq \infty$ . We present an element proof of the generalized Clarkson's inequality in the real case.

We also pay attention to the triangle inequality. The triangle inequality is one of the most fundamental inequalities in analysis. Several authors have been treating its generalizations, improvements and reverse inequalities ([16, 31, 42, 43, 49] and so on). We consider another aspect of the triangle inequality. For a Hilbert space  $H$ , the parallelogram law implies that the parallelogram inequality

$$\|x + y\|^2 \leq 2(\|x\|^2 + \|y\|^2)$$

holds for all  $x, y \in H$ . Saitoh in [62] noted that the above inequality may be more suitable than the classical triangle inequality. Motivated by this, Belbachir et al.

[9] introduced the notion of  $q$ -norm ( $1 \leq q < \infty$ ). We introduce the notion of  $\psi$ -norm by considering the fact that an absolute normalized norm on  $\mathbb{R}^2$  corresponds to a function  $\psi$  on the interval  $[0, 1]$  with some conditions (cf. [11, 58]). This is a generalization of the notion of  $q$ -norm. We show that a  $\psi$ -norm is a norm in the usual sense.

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# 1 Some geometrical constants of absolute normalized norms on $\mathbb{R}^2$

## 1.1 Introduction and preliminaries

Let  $X$  be a Banach space with  $\dim X \geq 2$ . We denote the unit sphere and the unit ball of a Banach space  $X$  by  $S_X$  and  $B_X$ , respectively. Many geometrical constants of a Banach space  $X$  have been investigated. In this chapter we shall consider the following constants:

$$C_{NJ}(X) = \sup \left\{ \frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} \mid x, y \in X, (x, y) \neq (0, 0) \right\},$$

$$C'_{NJ}(X) = \sup \left\{ \frac{\|x+y\|^2 + \|x-y\|^2}{4} \mid x, y \in S_X \right\},$$

$$C_Z(X) = \sup \left\{ \frac{\|x+y\|\|x-y\|}{\|x\|^2 + \|y\|^2} \mid x, y \in X, (x, y) \neq (0, 0) \right\}.$$

The constant  $C_{NJ}(X)$ , called the von Neumann-Jordan constant (hereafter referred to as NJ constant) have been considered in many papers ([13, 26, 28, 37, 63, 66, 69] and so on). The constant  $C'_{NJ}(X)$ , called the modified von Neumann-Jordan constant (shortly, modified NJ constant) was introduced by Gao in [18] and does not necessarily coincide with  $C_{NJ}(X)$  (cf. [4, 20]). The constant  $C_Z(X)$  was introduced by Zbăganu [70] and was conjectured that  $C_Z(X)$  always coincides with the NJ constant  $C_{NJ}(X)$ , but Alonso and Martin [3] gave an example that  $C_{NJ}(X) \neq C_Z(X)$  (cf.[21, 41]).

A norm  $\|\cdot\|$  on  $\mathbb{R}^2$  is said to be absolute if  $\|(x, y)\| = \||x|, |y|\|$  for any  $(x, y) \in \mathbb{R}^2$ , and normalized if  $\|(1, 0)\| = \|(0, 1)\| = 1$ . The  $\ell_p$ -norms  $\|\cdot\|_p$  are such examples:

$$\|(x, y)\|_p = \begin{cases} (|x|^p + |y|^p)^{\frac{1}{p}} & (1 \leq p < \infty), \\ \max\{|x|, |y|\} & (p = \infty). \end{cases}$$

Let  $AN_2$  denote the family of all absolute normalized norms on  $\mathbb{R}^2$ , and  $\Psi_2$  denote the family of all continuous convex functions  $\psi$  on  $[0, 1]$  such that  $\psi(0) = \psi(1) = 1$  and  $\max\{1-t, t\} \leq \psi(t) \leq 1$  for all  $0 \leq t \leq 1$ . As in [11], it is well known that  $AN_2$  and  $\Psi_2$  are in a one-to-one correspondence under the equation  $\psi(t) = \|(1-t, t)\|$

( $0 \leq t \leq 1$ ). Let  $\|\cdot\|_\psi$  be an absolute normalized norm associated with a convex function  $\psi \in \Psi_2$ .

For  $\psi, \varphi \in \Psi_2$ , we denote  $\psi \leq \varphi$  if  $\psi(t) \leq \varphi(t)$  for any  $0 \leq t \leq 1$ . Let

$$M_1 = \max_{0 \leq t \leq 1} \frac{\psi(t)}{\psi_2(t)} \quad \text{and} \quad M_2 = \max_{0 \leq t \leq 1} \frac{\psi_2(t)}{\psi(t)},$$

where

$$\psi_2(t) = \|(1-t, t)\|_2 = \sqrt{(1-t)^2 + t^2}$$

corresponds to the  $\ell_2$ -norm. In 2000, Saito, Kato and Takahashi [58] proved that, if  $\psi \geq \psi_2$  (resp.  $\psi \leq \psi_2$ ), then  $C_{NJ}(\mathbb{C}^2, \|\cdot\|_\psi) = M_1^2$  (resp.  $M_2^2$ ). The results obtained in [58] also hold on  $\mathbb{R}^2$ .

We put  $X = (\mathbb{R}^2, \|\cdot\|_\psi)$  for  $\psi \in \Psi_2$ . Our aim in this chapter is to study the conditions of  $\psi$  that  $C'_{NJ}(X) = C_{NJ}(X)$  or  $C_Z(X) = C_{NJ}(X)$ . In Section 1.2, we consider the modified NJ constant. We prove that if  $\psi \leq \psi_2$ , then  $C'_{NJ}(X) = C_{NJ}(X) = M_2^2$ . If  $\psi \geq \psi_2$ , then we present the necessarily and sufficient condition for that  $C'_{NJ}(X) = C_{NJ}(X) = M_1^2$ . Further, we consider the conditions that  $C'_{NJ}(X) = C_{NJ}(X) = M_1^2 M_2^2$ . In Section 1.3, we study the Zbăganu constant. First, we show that, if  $\psi \geq \psi_2$ , then  $C_Z(X) = C_{NJ}(X) = M_1^2$ . If  $\psi \leq \psi_2$ , then we give the necessarily and sufficient condition for that  $C_Z(X) = C_{NJ}(X) = M_2^2$ . Further we study the conditions that  $C_Z(X) = C_{NJ}(X) = M_1^2 M_2^2$ . In Section 1.4, we calculate modified NJ constant  $C'_{NJ}(X)$  and Zbăganu constant  $C_Z(X)$  for some Banach spaces.

## 1.2 The modified NJ constant of absolute normalized norms on $\mathbb{R}^2$

In this section, we consider the Banach space  $X = (\mathbb{R}^2, \|\cdot\|_\psi)$ . The modified NJ constant  $C'_{NJ}(X)$  of a Banach space  $X$  is defined by

$$C'_{NJ}(X) = \sup \left\{ \frac{\|x+y\|^2 + \|x-y\|^2}{4} \mid x, y \in S_X \right\}.$$

From the definition of the modified NJ constant, it is clear that  $C'_{NJ}(X) \leq C_{NJ}(X)$ . In this section, we consider the condition that  $C'_{NJ}(X) = C_{NJ}(X)$ .

**Proposition 1.2.1.** *Let  $\psi \in \Psi_2$ . If  $\psi \leq \psi_2$ , then  $C'_{NJ}(X) = C_{NJ}(X) = M_2^2$ .*



**Proof.** For any  $x, y \in S_X$ , by [58, Lemma 3],

$$\begin{aligned} \|x + y\|_\psi^2 + \|x - y\|_\psi^2 &\leq \|x + y\|_2^2 + \|x - y\|_2^2 \\ &= 2(\|x\|_2^2 + \|y\|_2^2) \\ &\leq 2M_2^2(\|x\|_\psi^2 + \|y\|_\psi^2) = 4M_2^2. \end{aligned}$$

Now let  $\psi_2/\psi$  attain the maximum  $M_2$  at  $t = t_0$  ( $0 \leq t_0 \leq 1$ ), and put

$$x = \frac{1}{\psi(t_0)}(1 - t_0, t_0), \quad y = \frac{1}{\psi(t_0)}(1 - t_0, -t_0).$$

Then  $x, y \in S_X$  and

$$\begin{aligned} \|x + y\|_\psi^2 + \|x - y\|_\psi^2 &= \frac{4(1 - t_0)^2 + 4t_0^2}{\psi(t_0)^2} \\ &= 4 \frac{\psi_2(t_0)^2}{\psi(t_0)^2} = 4M_2^2, \end{aligned}$$

which implies that  $C'_{NJ}(X) = M_2^2$ . By [58, Theorem 1], we have this proposition.  $\square$

If  $\psi \geq \psi_2$ , by [58, Theorem 1], then  $C_{NJ}(X) = M_1^2$ . We now give the necessarily and sufficient condition for that  $C'_{NJ}(X) = M_1^2$ .

**Theorem 1.2.2.** *Let  $\psi \in \Psi_2$  such that  $\psi \geq \psi_2$ . Then  $C'_{NJ}(X) = M_1^2$  if and only if there exist  $s, t \in [0, 1]$  ( $s < t$ ) satisfying one of the following conditions:*

(1)  $\psi(s) = \psi_2(s)$ ,  $\psi(t) = \psi_2(t)$  and, if we put

$$r = \frac{\psi(s)t + \psi(t)s}{\psi(s) + \psi(t)}, \quad \text{then} \quad \frac{\psi(r)}{\psi_2(r)} = \frac{\psi(1-r)}{\psi_2(1-r)} = M_1.$$

(2)  $\psi(s) = \psi_2(s)$ ,  $\psi(t) = \psi_2(t)$  and, if we put

$$r = \frac{\psi(t)s + \psi(s)t}{\psi(t) + \psi(s)(2t - 1)}, \quad \text{then} \quad \frac{\psi(r)}{\psi_2(r)} = \frac{\psi(1-r)}{\psi_2(1-r)} = M_1.$$

**Proof.** ( $\Rightarrow$ ) Suppose that  $C'_{NJ}(X) = M_1^2$ . First, for any  $x, y \in S_X$ , by [58, Lemma 3], we have

$$\begin{aligned} \|x + y\|_\psi^2 + \|x - y\|_\psi^2 &\leq M_1^2(\|x + y\|_2^2 + \|x - y\|_2^2) \\ &= 2M_1^2(\|x\|_2^2 + \|y\|_2^2) \\ &\leq 2M_1^2(\|x\|_\psi^2 + \|y\|_\psi^2) = 4M_1^2. \end{aligned}$$

Since  $X = (\mathbb{R}^2, \|\cdot\|_\psi)$  is a finite dimensional Banach space, we have

$$C'_{NJ}(X) = \max \left\{ \frac{\|x+y\|_\psi^2 + \|x-y\|_\psi^2}{4} \mid x, y \in S_X \right\}.$$

Therefore,  $C'_{NJ}(X) = M_1^2$  if and only if there exist  $x, y \in S_X$  ( $x \neq y$ ) such that

$$\|x+y\|_\psi^2 + \|x-y\|_\psi^2 = 4M_1^2.$$

From the above inequality, the elements  $x, y \in S_X$  satisfy  $\|x\|_\psi = \|x\|_2 = 1$ ,  $\|y\|_\psi = \|y\|_2 = 1$  and

$$\frac{\|x+y\|_\psi}{\|x+y\|_2} = \frac{\|x-y\|_\psi}{\|x-y\|_2} = M_1.$$

Since  $\|\cdot\|_\psi$  is absolute and  $x, y \in S_X$  satisfy  $\|x\|_2 = \|y\|_2 = 1$ , it is sufficient to consider the following three cases:

(i) There exist  $s, t \in [0, 1]$  ( $s \neq t$ ) satisfying

$$x = \frac{1}{\psi_2(s)}(1-s, s) \quad \text{and} \quad y = \frac{1}{\psi_2(t)}(1-t, t).$$

(ii) There exist  $s, t \in [0, 1]$  ( $s < t$ ) satisfying

$$x = \frac{1}{\psi_2(s)}(1-s, s) \quad \text{and} \quad y = \frac{1}{\psi_2(t)}(-1+t, t).$$

(iii) There exist  $s, t \in [0, 1]$  ( $s > t$ ) satisfying

$$x = \frac{1}{\psi_2(s)}(1-s, s) \quad \text{and} \quad y = \frac{1}{\psi_2(t)}(-1+t, t).$$

Case (i). We may suppose that  $s < t$ . Then there exist  $\alpha, \beta \in [0, \frac{\pi}{2}]$  ( $\alpha < \beta$ ) such that

$$x = \frac{1}{\psi_2(s)}(1-s, s) = (\cos \alpha, \sin \alpha), \quad y = \frac{1}{\psi_2(t)}(1-t, t) = (\cos \beta, \sin \beta).$$

Since  $\|x\|_2 = \|y\|_2 = 1$ , we have

$$x+y = \left( \frac{1-s}{\psi_2(s)} + \frac{1-t}{\psi_2(t)}, \frac{s}{\psi_2(s)} + \frac{t}{\psi_2(t)} \right) = \|x+y\|_2 \left( \cos \frac{\alpha+\beta}{2}, \sin \frac{\alpha+\beta}{2} \right).$$

By [67, Propositions 2a and 2b], we remark that

$$\frac{1-s}{\psi_2(s)} \geq \frac{1-t}{\psi_2(t)} \quad \text{and} \quad \frac{s}{\psi_2(s)} \leq \frac{t}{\psi_2(t)}.$$

Since  $x - y$  is orthogonal to  $x + y$  in the Euclidean space  $(\mathbb{R}^2, \|\cdot\|_2)$ , we have

$$\begin{aligned} x - y &= \left( \frac{1-s}{\psi_2(s)} - \frac{1-t}{\psi_2(t)}, \frac{s}{\psi_2(s)} - \frac{t}{\psi_2(t)} \right) \\ &= \|x - y\|_2 \left( \cos \frac{\alpha + \beta - \pi}{2}, \sin \frac{\alpha + \beta - \pi}{2} \right) \\ &= \|x - y\|_2 \left( \sin \frac{\alpha + \beta}{2}, -\cos \frac{\alpha + \beta}{2} \right). \end{aligned}$$

Thus we have

$$\begin{aligned} \|x + y\|_\psi &= \|x + y\|_2 \left\| \left( \cos \frac{\alpha + \beta}{2}, \sin \frac{\alpha + \beta}{2} \right) \right\|_\psi \\ &= \|x + y\|_2 \left( \cos \frac{\alpha + \beta}{2} + \sin \frac{\alpha + \beta}{2} \right) \psi \left( \frac{\sin \frac{\alpha + \beta}{2}}{\cos \frac{\alpha + \beta}{2} + \sin \frac{\alpha + \beta}{2}} \right). \end{aligned}$$

Since  $\|x + y\|_\psi = M_1 \|x + y\|_2$ , we have

$$M_1 = \left( \cos \frac{\alpha + \beta}{2} + \sin \frac{\alpha + \beta}{2} \right) \psi \left( \frac{\sin \frac{\alpha + \beta}{2}}{\cos \frac{\alpha + \beta}{2} + \sin \frac{\alpha + \beta}{2}} \right).$$

Putting

$$r = \frac{\sin \frac{\alpha + \beta}{2}}{\cos \frac{\alpha + \beta}{2} + \sin \frac{\alpha + \beta}{2}},$$

then it is clear that

$$r = \frac{\psi(s)t + \psi(t)s}{\psi(s) + \psi(t)} \quad \text{and} \quad M_1 = \frac{\psi(r)}{\psi_2(r)}.$$

We also have

$$\|x - y\|_\psi = \|x - y\|_2 \left( \sin \frac{\alpha + \beta}{2} + \cos \frac{\alpha + \beta}{2} \right) \psi \left( \frac{\cos \frac{\alpha + \beta}{2}}{\cos \frac{\alpha + \beta}{2} + \sin \frac{\alpha + \beta}{2}} \right).$$

Since  $\|x - y\|_\psi = M_1 \|x - y\|_2$ , we similarly have

$$M_1 = \left( \sin \frac{\alpha + \beta}{2} + \cos \frac{\alpha + \beta}{2} \right) \psi \left( \frac{\cos \frac{\alpha + \beta}{2}}{\sin \frac{\alpha + \beta}{2} + \cos \frac{\alpha + \beta}{2}} \right) = \frac{\psi(1-r)}{\psi_2(1-r)}.$$

Case (ii). There exist  $\alpha \in [0, \frac{\pi}{2}]$  and  $\beta \in [\frac{\pi}{2}, \pi]$  such that

$$x = \frac{1}{\psi_2(s)}(1-s, s) = (\cos \alpha, \sin \alpha), \quad y = \frac{1}{\psi_2(t)}(-1+t, t) = (\cos \beta, \sin \beta).$$

Since  $\|x\|_2 = \|y\|_2 = 1$ , we have

$$x + y = \left( \frac{1-s}{\psi_2(s)} - \frac{1-t}{\psi_2(t)}, \frac{s}{\psi_2(s)} + \frac{t}{\psi_2(t)} \right) = \|x + y\|_2 \left( \cos \frac{\alpha + \beta}{2}, \sin \frac{\alpha + \beta}{2} \right).$$

By [67, Propositions 2a and 2b], we remark that

$$\frac{1-s}{\psi_2(s)} \geq \frac{1-t}{\psi_2(t)} \quad \text{and} \quad \frac{s}{\psi_2(s)} \leq \frac{t}{\psi_2(t)}.$$

Since  $x - y$  is orthogonal to  $x + y$  in the Euclidean space  $(\mathbb{R}^2, \|\cdot\|_2)$ , we have

$$\begin{aligned} x - y &= \left( \frac{1-s}{\psi_2(s)} + \frac{1-t}{\psi_2(t)}, \frac{s}{\psi_2(s)} - \frac{t}{\psi_2(t)} \right) \\ &= \|x - y\|_2 \left( \cos \frac{\alpha + \beta - \pi}{2}, \sin \frac{\alpha + \beta - \pi}{2} \right) \\ &= \|x - y\|_2 \left( \sin \frac{\alpha + \beta}{2}, -\cos \frac{\alpha + \beta}{2} \right). \end{aligned}$$

Since  $\cos \frac{\alpha + \beta}{2} \geq 0$  and  $\sin \frac{\alpha + \beta}{2} \geq 0$ , we have

$$\begin{aligned} \|x + y\|_\psi &= \|x + y\|_2 \left\| \left( \cos \frac{\alpha + \beta}{2}, \sin \frac{\alpha + \beta}{2} \right) \right\|_\psi \\ &= \|x + y\|_2 \left( \cos \frac{\alpha + \beta}{2} + \sin \frac{\alpha + \beta}{2} \right) \psi \left( \frac{\sin \frac{\alpha + \beta}{2}}{\cos \frac{\alpha + \beta}{2} + \sin \frac{\alpha + \beta}{2}} \right). \end{aligned}$$

Since  $\|x + y\|_\psi = M_1 \|x + y\|_2$ , we have

$$M_1 = \left( \cos \frac{\alpha + \beta}{2} + \sin \frac{\alpha + \beta}{2} \right) \psi \left( \frac{\sin \frac{\alpha + \beta}{2}}{\cos \frac{\alpha + \beta}{2} + \sin \frac{\alpha + \beta}{2}} \right).$$

Putting

$$r = \frac{\sin \frac{\alpha + \beta}{2}}{\cos \frac{\alpha + \beta}{2} + \sin \frac{\alpha + \beta}{2}},$$

then it is clear that

$$r = \frac{\psi(t)s + \psi(s)t}{\psi(t) + \psi(s)(2t - 1)} \quad \text{and} \quad M_1 = \frac{\psi(r)}{\psi_2(r)}.$$

We also have

$$\|x - y\|_\psi = \|x - y\|_2 \left( \sin \frac{\alpha + \beta}{2} + \cos \frac{\alpha + \beta}{2} \right) \psi \left( \frac{\cos \frac{\alpha + \beta}{2}}{\cos \frac{\alpha + \beta}{2} + \sin \frac{\alpha + \beta}{2}} \right).$$

Since  $\|x - y\|_\psi = M_1\|x - y\|_2$ , we similarly have

$$M_1 = \left( \sin \frac{\alpha + \beta}{2} + \cos \frac{\alpha + \beta}{2} \right) \psi \left( \frac{\cos \frac{\alpha + \beta}{2}}{\sin \frac{\alpha + \beta}{2} + \cos \frac{\alpha + \beta}{2}} \right) = \frac{\psi(1 - r)}{\psi_2(1 - r)}.$$

Case (iii). There exist  $s, t \in [0, 1]$  ( $s > t$ ) satisfying

$$x = \frac{1}{\psi_2(s)}(1 - s, s) \quad \text{and} \quad y = \frac{1}{\psi_2(t)}(-1 + t, t).$$

Then, we put  $s_0 = t$  and  $t_0 = s$ . We define  $x_0, y_0$  in  $S_X$  by

$$x_0 = \frac{1}{\psi(s_0)}(1 - s_0, s_0), \quad y_0 = \frac{1}{\psi(t_0)}(-1 + t_0, t_0).$$

Then we can reduce Case (ii).

( $\Leftarrow$ ) If we suppose (1) (resp. (2)), then we put

$$x = \frac{1}{\psi_2(s)}(1 - s, s) \quad \left( \text{resp. } x = \frac{1}{\psi_2(s)}(1 - s, s) \right)$$

and

$$y = \frac{1}{\psi_2(t)}(1 - t, t) \quad \left( \text{resp. } y = \frac{1}{\psi_2(t)}(-1 + t, t) \right).$$

Then we have  $\|x\|_\psi = \|x\|_2 = 1$ ,  $\|y\|_\psi = \|y\|_2 = 1$ ,  $\|x + y\|_\psi = M_1\|x + y\|_2$  and  $\|x - y\|_\psi = M_1\|x - y\|_2$ . Hence it is clear to prove that  $C'_{NJ}(X) = M_1^2$ .  $\square$

We next study the modified NJ constant in the general case. If  $\psi \in \Psi_2$ , then by [58, Theorem 2] we have

$$\max\{M_1^2, M_2^2\} \leq C_{NJ}(X) \leq M_1^2 M_2^2.$$

However, by Theorem 1.2.2, there exist many  $\psi \in \Psi_2$  satisfying  $\psi \geq \psi_2$  such that

$$C'_{NJ}(X) < \max\{M_1^2, M_2^2\} = C_{NJ}(X).$$

From [58, Theorem 3],  $C_{NJ}(X) = M_1^2 M_2^2$  if either  $\psi/\psi_2$  or  $\psi_2/\psi$  attains a maximum at  $t = 1/2$ . At first, we have the following

**Proposition 1.2.3.** *Let  $\psi \in \Psi_2$  and let  $\psi(t) = \psi(1 - t)$  for all  $t \in [0, 1]$ . If  $\psi/\psi_2$  attains a maximum  $M_1$  at  $t = 1/2$ , then  $C'_{NJ}(X) = C_{NJ}(X) = M_1^2 M_2^2$ .*

**Proof.** Suppose first  $M_1 = \psi(1/2)/\psi_2(1/2)$ . Take an arbitrary  $t \in [0, 1]$  and put

$$x = \frac{1}{\psi(t)}(t, 1-t) \quad \text{and} \quad y = \frac{1}{\psi(t)}(1-t, t).$$

Then we have  $x, y \in S_X$  and

$$\|x + y\|_\psi = \frac{4}{\psi(t)}\psi\left(\frac{1}{2}\right), \quad \|x - y\|_\psi = \frac{4|2t - 1|}{\psi(t)}\psi\left(\frac{1}{2}\right).$$

Therefore we have

$$\begin{aligned} \frac{\|x + y\|_\psi^2 + \|x - y\|_\psi^2}{4} &= \{(2t - 1)^2 + 1\} \frac{\psi(1/2)^2}{\psi(t)^2} \\ &= 2\psi_2(t)^2 \frac{\psi(1/2)^2}{\psi(t)^2} \\ &= \frac{\psi_2(t)^2}{\psi(t)^2} \frac{\psi(1/2)^2}{\psi_2(1/2)^2} = M_1^2 \frac{\psi_2(t)^2}{\psi(t)^2}. \end{aligned}$$

Since  $t$  is arbitrary, we have  $C'_{NJ}(X) \geq M_1^2 M_2^2$  which prove that  $C'_{NJ}(X) = M_1^2 M_2^2$ .  $\square$

In the case that  $M_2 = \psi_2(1/2)/\psi(1/2)$ ,  $C'_{NJ}(X)$  does not necessarily coincide with  $M_1^2 M_2^2$ . However, we have the following

**Theorem 1.2.4.** *Let  $\psi \in \Psi_2$  and let  $\psi(t) = \psi(1-t)$  for all  $t \in [0, 1]$ . Assume that  $M_2 = \psi_2(1/2)/\psi(1/2)$  and  $M_1 > 1$ . Then  $C'_{NJ}(X) = M_1^2 M_2^2$  if and only if there exist  $s, t \in [0, 1]$  ( $s < t$ ) satisfying one of the following conditions:*

(1)  $\psi_2(s) = M_2\psi(s)$ ,  $\psi_2(t) = M_2\psi(t)$  and, if we put

$$r = \frac{\psi(s)t + \psi(t)s}{\psi(s) + \psi(t)}, \quad \text{then} \quad \psi(r) = M_1\psi_2(r).$$

(2)  $\psi_2(s) = M_2\psi(s)$ ,  $\psi_2(t) = M_2\psi(t)$  and, if we put

$$r = \frac{\psi(t)s + \psi(s)t}{\psi(t) + \psi(s)(2t - 1)}, \quad \text{then} \quad \psi(r) = M_1\psi_2(r).$$

**Proof.** For all  $x, y \in S_X$ , we have

$$\begin{aligned} \|x + y\|_\psi^2 + \|x - y\|_\psi^2 &\leq M_1^2 (\|x + y\|_2^2 + \|x - y\|_2^2) \\ &= 2M_1^2 (\|x\|_2^2 + \|y\|_2^2) \\ &\leq 2M_1^2 M_2^2 (\|x\|_\psi^2 + \|y\|_\psi^2) = 4M_1^2 M_2^2. \end{aligned}$$

From this inequality,  $C'_{NJ}(X) = M_1^2 M_2^2$  if and only if there exist  $x, y \in S_X$  ( $x \neq y$ ) such that

$$\|x + y\|_\psi^2 + \|x - y\|_\psi^2 = 4M_1^2 M_2^2.$$

Thus, the elements  $x, y \in S_X$  satisfy

$$\|x\|_2 = \|y\|_2 = M_2, \quad \|x + y\|_\psi = M_1 \|x + y\|_2, \quad \|x - y\|_\psi = M_1 \|x - y\|_2.$$

Since  $\|\cdot\|_\psi$  is absolute, it is sufficient to consider the following three cases:

(i) There exist  $s, t \in [0, 1]$  ( $s \neq t$ ) satisfying

$$x = \frac{1}{\psi_2(s)}(1 - s, s) \quad \text{and} \quad y = \frac{1}{\psi_2(t)}(1 - t, t).$$

(ii) There exist  $s, t \in [0, 1]$  ( $s < t$ ) satisfying

$$x = \frac{1}{\psi_2(s)}(1 - s, s) \quad \text{and} \quad y = \frac{1}{\psi_2(t)}(-1 + t, t).$$

(iii) There exist  $s, t \in [0, 1]$  ( $s > t$ ) satisfying

$$x = \frac{1}{\psi_2(s)}(1 - s, s) \quad \text{and} \quad y = \frac{1}{\psi_2(t)}(-1 + t, t).$$

As in the proof of Theorem 1.2.2, we can prove this theorem. □

### 1.3 The Zbăganu constant of absolute normalized norms on $\mathbb{R}^2$

The Zbăganu constant  $C_Z(X)$  in [70] is defined by

$$C_Z(X) = \sup \left\{ \frac{\|x + y\| \|x - y\|}{\|x\|^2 + \|y\|^2} \mid x, y \in X, (x, y) \neq (0, 0) \right\}.$$

Then it is clear that  $C_Z(X) \leq C_{NJ}(X)$  for any Banach space  $X$ . In this section, we consider the condition that  $C_Z(X) = C_{NJ}(X)$  for  $X = (\mathbb{R}^2, \|\cdot\|_\psi)$ . Then, we have the following

**Proposition 1.3.1.** *Let  $\psi \in \Psi_2$ . If  $\psi \geq \psi_2$ , then  $C_Z(X) = C_{NJ}(X) = M_1^2$ .*

**Proof.** First, by [58, Lemma 3], we have

$$\begin{aligned} 2\|x + y\|_\psi \|x - y\|_\psi &\leq \|x + y\|_\psi^2 + \|x - y\|_\psi^2 \\ &\leq M_1^2 (\|x + y\|_2^2 + \|x - y\|_2^2) \\ &= 2M_1^2 (\|x\|_2^2 + \|y\|_2^2) \leq 2M_1^2 (\|x\|_\psi^2 + \|y\|_\psi^2). \end{aligned}$$

for any  $x, y \in X$ .

Now let  $\psi/\psi_2$  attain the maximum  $M_1$  at  $t = t_0$  ( $0 \leq t_0 \leq 1$ ). Put  $x = (1 - t_0, 0)$  and  $y = (0, t_0)$ , respectively. Then we have

$$\begin{aligned} \|x + y\|_\psi^2 + \|x - y\|_\psi^2 &= 2\psi(t_0)^2 \\ &= 2M_1^2 \psi_2(t_0)^2 \\ &= 2M_1^2 (\|x\|_\psi^2 + \|y\|_\psi^2). \end{aligned}$$

Since  $\|x + y\|_\psi = \psi(t_0) = \|x - y\|_\psi$ , we have

$$2\|x + y\|_\psi \|x - y\|_\psi = \|x + y\|_\psi^2 + \|x - y\|_\psi^2 = 2M_1^2 (\|x\|_\psi^2 + \|y\|_\psi^2).$$

Therefore we have

$$\frac{\|x + y\|_\psi \|x - y\|_\psi}{\|x\|_\psi^2 + \|y\|_\psi^2} = M_1^2,$$

which implies that  $C_Z(X) = M_1^2$ . □

We next consider the case that  $\psi \leq \psi_2$ . We remark that the Zbăganu constant  $C_Z(X)$  can be in the following form:

$$C_Z(X) = \sup \left\{ \frac{4\|x\|\|y\|}{\|x + y\|^2 + \|x - y\|^2} \mid x, y \in X, (x, y) \neq (0, 0) \right\}.$$

Then we have the following

**Theorem 1.3.2.** *Let  $\psi \in \Psi_2$ . Assume that  $\psi \leq \psi_2$ . Then  $C_Z(X) = M_2^2$  if and only if there exist  $s, t \in [0, 1]$  ( $s < t$ ) satisfying one of the following conditions:*

(1)  $\psi(s) = \psi_2(s)$ ,  $\psi(t) = \psi_2(t)$  and, if we put

$$r = \frac{\psi(s)t + \psi(t)s}{\psi(s) + \psi(t)}, \quad \text{then} \quad \frac{\psi_2(r)}{\psi(r)} = \frac{\psi_2(1-r)}{\psi(1-r)} = M_2.$$



(2)  $\psi(s) = \psi_2(s)$ ,  $\psi(t) = \psi_2(t)$  and, if we put

$$r = \frac{\psi(t)s + \psi(s)t}{\psi(t) + \psi(s)(2t - 1)}, \quad \text{then} \quad \frac{\psi_2(r)}{\psi(r)} = \frac{\psi_2(1-r)}{\psi(1-r)} = M_2.$$

**Proof.** First, by [58, Lemma 3], for any  $x, y \in X$ ,

$$\begin{aligned} 4\|x\|_\psi\|y\|_\psi &\leq 2(\|x\|_\psi^2 + \|y\|_\psi^2) \\ &\leq 2(\|x\|_2^2 + \|y\|_2^2) \\ &= \|x+y\|_2^2 + \|x-y\|_2^2 \\ &\leq 2M_2^2(\|x+y\|_\psi^2 + \|x-y\|_\psi^2). \end{aligned}$$

Since  $X = (\mathbb{R}^2, \|\cdot\|_\psi)$  is a finite dimensional Banach space,

$$C_Z(X) = \max \left\{ \frac{4\|x\|_\psi\|y\|_\psi}{\|x+y\|_\psi^2 + \|x-y\|_\psi^2} \mid x, y \in X, (x, y) \neq (0, 0) \right\}.$$

Then  $C_Z(X) = M_2^2$  if and only if there exist  $x, y \in S_X$  ( $x \neq y$ ) such that

$$\frac{4\|x\|_\psi\|y\|_\psi}{\|x+y\|_\psi^2 + \|x-y\|_\psi^2} = M_2^2.$$

From the above inequality,  $\|x\|_2 = \|x\|_\psi = \|y\|_\psi = \|y\|_2$  and

$$\frac{\|x+y\|_2}{\|x+y\|_\psi} = \frac{\|x-y\|_2}{\|x-y\|_\psi} = M_2^2.$$

Hence we may assume that  $\|x\|_2 = \|x\|_\psi = \|y\|_\psi = \|y\|_2 = 1$ . As in the proof of Theorem 1.2.2, it is sufficient to consider the following three cases:

(i) There exist  $s, t \in [0, 1]$  ( $s \neq t$ ) satisfying

$$x = \frac{1}{\psi_2(s)}(1-s, s) \quad \text{and} \quad y = \frac{1}{\psi_2(t)}(1-t, t).$$

(ii) There exist  $s, t \in [0, 1]$  ( $s < t$ ) satisfying

$$x = \frac{1}{\psi_2(s)}(1-s, s) \quad \text{and} \quad y = \frac{1}{\psi_2(t)}(-1+t, t).$$

(iii) There exist  $s, t \in [0, 1]$  ( $s > t$ ) satisfying

$$x = \frac{1}{\psi_2(s)}(1-s, s) \quad \text{and} \quad y = \frac{1}{\psi_2(t)}(-1+t, t).$$

As in the proof of Theorem 1.2.2, we can similarly prove this theorem.  $\square$

We next study the Zbăganu constant  $C_Z(X)$  in general case. If  $\psi \in \Psi_2$ , by [58, Theorem 2], then we have

$$\max\{M_1^2, M_2^2\} \leq C_{NJ}(X) \leq M_1^2 M_2^2.$$

However, by Theorem 1.3.2, there exist many  $\psi \in \Psi_2$  satisfying  $\psi \leq \psi_2$  such that

$$C_Z(X) < C_{NJ}(X) \leq \max\{M_1^2, M_2^2\}.$$

From [58, Theorem 3],  $C_{NJ}(X) = M_1^2 M_2^2$  if either  $\psi/\psi_2$  or  $\psi_2/\psi$  attains a maximum at  $t = 1/2$ . At first, we have the following

**Proposition 1.3.3.** *Let  $\psi \in \Psi_2$  and let  $\psi(t) = \psi(1-t)$  for all  $t \in [0, 1]$ . If  $\psi_2/\psi$  attains a maximum  $M_2$  at  $t = 1/2$ , then  $C_Z(X) = C_{NJ}(X) = M_1^2 M_2^2$ .*

**Proof.** First, we know  $C_Z(X) \leq C_{NJ}(X) = M_1^2 M_2^2$ . Take an arbitrary  $t \in [0, 1]$  and put  $x = (t, 1-t)$  and  $y = (1-t, t)$ . Then  $\|x\|_\psi = \|y\|_\psi = \psi(t)$ ,  $\|x+y\|_\psi = \|(1, 1)\|_\psi = 2\psi(1/2)$  and

$$\|x-y\|_\psi = \|(2t-1, 1-2t)\|_\psi = 2|2t-1|\psi(1/2).$$

Hence we have

$$\begin{aligned} \frac{4\|x\|_\psi\|y\|_\psi}{\|x+y\|_\psi^2 + \|x-y\|_\psi^2} &= \frac{2(\|x\|_\psi^2 + \|y\|_\psi^2)}{\|x+y\|_\psi^2 + \|x-y\|_\psi^2} \\ &= \frac{\psi(t)^2}{(1+(2t-1)^2)\psi(1/2)^2} \\ &= \frac{\psi(t)^2}{2\psi_2(t)^2\psi(1/2)^2} \\ &= \frac{\psi(t)^2}{\psi_2(t)^2} \frac{\psi_2(1/2)^2}{\psi(1/2)^2} = M_2^2 \frac{\psi(t)^2}{\psi_2(t)^2}. \end{aligned}$$

Since  $t$  is arbitrary, we have  $C_Z(X) \geq M_1^2 M_2^2$ . Therefore we have  $C_Z(X) = M_1^2 M_2^2$ .  $\square$

In case that  $M_1 = \psi(1/2)/\psi_2(1/2)$ , we have the following theorem as in the proof of Theorem 1.2.2 and so omit the proof.

**Theorem 1.3.4.** *Let  $\psi \in \Psi_2$  and let  $\psi(t) = \psi(1 - t)$  for all  $t \in [0, 1]$ . If  $M_1 = \psi(1/2)/\psi_2(1/2)$  and  $M_2 > 1$ , then  $C_Z(X) = M_1^2 M_2^2$  if and only if there exist  $s, t \in [0, 1]$  ( $s < t$ ) satisfying one of the following conditions:*

(1)  $\psi_2(s) = M_2\psi(s)$ ,  $\psi_2(t) = M_2\psi(t)$  and, if we put

$$r = \frac{\psi(s)t + \psi(t)s}{\psi(s) + \psi(t)}, \quad \text{then } \psi(r) = M_1\psi_2(r).$$

(2)  $\psi_2(s) = M_2\psi(s)$ ,  $\psi_2(t) = M_2\psi(t)$  and, if we put

$$r = \frac{\psi(t)s + \psi(s)t}{\psi(t) + \psi(s)(2t - 1)}, \quad \text{then } \psi(r) = M_1\psi_2(r).$$

## 1.4 Examples

In this section, we shall calculate  $C'_{NJ}(X)$  and  $C_Z(X)$  for some Banach spaces  $X = (\mathbb{R}^2, \|\cdot\|_\psi)$ , where  $\psi \in \Psi_2$ . First, we consider the case that  $\psi = \psi_p$ .

**Example 1.4.1.** *Let  $1 \leq p \leq \infty$  and  $1/p + 1/p' = 1$ . We put  $t = \min\{p, p'\}$ . Then*

$$C'_{NJ}((\mathbb{R}^2, \|\cdot\|_p)) = C_Z((\mathbb{R}^2, \|\cdot\|_p)) = C_{NJ}((\mathbb{R}^2, \|\cdot\|_p)) = 2^{2/t-1}.$$

**Proof.** Suppose that  $1 \leq p \leq 2$ . Since  $\psi_p \geq \psi_2$ , we have  $C_Z((\mathbb{R}^2, \|\cdot\|_p)) = 2^{2/p-1}$  by Proposition 1.3.1. On the other hand, as in Theorem 1.2.2, we take  $s = 0$  and  $t = 1$ . Since

$$r = \frac{\psi(0) \cdot 1 + \psi(1) \cdot 0}{\psi(0) + \psi(1)} = \frac{1}{2} \quad \text{and} \quad M_1 = \frac{\psi_p(1/2)}{\psi_2(1/2)} = 2^{1/p-1/2},$$

we have

$$C'_{NJ}((\mathbb{R}^2, \|\cdot\|_p)) = M_1^2 = 2^{2/p-1}.$$

If  $2 \leq p \leq \infty$ , then we similarly have, by Proposition 1.2.1 and Theorem 1.3.2,

$$C'_{NJ}((\mathbb{R}^2, \|\cdot\|_p)) = C_Z((\mathbb{R}^2, \|\cdot\|_p)) = C_{NJ}((\mathbb{R}^2, \|\cdot\|_p)) = 2^{2/p'-1}. \quad \square$$

In [68, Example], Yang and Li calculated the modified NJ constant of the following Banach space. From our theorems, we have

**Example 1.4.2.** *Let  $\lambda > 0$  and let  $X_\lambda$  be the space  $\mathbb{R}^2$  endowed with norm*

$$\|(x, y)\|_\lambda = (\|(x, y)\|_p^2 + \lambda\|(x, y)\|_q^2)^{\frac{1}{2}}.$$

(i) If  $2 \leq p \leq q \leq \infty$ , then

$$C_{NJ}(X_\lambda) = C'_{NJ}(X_\lambda) = C_Z(X_\lambda) = \frac{2(\lambda + 1)}{2^{2/p} + \lambda 2^{2/q}}.$$

(ii) If  $1 \leq p \leq q \leq 2$ , then

$$C_{NJ}(X_\lambda) = C'_{NJ}(X_\lambda) = C_Z(X_\lambda) = \frac{2^{2/p} + \lambda 2^{2/q}}{2(\lambda + 1)}.$$

**Proof.** First, we remark that  $(p, q)$  is not necessarily a Hölder pair. We define a normalized norm  $\|\cdot\|_\lambda^0$  by

$$\|(x, y)\|_\lambda^0 = \frac{\|(x, y)\|_\lambda}{\sqrt{\lambda + 1}}.$$

Then  $\|\cdot\|_\lambda^0 \in AN_2$  and so put the corresponding function  $\psi_\lambda(t) = \|(1-t, t)\|_\lambda^0$ .

(i) Suppose that  $2 \leq p \leq q \leq \infty$ . Since  $\psi_\lambda \leq \psi_2$ , by Proposition 1.2.1, we have

$$C_{NJ}(X_\lambda) = C'_{NJ}(X_\lambda) = M_2^2 = \frac{2(\lambda + 1)}{2^{2/p} + \lambda 2^{2/q}}.$$

On the other hand, as in Theorem 1.3.2, we take  $s = 0$  and  $t = 1$ . Then we have  $r = 1/2$  and  $\psi_2(1/2)/\psi_\lambda(1/2) = M_2$ . Thus we have

$$C_Z(X_\lambda) = M_2^2 = \frac{2(\lambda + 1)}{2^{2/p} + \lambda 2^{2/q}}.$$

(ii) Suppose that  $1 \leq p \leq q \leq 2$ . Since  $\psi_\lambda \geq \psi_2$ , by Theorem 1.2.2 and Proposition 1.3.1, we similarly have (ii).  $\square$

**Example 1.4.3.** Put

$$\psi(t) = \begin{cases} \psi_2(t) & (0 \leq t \leq 1/2), \\ (2 - \sqrt{2})t + \sqrt{2} - 1 & (1/2 \leq t \leq 1). \end{cases}$$

Then

$$C'_{NJ}(\mathbb{R}^2, \|\cdot\|_\psi) < C_Z(\mathbb{R}^2, \|\cdot\|_\psi) = C_{NJ}(\mathbb{R}^2, \|\cdot\|_\psi) = 2\sqrt{2}(\sqrt{2} - 1).$$

**Proof.** In fact,  $\psi \in \Psi_2$  and the norm  $\|\cdot\|_\psi$  is given by

$$\|(x, y)\|_\psi = \begin{cases} \sqrt{|x|^2 + |y|^2} & (|x| \geq |y|), \\ (\sqrt{2} - 1)|x| + |y| & (|x| \leq |y|). \end{cases}$$

Since  $\psi \geq \psi_2$ , by Proposition 1.3.1, we have

$$C_Z((\mathbb{R}^2, \|\cdot\|_\psi)) = M_1^2 = 2\sqrt{2}(\sqrt{2} - 1).$$

We assume that  $C'_{NJ}((\mathbb{R}^2, \|\cdot\|_\psi)) = M_1^2$ . By Theorem 1.2.2, we can choose  $r \in [0, 1]$  such that

$$\frac{\psi(r)}{\psi_2(r)} = \frac{\psi(1-r)}{\psi_2(1-r)} = M_1.$$

This is impossible by the definition of  $\psi$ . Therefore we have  $C'_{NJ}((\mathbb{R}^2, \|\cdot\|_\psi)) < M_1^2$ .  $\square$

**Example 1.4.4.** Let  $1/2 \leq \beta \leq 1$ . We define a convex function  $\psi_\beta \in \Psi_2$  by  $\psi_\beta(t) = \max\{1-t, t, \beta\}$ .

(i) If  $1/2 \leq \beta < 1/\sqrt{2}$ , then

$$C_Z((\mathbb{R}^2, \|\cdot\|_{\psi_\beta})) < C_{NJ}((\mathbb{R}^2, \|\cdot\|_{\psi_\beta})) = C'_{NJ}((\mathbb{R}^2, \|\cdot\|_{\psi_\beta})) = \frac{\beta^2 + (1-\beta)^2}{\beta^2}.$$

(ii) If  $1/\sqrt{2} \leq \beta \leq 1$ , then

$$C_Z((\mathbb{R}^2, \|\cdot\|_{\psi_\beta})) = C_{NJ}((\mathbb{R}^2, \|\cdot\|_{\psi_\beta})) = C'_{NJ}((\mathbb{R}^2, \|\cdot\|_{\psi_\beta})) = 2\{\beta^2 + (1-\beta)^2\}.$$

**Proof.** By [58, Example 4], we have

$$C_{NJ}((\mathbb{R}^2, \|\cdot\|_{\psi_\beta})) = \begin{cases} \frac{\beta^2 + (1-\beta)^2}{\beta^2} & (1/2 \leq \beta \leq 1/\sqrt{2}), \\ 2\{\beta^2 + (1-\beta)^2\} & (1/\sqrt{2} \leq \beta \leq 1). \end{cases}$$

Indeed,

$$M_1 = \begin{cases} 1 & (1/2 \leq \beta \leq 1/\sqrt{2}), \\ \frac{\psi_\beta(1/2)}{\psi_2(1/2)} = \frac{\beta}{1/\sqrt{2}} = \sqrt{2}\beta & (1/\sqrt{2} \leq \beta \leq 1). \end{cases}$$

and

$$M_2 = \frac{\psi_2(\beta)}{\psi_\beta(\beta)} = \frac{1}{\beta}\{(1-\beta)^2 + \beta^2\}^{1/2}.$$

Suppose that  $1/2 \leq \beta \leq 1/\sqrt{2}$ . Then  $\psi_\beta \leq \psi_2$  and so, by Proposition 1.2.1, we have

$$C'_{NJ}((\mathbb{R}^2, \|\cdot\|_{\psi_\beta})) = M_2^2 = \frac{\beta^2 + (1-\beta)^2}{\beta^2}.$$

If  $1/2 \leq \beta < 1/\sqrt{2}$ , then by Theorem 1.3.2, we have  $C_Z((\mathbb{R}^2, \|\cdot\|_{\psi_\beta})) < M_2^2$ . If  $\beta = 1/\sqrt{2}$ , then we have

$$\psi_\beta(1/2) = 1/\sqrt{2} = \psi_2(1/2).$$

As in Theorem 1.3.2, we take  $s = 0$  and  $t = 1/2$ . Since

$$r = \frac{\psi_\beta(0) \cdot 1/2 + \psi_\beta(1/2) \cdot 0}{\psi_\beta(0) + \psi_\beta(1/2)} = 1 - 1/\sqrt{2}$$

and

$$\frac{\psi_2(1 - 1/\sqrt{2})}{\psi_\beta(1 - 1/\sqrt{2})} = \frac{\psi_2(1/\sqrt{2})}{\psi_\beta(1/\sqrt{2})} = M_2,$$

we have

$$C_Z((\mathbb{R}^2, \|\cdot\|_{\psi_\beta})) = M_2^2 = \frac{\beta^2 + (1 - \beta)^2}{\beta^2} = 2\sqrt{2}(\sqrt{2} - 1) = 2\{\beta^2 + (1 - \beta)^2\}.$$

Assume that  $1/\sqrt{2} < \beta \leq 1$ . Since  $M_1 = \psi_\beta(1/2)/\psi_2(1/2)$ , we have, by Proposition 1.2.3,

$$C'_{NJ}((\mathbb{R}^2, \|\cdot\|_{\psi_\beta})) = M_1^2 M_2^2 = 2\{\beta^2 + (1 - \beta)^2\}.$$

On the other hand, we take  $s = \beta$  and  $t = 1 - \beta$  in Theorem 1.3.4. Then we have

$$r = \frac{\psi(\beta)(1 - \beta) + \psi(1 - \beta)\beta}{\psi(\beta) + \psi(1 - \beta)} = \frac{1}{2}.$$

Thus we have

$$C_Z((\mathbb{R}^2, \|\cdot\|_{\psi_\beta})) = M_1^2 M_2^2 = 2\{\beta^2 + (1 - \beta)^2\}. \quad \square$$

**Example 1.4.5.** We consider  $\psi_\beta$  in Example 1.4.4 in the case of  $\beta = 1/\sqrt{2}$ . Using this  $\psi_\beta$ , we define a convex function  $\varphi \in \Psi_2$  by

$$\varphi(t) = \begin{cases} \psi_\beta(t) & (0 \leq t \leq 1/2), \\ \psi_2(t) & (1/2 \leq t \leq 1). \end{cases}$$

Then, as in Example 1.4.3,

$$C_Z((\mathbb{R}^2, \|\cdot\|_\varphi)) < C'_{NJ}((\mathbb{R}^2, \|\cdot\|_\varphi)) = C_{NJ}((\mathbb{R}^2, \|\cdot\|_\varphi)) = M_2^2 = 2\sqrt{2}(\sqrt{2} - 1).$$

## 2 The characterization of the Dunkl-Williams constant

### 2.1 Introduction

In 1964, Dunkl and Williams [16] showed that for any nonzero elements  $x, y$  in a Banach space  $X$ ,

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq \frac{4\|x - y\|}{\|x\| + \|y\|}.$$

This inequality is called the Dunkl-Williams inequality and have been studied in many papers ([6, 31, 42, 43, 54, 60, 61] and so on). In [16], it was proved that the constant 4 can be replaced by 2 if  $X$  is a Hilbert space, and also that the value 4 is the best possible choice in the space  $(\mathbb{R}^2, \|\cdot\|_1)$ . A bit later, Kirk and Smiley [34] showed that a Banach space  $X$  is a Hilbert space if the inequality

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq \frac{2\|x - y\|}{\|x\| + \|y\|}$$

holds for any nonzero elements  $x, y$  in  $X$ . Thus, the smallest number which can replace the 4 in the Dunkl-Williams inequality measures how much the space is close (or far) to be a Hilbert one.

In 2008, Jiménez-Melado et al. [25] introduced the Dunkl-Williams constant  $DW(X)$  of a Banach space  $X$ :

$$DW(X) = \sup \left\{ \frac{\|x\| + \|y\|}{\|x - y\|} \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \mid x, y \in X, x, y \neq 0, x \neq y \right\}.$$

We summarize some basic properties of the Dunkl-Williams constant:

- (i) For any Banach space  $X$ ,  $2 \leq DW(X) \leq 4$  ([16]).
- (ii)  $DW(X) = 2$   $X$  is a Hilbert space ([34]).
- (iii)  $DW(X) < 4$  if and only if  $X$  is uniformly non-square, that is, there exists  $\delta > 0$  such that  $\|x + y\| \geq 2(1 - \delta)$  and  $\|x\| = \|y\| = 1$  imply  $\|x - y\| \leq 2(1 - \delta)$  ([7, 25]).

From (iii), we have that  $DW(X) = 4$  if and only if  $X$  is not uniformly non-square. However, to calculate the Dunkl-Williams constant is very difficult and so, except

for Hilbert spaces, there exists no uniformly non-square Banach space in which the Dunkl-Williams constant has been calculated.

For  $x, y \in X$ ,  $x$  is said to be Birkhoff orthogonal to  $y$  and denoted by  $x \perp_B y$  if  $\|x\| \leq \|x + \lambda y\|$  for any  $\lambda \in \mathbb{R}$  ([10, 22]). Birkhoff orthogonality coincides with the usual orthogonality in Hilbert spaces. Birkhoff orthogonality is always homogeneous, that is,  $x \perp_B y$  implies  $\alpha x \perp_B \beta y$  for every  $\alpha, \beta \in \mathbb{R}$ . However it is not symmetric, that is,  $x \perp_B y$  does not necessarily imply  $y \perp_B x$ . In a Banach space of three or more dimension, Birkhoff orthogonality is symmetric if and only if the norm is induced by an inner product. More results about this orthogonality can be found in [1, 2, 5, 10, 15, 22, 23].

The Day-James space  $\ell_p\text{-}\ell_q$  is defined for two numbers  $p, q$  with  $1 \leq p, q \leq \infty$  as the space  $\mathbb{R}^2$  with the norm

$$\|(x, y)\|_{p,q} = \begin{cases} \|(x, y)\|_p & \text{if } xy \geq 0, \\ \|(x, y)\|_q & \text{if } xy \leq 0. \end{cases}$$

James [22] considered the space  $\ell_p\text{-}\ell_q$  with  $1/p + 1/q = 1$  as an example of a two-dimensional Banach space where Birkhoff orthogonality is symmetric. Day [15] considered even more general spaces.

In this chapter, we shall characterize the Dunkl-Williams constant and calculate  $DW(\ell_2\text{-}\ell_\infty)$ . In Section 2.2, we first recall a characterization of the Dunkl-Williams constant and introduce some notations related to Birkhoff orthogonality. Then we present a characterization of the Dunkl-Williams constant. To calculate the Dunkl-Williams constant, we also obtain several results about convergent sequences. In Section 2.3, we define the frame of the unit ball of a Banach space which is related to the norming functionals for elements of the unit sphere. Then we improve the characterization in Section 2.2. We also consider the case of  $\dim X = 2$  and prove that the frame of the unit ball of a two-dimensional Banach space  $X$  coincides with the set of all extreme point of the unit ball. In Section 2.4, as an application, we calculate the Dunkl-Williams constant of the Day-James space  $\ell_2\text{-}\ell_\infty$ . We first see that  $DW((\mathbb{R}^2, \|\cdot\|_2)) = 2$  and  $DW((\mathbb{R}^2, \|\cdot\|_\infty)) = 4$ . Furthermore we show that  $DW(\ell_2\text{-}\ell_\infty) = 2\sqrt{2}$ .



## 2.2 A characterization by Birkoff orthogonality

Let  $X$  be a real Banach space with  $\dim X \geq 2$ . We denote the unit sphere and the unit ball of a Banach space  $X$  by  $S_X$  and  $B_X$ , respectively. Then we have a characterization of the Dunkl-Williams constant:

**Proposition 2.2.1.** *Let  $X$  be a real Banach space with  $\dim X \geq 2$ . Then*

$$\begin{aligned} DW(X) &= \sup \left\{ \frac{\|u+v\|}{\|(1-t)u+tv\|} \mid u, v \in S_X, u+v \neq 0, 0 \leq t \leq 1 \right\} \\ &= \sup \left\{ \frac{\|u+v\|}{\min_{0 \leq t \leq 1} \|(1-t)u+tv\|} \mid u, v \in S_X, u+v \neq 0 \right\}. \end{aligned}$$

For  $x, y \in S_X$ , let  $z(t) = (1-t)x + ty$  for all  $t \in \mathbb{R}$ .

**Lemma 2.2.2.** *Suppose that  $X$  is a Banach space with  $\dim X \geq 2$ . Let  $x, y \in S_X$  with  $x+y \neq 0$ . For  $t_0 \in [0, 1]$ , the following are equivalent:*

- (i)  $z(t_0) \perp_B x - y$ .
- (ii)  $\|z(t)\| \geq \|z(t_0)\|$  for all  $t \in \mathbb{R}$ .

**Proof.** Let  $\lambda \in \mathbb{R}$ . Then we have  $z(t_0) + \lambda(x-y) = z(t_0 - \lambda)$ . Hence  $z(t_0) \perp_B x - y$  if and only if  $\|z(t_0 - \lambda)\| \geq \|z(t_0)\|$  for any  $\lambda \in \mathbb{R}$ .  $\square$

Now we shall introduce some notations. For each  $x \in S_X$ , we define the subset  $V(x)$  of  $X$  by

$$V(x) = \{y \in X \mid x \perp_B y\}.$$

For each  $x \in S_X$  and each  $y \in V(x)$ , we put

$$\Gamma(x, y) = \left\{ \frac{\lambda + \mu}{2} \mid \lambda \leq 0 \leq \mu, \|x + \lambda y\| = \|x + \mu y\| \right\}$$

and  $m(x, y) = \sup\{\|x + \gamma y\| \mid \gamma \in \Gamma(x, y)\}$ . We define the positive number  $M(x)$  by  $M(x) = \sup\{m(x, y) \mid y \in V(x)\}$  for each  $x \in S_X$ .

**Proposition 2.2.3.** *Let  $X$  be a real Banach space with  $\dim X \geq 2$  and let  $x \in S_X$ . Then the following holds:*

- (i)  $0 \in V(x)$ .

(ii) If  $y \in V(x)$ , then  $\alpha y \in V(x)$  for any  $\alpha \in \mathbb{R}$ .

(iii)  $m(x, 0) = 1 \leq m(x, y)$  for each  $y \in V(x)$ .

(iv)  $m(x, \alpha y) = m(x, y)$  for each  $y \in V(x)$  and each  $\alpha \neq 0$ .

**Proof.** We directly have (i), (ii) and (iii).

(iv) For each  $y \in V(x)$  and each  $\alpha \neq 0$ , take an arbitrary element  $\gamma \in \Gamma(x, \alpha y)$ . Then there exist  $\lambda, \mu$  with  $\lambda \leq 0 \leq \mu$  such that  $\gamma = (\lambda + \mu)/2$  and  $\|x + \lambda(\alpha y)\| = \|x + \mu(\alpha y)\|$ . From the fact that  $\|x + (\alpha\lambda)y\| = \|x + (\alpha\mu)y\|$ , we have  $\alpha\gamma \in \Gamma(x, y)$ . Hence

$$\|x + \gamma(\alpha y)\| = \|x + (\alpha\gamma)y\| \leq m(x, y).$$

Thus we have  $m(x, \alpha y) \leq m(x, y)$ .

By the preceding paragraph, we also have

$$m(x, y) = m(x, \alpha^{-1}\alpha y) \leq m(x, \alpha y).$$

Therefore we obtain  $m(x, \alpha y) = m(x, y)$ .  $\square$

**Lemma 2.2.4.** Suppose that  $X$  is a real Banach space with  $\dim X \geq 2$ . Let  $x \in S_X$  and  $y \in V(x) \setminus \{0\}$ . There exist  $s_1, s_2$  with  $s_1 \leq 0 \leq s_2$  such that

$$\{t \in \mathbb{R} \mid \|x + ty\| = 1\} = [s_1, s_2].$$

Moreover,  $t \mapsto \|x + ty\|$  is strictly decreasing on  $(-\infty, s_1]$  and strictly increasing on  $[s_2, \infty)$ .

**Proof.** By the convexity and the continuity of the function  $t \mapsto \|x + ty\|$  and the fact that  $\min_{t \in \mathbb{R}} \|x + ty\| = 1$ ,  $\{t \in \mathbb{R} \mid \|x + ty\| = 1\}$  is a bounded closed convex subset of  $\mathbb{R}$ . Thus there exist  $s_1, s_2$  with  $s_1 \leq 0 \leq s_2$  such that  $\{t \in \mathbb{R} \mid \|x + ty\| = 1\} = [s_1, s_2]$ .

Assume that  $s > t > s_2$  and  $\|x + sy\| = \|x + ty\| > 1$ . Then there exists  $\alpha \in (0, 1)$  such that  $t = (1 - \alpha)s_2 + \alpha s$ . By the convexity of  $t \mapsto \|x + ty\|$ , we have

$$\|x + sy\| = \|x + ty\| \leq (1 - \alpha)\|x + s_2 y\| + \alpha\|x + sy\| < \|x + sy\|.$$

This is a contradiction. Therefore  $t \mapsto \|x + ty\|$  is strictly increasing on  $[s_2, \infty)$ .

One can similarly show that  $t \mapsto \|x + ty\|$  is strictly decreasing on  $(-\infty, s_1]$ .  $\square$

**Proposition 2.2.5.** *Let  $X, Y$  be real Banach spaces with  $\dim X = \dim Y \geq 2$  and let  $T$  be an isometric isomorphism from  $X$  onto  $Y$ . Then*

(i) *For any  $x \in S_X$  and any  $y \in V(x)$ ,  $m(Tx, Ty) = m(x, y)$ .*

(ii) *For any  $x \in S_X$ ,  $M(Tx) = M(x)$ .*

**Proof.** (i) Since  $\Gamma(Tx, Ty) = \Gamma(x, y)$ , we have

$$\begin{aligned} m(Tx, Ty) &= \sup\{\|Tx + \gamma Ty\| \mid \gamma \in \Gamma(x, y)\} \\ &= \sup\{\|x + \gamma y\| \mid \gamma \in \Gamma(x, y)\} = m(x, y). \end{aligned}$$

(ii) From (i) and the fact that  $V(Tx) = T(V(x))$ , we have

$$\begin{aligned} M(Tx) &= \sup\{m(Tx, Ty) \mid y \in V(x)\} \\ &= \sup\{m(x, y) \mid y \in V(x)\} = M(x). \end{aligned} \quad \square$$

Now we obtain a characterization of the Dunkl-Williams constant.

**Theorem 2.2.6.** *Let  $X$  be a real Banach space with  $\dim X \geq 2$ . Then*

$$DW(X) = 2 \sup\{M(x) \mid x \in S_X\}.$$

**Proof.** Take any  $x, y \in S_X$  with  $x + y \neq 0$ . Assume that  $\min_{0 \leq t \leq 1} \|z(t)\| = \|z(t_0)\|$  for some  $t_0 \in [0, 1]$ . Then  $z(t_0) \perp_B x - y$  by Lemma 2.2.2. Since Birkhoff orthogonality is homogeneous, we also have that  $z(t_0)/\|z(t_0)\| \perp_B x - y$ . Since  $x = z(t_0) + t_0(x - y)$  and  $y = z(t_0) + (t_0 - 1)(x - y)$ , we have that

$$\left\| \frac{z(t_0)}{\|z(t_0)\|} + \frac{t_0}{\|z(t_0)\|}(x - y) \right\| = \frac{1}{\|z(t_0)\|} = \left\| \frac{z(t_0)}{\|z(t_0)\|} + \frac{t_0 - 1}{\|z(t_0)\|}(x - y) \right\|.$$

Thus we have

$$\frac{1}{2} \left( \frac{t_0 - 1}{\|z(t_0)\|} + \frac{t_0}{\|z(t_0)\|} \right) \in \Gamma \left( \frac{z(t_0)}{\|z(t_0)\|}, x - y \right)$$

and

$$\begin{aligned} \frac{\|x + y\|}{\|z(t_0)\|} &= 2 \left\| \frac{z(t_0)}{\|z(t_0)\|} + \frac{1}{2} \left( \frac{t_0 - 1}{\|z(t_0)\|} + \frac{t_0}{\|z(t_0)\|} \right) (x - y) \right\| \\ &\leq 2m \left( \frac{z(t_0)}{\|z(t_0)\|}, x - y \right) \\ &\leq 2M \left( \frac{z(t_0)}{\|z(t_0)\|} \right). \end{aligned}$$

By Proposition 2.2.1, we obtain  $DW(X) \leq 2 \sup\{M(x) \mid x \in S_X\}$ .

For each  $x \in S_X$ , take any  $y \in V(x)$ . Let  $\lambda, \mu$  with  $\lambda \leq 0 \leq \mu$  such that  $\|x + \lambda y\| = \|x + \mu y\|$ . If  $\lambda = 0$  or  $\mu = 0$ , then we have  $\|x + \lambda y\| = \|x + \mu y\| = 1$  and hence

$$2 \left\| x + \frac{\lambda + \mu}{2} y \right\| = 2 \leq DW(X).$$

We assume that  $\lambda < 0 < \mu$ . Let  $t_1 = -\lambda/(\mu - \lambda) \in (0, 1)$ . Put

$$z = \frac{x + \lambda y}{\|x + \lambda y\|} \quad \text{and} \quad w = \frac{x + \mu y}{\|x + \mu y\|}.$$

Then

$$(1 - t_1)z + t_1 w = \frac{x}{\|x + \lambda y\|} \quad \text{and hence} \quad \|(1 - t_1)z + t_1 w\| = \frac{1}{\|x + \lambda y\|}.$$

Thus we have

$$\begin{aligned} 2 \left\| x + \frac{\lambda + \mu}{2} y \right\| &= \|x + \lambda y\| \left\| \frac{x + \lambda y}{\|x + \lambda y\|} + \frac{x + \mu y}{\|x + \mu y\|} \right\| \\ &= \frac{\|z + w\|}{\|(1 - t_1)z + t_1 w\|} \\ &\leq DW(X). \end{aligned}$$

Therefore  $DW(X) \geq 2 \sup\{M(x) \mid x \in S_X\}$ . □

**Lemma 2.2.7.** *Let  $X$  be a real Banach space with  $\dim X \geq 2$ . Then  $\Gamma(x, y)$  is a bounded subset of  $\mathbb{R}$  for any  $x \in S_X$  and any  $y \in V(x) \setminus \{0\}$ .*

**Proof.** Let  $x \in S_X$  and  $y \in V(x) \setminus \{0\}$ . By Theorem 2.2.6 and the fact that  $DW(X) \leq 4$ , we have  $m(x, y) \leq 2$ . Hence we have  $\|x + \gamma y\| \leq 2$  for any  $\gamma \in \Gamma(x, y)$ . Thus  $\Gamma(x, y)$  is bounded. □

**Proposition 2.2.8.** *Let  $X$  be a real Banach space with  $\dim X \geq 2$ . For each  $x \in S_X$  and each  $y \in V(x) \setminus \{0\}$ ,*

$$m(x, y) = \max\{\|x + \alpha y\|, \|x + \beta y\|\},$$

where  $\alpha = \inf \Gamma(x, y)$  and  $\beta = \sup \Gamma(x, y)$ .

**Proof.** Let  $x \in S_X$  and  $y \in V(x) \setminus \{0\}$ . Take an arbitrary element  $\gamma \in \Gamma(x, y)$ . Then there exists  $t \in [0, 1]$  such that  $\gamma = (1 - t)\alpha + t\beta$ . Thus we have

$$\|x + \gamma y\| \leq (1 - t)\|x + \alpha y\| + t\|x + \beta y\| \leq \max\{\|x + \alpha y\|, \|x + \beta y\|\}.$$

On the other hand, there exists  $\{\alpha_n\}$  in  $\Gamma(x, y)$  such that  $\alpha_n \rightarrow \alpha$ . Then we have

$$\|x + \alpha y\| = \lim_{n \rightarrow \infty} \|x + \alpha_n y\| \leq m(x, y).$$

We have  $\|x + \beta y\| \leq m(x, y)$  similarly.  $\square$

To calculate the Dunkl-Williams constant, we need the following result.

**Theorem 2.2.9.** *Let  $X$  be a real Banach space with  $\dim X \geq 2$ ,  $x \in S_X$  and  $y \in V(x)$ . Suppose that  $\{x_n\}$  in  $S_X$  converges to  $x$ . If there exists  $\{y_n\}$  with  $y_n \in V(x_n)$  for all  $n \in \mathbb{N}$  which converges to  $y$ , then*

$$m(x, y) \leq \varliminf_{n \rightarrow \infty} m(x_n, y_n).$$

**Proof.** By Proposition 2.2.3 (iii), it is clear if  $m(x, y) = 1$ , and so we may assume that  $m(x, y) > 1$ . For any  $\varepsilon \in (0, 2(m(x, y) - 1))$ , there exist real numbers  $\lambda_\varepsilon, \mu_\varepsilon$  with  $\lambda_\varepsilon < 0 < \mu_\varepsilon$  such that  $\|x + \lambda_\varepsilon y\| = \|x + \mu_\varepsilon y\|$  and

$$\left\| x + \frac{\lambda_\varepsilon + \mu_\varepsilon}{2} y \right\| > m(x, y) - \frac{\varepsilon}{2} > 1.$$

For arbitrary  $n \in \mathbb{N}$ , there exists a nonnegative number  $\mu_n$  such that

$$\|x_n + \lambda_\varepsilon y_n\| = \|x_n + \mu_n y_n\|.$$

Since

$$\left| \|x_n + \mu_n y_n\| - \|x + \mu_\varepsilon y\| \right| = \left| \|x_n + \lambda_\varepsilon y_n\| - \|x + \lambda_\varepsilon y\| \right| \rightarrow 0,$$

it follows that  $\|x_n + \mu_n y_n\| \rightarrow \|x + \mu_\varepsilon y\|$ . From the fact that

$$\begin{aligned} & \left| \|x + \mu_n y\| - \|x + \mu_\varepsilon y\| \right| \\ & \leq \left| \|x + \mu_n y\| - \|x_n + \mu_n y_n\| \right| + \left| \|x_n + \mu_n y_n\| - \|x + \mu_\varepsilon y\| \right| \\ & \leq \|x - x_n\| + |\mu_n| \|y_n - y\| + \left| \|x_n + \mu_n y_n\| - \|x + \mu_\varepsilon y\| \right| \rightarrow 0, \end{aligned}$$

we also have  $\|x + \mu_n y\| \rightarrow \|x + \mu_\varepsilon y\|$ . Thus, by Lemma 2.2.4 and the fact  $\|x + \mu_\varepsilon y\| > 1$ , we obtain  $\mu_n \rightarrow \mu_\varepsilon$ .

It follows from

$$\left\| x_n + \frac{\lambda_\varepsilon + \mu_n}{2} y_n \right\| \rightarrow \left\| x + \frac{\lambda_\varepsilon + \mu_\varepsilon}{2} y \right\|$$

that there exists  $n_0 \in \mathbb{N}$  such that

$$\left\| x_n + \frac{\lambda_\varepsilon + \mu_n}{2} y_n \right\| > \left\| x + \frac{\lambda_\varepsilon + \mu_\varepsilon}{2} y \right\| - \frac{\varepsilon}{2} > m(x, y) - \varepsilon$$

for any  $n \geq n_0$ . Therefore we obtain

$$\underline{\lim}_{n \rightarrow \infty} m(x_n, y_n) \geq \inf_{n \geq n_0} m(x_n, y_n) \geq m(x, y) - \varepsilon. \quad \square$$

**Corollary 2.2.10.** *Let  $X$  be a real Banach space with  $\dim X \geq 2$  and let  $x \in S_X$ . Suppose that  $C$  is a subset of  $V(x)$  satisfying  $M(x) = \sup\{m(x, y) \mid y \in C\}$  and that  $D$  is a dense subset of  $C$ . Then  $M(x) = \sup\{m(x, y) \mid y \in D\}$ .*

**Proof.** For any  $y_0 \in C$ , there exists a sequence  $\{y_n\}$  in  $D$  converging to  $y_0$ . Thus

$$m(x, y_0) \leq \underline{\lim}_{n \rightarrow \infty} m(x, y_n) \leq \sup\{m(x, y) \mid y \in D\}$$

by Theorem 2.2.9. Therefore we obtain  $M(x) \leq \sup\{m(x, y) \mid y \in D\}$ .  $\square$

**Corollary 2.2.11.** *Suppose that  $X$  is a real Banach space with  $\dim X \geq 2$ . Let  $x \in S_X$  and let  $\{x_n\}$  in  $S_X$  converging to  $x$ . Assume that, for any  $y \in V(x)$ , there exists  $\{y_n\}$  with  $y_n \in V(x_n) \setminus \{0\}$  for all  $n \in \mathbb{N}$ , converging to  $y$ . Then*

$$M(x) \leq \underline{\lim}_{n \rightarrow \infty} M(x_n).$$

**Proof.** Take an arbitrary element  $y \in V(x)$ . By the assumption, there exists  $\{y_n\}$  such that  $y_n \in V(x_n) \setminus \{0\}$  for all  $n \in \mathbb{N}$  and  $y_n \rightarrow y$ . By Theorem 2.2.9, we have

$$m(x, y) \leq \underline{\lim}_{n \rightarrow \infty} m(x_n, y_n) \leq \underline{\lim}_{n \rightarrow \infty} M(x_n).$$

Thus we obtain  $M(x) \leq \underline{\lim}_{n \rightarrow \infty} M(x_n)$ .  $\square$

## 2.3 A characterization by the frame of the unit ball

The dual space of  $X$  is denoted by  $X^*$ . In [23], James proved the following result about Birkhoff orthogonality.

**Lemma 2.3.1** ([23]). *Let  $X$  be a real Banach space with  $\dim X \geq 2$ . For  $x, y \in X$ ,  $x \perp_B y$  if and only if there exists  $f \in S_{X^*}$  such that  $f(x) = \|x\|$  and  $f(y) = 0$ .*

In this section, we characterize the Dunkl-Williams constant with the norming functionals. For each  $x_0 \in S_X$ , a functional  $f \in S_{X^*}$  is said to be norming if  $f(x_0) = \|x_0\| = 1$ . Let  $\nu(x_0) = \{f \in S_{X^*} \mid f(x_0) = 1\}$ . We note that, from the Hahn-Banach theorem,  $\nu(x) \neq \emptyset$  for any  $x \in S_X$ .

For each  $x_0$  and each  $f \in \nu(x_0)$ , let  $F(x_0, f) = S_X \cap (x_0 + \ker f)$  and let  $E(x_0, f)$  be the relative boundary of  $F(x_0, f)$  in  $x_0 + \ker f$ . We define the frame  $\text{fr}(B_X)$  of  $B_X$  by

$$\text{fr}(B_X) = \bigcup \{E(x, f) \mid x \in S_X, f \in \nu(x)\}.$$

**Lemma 2.3.2.** *Suppose that  $X$  is a real Banach space with  $\dim X \geq 2$ . Let  $x_0 \in S_X$  and  $f \in \nu(x_0)$ . Then  $\ker f \subseteq V(x)$  for any  $x \in F(x_0, f)$ .*

**Proof.** Let  $x \in F(x_0, f)$ . Take an arbitrary element  $y \in \ker f$ . Then we have

$$\|x\| = 1 = f(x) = f(x + \lambda y) \leq \|x + \lambda y\|$$

for all  $\lambda \in \mathbb{R}$ . Thus  $y \in V(x)$ . This means that  $\ker f \subseteq V(x)$ .  $\square$

Let  $x_0 \in S_X$  and  $f \in \nu(x_0)$ . By Lemmas 2.2.4 and 2.3.2, for each  $x \in F(x_0, f)$  and each  $y \in \ker f \setminus \{0\}$ , we have  $s_1, s_2$  with  $s_1 \leq 0 \leq s_2$  such that

$$\{t \in \mathbb{R} \mid \|x + ty\| = 1\} = [s_1, s_2].$$

**Lemma 2.3.3.** *Suppose that  $X$  is a real Banach space with  $\dim X \geq 2$ . Let  $x_0 \in S_X$  and  $f \in \nu(x_0)$ . Assume that  $x \in F(x_0, f)$  and that there exists  $y \in \ker f \setminus \{0\}$  such that  $s_1 = 0$  or  $s_2 = 0$  for  $s_1, s_2$  with  $\{t \in \mathbb{R} \mid \|x + ty\| = 1\} = [s_1, s_2]$ . Then  $x \in E(x_0, f)$ .*

**Proof.** Suppose that  $s_2 = 0$ . Then, for any  $t > 0$ , we have  $\|x + ty\| > 1$  and hence  $x + ty \in (x_0 + \ker f) \setminus F(x_0, f)$ . Thus we obtain  $x \in E(x_0, f)$ . If  $s_1 = 0$ , we similarly have  $x \in E(x_0, f)$ .  $\square$

**Remark 2.3.4.** A Banach space  $X$  is said to be strictly convex if  $x, y \in S_X$  and  $x \neq y$  imply  $\|x + y\| < 2$ . In a strictly convex Banach space  $X$ ,

$$F(x_0, f) = E(x_0, f) = \{x_0\}$$

holds for each  $x_0 \in S_X$  and each  $f \in \nu(x_0)$  (cf. [46]), and hence  $\text{fr}(B_X) = S_X$ .

An element  $x \in S_X$  is called an extreme point of  $B_X$  if  $y, z \in S_X$  and  $x = (y+z)/2$  imply  $x = y = z$ . The set of all extreme points of  $B_X$  is denoted by  $\text{ext}(B_X)$ .

**Proposition 2.3.5.** Let  $X$  be a real Banach space with  $\dim X \geq 2$ . Then  $\text{ext}(B_X) \subseteq \text{fr}(B_X)$ .

**Proof.** Let  $x \in S_X \setminus \text{fr}(B_X)$  and let  $f \in \nu(x)$ . Since  $x \notin \text{fr}(B_X)$ , we have  $x \in F(x, f) \setminus E(x, f)$ . Let  $y \in \ker f \setminus \{0\}$ . Then, by Lemmas 2.2.4 and 2.3.3, there exist  $s_1, s_2$  with  $s_1 < 0 < s_2$  such that  $\{t \in \mathbb{R} \mid \|x + ty\| = 1\} = [s_1, s_2]$ . Putting  $s_0 = \min\{|s_1|, |s_2|\}$ , then we have  $x \pm s_0 y \in S_X$  and

$$x = \frac{1}{2}\{(x + s_0 y) + (x - s_0 y)\}.$$

Therefore  $x \notin \text{ext}(B_X)$ . □

**Lemma 2.3.6.** Suppose that  $X$  is a real Banach space with  $\dim X \geq 2$ . Let  $x_0 \in S_X$  and  $f \in \nu(x_0)$ . If  $x_0 \in F(x_0, f) \setminus E(x_0, f)$ , then  $V(x_0) = \ker f$ .

**Proof.** By Lemma 2.3.2 and the fact that  $x_0 \in F(x_0, f)$ , we have  $\ker f \subseteq V(x_0)$ . Conversely, take any  $y \in V(x_0)$ . Letting  $z = y - f(y)x_0$ , then  $z \neq 0$  and  $z \in \ker f$ . Since  $x_0 \in F(x_0, f) \setminus E(x_0, f)$ , by Lemmas 2.2.4 and 2.3.3, we have  $s_1, s_2$  with  $s_1 < 0 < s_2$  such that  $\{t \in \mathbb{R} \mid \|x_0 + tz\| = 1\} = [s_1, s_2]$ . For any  $t \in [s_1, s_2]$ , we have  $(1 - tf(y))x_0 \perp_B y$  and hence

$$|1 - tf(y)| \leq \|(1 - tf(y))x_0 + ty\| = \|x_0 + tz\| = 1.$$

Thus  $f(y) = 0$ . Therefore we obtain  $V(x_0) = \ker f$ . □

**Theorem 2.3.7.** Let  $X$  be a real Banach space with  $\dim X \geq 2$ . Then

$$DW(X) = 2 \sup\{M(x) \mid x \in \text{fr}(B_X)\}.$$



**Proof.** Let  $x \in S_X \setminus \text{fr}(B_X)$  and let  $f \in \nu(x)$ . Since  $x \notin \text{fr}(B_X)$ , we have  $x \in F(x, f) \setminus E(x, f)$ . Let  $y \in V(x) \setminus \{0\}$ . By Lemmas 2.2.4 and 2.3.3, we have  $s_1, s_2$  with  $s_1 < 0 < s_2$  such that  $\{t \in \mathbb{R} \mid \|x + ty\| = 1\} = [s_1, s_2]$ . For this  $s_2$ , we have  $y \in V(x + s_2y)$  by Lemmas 2.3.2 and 2.3.6. It follows from

$$\{t \in \mathbb{R} \mid \|(x + s_2y) + ty\| = 1\} = [s_1 - s_2, 0]$$

that  $x + s_2y \in E(x, f) \subseteq \text{fr}(B_X)$ . We prove that  $m(x, y) \leq m(x + s_2y, y)$ . Let  $\lambda, \mu$  be real numbers with  $\lambda \leq 0 \leq \mu$  such that  $\|x + \lambda y\| = \|x + \mu y\| = 1$ .

Case 1. Suppose that  $0 \leq \mu \leq s_2$ . From the fact that  $\|x + \lambda y\| = \|x + \mu y\| = 1$ , we have  $s_1 \leq \lambda \leq 0$ . Thus we obtain  $(\lambda + \mu)/2 \in [s_1, s_2]$  and hence

$$\left\|x + \frac{\lambda + \mu}{2}y\right\| = 1 \leq m(x + s_2y, y).$$

Case 2. If  $s_2 < \mu$ , then we have  $\lambda - s_2 < 0 < \mu - s_2$ . From the fact that

$$\|x + s_2y + (\lambda - s_2)y\| = \|x + s_2y + (\mu - s_2)y\|,$$

we have

$$\frac{1}{2}\{(\lambda - s_2) + (\mu - s_2)\} \in \Gamma(x + s_2y, y).$$

Thus

$$\begin{aligned} \left\|x + \frac{\lambda + \mu}{2}y\right\| &= \left\|x + s_2y + \frac{1}{2}\{(\lambda - s_2) + (\mu - s_2)\}y\right\| \\ &\leq m(x + s_2y, y). \end{aligned}$$

Hence we have

$$m(x, y) \leq m(x + s_2y, y) \leq M(x + s_2y) \leq \sup\{M(x) \mid x \in \text{fr}(B_X)\}.$$

Thus

$$\sup\{M(x) \mid x \in S_X\} \leq \sup\{M(x) \mid x \in \text{fr}(B_X)\}$$

and hence

$$\sup\{M(x) \mid x \in S_X\} = \sup\{M(x) \mid x \in \text{fr}(B_X)\}.$$

Therefore, by Theorem 2.2.6, we obtain  $DW(X) = 2 \sup\{M(x) \mid x \in \text{fr}(B_X)\}$ .  $\square$

Now we consider the case of  $\dim X = 2$ .

**Lemma 2.3.8.** *Let  $X$  be a two-dimensional real Banach space. Then  $\text{ext}(B_X) = \text{fr}(B_X)$ .*

**Proof.** Let  $x \in S_X \setminus \text{ext}(B_X)$ . Then there exist distinct elements  $y, z \in S_X$  such that  $x = (y + z)/2$ . For any  $f \in \nu(x)$ , it follows from

$$1 = f(x) = \frac{1}{2}(f(y) + f(z))$$

that  $f(y) = f(z) = 1$  and hence  $y, z \in F(x, f)$ . Thus we have that  $\|x + t(y - z)\| = 1$  for all  $t \in [-1/2, 1/2]$ .

On the other hand, from the fact  $\dim \ker f = 1$ , we have  $\ker f = [y - z]$  which is the closed linear span of  $\{y - z\}$ . Hence, for any  $w \in \ker f \setminus \{0\}$ , there exists a unique nonzero real number  $\alpha$  such that  $w = \alpha(y - z)$ . Thus we have

$$\left[-\frac{1}{2|\alpha|}, \frac{1}{2|\alpha|}\right] \subset \{t \in \mathbb{R} \mid \|x + tw\| = 1\}$$

and so we obtain  $x \notin E(x, f)$ . Therefore  $x \notin \text{fr}(B_X)$ .  $\square$

By Theorem 2.3.7 and Lemma 2.3.8, we have the following result.

**Theorem 2.3.9.** *Let  $X$  be a two-dimensional real Banach space. Then*

$$DW(X) = 2 \sup\{M(x) \mid x \in \text{ext}(B_X)\}.$$

## 2.4 Examples

In this section, as an application of the results in above sections, we calculate the Dunkl-Williams constant of the Day-James space  $\ell_2\text{-}\ell_\infty$ . We first see that  $DW((\mathbb{R}^2, \|\cdot\|_2)) = 2$  and  $DW((\mathbb{R}^2, \|\cdot\|_\infty)) = 4$ .

**Example 2.4.1.**  $DW((\mathbb{R}^2, \|\cdot\|_2)) = 2$  ([16, 34]).

**Proof.** From Remark 2.3.4, we have  $\text{ext}(B_{(\mathbb{R}^2, \|\cdot\|_2)}) = S_{(\mathbb{R}^2, \|\cdot\|_2)}$ . Letting  $e_1 = (1, 0)$ , by Proposition 2.2.5 (i) and the fact that the mapping

$$(r \cos \alpha, r \sin \alpha) \mapsto (r \cos(\alpha + \theta), r \sin(\alpha + \theta))$$

is isometric isomorphism from  $(\mathbb{R}^2, \|\cdot\|_2)$  onto itself for all  $\theta \in [0, 2\pi]$ , it is enough to consider  $M(e_1)$ . From the fact that, for  $x, y \in (\mathbb{R}^2, \|\cdot\|_2)$ ,  $x \perp_B y$  if and only if  $x \perp y$  in the usual sense,

$$V(e_1) = \{(0, t) \mid t \in \mathbb{R}\}.$$

Putting  $e_2 = (0, 1)$ , we have  $M(e_1) = m(e_1, e_2)$  by Proposition 2.2.3 (iv). It is easy to check that  $m(e_1, e_2) = 1$ . Thus, by Theorem 2.2.6, we obtain  $DW((\mathbb{R}^2, \|\cdot\|_2)) = 2$ .  $\square$

**Example 2.4.2.**  $DW((\mathbb{R}^2, \|\cdot\|_\infty)) = 4$ . ([7, 25])

**Proof.** Let  $x_0 = (1, 1)$ . Then it is clear that  $x_0 \in \text{ext}(B_{(\mathbb{R}^2, \|\cdot\|_\infty)})$ . For  $t \in (0, 1)$ , letting  $y_t = (1, -t)$ , then  $y_t \in V(x_0)$ . For  $\lambda \leq 0$  and  $\mu \geq 0$ , we have

$$\|x_0 + \lambda y_t\|_\infty = \begin{cases} 1 - \lambda t & (-\frac{2}{1-t} \leq \lambda \leq 0), \\ -(1 + \lambda) & (\lambda \leq -\frac{2}{1-t}) \end{cases}$$

and  $\|x_0 + \mu y_t\|_\infty = 1 + \mu$ . We note that, for any  $\mu \geq 0$ , there exists a unique  $\lambda \leq 0$  such that  $\|x_0 + \lambda y_t\|_\infty = \|x_0 + \mu y_t\|_\infty$ .

For  $\mu \leq 2t/(1-t)$ , put  $\lambda = -\mu/t$ . Then  $\|x_0 + \lambda y_t\|_\infty = \|x_0 + \mu y_t\|_\infty$ . We have

$$\frac{\lambda + \mu}{2} = -\frac{1-t}{2t}\mu \quad \text{and} \quad \left\|x_0 + \frac{\lambda + \mu}{2}y_t\right\|_\infty = 1 + \frac{1-t}{2}\mu.$$

For  $\mu \geq 2t/(1-t)$ , put  $\lambda = -(2 + \mu)$ . Then  $\|x_0 + \lambda y_t\|_\infty = \|x_0 + \mu y_t\|_\infty$ . We have

$$\frac{\lambda + \mu}{2} = -1 \quad \text{and} \quad \left\|x_0 + \frac{\lambda + \mu}{2}y_t\right\|_\infty = 1 + t.$$

Thus  $m(x_0, y_t) = 1 + t$  for any  $t \in (0, 1)$  and hence, by Theorem 2.3.9,

$$DW((\mathbb{R}^2, \|\cdot\|_\infty)) \geq 2M(x_0) \geq 2 \sup\{1 + t \mid 0 < t < 1\} = 4.$$

From the fact that  $DW((\mathbb{R}^2, \|\cdot\|_\infty)) \leq 4$ , we have  $DW((\mathbb{R}^2, \|\cdot\|_\infty)) = 4$ .  $\square$

The Day-James space  $\ell_2\text{-}\ell_\infty$  is defined as the space  $\mathbb{R}^2$  with the norm

$$\|(x, y)\|_{2,\infty} = \begin{cases} \|(x, y)\|_2 & (xy \geq 0), \\ \|(x, y)\|_\infty & (xy \leq 0). \end{cases}$$

In the rest of this chapter, we shall consider the space  $\ell_2\text{-}\ell_\infty$  and calculate the Dunkl-Williams constant  $DW(\ell_2\text{-}\ell_\infty)$ . From [55, Theorem 2.5], we have the following:

**Lemma 2.4.3.** *There is an isometric isomorphism that identifies  $(\ell_2\text{-}\ell_\infty)^*$  with  $\ell_2\text{-}\ell_1$  such that if  $f \in (\ell_2\text{-}\ell_\infty)^*$  is identified with the element  $(u, v) \in \ell_2\text{-}\ell_1$ , then  $f(x, y) = ux + vy$  for all  $(x, y) \in \ell_2\text{-}\ell_\infty$ .*

**Lemma 2.4.4.** For any  $a \in [0, 1]$ , put  $b = \sqrt{1 - a^2}$ . Then

$$V((a, b)) = \{\alpha(b, -a) \mid \alpha \in \mathbb{R}\}.$$

**Proof.** It is clear that  $(a, b) \perp_B (b, -a)$  and hence

$$\{\alpha(b, -a) \mid \alpha \in \mathbb{R}\} \subseteq V((a, b))$$

by Proposition 2.2.3 (ii).

Conversely, take any  $y = (y_1, y_2) \in V((a, b))$ . Then, by Lemma 2.3.1, there exists  $f \in S_{(\ell_2 - \ell_\infty)^*}$  such that  $f((a, b)) = 1$  and  $f((y_1, y_2)) = 0$ . For this  $f$ , by Lemma 2.4.3, there exists  $(u, v) \in \ell_2 - \ell_1$  such that  $\|(u, v)\|_{2,1} = \|f\|_{(\ell_2 - \ell_\infty)^*} = 1$  and  $f((x, y)) = ux + vy$  for all  $(x, y) \in \ell_2 - \ell_\infty$ .

By the Cauchy-Schwarz inequality, we have

$$1 = f((a, b)) = ua + vb \leq \|(u, v)\|_2 \|(a, b)\|_2 \leq \|(u, v)\|_{2,1} \|(a, b)\|_{2,\infty} = 1$$

and hence  $(u, v) = (a, b)$ . From the fact that  $f((y_1, y_2)) = 0$ , we obtain  $ay_1 + by_2 = 0$ .

Thus

$$y = (y_1, y_2) \in \{\alpha(b, -a) \mid \alpha \in \mathbb{R}\}. \quad \square$$

**Lemma 2.4.5.**  $V((1, -1)) = \{(a, b) \mid ab \geq 0\}$ .

**Proof.** Take any  $y = (y_1, y_2) \in V((1, -1))$ . Then, by Lemma 2.3.1, there exists  $f \in S_{(\ell_2 - \ell_\infty)^*}$  such that  $f((1, -1)) = 1$  and  $f((y_1, y_2)) = 0$ . For this  $f$ , by Lemma 2.4.3, there exists  $(u, v) \in \ell_2 - \ell_1$  such that  $\|(u, v)\|_{2,1} = \|f\|_{(\ell_2 - \ell_\infty)^*} = 1$  and  $f((x, y)) = ux + vy$  for all  $(x, y) \in \ell_2 - \ell_\infty$ . From the fact that  $f((1, -1)) = 1$ , we have  $v = -(1 - u)$ . We note that  $0 \leq u \leq 1$ , since  $\|(u, -(1 - u))\|_{2,1} = 1$ . It follows from  $f((y_1, y_2)) = 0$  that  $uy_1 - (1 - u)y_2 = 0$ . If  $u = 0$ , then  $y_2 = 0$  and hence  $y_1y_2 = 0$ . If  $u \neq 0$ , then we have

$$y_1y_2 = \frac{1 - u}{u}y_2^2 \geq 0.$$

Thus

$$y = (y_1, y_2) \in \{(a, b) \mid ab \geq 0\} \quad \text{and hence} \quad V((1, -1)) \subseteq \{(a, b) \mid ab \geq 0\}.$$

Conversely, take any  $(a, b)$  with  $ab \geq 0$ . If  $ab = 0$ , then we have  $(a, b) \in V((1, -1))$ , immediately. Hence we may assume that  $ab > 0$ . We define a linear functional  $f$  by

$$f((s, t)) = \frac{b}{a + b}s - \frac{a}{a + b}t.$$

Then we have  $f((1, -1)) = 1$ ,  $f((a, b)) = 0$  and

$$\|f\|_{(\ell_2-\ell_\infty)^*} = \left\| \left( \frac{b}{a+b}, -\frac{a}{a+b} \right) \right\|_{2,1} = \frac{b}{a+b} + \frac{a}{a+b} = 1$$

Thus, by Lemma 2.3.1,  $(a, b) \in V((1, -1))$  and hence

$$\{(a, b) \mid ab \geq 0\} \subseteq V((1, -1)). \quad \square$$

**Lemma 2.4.6.**

$$DW(\ell_2-\ell_\infty) = 2 \max\{\sup\{m((a, b), (b, -a)) \mid a^2 + b^2 = 1, 0 < b < 1/\sqrt{2} < a < 1\}, \\ \sup\{m((1, -1), (a, b)) \mid a^2 + b^2 = 1, 0 < b < 1/\sqrt{2} < a < 1\}\}.$$

**Proof.** By the definition of the norm  $\|\cdot\|_{2,\infty}$ , we clearly have

$$\text{ext}(B_{\ell_2-\ell_\infty}) = \{\pm(a, b) \mid a^2 + b^2 = 1, 0 \leq a, b \leq 1\} \cup \{\pm(1, -1)\}.$$

Since the mapping  $x \mapsto -x$  is an isometric isomorphism from  $\ell_2-\ell_\infty$  onto itself, by Proposition 2.2.5 (ii), we have  $M(-x) = M(x)$  for all  $x \in \text{ext}(B_{\ell_2-\ell_\infty})$ . By Theorem 2.3.9, we have

$$DW(\ell_2-\ell_\infty) = 2 \sup\{M(x) \mid x \in \text{ext}(B_{\ell_2-\ell_\infty})\} \\ = 2 \sup\{M(x) \mid x \in \{(a, b) \mid a^2 + b^2 = 1, 0 \leq a, b \leq 1\} \cup \{(1, -1)\}\} \\ = 2 \max\{\sup\{M((a, b)) \mid a^2 + b^2 = 1, 0 \leq a, b \leq 1\}, M((1, -1))\}.$$

Take an arbitrary element  $(a, b)$  with  $a^2 + b^2 = 1$  and  $0 \leq a, b \leq 1$ . Then we have

$$V((a, b)) = \{\alpha(b, -a) \mid \alpha \in \mathbb{R}\}$$

by Lemma 2.4.4. Hence we have  $M((a, b)) = m((a, b), (b, -a))$  by Proposition 2.2.3 (iv). Thus

$$\sup\{M((a, b)) \mid a^2 + b^2 = 1, 0 \leq a, b \leq 1\} \\ = \sup\{m((a, b), (b, -a)) \mid a^2 + b^2 = 1, 0 \leq a, b \leq 1\}.$$

Since the mapping  $(s, t) \mapsto (t, s)$  is an isometric isomorphism from  $\ell_2-\ell_\infty$  onto itself, by Proposition 2.2.5 (i), we have

$$m((b, a), (a, -b)) = m((a, b), (b, -a))$$

for all  $(a, b)$  satisfying  $a^2 + b^2 = 1$  and  $0 \leq a, b \leq 1$ . Thus

$$\begin{aligned} & \sup\{m((a, b), (b, -a)) \mid a^2 + b^2 = 1, 0 \leq a, b \leq 1\} \\ &= \sup\{m((a, b), (b, -a)) \mid a^2 + b^2 = 1, 0 \leq b \leq 1/\sqrt{2} \leq a \leq 1\}. \end{aligned}$$

By Corollary 2.2.11, we obtain

$$\begin{aligned} & \sup\{M((a, b)) \mid a^2 + b^2 = 1, 0 \leq a, b \leq 1\} \\ &= \sup\{m((a, b), (b, -a)) \mid a^2 + b^2 = 1, 0 < b < 1/\sqrt{2} < a < 1\}. \end{aligned}$$

For all  $(a, b) \neq (0, 0)$  with  $ab \geq 0$ , by Proposition 2.2.3 (iv), we have

$$m((1, -1), (a, b)) = m\left((1, -1), \frac{(a, b)}{\sqrt{a^2 + b^2}}\right).$$

Thus

$$M((1, -1)) = \sup\{m((1, -1), (a, b)) \mid a^2 + b^2 = 1, 0 \leq a, b \leq 1\}.$$

Take an arbitrary element  $(a, b)$  with  $a^2 + b^2 = 1$  and  $0 \leq a, b \leq 1$ . Since the mapping  $(s, t) \mapsto (-t, -s)$  is an isometric isomorphism from  $\ell_2$ - $\ell_\infty$  onto itself, we have

$$m((1, -1), (a, b)) = m((1, -1), (-b, -a))$$

by Proposition 2.2.5 (i). Hence we have

$$m((1, -1), (a, b)) = m((1, -1), (b, a))$$

by Proposition 2.2.3 (iv). Thus

$$\begin{aligned} & \sup\{m((1, -1), (a, b)) \mid a^2 + b^2 = 1, 0 \leq a, b \leq 1\} \\ &= \sup\{m((1, -1), (a, b)) \mid a^2 + b^2 = 1, 0 \leq b \leq 1/\sqrt{2} \leq a \leq 1\}. \end{aligned}$$

By Corollary 2.2.10, we obtain

$$M(1, -1) = \sup\{m((1, -1), (a, b)) \mid a^2 + b^2 = 1, 0 < b < 1/\sqrt{2} < a < 1\}. \quad \square$$

Henceforth, for  $a \in (1/\sqrt{2}, 1)$ , we put  $b = \sqrt{1 - a^2}$  and let  $x_a = (a, b)$ ,  $y_a = (b, -a)$ . We also put  $u = (1, -1)$ .

**Lemma 2.4.7.** *The following holds.*

(i) *For each  $\mu \geq 0$ , there exists a unique  $\lambda \leq 0$  such that*

$$\|x_a + \lambda y_a\|_{2,\infty} = \|x_a + \mu y_a\|_{2,\infty}.$$

(ii) *For each  $\mu \geq 0$ , there exists a unique  $\lambda \leq 0$  such that*

$$\|u + \lambda x_a\|_{2,\infty} = \|u + \mu x_a\|_{2,\infty}.$$

**Proof.** (i) We define a function  $f$  from  $\mathbb{R}$  into  $\mathbb{R}$  by  $f(t) = \|x_a + ty_a\|_{2,\infty}$ . Then we have that  $\min_{t \in \mathbb{R}} f(t) = 1$  is attained at only  $t = 0$ . Hence, by Lemma 2.2.4,  $f(t)$  is strictly increasing on  $[0, \infty)$  and strictly decreasing on  $(-\infty, 0]$ . Thus we obtain (i).

The proof of (ii) is similar and so we omit it.  $\square$

**Lemma 2.4.8.**

$$\Gamma(x_a, y_a) = \begin{cases} \left[0, \frac{b}{a}\right] & (a \leq 2b), \\ \left[0, \frac{a+b-\sqrt{2ab}}{a-b}\right] & (a > 2b). \end{cases}$$

**Proof.**  $\Gamma(x_a, y_a)$  is defined by

$$\Gamma(x_a, y_a) = \left\{ \frac{\lambda + \mu}{2} \mid \lambda \leq 0 \leq \mu, \|x_a + \lambda y_a\|_{2,\infty} = \|x_a + \mu y_a\|_{2,\infty} \right\}.$$

Hence, by Lemma 2.4.7 (i), for each  $\mu \geq 0$ , it is enough to find  $\lambda \leq 0$  such that  $\|x_a + \lambda y_a\|_{2,\infty} = \|x_a + \mu y_a\|_{2,\infty}$  and calculate  $(\lambda + \mu)/2$ .

For  $\lambda \leq 0$  and  $\mu \geq 0$ , we have

$$\|x_a + \lambda y_a\|_{2,\infty} = \begin{cases} b - \lambda a & (\lambda \leq -\frac{a}{b}), \\ \sqrt{1 + \lambda^2} & (-\frac{a}{b} \leq \lambda \leq 0), \end{cases}$$

$$\|x_a + \mu y_a\|_{2,\infty} = \begin{cases} \sqrt{1 + \mu^2} & (0 \leq \mu \leq \frac{b}{a}), \\ a + \mu b & (\frac{b}{a} \leq \mu \leq \frac{a+b}{a-b}), \\ \mu a - b & (\frac{a+b}{a-b} \leq \mu). \end{cases}$$

Related to

$$\left\| x_a - \frac{a}{b} y_a \right\|_{2,\infty} = \frac{1}{b} \quad \text{and} \quad \left\| x_a + \frac{a+b}{a-b} y_a \right\|_{2,\infty} = \frac{1}{a-b},$$

we have to consider the following two cases.

Case 1.  $a \leq 2b$ . Then

$$\left\| x_a - \frac{a}{b} y_a \right\|_{2,\infty} = \frac{1}{b} \leq \frac{1}{a-b} = \left\| x_a + \frac{a+b}{a-b} y_a \right\|_{2,\infty}$$

and we obtain that  $b/a \leq (1-ab)/b^2 \leq (a+b)/(a-b)$  and

$$\left\| x_a + \frac{1-ab}{b^2} y_a \right\|_{2,\infty} = \frac{1}{b} = \left\| x_a - \frac{a}{b} y_a \right\|_{2,\infty}.$$

Thus we consider the following four subcases.

Subcase 1.1.  $0 \leq \mu \leq b/a$ . Put  $\lambda = -\mu$ . Then we have  $-a/b \leq \lambda \leq 0$ ,

$$\|x_a + \lambda y_a\|_{2,\infty} = \sqrt{1 + \lambda^2} = \sqrt{1 + \mu^2} = \|x_a + \mu y_a\|_{2,\infty}$$

and  $(\lambda + \mu)/2 = 0$ .

Subcase 1.2.  $b/a \leq \mu \leq (1-ab)/b^2$ . Put  $\lambda = -\sqrt{2ab\mu + (\mu^2 - 1)b^2}$ . Then we have  $-a/b \leq \lambda \leq 0$ ,

$$\begin{aligned} \|x_a + \lambda y_a\|_{2,\infty} &= \sqrt{1 + 2ab\mu + (\mu^2 - 1)b^2} \\ &= \sqrt{a^2 + 2ab\mu + \mu^2 b^2} = a + \mu b = \|x_a + \mu y_a\|_{2,\infty} \end{aligned}$$

and

$$\frac{\lambda + \mu}{2} = \frac{\mu - \sqrt{2ab\mu + (\mu^2 - 1)b^2}}{2}.$$

Subcase 1.3.  $(1-ab)/b^2 \leq \mu \leq (a+b)/(a-b)$ . Put  $\lambda = -(a-b + \mu b)/a$ . Then we have  $\lambda \leq -a/b$ ,

$$\begin{aligned} \|x_a + \lambda y_a\|_{2,\infty} &= b - \left( -\frac{a-b + \mu b}{a} \right) a \\ &= b + (a-b + \mu b) = a + \mu b = \|x + \mu y_a\|_{2,\infty} \end{aligned}$$

and

$$\frac{\lambda + \mu}{2} = \frac{(a-b)(\mu - 1)}{2a}.$$

Notice that

$$\mu - 1 \geq \frac{1-ab}{b^2} - 1 = \frac{a(a-b)}{b^2} \geq 0.$$



Subcase 1.4.  $(a+b)/(a-b) \leq \mu$ . Put  $\lambda = -(\mu a - 2b)/a$ . Then we have  $\lambda \leq -a/b$ ,

$$\begin{aligned}\|x_a + \lambda y_a\|_{2,\infty} &= b - \left(-\frac{\mu a - 2b}{a}\right) a \\ &= b + (\mu a - 2b) = \mu a - b = \|x + \mu y_a\|_{2,\infty}\end{aligned}$$

and  $(\lambda + \mu)/2 = b/a$ .

We define a function  $f$  on  $\mathbb{R}_+$  by

$$f(\mu) = \mu - \sqrt{2ab\mu + (\mu^2 - 1)b^2}.$$

Then it is easy to check that  $f(\mu)$  is increasing in the interval  $[b/a, (1-ab)/b^2]$ .

Hence we have  $\Gamma(x_a, y_a) = [0, b/a]$ .

Case 2.  $a > 2b$ . Then

$$\left\|x_a - \frac{a}{b}y_a\right\|_{2,\infty} = \frac{1}{b} > \frac{1}{a-b} = \left\|x_a + \frac{a+b}{a-b}\right\|_{2,\infty}$$

and we obtain that  $(1+b^2)/ab \geq (a+b)/(a-b)$  and

$$\left\|x_a + \frac{1+b^2}{ab}y_a\right\|_{2,\infty} = \frac{1}{b} = \left\|x_a - \frac{a}{b}y_a\right\|_{2,\infty}.$$

Thus we consider the following four subcases.

Subcase 2.1.  $0 \leq \mu \leq b/a$ . Similarly to Subcase 1.1, we have  $(\lambda + \mu)/2 = 0$ .

Subcase 2.2.  $b/a \leq \mu \leq (a+b)/(a-b)$ . Similarly to Subcase 1.2, we have

$$\frac{\lambda + \mu}{2} = \frac{\mu - \sqrt{2ab\mu + (\mu^2 - 1)b^2}}{2}.$$

Subcase 2.3.  $(a+b)/(a-b) \leq \mu \leq (1+b^2)/ab$ . Put  $\lambda = -\sqrt{-2ab\mu + (\mu^2 - 1)a^2}$ .

Then we have  $-a/b \leq \lambda \leq 0$ ,

$$\begin{aligned}\|x_a + \lambda y_a\|_{2,\infty} &= \sqrt{1 + \lambda^2} \\ &= \sqrt{1 - 2ab\mu + (\mu^2 - 1)a^2} \\ &= \sqrt{b^2 - 2ab\mu + \mu^2 a^2} = b - \mu a = \|x_a + \mu y_a\|_{2,\infty}\end{aligned}$$

and

$$\frac{\lambda + \mu}{2} = \frac{\mu - \sqrt{-2ab\mu + (\mu^2 - 1)a^2}}{2}.$$

Subcase 2.4.  $(1 + b^2)/ab \leq \mu$ . Similarly to Subcase 1.4, we have  $(\lambda + \mu)/2 = b/a$ .

We define a function  $g$  on  $\mathbb{R}_+$  by

$$g(\mu) = \mu - \sqrt{-2ab\mu + (\mu^2 - 1)a^2}.$$

Then it is clear that  $g(\mu)$  is decreasing in the interval  $[(a + b)/(a - b), (1 + b^2)/ab]$ . Hence we have

$$\Gamma(x_a, y_a) = \left[ 0, \frac{g\left(\frac{a+b}{a-b}\right)}{2} \right] = \left[ 0, \frac{a + b - \sqrt{2ab}}{a - b} \right]. \quad \square$$

**Lemma 2.4.9.**  $\Gamma(u, x_a) = [-(a - b), 0]$ .

**Proof.**  $\Gamma(u, x_a)$  is defined by

$$\Gamma(u, x_a) = \left\{ \frac{\lambda + \mu}{2} \mid \lambda \leq 0 \leq \mu, \|u + \lambda x_a\|_{2,\infty} = \|u + \mu x_a\|_{2,\infty} \right\}.$$

Hence, by Lemma 2.4.7 (ii), for each  $\mu \geq 0$ , it is enough to find  $\lambda \leq 0$  such that  $\|u + \lambda x_a\|_{2,\infty} = \|u + \mu x_a\|_{2,\infty}$  and calculate  $(\lambda + \mu)/2$ . For  $\lambda \leq 0$  and  $\mu \geq 0$ , we have

$$\|u + \lambda x_a\|_{2,\infty} = \begin{cases} \sqrt{2 + \lambda^2 + 2(a - b)\lambda} & (\lambda \leq -1/a), \\ 1 - \lambda b & (-1/a \leq \lambda \leq 0), \end{cases}$$

$$\|u + \mu x_a\|_{2,\infty} = \begin{cases} 1 + \mu a & (0 \leq \mu \leq 1/b), \\ \sqrt{2 + \mu^2 + 2(a - b)\mu} & (1/b \leq \mu). \end{cases}$$

For any  $1/\sqrt{2} < a < 1$ ,

$$\left\| u - \frac{1}{a}x_a \right\|_{2,\infty} = 1 + \frac{b}{a} < 1 + \frac{a}{b} = \left\| u + \frac{1}{b}x_a \right\|_{2,\infty}$$

holds and so we obtain  $0 \leq b/a^2 \leq 1/b$  and

$$\left\| u + \frac{b}{a^2}x_a \right\|_{2,\infty} = 1 + \frac{b}{a} = \left\| u - \frac{1}{a}x_a \right\|_{2,\infty}.$$

Thus we consider the following three cases.

Case 1.  $0 \leq \mu \leq b/a^2$ . Put  $\lambda = -a\mu/b$ . Then we have  $-1/a \leq \lambda \leq 0$ ,

$$\|u + \lambda x_a\|_{2,\infty} = 1 - \left(-\frac{a}{b}\mu\right)b = 1 + \mu a = \|u + \mu x_a\|_{2,\infty}$$

and

$$\frac{\lambda + \mu}{2} = -\frac{a-b}{2b}\mu.$$

Case 2.  $b/a^2 \leq \mu \leq 1/b$ . Put  $\lambda = -\sqrt{a^2\mu^2 + 2a\mu - 2ab} - (a-b)$ . Then we have  $\lambda \leq -1/a$ ,

$$\|u + \lambda x_a\|_{2,\infty} = \sqrt{2 + \lambda^2 + 2(a-b)\lambda} = 1 + \mu a = \|u + \mu x_a\|_{2,\infty}$$

and

$$\frac{\lambda + \mu}{2} = \frac{\mu - \sqrt{a^2\mu^2 + 2a\mu - 2ab} - (a-b)}{2}.$$

Case 3.  $1/b \leq \mu$ . Put  $\lambda = -2(a-b) - \mu$ . Then we have  $\lambda \leq -1/a$ ,

$$\|u + \lambda x_a\|_{2,\infty} = \sqrt{2 + \lambda^2 + 2(a-b)\lambda} = \sqrt{2 + \mu^2 + 2(a-b)\mu} = \|u + \mu x_a\|_{2,\infty}$$

and  $(\lambda + \mu)/2 = -(a-b)$ .

We define a function  $h$  on  $\mathbb{R}_+$  by

$$h(\mu) = \mu - \sqrt{a^2\mu^2 + 2a\mu - 2ab}.$$

Then it is clear that  $h(\mu)$  is decreasing in the interval  $[b/a^2, 1/b]$ . Hence we have  $\Gamma(u, x_a) = [-(a-b), 0]$ .  $\square$

**Theorem 2.4.10.**

$$DW(\ell_2\text{-}\ell_\infty) = 2\sqrt{2}.$$

**Proof.** From Lemma 2.4.6, we have

$$\begin{aligned} & DW(\ell_2\text{-}\ell_\infty) \\ &= \max \left\{ \sup \left\{ m(x_a, y_a) \mid \frac{1}{\sqrt{2}} < a < 1 \right\}, \sup \left\{ m(u, x_a) \mid \frac{1}{\sqrt{2}} < a < 1 \right\} \right\}. \end{aligned}$$

If  $a \leq 2b$ , then  $\Gamma(x_a, y_a) = [0, b/a]$  by Lemma 2.4.8. Hence, by Proposition 2.2.8, we have

$$\begin{aligned} m(x_a, y_a) &= \max \left\{ \|x_a\|_{2,\infty}, \left\| x_a + \frac{b}{a}y_a \right\|_{2,\infty} \right\} \\ &= \left\| x_a + \frac{b}{a}y_a \right\|_{2,\infty} \\ &= \left\| \left( a + \frac{b^2}{a}, 0 \right) \right\|_{2,\infty} = \frac{1}{a}. \end{aligned}$$

If  $a > 2b$ , then

$$\Gamma(x_a, y_a) = \left[ 0, \frac{a+b-\sqrt{2ab}}{a-b} \right]$$

by Lemma 2.4.8. From the assumption  $a > 2b$ , we have  $\sqrt{2ab} > 2b$  which implies that

$$\frac{a+b-\sqrt{2ab}}{a-b} < \frac{a+b-2b}{a-b} = 1.$$

On the other hand, from Lemma 2.2.4, we have that the function  $t \mapsto \|x_a + ty_a\|_{2,\infty}$  is strictly increasing in  $[0, \infty)$ . Thus, by Proposition 2.2.8, we obtain

$$\begin{aligned} m(x_a, y_a) &< \|x_a + y_a\|_{2,\infty} \\ &= \|(a+b, b-a)\|_{2,\infty} \\ &= \|(a+b, -(a-b))\|_\infty \\ &= a+b \\ &< \sqrt{2}\sqrt{a^2+b^2} = \sqrt{2}. \end{aligned}$$

By Lemma 2.4.9, we have  $\Gamma(u, x_a) = [-(a-b), 0]$ . Hence, by Proposition 2.2.8, we have

$$\begin{aligned} m(u, x_a) &= \max\{\|u - (a-b)x_a\|_{2,\infty}, \|u\|_{2,\infty}\} \\ &= \|u - (a-b)x_a\|_{2,\infty} \\ &= \|(b(a+b), -a(a+b))\|_\infty \\ &= a(a+b) \\ &< \sqrt{2}\sqrt{a^2+b^2} = \sqrt{2}. \end{aligned}$$

Therefore, from the fact that  $\sup\{1/a \mid 1/\sqrt{2} < a < 1, a \geq 2b\} = \sqrt{2}$ , we obtain  $DW(\ell_2\text{-}\ell_\infty) = 2\sqrt{2}$ .  $\square$

## 3 On some norm inequalities in Banach spaces

### 3.1 Introduction

In this chapter, we shall describe some remarks on Clarkson's inequalities and triangle inequality. We consider the generalized Clarkson's inequalities in the real and complex cases. In 1935, Jordan and von Neumann [26] characterized Hilbert spaces as Banach spaces satisfying the parallelogram law. In the next year 1936, as a generalization of the parallelogram law, Clarkson [12] proved famous norm inequalities for  $L_p$  so called Clarkson's inequalities which can be written in one formula:

if  $1 \leq p \leq \infty$ ,  $1/p + 1/p' = 1$  and  $r = \max\{p, p'\}$ , then

$$(\|f + g\|_p^r + \|f - g\|_p^r)^{\frac{1}{r}} \leq 2^{\frac{1}{r}} (\|f\|_p^{r'} + \|g\|_p^{r'})^{\frac{1}{r'}}$$

for all  $f, g \in L_p$ . The key to prove Clarkson's inequality is the following inequality: if  $1 \leq p \leq 2$  and  $1/p + 1/p' = 1$ , then

$$(|z + w|^{p'} + |z - w|^{p'})^{\frac{1}{p'}} \leq 2^{\frac{1}{p'}} (|z|^p + |w|^p)^{\frac{1}{p}} \quad (1)$$

for all  $z, w \in \mathbb{C}$ . This inequality is called the classical Clarkson's inequality in the complex case. To study the generalizations of Clarkson's inequalities, for two numbers  $p, q$  with  $0 < p, q \leq \infty$ , the generalized Clarkson's inequality in the complex case have been considered:

$$(|z + w|^q + |z - w|^q)^{\frac{1}{q}} \leq C(|z|^p + |w|^p)^{\frac{1}{p}}$$

for all  $z, w \in \mathbb{C}$ . The smallest real number for which this inequality holds is called the best constant in the generalized Clarkson's inequality in the complex case, and denoted by  $C_{p,q}(\mathbb{C})$ . Clarkson [12] proved that  $C_{p,p'}(\mathbb{C}) = 2^{1/p'}$  for  $1 \leq p \leq 2$ . Later on the best constants  $C_{p,q}(\mathbb{C})$  for the remaining pairs of  $p$  and  $q$  such that  $0 < p, q \leq \infty$  were found by Koskela [40], Maligranda and Persson [44]. In 1997, Kuriyama et al. [35] obtained an elementary proof of the generalized Clarkson's inequality in the complex case.

Moreover we can consider the generalized Clarkson's inequality in the real case:

$$(|a + b|^q + |a - b|^q)^{\frac{1}{q}} \leq C(|a|^p + |b|^p)^{\frac{1}{p}}$$

for all  $a, b \in \mathbb{R}$ . As in the complex case, the smallest real number for which this inequality holds is called the best constant in the generalized Clarkson's inequality

in the real case, and denoted by  $C_{p,q}(\mathbb{R})$ . In 2007, Maligranda and Sabourova [45] computed the best constant  $C_{p,q}(\mathbb{R})$  for all  $0 < p, q \leq \infty$ .

In Section 3.2, we look at the generalized Clarkson's inequality in the complex case. In Section 3.3, we consider the generalized Clarkson's inequality in the real case and present an elementary proof.

We also pay attention to the triangle inequality. The triangle inequality is one of the most fundamental inequalities in analysis. Several authors have been treating its generalizations, improvements and reverse inequalities ([16, 31, 42, 43, 49] and so on). We consider another aspect of the triangle inequality. For a Hilbert space  $H$ , the parallelogram law implies that the parallelogram inequality

$$\|x + y\|^2 \leq 2(\|x\|^2 + \|y\|^2)$$

holds for all  $x, y \in H$ . Saitoh in [62] noted that the above inequality may be more suitable than the classical triangle inequality and used the inequality to the setting of natural sum Hilbert space for two arbitrary Hilbert spaces. In general, for any normed linear space  $X$ , we easily have

$$\|x + y\|^2 \leq 2(\|x\|^2 + \|y\|^2)$$

for all  $x, y \in X$ . Belbachir et al. introduced the notion of  $q$ -norm ( $1 \leq q < \infty$ ) in a vector space  $X$  over  $\mathbb{K}(= \mathbb{R} \text{ or } \mathbb{C})$ .

In Section 3.4, we shall generalize the notion of  $q$ -norm, that is, we introduce the notion of  $\psi$ -norm by considering the fact that an absolute normalized norm on  $\mathbb{R}^2$  corresponds to a function  $\psi$  on the interval  $[0, 1]$  with some conditions. We show that a  $\psi$ -norm is a norm in the usual sense.

## 3.2 The generalized Clarkson's inequality in the complex case

In this section, we consider the generalized Clarkson's inequality in the complex case:

$$(|z + w|^q + |z - w|^q)^{\frac{1}{q}} \leq C(|z|^p + |w|^p)^{\frac{1}{p}} \quad (2)$$

for all  $z, w \in \mathbb{C}$ . The smallest real number for which the inequality (2) holds is called the best constant in the generalized Clarkson's inequality in the complex

case, and denoted by  $C_{p,q}(\mathbb{C})$ . We note that the best constant  $C_{p,q}(\mathbb{C})$  in (2) is also the norm of an operator given by the rule  $T(a, b) = (a + b, a - b)$  and acting between two-dimensional complex Banach spaces  $(\mathbb{C}^2, \|\cdot\|_p)$  and  $(\mathbb{C}^2, \|\cdot\|_q)$ , that is,  $\|T\|_{p,q} = C_{p,q}(\mathbb{C})$ .

We first consider the proof of the classical Clarkson's inequality in complex case (1). In order to prove (1), Clarkson [12] used the generalized binomial theorem. After his paper, several different proof of the inequality appeared in literature ([17, 35, 44] and so on).

**Lemma 3.2.1** ([12]). *Let  $1 \leq p \leq 2$  and  $1/p + 1/p' = 1$ . Then*

$$(|z + w|^{p'} + |z - w|^{p'})^{\frac{1}{p'}} \leq 2^{\frac{1}{p'}} (|z|^p + |w|^p)^{\frac{1}{p}}$$

*holds for all  $z, w \in \mathbb{C}$ .*

*Proof.* We can calculate the norm of operator  $T$  in two easy cases:

$T : (\mathbb{C}^2, \|\cdot\|_1) \mapsto (\mathbb{C}^2, \|\cdot\|_\infty)$  has the norm

$$\|T\|_{1,\infty} = \sup\{\|(z + w, z - w)\|_\infty \mid \|(z, w)\|_1 = 1\} = 1$$

and  $T : (\mathbb{C}^2, \|\cdot\|_2) \mapsto (\mathbb{C}^2, \|\cdot\|_2)$  has the norm

$$\|T\|_{2,2} = \sup\{\|(z + w, z - w)\|_2 \mid \|(z, w)\|_2 = 1\} = \sqrt{2}.$$

Putting  $\theta = 2/p'$ , we have

$$\frac{1}{p} = \frac{1 - \theta}{1} + \frac{\theta}{2} \quad \text{and} \quad \frac{1}{p'} = \frac{1 - \theta}{\infty} + \frac{\theta}{2}.$$

Therefore, by the complex Riesz-Thorin interpolation theorem,  $T : (\mathbb{C}^2, \|\cdot\|_p) \mapsto (\mathbb{C}^2, \|\cdot\|_{p'})$  ( $1 < p \leq 2$ ) is bounded with the norm

$$\|T\|_{p,p'} \leq \|T\|_{1,\infty}^{1-\theta} \|T\|_{2,2}^\theta = 2^{1/p'}.$$

□

Now we obtain the best constant  $C_{p,q}(\mathbb{C})$  for all  $0 < p, q \leq \infty$ .

**Theorem 3.2.2** ([40, 44]). *Let  $0 < p, q \leq \infty$ . Then the best constant  $C_{p,q}(\mathbb{C})$  in inequality (2) is:*

I *If  $0 < q \leq 2 \leq p \leq \infty$ , then  $C_{p,q}(\mathbb{C}) = 2^{1/q-1/p+1/2}$ .*

II If  $2 \leq q < \infty$  and  $1/p + 1/q \leq 1$ , then  $C_{p,q}(\mathbb{C}) = 2^{1-1/p}$ .

III If either  $0 < p, q \leq 2$  or  $2 \leq q < \infty$ ,  $1/p + 1/q \geq 1$ , then  $C_{p,q}(\mathbb{C}) = 2^{1/q}$ .

**Proof.** We note that if  $0 < s \leq t$ , then

$$(a^t + b^t)^{\frac{1}{t}} \leq (a^s + b^s)^{\frac{1}{s}} \leq 2^{1/s-1/t}(a^t + b^t)^{\frac{1}{t}}$$

holds for all nonnegative real numbers  $a, b$ .

We shall consider the following four cases for  $p$  and  $q$ :

I. If  $0 < q \leq 2 \leq p \leq \infty$ , then by the parallelogram law, we have

$$\begin{aligned} (|z + w|^q + |z - w|^q)^{\frac{1}{q}} &\leq 2^{1/q-1/2}(|z + w|^2 + |z - w|^2)^{\frac{1}{2}} \\ &= 2^{1/q}(|z|^2 + |w|^2)^{\frac{1}{2}} \leq 2^{1/q-1/p+1/2}(|z|^p + |w|^p)^{\frac{1}{p}} \end{aligned}$$

for all  $z, w \in \mathbb{C}$ .

The equality holds for  $(z, w) = (1, i)$ . Indeed, we have

$$(|1 + i|^q + |1 - i|^q)^{\frac{1}{q}} = 2^{1/q+1/2}, \quad (|1|^p + |i|^p)^{\frac{1}{p}} = 2^{1/p}$$

and hence

$$(|1 + i|^q + |1 - i|^q)^{\frac{1}{q}} = 2^{1/q-1/p+1/2}(|1|^p + |i|^p)^{\frac{1}{p}}.$$

II. If  $2 \leq q < \infty$  and  $1/p + 1/q \leq 1$ , then by Lemma 3.2.1, we have

$$\begin{aligned} (|z + w|^q + |z - w|^q)^{\frac{1}{q}} &\leq 2^{1/q}(|z|^{q'} + |w|^{q'})^{\frac{1}{q'}} \\ &\leq 2^{1/q+1/q'-1/p}(|z|^p + |w|^p)^{\frac{1}{p}} = 2^{1-1/p}(|z|^p + |w|^p)^{\frac{1}{p}} \end{aligned}$$

for all  $z, w \in \mathbb{C}$ .

The equality holds for  $(z, w) = (1, 1)$ .

IIIa. If  $0 < p, q \leq 2$ , then by the parallelogram law, we have

$$\begin{aligned} (|z + w|^q + |z - w|^q)^{\frac{1}{q}} &\leq 2^{1/q-1/2}(|z + w|^2 + |z - w|^2)^{\frac{1}{2}} \\ &= 2^{1/q}(|z|^2 + |w|^2)^{\frac{1}{2}} \leq 2^{1/q}(|z|^p + |w|^p)^{\frac{1}{p}} \end{aligned}$$

for all  $z, w \in \mathbb{C}$ .

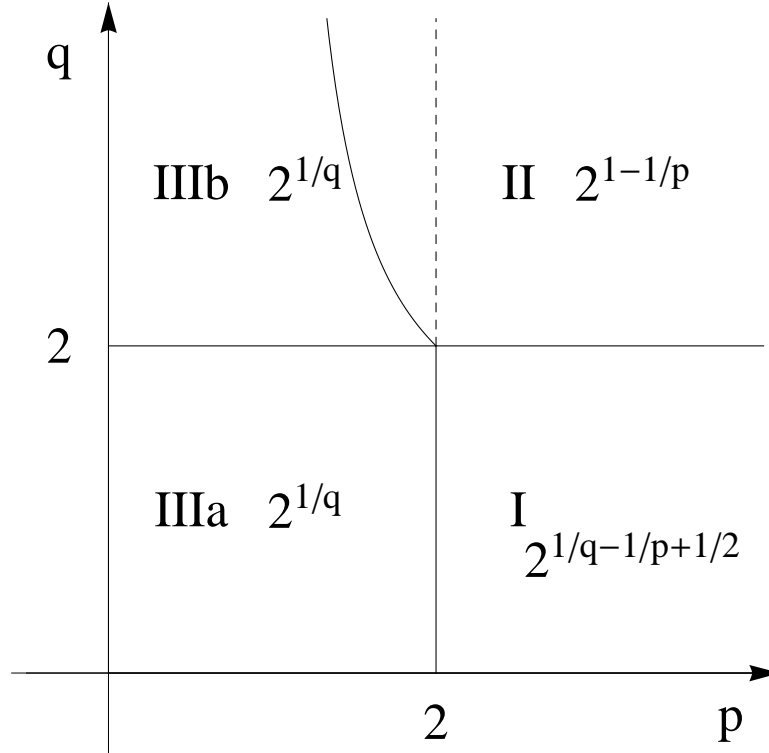


IIIb. If  $2 \leq q < \infty$  and  $1/p + 1/q \geq 1$ , then by Lemma 3.2.1, we have

$$(|z + w|^q + |z - w|^q)^{\frac{1}{q}} \leq 2^{1/q}(|z|^{q'} + |w|^{q'})^{\frac{1}{q'}} \leq 2^{1/q}(|z|^p + |w|^p)^{\frac{1}{p}}$$

for all  $z, w \in \mathbb{C}$ .

The equalities hold in III a,b for  $(z, w) = (1, 0)$ . □



**Figure 1.** The best constant  $C_{p,q}(\mathbb{C})$  in the generalized Clarkson's inequality in the complex case

In [35], Kuriyama et al. gave the elementary proof of Lemma 3.2.1 and Theorem 3.2.2.

### 3.3 The generalized Clarkson's inequality in the real case

In this section, we consider the generalized Clarkson's inequality in the real case:

$$(|a + b|^q + |a - b|^q)^{\frac{1}{q}} \leq C(|a|^p + |b|^p)^{\frac{1}{p}} \quad (3)$$

for all  $a, b \in \mathbb{C}$ . As in the complex case, the smallest real number for which the inequality (3) holds is called the best constant in the generalized Clarkson's inequality in the real case, and denoted by  $C_{p,q}(\mathbb{R})$ . In [45], Maligranda and Sabourova computed the best constant  $C_{p,q}(\mathbb{R})$  for all  $0 < p, q \leq \infty$ . By [45, Theorem 2.1], we have

**Theorem 3.3.1** ([45]). *Let  $0 < p, q \leq \infty$ . Then the best constant  $C_{p,q}(\mathbb{R})$  in inequality (3) is:*

- Ia. *If  $2 < p \leq \infty$  and  $0 < q \leq 1$ , then  $C_{p,q}(\mathbb{R}) = 2^{1/q}$ .*
- Ib. *If  $1 < q < 2 < p < \infty$ , then  $\max\{2^{1-1/p}, 2^{1/q}\} < C_{p,q}(\mathbb{R}) < 2^{1/q-1/p+1/2}$ .*
- II. *If  $2 \leq q \leq \infty$  and  $1/p + 1/q \leq 1$ , then  $C_{p,q}(\mathbb{R}) = 2^{1-1/p}$ .*
- III. *If either  $0 < p, q \leq 2$  or  $2 \leq q \leq \infty, 1/p + 1/q \geq 1$ , then  $C_{p,q}(\mathbb{R}) = 2^{1/q}$ .*

In [35, Theorem 2.5], Kuriyama et al. gave the elementary proof of II and III. Our aim in this section is to present an elementary proof of Theorem 3.3.1 Ib. In the case of  $p = \infty$ , we have

$$C_{p,q}(\mathbb{R}) = \sup_{a,b \in \mathbb{R}, \max\{|a|, |b|\}=1} (|a + b|^q + |a - b|^q)^{\frac{1}{q}} = \max\{2, 2^{\frac{1}{q}}\}.$$

Suppose that  $0 < q < 2 < p < \infty$ , then

$$C_{p,q}(\mathbb{R}) = \sup_{a,b \in \mathbb{R}, |a|^p + |b|^p \neq 0} \frac{(|a + b|^q + |a - b|^q)^{\frac{1}{q}}}{(|a|^p + |b|^p)^{\frac{1}{p}}} = \sup_{t \in [0,1]} \frac{((1+t)^q + (1-t)^q)^{\frac{1}{q}}}{(1+t^p)^{\frac{1}{p}}}.$$

We define a function  $f$  from  $[0, 1]$  into  $\mathbb{R}$  by

$$f(t) = \frac{((1+t)^q + (1-t)^q)^{\frac{1}{q}}}{(1+t^p)^{\frac{1}{p}}}$$

for  $t$  with  $0 \leq t \leq 1$ . Then the derivative of  $f$  is

$$\begin{aligned} f'(t) &= \frac{((1+t)^q + (1-t)^q)^{\frac{1}{q}-1}}{(1+t^p)^{1+\frac{1}{p}}} \\ &\quad \times \left\{ ((1+t)^{q-1} - (1-t)^{q-1})(1+t^p) - x^{p-1}((1+t)^q + (1-t)^q) \right\}. \end{aligned}$$

If  $0 < q \leq 1$ , then  $f'(t) < 0$  and so  $f$  is decreasing in  $[0, 1]$ . Therefore

$$C_{p,q}(\mathbb{R}) = f(0) = 2^{1/q}.$$

In the case of  $1 < q < 2 < p < \infty$ , we prove that there exists a unique  $t_0 \in (0, 1)$  at which  $f$  has its maximum.

**Lemma 3.3.2.** *If  $1 < q < 2 < p < \infty$ , then there exists a unique  $t_0 \in (0, 1)$  such that  $C_{p,q}(\mathbb{R}) = f(t_0)$ . Moreover,  $\max\{2^{1-1/p}, 2^{1/q}\} < C_{p,q}(\mathbb{R}) < 2^{1/q-1/p+1/2}$ .*

**Proof.** It is clear that the derivative of  $f$  is

$$f'(t) = \frac{((1+t)^q + (1-t)^q)^{\frac{1}{q}}}{(1+t^p)^{\frac{1}{p}}} \{(1+t)^{q-1}(1-t^{p-1}) - (1-t)^{q-1}(1+t^{p-1})\}.$$

For simplicity, we put  $\alpha = p - 1$  and  $\beta = q - 1$ , respectively. We define a function  $f_1$  from  $[0, 1]$  into  $\mathbb{R}$  by

$$f_1(x) = (1+t)^\beta(1-t^\alpha) - (1-t)^\beta(1+t^\alpha)$$

for  $t$  with  $0 \leq t \leq 1$ . We also define

$$f_2(t) = \log\{(1+t)^\beta(1-t^\alpha)\} - \log\{(1-t)^\beta(1+t^\alpha)\}$$

for  $t$  with  $0 \leq t \leq 1$ . Note that for any  $t$ ,  $f_2(t) \geq 0$  if and only if  $f'(t) \geq 0$ . Since

$$f_2(t) = \beta \log(1+t) + \log(1-t^\alpha) - \beta \log(1-t) - \log(1+t^\alpha),$$

we have  $f_2(0) = 0$  and  $\lim_{t \rightarrow 1-0} f_2(t) = -\infty$ .

Since the derivative of  $f_2$  is

$$f_2'(t) = \frac{2(\beta - \beta t^{2\alpha} - \alpha t^{2\alpha-1} + \alpha t^{2\alpha+1})}{(1+t)(1-t)(1-t^\alpha)(1+t^\alpha)},$$

we put

$$f_3(t) = \beta - \beta t^{2\alpha} - \alpha t^{2\alpha-1} + \alpha t^{2\alpha+1}.$$

Then the derivative of  $f_3$  is

$$f_3'(t) = \alpha t^{\alpha-2} \{(2\alpha + 1)t^2 - 2\beta t - (2\alpha - 1)\}.$$

Table 1

$t$	0		$t_2$		$t_1$		1
$f_3'(t)$		-		-	0	+	
$f_3(t)$	$\beta$	+	0	-		-	0
$f_2(t)$	0	$\nearrow$	max	$\searrow$		$\searrow$	$-\infty$

We put

$$f_4(t) = (2\alpha + 1)t^2 - 2\beta t - (2\alpha - 1).$$

Since  $f_4(0) = -2\alpha + 1 < 0$  and  $f_4(1) = 2(1 - \beta) > 0$ , there exists a unique  $t_1 \in (0, 1)$  such that  $f_4(t_1) = f_4'(t_1) = 0$ . Since  $f_3(0) = \beta$  and  $f_3(1) = 0$ , the function  $f_3$  has a minimum at  $t_1$  and we have  $f_3'(t) < 0$  on  $(0, t_1)$ ,  $f_3'(t) > 0$  on  $(t_1, 1)$ . Since  $f_3(t_1) < 0$ , there exists a unique  $t_2 \in (0, t_1)$  such that  $f_3(t_2) = 0$ . Since  $f_2'(t_2) = f_3(t_2) = 0$ , by Table 1,  $f_2$  has a unique maximum at  $t_2$ . Since  $f_2(0) = 0$  and  $\lim_{t \rightarrow 1-0} f_2(t) = -\infty$ , there exists  $t_0 \in (t_2, 1)$  such that  $f_2(t_0) = 0$ . From the fact that  $f_1$  and  $f_2$  have the same signature on  $[0, 1)$ , we have  $f_1(t_0) = 0$ ,  $f_1(t) > 0$  on  $(0, t_0)$  and  $f_1(t) < 0$  on  $(t_0, 1)$ .

Table 2

$t$	0		$t_0$		1
$f_2(t)$	0	+	0	-	$-\infty$
$f_1(t)$	0	+	0	-	0
$f'(t)$	0	+	0	-	0
$f(t)$	$2^{1/q}$	$\nearrow$	$f(t_0)$	$\searrow$	$2^{1-1/p}$

Thus  $f$  is increasing on  $[0, t_0]$  and decreasing on  $[t_0, 1]$ . This implies that  $f$  has the unique maximum at  $t_0$ . Therefore we obtain

$$C_{p,q}(\mathbb{R}) = \sup_{a,b \in \mathbb{R}, |a|^p + |b|^p \neq 0} \frac{(|a+b|^q + |a-b|^q)^{\frac{1}{q}}}{(|a|^p + |b|^p)^{\frac{1}{p}}} = \max_{0 \leq t \leq 1} f(t) = f(t_0)$$

and

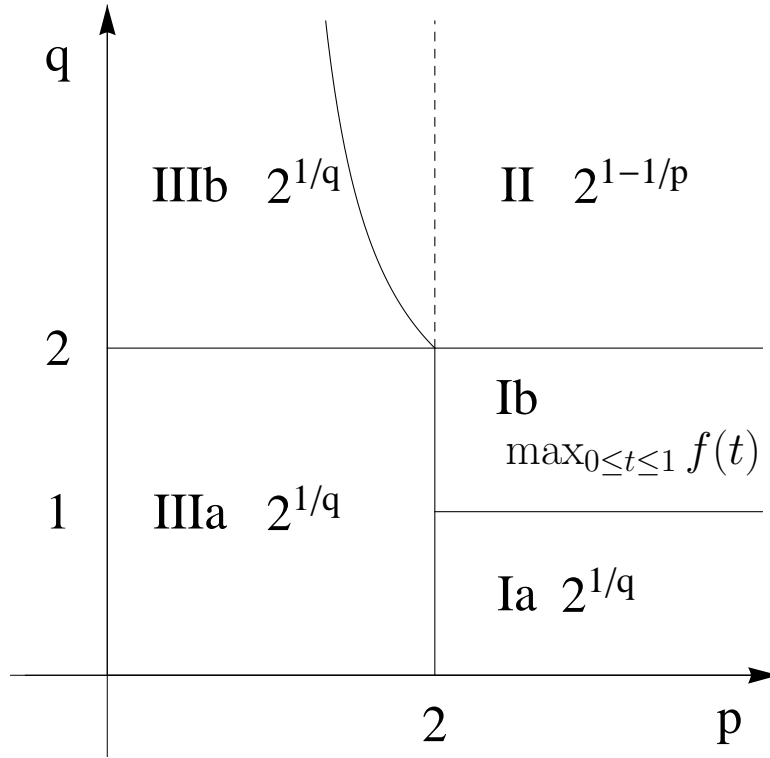
$$\max\{2^{1/q}, 2^{1-1/p}\} = \max\{f(0), f(1)\} < f(t_0) = C_{p,q}(\mathbb{R}).$$

Since  $1 < q < 2 < p < \infty$ , we have for  $t \in (0, 1)$ , by the Hölder inequality,

$$\begin{aligned} \{(1+t)^q + (1-t)^q\}^{\frac{1}{q}} &< 2^{1/q-1/2} \{(1+t)^2 + (1-t)^2\}^{\frac{1}{2}} \\ &= 2^{1/q} (1+t^2)^{\frac{1}{2}} \\ &< 2^{1/q-1/p+1/2} (1+t^p)^{\frac{1}{p}}. \end{aligned}$$

Thus, we have  $C_{p,q}(\mathbb{R}) = f(t_0) < 2^{1/q-1/p+1/2}$ .

□



**Figure 2.** The best constant  $C_{p,q}(\mathbb{R})$  in the generalized Clarkson's inequality in the real case

In the case of  $0 < q < 2 < p < \infty$ , we have the difference of the best constant in the generalized Clarkson's inequality between complex case and real case.

### 3.4 Another aspect of triangle inequality

In this section, we consider another aspect of the triangle inequality. For any normed linear space  $X$ , we easily have

$$\|x + y\|^2 \leq 2(\|x\|^2 + \|y\|^2)$$

for all  $x, y \in X$ . Belbachir et al. [9] introduced the notion of  $q$ -norm ( $1 \leq q < \infty$ ) in a vector space  $X$  over  $\mathbb{K}(= \mathbb{R} \text{ or } \mathbb{C})$ .

**Definition 3.4.1** ([9]). *Let  $X$  be a vector space and  $1 \leq q < \infty$ . Then the mapping  $\|\cdot\|$  from  $X$  into  $\mathbb{R}_+(= \{a \in \mathbb{R} \mid a \geq 0\})$  is called  $q$ -norm on  $X$  if it satisfies the following conditions:*

- (1)  $\|x\| = 0$  if and only if  $x = 0$ ,
- (2)  $\|\alpha x\| = |\alpha|\|x\|$  for any  $x \in X$  and any  $\alpha \in \mathbb{K}$ ,
- (3)  $\|x + y\|^q \leq 2^{q-1}(\|x\|^q + \|y\|^q)$  for any  $x, y \in X$ .

It is easy to show that every norm is a  $q$ -norm. Conversely, they proved that for all  $q$  with  $1 \leq q < \infty$  every  $q$ -norm is a norm in the usual sense. In this section, we generalize the notion of  $q$ -norm, that is, we introduce the notion of  $\psi$ -norm by considering the fact that an absolute normalized norm on  $\mathbb{R}^2$  corresponds to a continuous convex function  $\psi$  on the interval  $[0, 1]$  with some conditions (cf. [11, 58]). We show that a  $\psi$ -norm is a norm in the usual sense.

We recall some properties of absolute normalized norms on  $\mathbb{R}^2$ . A norm  $\|\cdot\|$  on  $\mathbb{R}^2$  is said to be absolute if  $\|(x, y)\| = \||x|, |y|\|$  for any  $(x, y) \in \mathbb{R}^2$ , and normalized if  $\|(1, 0)\| = \|(0, 1)\| = 1$ . The  $\ell_p$ -norms  $\|\cdot\|_p$  are such examples:

$$\|(x, y)\|_p = \begin{cases} (|x|^p + |y|^p)^{\frac{1}{p}} & (1 \leq p < \infty), \\ \max\{|x|, |y|\} & (p = \infty). \end{cases}$$

Let  $AN_2$  denote the family of all absolute normalized norms on  $\mathbb{R}^2$ . It is well known that  $AN_2$  is in a one-to-one correspondence with the set  $\Psi_2$  of all continuous convex functions  $\psi$  on  $[0, 1]$  satisfying  $\max\{1 - t, t\} \leq \psi(t) \leq 1$  for all  $0 \leq t \leq 1$  (see [11, 58, 59]). The correspondence is given by the equation  $\psi(t) = \|(1 - t, t)\|_\psi$ . Indeed, for all  $\psi \in \Psi_2$ , we define the norm  $\|\cdot\|_\psi$  as

$$\|(x, y)\|_\psi = \begin{cases} (|x| + |y|)\psi\left(\frac{|y|}{|x| + |y|}\right) & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0). \end{cases}$$

Then  $\|\cdot\|_\psi \in AN_2$  and satisfies  $\psi(t) = \|(1 - t, t)\|_\psi$ . The functions which correspond to the  $\ell_p$ -norms  $\|\cdot\|_p$  on  $\mathbb{R}^2$  are  $\psi_p(t) = \{(1 - t)^p + t^p\}^{1/p}$  if  $1 \leq p < \infty$  and  $\psi_\infty(t) = \max\{1 - t, t\}$  if  $p = \infty$ .

**Definition 3.4.2.** *Let  $X$  be a vector space and  $\psi \in \Psi_2$ . Then the mapping  $\|\cdot\|$  from  $X$  into  $\mathbb{R}_+$  is called  $\psi$ -norm on  $X$  if it satisfies the following conditions:*

- (1)  $\|x\| = 0$  if and only if  $x = 0$ ,
- (2)  $\|\alpha x\| = |\alpha|\|x\|$  for any  $x \in X$  and any  $\alpha \in \mathbb{K}$ ,

$$(3) \quad \|x + y\| \leq \frac{1}{\min_{0 \leq t \leq 1} \psi(t)} \|(\|x\|, \|y\|)\|_\psi \text{ for any } x, y \in X.$$

Note that for all  $q$  with  $1 \leq q < \infty$ ,  $\psi_q$ -norm  $\|\cdot\|$  is a just  $q$ -norm. Indeed, since the function  $\psi_q$  takes the minimum at  $t = 1/2$  and

$$\psi_q\left(\frac{1}{2}\right) = \left\{ \left(\frac{1}{2}\right)^q + \left(\frac{1}{2}\right)^q \right\}^{\frac{1}{q}} = 2^{1/q-1},$$

the condition (3) of Definition 3.4.2 implies

$$\|x + y\| \leq \frac{1}{\psi_q(1/2)} \|(\|x\|, \|y\|)\|_{\psi_q} = 2^{1-1/q} (\|x\|^q + \|y\|^q)^{1/q}.$$

Thus we have  $\|x + y\|^q \leq 2^{q-1} (\|x\|^q + \|y\|^q)$  and hence  $\|\cdot\|$  becomes  $q$ -norm.

If  $\psi = \psi_1$ , then the condition (3) of Definition 3.4.2 is just a triangle inequality. Thus we suppose that  $\psi \neq \psi_1$ .

**Proposition 3.4.3.** *Let  $X$  be a vector space and  $\psi \in \Psi_2$  with  $\psi \neq \psi_1$ . Then every norm on  $X$  in the usual sense is a  $\psi$ -norm.*

*Proof.* Let  $\|\cdot\|$  be a norm on  $X$  and  $x, y \in X$ . Since  $\psi \leq \psi_1$ , by [58, Lemma 3], we have

$$\begin{aligned} \|x + y\| &\leq \|x\| + \|y\| = \|(\|x\|, \|y\|)\|_1 \\ &\leq \max_{0 \leq t \leq 1} \frac{1}{\psi(t)} \|(\|x\|, \|y\|)\|_\psi \\ &= \frac{1}{\min_{0 \leq t \leq 1} \psi(t)} \|(\|x\|, \|y\|)\|_\psi. \end{aligned}$$

Thus  $\|\cdot\|$  is a  $\psi$ -norm on  $X$ . □

We show that every  $\psi$ -norm is a norm of the usual sense. To do this, we need the following lemma given in [9]

**Lemma 3.4.4** ([9]). *Let  $X$  be a vector space. Let  $\|\cdot\|$  from  $X$  into  $\mathbb{R}_+$  be satisfying the condition (1) and (2) in Definition 3.4.2. Then  $\|\cdot\|$  is a norm if and only if the set  $B_X = \{x \in X \mid \|x\| \leq 1\}$  is convex.*

*Proof.* Suppose that  $B_X$  is convex. Then for every  $x, y \in X$  with  $x, y \neq 0$ , we have

$$\left\| \frac{x}{\|x\| + \|y\|} + \frac{y}{\|x\| + \|y\|} \right\| = \left\| \frac{x}{\|x\|} \frac{\|x\|}{\|x\| + \|y\|} + \frac{y}{\|y\|} \frac{\|y\|}{\|x\| + \|y\|} \right\| \leq 1. \quad \square$$

Put  $t_0$  with  $0 < t_0 < 1$  such that  $\min_{0 \leq t \leq 1} \psi(t) = \psi(t_0)$ . Then we have the following lemma.

**Lemma 3.4.5.** *Let  $\|\cdot\|$  be a  $\psi$ -norm on  $X$ . Then for every  $x, y \in B_X$  we have  $(1 - t_0)x + t_0y \in B_X$ .*

*Proof.* Let  $x, y \in B_X$ . We may assume that  $x \neq y$  and  $x, y \neq 0$ . From the definition of a  $\psi$ -norm and [67, Lemma 1], we have

$$\begin{aligned} \|(1 - t_0)x + t_0y\| &\leq \frac{1}{\psi(t_0)} \|((1 - t_0)\|x\|, t_0\|y\|)\|_\psi \\ &\leq \frac{1}{\psi(t_0)} \|(1 - t_0, t_0)\|_\psi = 1, \end{aligned}$$

which implies  $(1 - t_0)x + t_0y \in B_X$ . □

Here we define the set  $A_n$  for all  $n = 0, 1, \dots$ , by

$$A_0 = \{0, 1\}, \quad A_n = \{(1 - t_0)a + t_0b \mid a, b \in A_{n-1}\} \quad (n = 1, 2, \dots).$$

Put  $A = \bigcup_{n=0}^{\infty} A_n$ . It is clear that  $\bar{A} = [0, 1]$ . We also define a function  $f$  by  $f(x, y, t) = (1 - t)x + ty$  for all  $x, y \in B_X$  and all  $t \in [0, 1]$ .

**Lemma 3.4.6.** *For every  $x, y \in B_X$ , we have  $f(x, y, t) \in B_X$  for all  $t \in A$ .*

*Proof.* Let  $x, y \in B_X$ . It is clear that  $f(x, y, t) \in B_X$  for all  $t \in A_0$ . We suppose that  $f(x, y, t) \in B_X$  for all  $t \in A_{n-1}$ . Then for all  $t \in A_n$ , there exist  $a, b \in A_{n-1}$  such that  $t = (1 - t_0)a + t_0b$ . Hence

$$\begin{aligned} f(x, y, t) &= (1 - t)x + ty \\ &= \{1 - ((1 - t_0)a + t_0b)\}x + ((1 - t_0)a + t_0b)y \\ &= (1 - t_0)((1 - a)x + ay) + t_0((1 - b)t + by) \\ &= (1 - t_0)f(x, y, a) + t_0f(x, y, b). \end{aligned}$$

Since  $f(x, y, a), f(x, y, b) \in B_X$ , from Lemma 3.4.5,  $f(x, y, t) \in B_X$  for all  $t \in A_n$ . Thus  $f(x, y, t) \in B_X$  for all  $t \in A$  □

**Theorem 3.4.7.** *Let  $X$  be a vector space and  $\psi \in \Psi_2$  with  $\psi \neq \psi_1$ . Then every  $\psi$ -norm on  $X$  is a norm in the usual sense.*



*Proof.* Let  $x, y \in B_X$  and  $\lambda$  with  $0 < \lambda < 1$ . Let  $z = (1 - \lambda)x + \lambda y$ . Take a strictly decreasing sequence  $\{\gamma_n\}$  in  $A$  such that  $\gamma_n \searrow \lambda$ . For each  $n$ , we define

$$\beta_n = (1 - \gamma_n)/(1 - \lambda).$$

Then  $0 < \beta_n < 1$  and  $\beta_n \nearrow 1$ . Since  $0 < \lambda\beta_n/\gamma_n < 1$ , we have  $(\lambda\beta_n/\gamma_n)y \in B_X$ . By Lemma 3.4.6,

$$\begin{aligned} \beta_n z &= (1 - \lambda)\beta_n x + \lambda\beta_n y = (1 - \gamma_n)x + \gamma_n \frac{\lambda\beta}{\gamma_n} y \\ &= f\left(x, \frac{\lambda\beta}{\gamma_n} y, \gamma_n\right) \in B_X. \end{aligned}$$

Since  $\beta_n \|z\| = \|\beta_n z\| \leq 1$ , we obtain  $z \in B_X$ . Thus  $B_X$  is convex. By Lemma 3.4.4,  $\|\cdot\|$  becomes a norm in the usual sense.  $\square$

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