Study on mappings in partially ordered vector spaces

by

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Preface

In this thesis, we study a theory of mappings that take values in a partially ordered vector space. The studies of these areas are important in non-linear and linear analysis. Particularly, we study mainly two topics. One is non-additive measures that take value in a partially ordered vector space and the another is Hahn-Banach's theorem with respect to functions that take value in a partially ordered vector space. For the former, we consider the measurable space and the non-additive measures that take values in a partially ordered vector space. We consider the Egoroff's theorem and Lusin's theorem which are established for real-valued measurable functions on the measurable space. In general, one of important purpose of study in non-additive measure theory is to find the sufficient conditions, if possible, to find the necessary and sufficient conditions to ensure the establishment of the theorems in measurable spaces with non-additivity.

For the most important theorems are Egoroff's theorem and Lusin's theorem for non-additive measures. In general, both theorems need additivity for their proof for additive measures. For the Egoroff's theorem, in [32], T. Murofushi, K. Uchino and S. Asahina find the necessary and sufficient condition called Egoroff condition, which assures that the Egoroff theorem remains valid for real-valued non-additive measures; see also J. Li, M. Yasuda and et al. [22, 23, 28, 29]. For information on real-valued non-additive measures introduced in [7, 36, 40]. In the real-valued measure case, the convergence structure with respect to the order and the topology coincide. However, in the vector-valued case, the convergence structure with respect to them are different. In [16, 17], J. Kawabe studies Egoroff's theorem for Riesz space-valued non-additive measure using the convergence structure with respect to a partially order. In [16], he introduces and imposes a smoothness condition called the asymptotic Egoroff property on Riesz space and he shows that the Egoroff theorem remains valid if the measure is continuous from above and below.

In [17], he also shows that the Egoroff theorem remains valid for any Riesz space-valued non-additive measure in the three cases. The first case is that the measure is strongly order totally continuous, the second case is that the measure is strongly order continuous together with some property by assuming that the Riesz space has the Egoroff property, the third case is that the measure is uniformly autocontinuous from above, continuous from below and strong order continuous by assuming that the Riesz space is Dedekind σ-complete and has the weak σ-distributivity.

Acknowledgments: I would like to express my heartfelt gratitude to Professor Tamaki Tanaka of Niigata University for his invaluable suggestions and encouragement. I have studied under him during my undergraduate, a master's course, and a doctoral course at Niigata University. During that time, he gave me much advice on my study.

Also, I am grateful to Professor Eiichi Isogai of Niigata University, Professor Kitisuke Saitoo of Niigata University, Professor Osamu Hatori of Niigata University, Professor Teruo Takeuchi of Niigata University and Associate Professor Syuuji Yamada of Niigata University for many valuable suggestions and comments.

I would like to thank Professor Shizu Nakanishi of Osaka Prefecture University and Professor Yasjiro Nagakura of Tokyo University of Science for their valuable comments and remarks.

I would like to thank Professor Wataru Takahashi of Tokyo Institute of Technology and Professor Masashi Toyoda of Tamagawa University for their valuable comments and remarks.

Furthemore, I wish to thank Doctor Issei Kuwano of of Niigata University, Doctor Toshiharu Kawasaki of of Niigata University and Mr. Ichiro Suzuki of Hachouji High School for their helpful comments and encouragement.

I would like to express my heartfelt thanks to Professor Youji Uesaka, Professor Noriko Koono, Professor Noriaki Yoshikai, Professor Syuniti Kurino, Professor Satoko Komurasaki, Mr. Tatuya Shinozawa and Mrs. Emiko Baba of College of Science and Technology, Nihon University.

I would like to express my heartfelt thanks to Professor Eiji Nunohiro, Professor Kazuhisa Kitakaze, Professor Kiyotoshi Hiratuka, and Professor Fumio Masuda of Tokyo University of Infomation Science.

Finally, I would like to express my highest gratitude to my parents Jiro Watanabe, Eiko Watanabe and Kuro Watanabe and all of my family for their encouragement and support.

Niigata, Japan
September 2012 **Toshikazu Watanabe**

Contents

Chapter 1

Introduction

In this thesis, in Chapter 2, we mention that the Egoroff condition is the necessary and sufficient condition of the establishment of Egoroff's theorem for any ordered vector space-valued non-additive measure. We also show that Egoroff's theorem remains valid for vector-valued non-additive measure in a partially ordered vector space if the measure is multiple continuous from above and continuous from below.

On the other hand, when we consider the convergence structure, the topological structure is are useful. In Chapter 3, we consider an ordered vector space endowed with a locally full topology, which is called an ordered topological vector space [5] in this thesis, and we show that the Egoroff theorem remains valid for any ordered topological for vector-valued non-additive measure in a partially ordered vector space in the following four cases. The first case is that the measure is strongly order totally continuous; the second case is that the measure is strongly order continuous together with property (S) when the ordered topological vector space has a certain property; the third case is that the measure is continuous from above and below when the topology is locally convex; and the last case is that the measure is uniformly autocontinuous from above, continuous from below and strong order continuous when the topology is locally convex. We give examples our methods are applicable.

In Chapter 4, we treat regularity for measure that takes value in partially ordered vector spaces. Lusin's theorem is one of the most fundamental theorems in classical measure theory and does not hold in non-additive measure theory without additional conditions. In [44], Wu and Ha generalize Lusin's theorem from a classical measure space to a finite autocontinuous fuzzy measure space. Jiang and Suzuki [13] extend the result of [44] to a σ-finite fuzzy measure space. In [37], Song and Li investigate the regularity of null-additive fuzzy measures on a metric space and prove that Lusin's theorem remains valid for the real-valued fuzzy measures on a metric space under the null-additivity condition. In [27], Li and Yasuda prove that Lusin's theorem remains valid for the real-valued fuzzy Borel measures on a metric space under the weakly null-additivity condition. Recently, in [25], Li and Mesiar show several sets of sufficient conditions for Lusin's theorem on monotone measure spaces. For the regularity of fuzzy measures, see also Pap [36], Jiang et al. [14], and Wu and Wu [44]. For real-valued non-additive measures, see [7, 36, 40]. In [18], by means of an order structure, Kawabe proves that Lusin's theorem remains valid for any Riesz space-valued fuzzy measures on a metric space. He introduces and imposes smoothness conditions called the asymptotic Egoroff property and multiple Egoroff property on Riesz spaces. Moreover, he shows that Lusin's theorem remains valid for any Riesz space-valued fuzzy Borel measures on a metric space under the weakly null-additivity condition.

In this paper, as a same way in Chapter 3, we consider an ordered vector space endowed with a locally full topology, which is denoted by E. We show that Lusin's theorem remains valid for E-valued non-additive measures. Firstly, we prove Lusin type theorem for weakly null-additive Borel measures that are continuous from above together with a property suggested by Sun [38] on a metric space when E is a Hausdorff space, satisfies the first axiom of countability and has suitable property. Secondly, we prove another Lusin type theorem for weakly null-additive fuzzy Borel measures on a metric space when E is a Hausdorff locally convex space and satisfies the first axiom of countability. Our results are applicable to several ordered topological vector spaces.

In Chapter 5, we give a new proof of the Hahn-Banach theorem for the mapping that takes value in a partially ordered vector space.

The Hahn-Banach theorem is one of the most fundamental theorems in the functional analysis theory and the separation theorem is one of the most fundamental theorems in the optimization theory.

It is known that the Hahn-Banach theorem establishes in the case where the range space is a Dedekind complete Riesz space; see [5, 35, 46] and the separation theorem establishes in the Cartesian product space of a vector space and a Dedekind complete ordered vector space; see [8, 9, 33, 34].

The Hahn-Banach theorem is proved often using the Zorn lemma. For the proof of the Hahn-Banach theorem, there exist several approaches. For instance, Hirano, Komiya, and Takahashi [11] showed the Hahn-Banach theorem by using the Markov-Kakutani fixed point theorem [15] in the case where the range space is the real number system.

In Section 2, using the Bourbaki-Kneser fixed point theorem, we give a new proof of the Hahn-Banach theorem and the Mazur-Orlicz theorem in the case where the range space is a Dedekind complete partially ordered vector space (Theorem 5.3 and Theorem 5.4). In Section 4, we give a new proof of the separation theorem in the Cartesian product of a vector space and a Dedekind complete partially ordered vector space (Theorem 5.5); see [8, 9, 33, 34]. Recently, T.C.Lim[31] proved that for the Bourbaki-Kneser fixed point theorem, minimality is hold ; see [31]. Therefore the Hahn-Banach theorem is proved.

Moreover, using a fixed point theorem in a partially ordered set, we give a proof of the Hahn-Banach theorem for mapping that takes value in a Dedekind complete partially ordered vector space. (Theorem 5.3)

Moreover, we show the Mazur-Orlicz theorem in a Dedekind complete partially ordered vector space (Theorem 5.4) and Separation Theorem in a Dedekind complete partially ordered vector space (Theorem 5.5).

Chapter 2

Egoroff's theorem in a partially ordered vector space

2.1 Preliminaries

In this chapter, we treated Egoroff's theorem for non-additive measure that take values in a partially ordered vector space.

Throughout this paper, let R be the set of real numbers and N the set of natural numbers. Denote by Θ the set of all mappings from N into N. Let E be an ordered vector space and $(X, \mathcal{F})$ a measurable space.

Definition 2.1 *A set function $\mu : \mathcal{F} \to E$ is called a non-additive measure if it satisfies the following two conditions.*
(1) $\mu(\emptyset) = 0$.
(2) $\mu(A) \leq \mu(B)$ *whenever A, $B \in \mathcal{F}$ and $A \subset B$.*

Definition 2.2 *Let $\mu : \mathcal{F} \to E$ be a non-additive measure.*
(1) *A double sequence $\{A_{m,n}\} \subset \mathcal{F}$ is called a μ-regulator if it satisfies the following two conditions.*

(D1) $A_{m,n} \supset A_{m,n'}$ *whenever* $n \leq n'$.

(D2) $\mu(\cup_{m=1}^{\infty} \cap_{n=1}^{\infty} A_{m,n}) = 0$.

(2) μ *satisfies the Egoroff condition if* $\inf_{\theta \in \Theta} \mu(\cup_{m=1}^{\infty} A_{m,\theta(m)}) = 0$ *for any* μ*-regulator* $\{A_{m,n}\}$.

Remark 2.1 *A non-additive measure* μ *satisfies the Egoroff condition if (and only if), for any double sequence* $\{A_{m,n}\} \subset \mathcal{F}$ *satisfying* (D2) *and the following* (D1$'$), *it holds that* $\inf_{\theta \in \Theta} \mu(\cup_{m=1}^{\infty} A_{m,\theta(m)}) = 0$.

(D1$'$) $A_{m,n} \supset A_{m',n'}$ *whenever* $m \geq m'$ *and* $n \leq n'$.

Definition 2.3 *A double sequence* $\{u_{m,n}\} \subset E$ *is called a regulator if it is order bounded and* $u_{m,n} \downarrow 0$ *as* $n \to 0$ *for any* $m \in N$, *that is,* $u_{m,n} \geq u_{m,n+1}$ *for any* $m, n \in N$ *and* $\inf_{n \in N} u_{m,n} = 0$ *for any* $m \in N$.

2.2 Egoroff's theorem

Definition 2.4 *Let* $\mu : \mathcal{F} \to E$ *be a non-additive measure. Let* $\{f_n\}$ *be a sequence of* $\mathcal{F}$*-measurable real valued functions on* X *and* f *also such a function.*

(1) $\{f_n\}$ *is said to be convergent* μ*-a.e. to* f *if there exists an* $A \in \mathcal{F}$ *with* $\mu(A) = 0$ *such that* $\{f_n\}$ *converges to* f *on* $X - A$.

(2) $\{f_n\}$ *is said to be* μ*-almost uniformly convergent to* f *if there exists an upward directed set* Γ *and a decreasing net* $\{B_\gamma; \gamma \in \Gamma\} \subset \mathcal{F}$ *with* $\mu(B_\gamma) \downarrow 0$ *such that* $\{f_n\}$ *converges to* f *uniformly on each set* $X - B_\gamma$.

(3) *We say that the Egoroff theorem holds for* μ *if* $\{f_n\}$ *converges* μ*-almost uniformly to* f *whenever it converges* μ*-a.e. to the same limit.*

The following theorem holds for any ordered vector space-valued non-additive measure.

Theorem 2.1 *Let $\mu : \mathcal{F} \to E$ be a non-additive measure. Then the following two conditions are equivalent.*

(1) *μ satisfies the Egoroff condition.*

(2) *The Egoroff theorem holds for μ.*

Proof. (i) $\to$ (ii): Let $\{f_n\}$ be a sequence of $\mathcal{F}$-measurable, real-valued functions on X and f also such a function. Assume that $\{f_n\}$ converges μ-almost everywhere to f. For each m, $n \in N$, put

$$A_{m,n} := \bigcup_{i=n}^{\infty} \left\{ x \in X \mid |f_i(x) - f(x)| \geq \frac{1}{m} \right\}. \tag{2.1}$$

It is readily seen that the double sequence $\{A_{m,n}\}$ is a μ-regulator in $\mathcal{F}$, and hence it holds that

$$\inf_{\theta \in \Theta} \mu \left(\bigcup_{m=1}^{\infty} A_{m,\theta(m)} \right) = 0$$

For each $\theta \in \Theta$, put $E_\theta := \cup_{m=1}^{\infty} A_{m,\theta(m)}$. Note that Θ is ordered and directed upwards by pointwise partial ordering. It follows from (2.1) that the net $\{E_\theta\}_{\theta \in \Theta}$ satisfies $\mu(E_\theta) \downarrow 0$ and it is a routine to prove that $\{f_n\}$ converges to f uniformly on each set $X \setminus E_\theta$. Thus the Egoroff theorem holds for μ.

(ii) $\to$ (i): Let $\{A_{m,n}\}$ be a μ-regulator in $\mathcal{F}$. By Remark 2.1, we may assume without loss of generality that $A_{m,n} \supset A_{m',n'}$ if $m \geq m'$ and $n \leq n'$.

For each $n \in N$, put $f_n := \sup_{k \in N}(\frac{1}{k})\chi_{A_{k,n}}$. Here χ_E denotes the characteristic function of a set E. Then we have

$$A_{m,n} = \{x \in X \mid f_n(x) \geq \frac{1}{m}\} = \bigcup_{i=n}^{\infty} \{x \in X \mid f_i(x) \geq \frac{1}{m}\}$$

for all $m, n \in N$, so that it follows from the μ-regularity of $\{A_{m,n}\}$ that

$$\mu(\bigcup_{m=1}^{\infty} \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} \left\{ x \in X : f_i(x) \geq \frac{1}{m} \right\}) = 0.$$

This implies that $\{f_n\}$ converges μ-almost everywhere to 0 and hence, by assumption, f_n converges μ-almost uniformly to 0, that is, there is a decreasing net $\{E_\alpha\}_{\alpha \in \Gamma} \subset \mathcal{F}$with

$\mu(E_\alpha) \downarrow 0$ such that f_n converges to 0 uniformly on each set $X \setminus E_\alpha$. For each $\alpha \in \Gamma$, one can find $\theta_\alpha \in \Theta$ such that $\bigcup_{m=1}^{\infty} A_{m,\theta_\alpha(m)} \subset E_\alpha$. This implies $\inf_{\theta \in \Theta} \mu\left(\bigcup_{m=1}^{\infty} A_{m,\theta(m)}\right) = 0$ and hence μ has the Egoroff condition. $\square$

2.3 Main result

Definition 2.5 *Let $\mu : \mathcal{F} \to E$ be a non-additive measure.*
(1) μ is said to be continuous from above if $\mu(A_n) \downarrow \mu(A)$ whenever $\{A_n\} \subset \mathcal{F}$ and $A \in \mathcal{F}$ satisfy $A_n \downarrow A$ as $n \to \infty$.
(2) μ is said to be continuous from below if $\mu(A_n) \uparrow \mu(A)$ whenever $\{A_n\} \subset \mathcal{F}$ and $A \in \mathcal{F}$ satisfy $A_n \uparrow A$ as $n \to \infty$.

Definition 2.6 *A regulator $\{u_{m,n}\}$ is said to be summative if for any $m, n \in N$, it holds that $u_{m+1,n} + u_{m+1,n} \leq u_{m,n}$.*

Definition 2.7 *Let $\mu : \mathcal{F} \to E$ be a non-additive measure. μ is said to be multiple continuous from above if for any μ-regulator $\{A_{m,n}\}$ there exists a summative regulator $\{u_{m,i}\}$ such that for any $i \in N$ there exists a $\theta_i \in \Theta$ such that for any $m \in N$ and $(n_1, n_2, \cdots, n_m) \in N^m$, $\mu(\cup_{j=1}^{m} A_{j,n_j} \cup A_{m+1,n} \cup A) - \mu(\cup_{j=1}^{m} A_{j,n_j} \cup A) \leq u_{m+2,i}$ for any $n \geq \theta_i(m+1)$ whenever $A \in \mathcal{F}$ satisfies $\cup_{j=1}^{m} A_{j,n_j} \cup A_{m+1,n} \cup A \downarrow \cup_{j=1}^{m} A_{j,n_j} \cup A$ as $n \to \infty$.*

Example 2.1 *We exhibit an example of non-additive mesure which are multiple continuous from above. We assume that an ordered vector space E satisfies the following condition (A).*

> *(A) For any regulator $\{u_{m,n}\}$ there exists a positive sequence $\{\lambda_m\} \subset R$ with $\lambda_{m+1} + \lambda_{m+1} \leq \lambda_m$ and a single sequence $\{v_i\} \subset E$ with $v_i \downarrow 0$ as*

$i \to \infty$ such that for any $m, i \in N$ there exists an $n(m, i) \in N$ such that $u_{m,n(m,i)} \leq \lambda_m v_i$.

Let $(X, \mathcal{F})$ be a measure space and $\mu : \mathcal{F} \to E$ a non-additive measure. Assume that μ is subadditive, that is, $\mu(A \cup B) \leq \mu(A) + \mu(B)$ for any $A, B \in \mathcal{F}$, and continuous from above. Let $\{A_{m,n}\}$ be a μ-regulator. Since $\mu(\cup_{m=1}^{\infty} \cap_{n=1}^{\infty} A_{m,n}) = 0$ and μ is continuous from above, there exists a regulator $\{u_{m,n}\}$ such that $\mu(A_{m,n}) \leq u_{m+1,n}$ for any $m, n \in N$. By assumption (A), there exists a positive sequence $\{\lambda_m\} \subset R$ and a single sequence $\{v_i\} \subset E$ with the property that $\{\lambda_m v_i\}$ is a summative regulator and for any $m, i \in N$ there exists an $n(m, i) \in N$ with $u_{m,n(m,i)} \leq \lambda_m v_i$. Fix $i \in N$ and put $\theta_i(m) = n(m+1, i)$ for all $m \in N$. Fix $m \in N$ and $(n_1, \cdots, n_m) \in N^m$. Since μ is subadditive, for any $A \in \mathcal{F}$ and any $n \geq \theta_i(m+1)$ we have $\mu(\cup_{j=1}^{m} A_{j,n_j} \cup A_{m+1,n} \cup A) - \mu(\cup_{j=1}^{m} A_{j,n_j} \cup A) \leq \mu(A_{m+1,n}) \leq \lambda_{m+2} v_i$. Thus μ is multiple continuous from above.

Remark 2.2 *Condition (A) is satisfied when E has the strong Egoroff property, that is, for any double sequence $\{u_{m,n}\} \subset E$ with $u_{m,n} \downarrow 0$ as $n \to \infty$ there exists a single sequence $\{v_i\} \subset R$ with $v_i \downarrow 0$ as $i \to \infty$ such that for any $m, i \in N$ there exists an $n(m, i) \in N$ such that $u_{m,n(m,i)} \leq v_i$. In fact, let $\{u_{m,n}\}$ be a regulator and take a positive sequence $\{\lambda_m\} \subset R$ with $\lambda_{m+1} + \lambda_{m+1} \leq \lambda_m$. Since $u_{m,n} \downarrow 0$ as $n \to \infty$, $\{\lambda_m^{-1} u_{m,n}\}$ is a double sequence in E with $\lambda_m^{-1} u_{m,n} \downarrow 0$ as $n \to \infty$. Since E has the strong Egoroff property, there exists a single sequence $\{v_i\} \subset E$ with $v_i \downarrow 0$ such that for any $m, i \in N$ there exists an $n(m, i) \in N$ such that $\lambda_m^{-1} u_{m,n(m,i)} \leq v_i$. Then condition (A) is satisfied.*

Theorem 2.2 *If μ is multiple continuous from above and continuous from below, then μ satisfies the Egoroff condition.*

Proof. Let $\{A_{m,n}\}$ be a μ-regulator and put $D = \cup_{m=1}^{\infty} \cap_{n=1}^{\infty} A_{m,n}$. Then for any $m \in N$ and $(n_1, \cdots, n_m) \in N^m$ it holds that $A_{1,n} \cup D \downarrow D$, $A_{1,n_1} \cup A_{2,n} \cup D \downarrow A_{1,n_1} \cup D, \cdots,$ and $\cup_{j=1}^{m} A_{j,n_j} \cup A_{m+1,n} \cup D \downarrow \cup_{j=1}^{m} A_{j,n_j} \cup D$ as $n \to \infty$. Since μ is multiple continuous from above, there exists a summative regulator $\{u_{m,i}\}$ such that for any $i \in N$ there exists a $\theta_i \in \Theta$ such that for any $m \in N$ and $(n_1, n_2, \cdots, n_m) \in N^m$, we have $\mu(A_{1,n} \cup D) \le u_{2,i}$ for any $n \ge \theta_i(1)$, $\mu(A_{1,n_1} \cup A_{2,n} \cup D) \le \mu(A_{1,n_1} \cup D) + u_{3,i}$ for any $n \ge \theta_i(2)$, $\cdots$, and $\mu(\cup_{j=1}^m A_{j,n_j} \cup A_{m+1,n} \cup D) \le \mu(\cup_{j=1}^m A_{j,n_j} \cup D) + u_{m+2,i}$ for any $n \ge \theta_i(m+1)$. In particular, taking $n_j = \theta_i(j)$ $(j = 1, \cdots, m)$, then, since $\{u_{m,i}\}$ is summative, we have $\mu(A_{1,\theta_i(1)} \cup D) \le u_{2,i}$, $\mu(A_{1,\theta_i(1)} \cup A_{2,\theta_i(2)} \cup D) \le \mu(A_{1,\theta_i(1)} \cup D) + u_{3,i} \le u_{2,i} + u_{3,i} \le u_{2,i} + u_{2,i} \le u_{1,i}, \cdots$, and $\mu(\cup_{j=1}^m A_{j,\theta_i(j)} \cup D) \le \mu(\cup_{j=1}^{m-1} A_{j,\theta_i(j)} \cup D) + u_{m+1,i} \le u_{2,i} + \cdots + u_{m,i} + u_{m+1,i} \le \cdots \le u_{2,i} + u_{2,i} \le u_{1,i}$. Therefore $\mu(\cup_{j=1}^m A_{j,\theta_i(j)}) \le u_{1,i}$. Next by the monotonicity and the continuity of μ from below, we have $\mu(\cup_{j=1}^\infty A_{j,\theta_i(j)}) \le u_{1,i}$. Since $u_{1,i} \downarrow 0$ as $i \to \infty$, we have $\inf_{\theta \in \Theta} \mu(\cup_{j=1}^\infty A_{j,\theta(j)}) = 0$. Then μ satisfies the Egoroff condition. $\square$

By Theorem 2.2, the Egoroff theorem remains valid for any ordered vector space-valued non-additive measure if the measure is multiple continuous from above and continuous from below. This result contains the case which is not treated in [16, 17].

Chapter 3

Egoroff's theorem in a partially ordered topological vector space

3.1 Preliminaries

In this chapter, we treated the Egoroff's theorem for non-additive measure that take values in a partially ordered topological vector space.

First, we introduce some basic definitions for non-additive measure that take values in a partially ordered vector space which will be used in this chapter.

A topology on a vector space E is called a vector topology if the mappings $(x, y) \mapsto x+y$ and $(\alpha, x) \mapsto \alpha x$, where $x, y \in E$ and $\alpha \in R$, are continuous. Let E be an ordered vector space. A subset F of an ordered vector space E is said to be full if $x_1, x_2 \in F$ and $x_1 \leq x_2$ implies $[x_1, x_2] \subset F$. We consider a vector topology on E and let $\mathcal{B}_0$ be a neighborhood of $0 \in E$. The vector topology on E is called a locally full topology, if there exists a basis of $\mathcal{B}_0$ consisting of full sets. An ordered vector space endowed with this topology is called an ordered topological vector space. Let $\{u_n\}$ be a sequence in E and $u \in E$. We write $u_n \to u$ if u_n converges to u with respect to the vector topology on E, that is, for any $U \in B_0$ there exists an $n_0 \in N$ such that $u_n - u \in U$ for any $n \geq n_0$.

In what follows, let E be an ordered topological vector space and $(X, \mathcal{F})$ a measurable space.

Definition 3.1 *A set function $\mu : \mathcal{F} \to E$ is called a non-additive measure if it satisfies the following two conditions.*
(1) $\mu(\emptyset) = 0$.
(2) *If A, $B \in \mathcal{F}$ and $A \subset B$, then $\mu(A) \le \mu(B)$.*

Definition 3.2 *Let $\mu : \mathcal{F} \to E$ be a non-additive measure.*
(1) *μ is said to be strongly order totally continuous if $\inf_{\gamma \in \Gamma} \mu(A_\gamma) = 0$, where Γ is an upword directed set, for any $\{A_\gamma\} \subset \mathcal{F}$ and $A \in \mathcal{F}$ satisfying $A_\gamma \downarrow A$ and $\mu(A) = 0$.*
(2) *μ is said to be continuous from above if for any $\{A_n\} \subset F$ and $A \in F$ satisfying $A_n \downarrow A$ it holds that $\mu(A_n) - \mu(A) \to 0$.*
(3) *μ is said to be continuous from below if for any $\{A_n\} \subset F$ and $A \in F$ satisfying $A_n \uparrow A$ it holds that $\mu(A) - \mu(A_n) \to 0$.*
(4) *μ is said to be strongly order continuous if it is continuous from above at measurable sets of measure 0, that is, for any $\{A_n\} \subset \mathcal{F}$ and $A \in \mathcal{F}$ satisfying $A_n \downarrow A$ and $\mu(A) = 0$ it holds that $\mu(A_n) \to 0$.*
(5) *μ is said to have property (S) if for any sequence $\{A_n\} \subset \mathcal{F}$ with $\mu(A_n) \to 0$, there exists a subsequence $\{A_{n_k}\}$ such that $\mu(\cap_{i=1}^{\infty} \cup_{k=i}^{\infty} A_{n_k}) = 0$.*
(6) *μ is said to be uniformly autocontinuous from above if for any $U \in B_0$ and any sequence $\{B_n\} \subset \mathcal{F}$ with $\mu(B_n) \to 0$, there exists an $n_0 \in N$ such that $\mu(A \cup B_n) - \mu(A) \in U$ for any $A \in \mathcal{F}$ and any $n \ge n_0$.*
(7) *μ is said to be uniformly autocontinuous from below if for any $U \in B_0$ and any sequence $\{B_n\} \subset \mathcal{F}$ with $\mu(B_n) \to 0$, there exists an $n_0 \in N$ such that $\mu(A) - \mu(A - B_n) \in U$ for any $A \in \mathcal{F}$ and any $n \ge n_0$.*

3.2 Egoroff's theorem

Definition 3.3 *Let $\mu : \mathcal{F} \to E$ be a non-additive measure.*

(1) *A double sequence $\{A_{m,n}\} \subset \mathcal{F}$ is called a μ-regulator if it satisfies the following two conditions.*

(D1) $A_{m,n} \supset A_{m,n'}$ *whenever* $n \leq n'$.

(D2) $\mu(\cup_{m=1}^{\infty} \cap_{n=1}^{\infty} A_{m,n}) = 0$.

(2) *μ satisfies the Egoroff condition if for any μ-regulator $\{A_{m,n}\}$ and any $U \in \mathcal{B}_0$ there exists a $\theta \in \Theta$ such that $\mu(\cup_{m=1}^{\infty} A_{m,\theta(m)}) \in U$.*

Remark 3.1 *A non-additive measure μ satisfies the Egoroff condition if (and only if), for any double sequence $\{A_{m,n}\} \subset \mathcal{F}$ satisfying* (D2) *and the following* (D1$'$), *it holds that for any $U \in \mathcal{B}_0$ there exists a $\theta \in \Theta$ such that $\mu(\cup_{m=1}^{\infty} A_{m,\theta(m)}) \in U$.*

(D1$'$) $A_{m,n} \supset A_{m',n'}$ *whenever* $m \geq m'$ *and* $n \leq n'$.

Definition 3.4 *Let $\mu : \mathcal{F} \to E$ be a non-additive measure. Let $\{f_n\}$ be a sequence of $\mathcal{F}$-measurable real valued functions on X and f also such a function.*

(1) *$\{f_n\}$ is said to be convergent μ-a.e. to f if there exists an $A \in \mathcal{F}$ with $\mu(A) = 0$ such that $\{f_n\}$ converges to f on $X - A$.*

(2) *$\{f_n\}$ is said to be μ-almost uniformly convergent to f if there exists an upward directed set Γ and a decreasing net $\{B_\gamma; \gamma \in \Gamma\} \subset \mathcal{F}$ such that for any $U \in \mathcal{B}_0$ there exists a $\gamma \in \Gamma$ such that $\mu(B_\gamma) \in U$ and $\{f_n\}$ converges to f uniformly on each set $X - B_\gamma$.*

(3) *We say that the Egoroff theorem holds for μ if $\{f_n\}$ converges μ-almost uniformly to f whenever it converges μ-a.e. to the same limit.*

Theorem 3.1 *Let $\mu : \mathcal{F} \to E$ be a non-additive measure. Then the following two conditions are equivalent.*

(1) *μ satisfies the Egoroff condition.*

(2) *The Egoroff theorem holds for μ.*

Proof. (1)→(2): Let $\{f_n\}$ be a sequence of $\mathcal{F}$-measurable real valued functions on X and f also such a function. Assume that $\{f_n\}$ converges μ-a.e. to f. For each $m, n \in N$, put $A_{m,n} = \cup_{j=n}^{\infty}\{x \in X; |f_j(x) - f(x)| \geq \frac{1}{m}\}$. It is easy to see that $\{A_{m,n}\}$ is a μ-regulator. By assumption, for any $U \in \mathcal{B}_0$, there exists a $\theta \in \Theta$ such that $\mu(\cup_{m=1}^{\infty} A_{m,\theta(m)}) \in U$. Note that Θ is upward directed by pointwise partial ordering. Put $B_\theta = \cup_{m=1}^{\infty} A_{m,\theta(m)}$, then $\mu(B_\theta) \in U$. And it is a routine work to prove that $f_n \to f$ uniformly on each set $X - B_\theta$.

(2)→(1): Let $\{A_{m,n}\}$ be a μ-regulator. By Remark 1, we are able to assume that $A_{m,n} \supset A_{m',n'}$ whenever $m \geq m'$ and $n \leq n'$. For each $n \in N$, put $f_n = \sup_{i \in N}\left((\frac{1}{i})\chi_{A_{i,n}}\right)$ where χ_B denotes the characteristic function of B. Then we have $A_{m,n} = \{x \in X; f_n(x) \geq \frac{1}{m}\} = \cup_{j=n}^{\infty}\{x \in X; f_j(x) \geq \frac{1}{m}\}$ for all $m, n \in N$. By (D2), we have $\mu(\cup_{m=1}^{\infty} \cap_{n=1}^{\infty} \cup_{j=n}^{\infty}\{x \in X; f_j(x) \geq \frac{1}{m}\}) = 0$. This implies that $\{f_n\}$ converges μ-a.e. to 0. By assumption, $\{f_n\}$ converges μ-almost uniformly to 0. Since E is an ordered topological vector space, for any $U \in \mathcal{B}_0$, there exists a $U_1 \in \mathcal{B}_0$ such that $U_1 \subset U$ and U_1 is full. Then there exists a decreasing net $\{B_\gamma; \gamma \in \Gamma\} \subset \mathcal{F}$ and there exists a $\gamma \in \Gamma$ such that $\mu(B_\gamma) \in U_1$ and $\{f_n\}$ converges to 0 uniformly on each set $X - B_\gamma$. Then we can find a $\theta \in \Theta$ such that $\cap_{m=1}^{\infty}(X - A_{m,\theta(m)}) \supset X - B_\gamma$. Thus we have $\mu(\cup_{m=1}^{\infty} A_{m,\theta(m)}) \leq \mu(B_\gamma)$. Since U_1 is full, we have $\mu(\cup_{m=1}^{\infty} A_{m,\theta(m)}) \in U_1 \subset U$. □

3.3 The case where μ is strongly order totally continuous

Theorem 3.2 *If μ is strongly order totally continuous, then μ satisfies the Egoroff condition.*

Proof. It is clear from the definition. □

3.4 The case where μ is strongly order continuous and property (S)

Definition 3.5 *The double sequence $\{r_{m,n}\}$ in E is called a topological regulator if it satisfies the following two conditions.*
(1) $r_{m,n} \geq r_{m,n+1}$ *for any* m, $n \in N$.
(2) *For any* $m \in N$, *it holds that* $r_{m,n} \to 0$.

Definition 3.6 *E has property (EP) if for any topological regulator $\{r_{m,n}\}$ in E there exists a sequence $\{p_k\}$ in E satisfying the following two conditions.*
(1) $p_k \to 0$.
(2) *For any $k \in N$ and any $m \in N$ there exists an $n_0(m, k) \in N$ such that $r_{m,n} \leq p_k$ for any $n \geq n_0(m, k)$.*

Theorem 3.3 *Assume that E has property (EP). If μ has property (S) and is strongly order continuous, then μ satisfies the Egoroff condition.*

Proof. Let $\{A_{m,n}\}$ be a μ-regulator. By Remark 1, we are able to assume that $A_{m,n} \supset A_{m',n'}$ whenever $m \geq m'$ and $n \leq n'$. Then for any $m \in N$, $A_{m,n} \downarrow \cap_{n=1}^{\infty} A_{m,n}$ and $\mu(\cap_{n=1}^{\infty} A_{m,n}) = 0$ hold. By the strongly order continuity of μ, $\{\mu(A_{m,n})\}$ is a topological regulator in E. For any $U \in \mathcal{B}_0$, there exists a $U_1 \in \mathcal{B}_0$ such that $U_1 \subset U$ and U_1 is full. Since E has property (EP), there exists a sequence $\{p_m\}$ with $p_m \to 0$ with the property that for any $m \in N$, there exists an $n_0(m) \in N$ such that $\mu(A_{m,n_0(m)}) \leq p_m$. Then there exists an $m_0 \in N$ such that for any $m \geq m_0$, we have $p_m \in U_1$, so that $\mu(A_{m,n_0(m)}) \in U_1 \subset U$ since U_1 is full. Since μ has property (S), there exists a strictly increasing sequence $\{m_i\} \subset N$ such that $\mu(\cap_{j=1}^{\infty} \cup_{i=j}^{\infty} A_{m_i, n_0(m_i)}) = 0$. By the strongly order continuity of μ, for any $U \in \mathcal{B}_0$, there exists a $j_0 \in N$ such that $\mu(\cup_{i=j_0}^{\infty} A_{m_i, n_0(m_i)}) \in U$. Define $\theta \in \Theta$ such that $\theta(m) = n_0(m_{j_0})$ if $1 \leq m \leq m_{j_0}$ and $\theta(m) = n_0(m_i)$ if $m_{i-1} < m \leq m_i$ for some $i > j_0$. Since $\{A_{m,n}\}$ is increasing for each $n \in N$, it holds that $\cup_{i=j_0}^{\infty} A_{m_i, n_0(m_i)} = \cup_{m=1}^{\infty} A_{m, \theta(m)}$. Then μ satisfies the Egoroff condition. □

3.5 The case where μ is continuous from above and below

In what follows, we assume that the topology of E is locally convex. Then E is said to be an ordered locally convex space.

Theorem 3.4 *If μ is continuous from above and below, then μ satisfies the Egoroff condition.*

Proof. For any $W \in \mathcal{B}_0$, there exist $U, V \in \mathcal{B}_0$ such that $U + V \subset W$ and U is full. For any $k \in N$, there exists a $V_k \in \mathcal{B}_0$ such that $2^k V_k \subset U$. Let $\{A_{m,n}\}$ be a μ-regulator and put $D = \cup_{m=1}^{\infty} \cap_{n=1}^{\infty} A_{m,n}$. Then for any $m \in N$ and $(n_1, \ldots, n_m) \in N^m$, $A_{1,n} \cup D \downarrow D$, $A_{1,n_1} \cup A_{2,n} \cup D \downarrow A_{1,n_1} \cup D$, $\ldots$, and $\cup_{j=1}^{m} A_{j,n_j} \cup A_{m+1,n} \cup D \downarrow \cup_{j=1}^{m} A_{j,n_j} \cup D$ hold as $n \to \infty$. Since μ is continuous from above, for V_1, there exists an $n_1 \in N$ such that $\mu(A_{1,n} \cup D) - \mu(D) \in V_1$ for any $n \geq n_1$. Since $\mu(D) = 0$, we have $\mu(A_{1,n_1} \cup D) \in V_1$. For this n_1, $A_{1,n_1} \cup A_{2,n} \cup D \downarrow A_{1,n_1} \cup D$ as $n \to \infty$. For V_2, there exists an $n_2 \in N$ such that $\mu(A_{1,n_1} \cup A_{2,n} \cup D) - \mu(A_{1,n_1} \cup D) \in V_2$ for any $n \geq n_2$. We have $\mu(A_{1,n_1} \cup A_{2,n_2} \cup D) \in \mu(A_{1,n_1} \cup D) + V_2 \subset V_1 + V_2$. Since the topology is locally convex, repeating the argument, for any $m \in N$, we have $\mu(\cup_{j=1}^{m} A_{i,n_j} \cup D) \in \sum_{j=1}^{m} V_j \subset U$. Since U is full, we have $\mu(\cup_{j=1}^{m} A_{j,n_j}) \in U$. Let $\theta \in \Theta$ satisfy $\theta(j) = n_j$ $(j = 1, 2, \ldots)$. We have $\cup_{j=1}^{m} A_{j,\theta(j)} \uparrow \cup_{j=1}^{\infty} A_{j,\theta(j)}$ as $m \to \infty$. Since μ is continuous from below and $\{A_{m,n}\}$ is a μ-regulator, for $V \in \mathcal{B}_0$, there exists an $m_0 \in N$ such that $\mu(\cup_{j=1}^{\infty} A_{j,\theta(j)}) - \mu(\cup_{j=1}^{m} A_{j,\theta(j)}) \in V$ for any $m \geq m_0$. Thus we have $\mu(\cup_{j=1}^{\infty} A_{j,\theta(j)}) \in \mu(\cup_{j=1}^{m} A_{j,\theta(j)}) + V \subset U + V \subset W$. Then μ satisfies the Egoroff condition. □

3.6 The case where μ is uniformly autocontinuous from above, strongly order continuous from above and continuous below

Theorem 3.5 *If μ is uniformly autocontinuous from above, strongly order continuous from above and continuous from below, then μ satisfies the Egoroff condition.*

Proof. For any $W \in \mathcal{B}_0$, there exist U, $V \in \mathcal{B}_0$ such that $U + V \subset W$. For any $k \in N$, there exists a $V_k \in \mathcal{B}_0$ such that $2^k V_k \subset U$. Let $\{A_{m,n}\}$ be a μ-regulator. By the strongly order continuity of μ, for any $m \in N$, we have $\mu(A_{m,n}) \to 0$. For V_1, there exists an $n_1 \in N$ such that $\mu(A_{1,n_1}) \in V_1$. For V_2, there exists an $n_2^{'} \in N$ such that $\mu(A_{2,n_2^{'}}) \in V_2$. Since μ is uniformly autocontinuous from above, there exists an $n_2^{''} \in N$ such that for any $A \in \mathcal{F}$, $\mu(A \cup A_{2,n_2^{''}}) - \mu(A) \in V_2$. Put $n_2 = \max(n_2^{'}, n_2^{''})$, then $\mu(A_{1,n_1} \cup A_{2,n_2}) \in V_1 + V_2$. Since the topology is locally convex, repeating the argument, we have $\mu(\cup_{j=1}^{m} A_{j,n_j}) \in \sum_{j=1}^{m} V_j \subset U$ for any $m \in N$. Let $\theta \in \Theta$ satisfy $\theta(j) = n_j$ $(j = 1, 2, \ldots)$. We have $\cup_{j=1}^{m} A_{j,\theta(j)} \uparrow \cup_{j=1}^{\infty} A_{j,\theta(j)}$ as $m \to \infty$. Since μ is continuous from below and $\{A_{m,n}\}$ is a μ-regulator, for $V \in \mathcal{B}_0$, there exists an $m_0 \in N$ such that $\mu(\cup_{j=1}^{\infty} A_{j,\theta(j)}) \in \mu(\cup_{j=1}^{m} A_{j,\theta(j)}) + V$ for any $m \geq m_0$. Thus we have $\mu(\cup_{j=1}^{\infty} A_{j,\theta(j)}) \in U + V \subset W$. Then μ satisfies the Egoroff condition. □

3.7 Examples of ordered topological spaces

In this section we give some examples of ordered topological vector spaces to which our methods are applicable.

(i) Let T be a Hausdorff space and $C(T)$ the space of all real continuous functions defined on T. Then $C(T)$ is an ordered vector space endowed with a pointwise order. The topology of compact convergence on $C(T)$ is a locally full topology, see [5, page 159]. Therefore we can apply to μ in case of the Section 4, 6 and 7.

(ii) Let T be a locally compact space and $C_0(T)$ the space of all continuous real functions with compact support defined on T. Then $C_0(T)$ is an ordered locally convex space endowed with the pointwise order and endowed with the inductive limit topology, see [5, pages 75 and 159]. Therefore we can apply to μ in case of the Section 4, 6 and 7.
(iii) Let $C^\infty(R^n)$ be the space of all real functions having continuous derivatives of any order. $C^\infty(R^n)$ is an ordered locally convex space but not a Riesz space, see [5, page 159]. Therefore we can apply to μ in case of the Section 4, 6 and 7.
(iv) Let $L_p([0,1])$ be a set of real valued Lebesgue measurable functions f defined on [0, 1] such that $\int_0^1 |f(x)|^p d\mu < \infty (0 < p < \infty)$. $L_p([0,1])$ is a Riesz space with the pointwise order. Moreover $L_p([0,1])$ is a locally solid space, see [1, e.g. 8.6]. If the topology of Riesz space is locally solid, then it is locally full, see [5, Chapter 7, Proposition 1]. Therefore $L_p([0,1])$ is an ordered topological vector space. However its topology is not usually locally convex. In addition, $L_p([0,1])$ satisfies the σ-Lebesgue property, that is, the sequence $\{u_n\}$ converge to 0 with respect to the topology whenever $\{u_n\}$ converge to 0 with respect to the order, see [1, e.g. 8.6]. And $L_p([0,1])$ has the Egoroff property, see [30, Section 71]. Therefore $L_p([0,1])$ has property (EP). Thus in this case we can apply to μ in case of the Section 4 and 5.

Chapter 4

Lusin's theorem for non-additive measure

4.1 Preliminaries

In this Chapter we use a notation and definition in Chapter 3. Let R be the set of real numbers and N the set of natural numbers. Denote by Θ the set of all mappings from N into N. Let X be a non-empty set and $\mathcal{F}$ a σ-field of X and E a ordered topological vector space, that is, an ordered vector space endowed with full topology; see [5]. We also assume that E is a Hausdorff space with the first axiom of countability.

Definition 4.1 (1) *μ is called weakly null-additive if $\mu(A \cup B) = 0$ whenever $A, B \in \mathcal{F}$ and $\mu(A) = \mu(B) = 0$; see* [40].

4.2 The case where μ is continuous from above and has property (S)

In the following lemma, we characterize the weakly null-additivity for the non-additive measure which is strong order continuous and has property (S) in the case where the range space is an ordered topological vector space. It is also a counterpart of [27, Lemma 1] for measures whose range space is an ordered topological vector space. The strongly order continuity plays an important role in discussing the convergence of measurable function sequence and of integral sequence. For instance, it is a necessary and sufficient condition for a generalized Lebesgue theorem. For more details, see [24].

Lemma 4.1 *Let $\mu : \mathcal{F} \to E$ be a non-additive measure which is strongly order continuous and has property* (S). *We assume that E has property* (EP). *Then the following two conditions are equivalent:*
(i) *μ is weakly null-additive.*
(ii) *For any $U \in \mathcal{B}_0$ and double sequence $\{A_{m,n}\} \subset \mathcal{F}$ satisfying that $A_{m,n} \downarrow D_m$ as $n \to \infty$ and $\mu(D_m) = 0$ for each $m \in N$, there exists a $\theta \in \Theta$ such that $\mu\left(\cup_{m=1}^{\infty} A_{m,\theta(m)}\right) \in U$.*

Proof. (i)$\to$ (ii): Let $\{A_{m,n}\}$ be a double sequence such that $A_{m,n} \downarrow D_m$ as $n \to \infty$ and $\mu(D_m) = 0$ for each $m \in N$. Put $B_{m,n} = \cup_{j=1}^{m} A_{j,n}$ and $F_m = \cup_{j=1}^{m} D_j$, then $\{B_{m,n}\}$ is increasing for each $n \in N$ and $B_{m,n} \downarrow F_m$ as $n \to \infty$. Since μ is weakly null-additive, $\mu(F_m) = 0$. Since μ is strongly order continuous, $\{\mu(B_{m,n})\}$ is a topological regulator in E. For any $U \in \mathcal{B}_0$ there exists a $U_1 \in \mathcal{B}_0$ such that $U_1 \subset U$ and U_1 is full. Since E has property (EP), there exists a sequence $\{p_m\}$ with $p_m \to 0$ such that for any $m \in N$, there exists an $n_0(m) \in N$ such that $\mu(B_{m,n_0(m)}) \leq p_m$. Then there exists an $m_0 \in N$ such that for any $m \geq m_0$, we have $p_m \in U_1$, so that $\mu(B_{m,n_0(m)}) \in U_1 \subset U$ since U_1 is full. Since μ has property (S), there exists a strictly increasing sequence $\{m_i\} \subset N$ such that $\mu(\cap_{j=1}^{\infty} \cup_{i=j}^{\infty} B_{m_i, n_0(m_i)}) = 0$. Since μ is strongly order continuous, there exists a $j_0 \in N$ such that $\mu(\cup_{i=j_0}^{\infty} B_{m_i, n_0(m_i)}) \in U_1$. Define $\theta \in \Theta$ such that $\theta(m) = n_0(m_{j_0})$

if $1 \leq m \leq m_{j_0}$ and $\theta(m) = n_0(m_i)$ if $m_{i-1} < m \leq m_i$ for some $i > j_0$. Since $\{B_{m,n}\}$ is increasing for each $n \in N$, we have $\cup_{i=j_0}^{\infty} B_{m_i, n_0(m_i)} = \cup_{m=1}^{\infty} B_{m, \theta(m)}$. Since $\cup_{m=1}^{\infty} A_{m, \theta(m)} \subset \cup_{m=1}^{\infty} B_{m, \theta(m)}$, we have $\mu(\cup_{m=1}^{\infty} A_{m, \theta(m)}) \leq \mu(\cup_{i=j_0}^{\infty} B_{m_i, n_0(m_i)})$. Since U_1 is full, (ii) holds.

(ii)$\rightarrow$(i): Let F, $G \in \mathcal{F}$ and $\mu(F) = \mu(G) = 0$. Define a double sequence $\{A_{m,n}\} \subset \mathcal{F}$ such that $A_{1,n} = F$, $A_{2,n} = G$ and $A_{m,n} = \emptyset$ $(m \geq 3)$ for any $n \in N$. Let $D_1 = F$, $D_2 = G$ and $D_m = \emptyset$ $(m \geq 3)$. By assumption, for any $U \in \mathcal{B}_0$ there exists a $\theta \in \Theta$ such that $\mu(\cup_{m=1}^{\infty} A_{m, \theta(m)}) \in U$. Since $\cup_{m=1}^{\infty} A_{m, \theta(m)} = F \cup G$, we have $\mu(F \cup G) \in U$. Then we have $\mu(F \cup G) = 0$. □

4.2.1 Regularity of measure

Let X be a Hausdorff space. Denote by $\mathcal{B}(X)$ the σ-field of all Borel subsets of X, that is, the σ-field generated by the open subsets of X. A non-additive measure defined on $\mathcal{B}(X)$ is called a non-additive Borel measure on X.

Definition 4.2 **([44])** *Let $\mu : \mathcal{B}(X) \to E$ be a non-additive Borel measure on X. μ is called regular if for any $U \in \mathcal{B}_0$ and $A \in \mathcal{B}(X)$, there exist a closed set F_U and an open set G_U such that $F_U \subset A \subset G_U$ and $\mu(G_U \setminus F_U) \in U$.*

Theorem 4.1 *Let X be a metric space and $\mathcal{B}(X)$ a σ-field of all Borel subsets of X. Let $\mu : \mathcal{B}(X) \to E$ be a non-additive Borel measure on X which is weakly null additive, continuous from above and has property* (S). *We assume that E has property* (EP). *Then μ is regular.*

Proof. Let $\mu : \mathcal{B}(X) \to E$ be a non-additive Borel measure. Denote by $\mathcal{E}$ the family of Borel subsets A of X with the property that for any $U \in \mathcal{B}_0$, there exist a closed set F_U and an open set G_U such that

$$F_U \subset A \subset G_U \text{ and } \mu(G_U \setminus F_U) \in U.$$

We first show that $\mathcal{E}$ is a σ-field. It is obvious that $\mathcal{E}$ is closed for complementation and contains $\emptyset$ and X. We show that $\mathcal{E}$ is closed for countable unions. Let $\{A_m\}$ be a sequence of $\mathcal{E}$ and put $A = \cup_{m=1}^{\infty} A_m$ on X. Since E is Hausdorff and satisfies the first axiom of countability, there exists $\{V_n\} \subset \mathcal{B}_0$ such that $\cap_{n=1}^{\infty} V_n = \{0\}$ and V_n is full. Then for each $m \in N$, there exist double sequences $\{F_{m,n}\}$ of closed sets and $\{G_{m,n}\}$ of open sets such that

$$F_{m,n} \subset A_m \subset G_{m,n} \text{ and } \mu(G_{m,n} \setminus F_{m,n}) \in V_n \text{ for all } n.$$

We may assume that, for each $m \in N$, $\{F_{m,n}\}$ is increasing and $\{G_{m,n}\}$ is decreasing without loss of generality. For each $m \in N$, put $D_m = \cap_{n=1}^{\infty}(G_{m,n} \setminus F_{m,n})$. Since

$$G_{m,n} \setminus F_{m,n} \downarrow D_m \text{ as } n \to \infty$$

and V_n is full, we have $\mu(D_m) \in V_n$. Moreover, since $\cap_{n=1}^{\infty} V_n = \{0\}$, we have $\mu(D_m) = 0$. For any $U \in \mathcal{B}_0$, take V, $W \in \mathcal{B}_0$ such that $V + W \subset U$ and V is full. By Lemma 4.1, there exists a $\theta \in \Theta$ such that

$$\mu\left(\bigcup_{m=1}^{\infty}\left(G_{m,\theta(m)} \setminus F_{m,\theta(m)}\right)\right) \in V.$$

Since

$$\bigcup_{m=1}^{\infty} G_{m,\theta(m)} \setminus \bigcup_{m=1}^{\infty} F_{m,\theta(m)} \subset \bigcup_{m=1}^{\infty}\left(G_{m,\theta(m)} \setminus F_{m,\theta(m)}\right),$$

we have

$$\mu\left(\bigcup_{m=1}^{\infty} G_{m,\theta(m)} \setminus \bigcup_{m=1}^{\infty} F_{m,\theta(m)}\right) \leq \mu\left(\bigcup_{m=1}^{\infty}\left(G_{m,\theta(m)} \setminus F_{m,\theta(m)}\right)\right).$$

Since V is full, we have

$$\mu\left(\bigcup_{m=1}^{\infty} G_{m,\theta(m)} \setminus \bigcup_{m=1}^{\infty} F_{m,\theta(m)}\right) \in V.$$

Since

$$\bigcup_{m=1}^{\infty} G_{m,\theta(m)} \setminus \bigcup_{m=1}^{N} F_{m,\theta(m)} \downarrow \bigcup_{m=1}^{\infty} G_{m,\theta(m)} \setminus \bigcup_{m=1}^{\infty} F_{m,\theta(m)}.$$

as $N \to \infty$, by the continuity from above and monotonicity of μ, for $W \in \mathcal{B}_0$, there exists an $N_0 \in N$ such that

$$\mu\left(\bigcup_{m=1}^{\infty} G_{m,\theta(m)} \setminus \bigcup_{m=1}^{N_0} F_{m,\theta(m)}\right) - \mu\left(\bigcup_{m=1}^{\infty} G_{m,\theta(m)} \setminus \bigcup_{m=1}^{\infty} F_{m,\theta(m)}\right) \in W.$$

Then we have

$$\begin{aligned}\mu\left(\bigcup_{m=1}^{\infty} G_{m,\theta(m)} \setminus \bigcup_{m=1}^{N_0} F_{m,\theta(m)}\right) &\in \mu\left(\bigcup_{m=1}^{\infty} G_{m,\theta(m)} \setminus \bigcup_{m=1}^{\infty} F_{m,\theta(m)}\right) + W \\ &\subset V + W \subset U.\end{aligned}$$

Denote $F_U = \cup_{m=1}^{N_0} F_{m,\theta(m)}$ and $G_U = \cup_{m=1}^{\infty} G_{m,\theta(m)}$, then F_U is closed, G_U is open and we have

$$F_U \subset A \subset G_U \text{ and } \mu\left(G_U \setminus F_U\right) \in U.$$

Therefore $A \in \mathcal{E}$. Thus $\mathcal{E}$ is a σ-field.

Next we verify that $\mathcal{E}$ contains all closed subsets of X. Let F be closed in X. Since X is a metric space, one can find a sequence $\{G_n\}$ of open subsets of X such that $G_n \downarrow F$, and hence $\mu(G_n \setminus F) \to 0$ by the continuity from above. Thus, we have $F \in \mathcal{E}$. Consequently, $\mathcal{E}$ is a σ-field which contains all closed subsets of X, so that it also contains all Borel subsets of X. Therefore μ is regular. □

Example 4.1 *Let $X = [0,1]$ be a metric space with the metric $d(x,y) = |x-y|$, $\mathcal{B}(X)$ a Borel measure of X and m the Lebesgue measure on $\mathcal{B}(X)$. Define*

$$\mu(A) = \begin{cases} a \cdot m(A) & \text{if } m(A) < 1, \\ 1 & \text{if } m(A) = 1, \end{cases}$$

where $0 < a < 1$. Then μ is a non-additive measure. It is easy to see that μ is continuous from above. In fact, let $\{A_n\} \subset \mathcal{B}(X)$ a sequence with $A_n \searrow A$ where $A \in \mathcal{B}(X)$. We consider the following cases:

(i) $m(A) = 1$,

(ii) $m(A) < 1$.

In cases (i), since $m(A_n) = 1$, $m(A_n) = 1 = m(A)$ *for all* n. *In cases (ii), we have* $m(A_n) < 1$ *for some* $n \in N$ *because the case that* $m(A_n) = 1$ *for all* n *is impossible. Hence* $\mu(A_n) = a \cdot m(A_n) \to a \cdot m(A) = \mu(A)$. *Thus* μ *is continuous from above. Since* m *is the Lebesgue measure, weakly null additivity and property (S) of* μ *hold. However,* μ *is not continuous from below. In fact, if we take* $A_n = \left[0, 1 - \frac{1}{n}\right] \cup \{1\}$, $n \in N$, *then* $A_n \nearrow X$. *Nevertheless, we have* $\mu(A_n) = a \cdot m(A_n) = a \cdot \left(1 - \frac{1}{n}\right) \nearrow a < 1 = \mu(X)$.

4.2.2 Egoroff's theorem

In this section, we show a version of Egoroff's theorem for a measure which is continuous from above and has property (S) defined on a metric space in the case where the range space is an ordered topological vector space. Egoroff's theorem for the real valued non-additive measure case, see [26], the real valued fuzzy measure case, see [27], and the Riesz space-valued fuzzy measure case, see [16]. For a measure which is strongly order continuous and has property (S), we have obtained the following result; see [42]:

Theorem 4.2 *Let* $\mu : \mathcal{F} \to E$ *be a non-additive measure which is strongly order continuous and has property* (S). *We assume that* E *has property* (EP). *Let* $\{f_n\}$ *be a sequence of* $\mathcal{F}$*-measurable real valued functions on* X *and* f *also such a function. If* $\{f_n\}$ *converges* μ*-a.e. to* f, *then* $\{f_n\}$ *converges* μ*-almost uniformly to* f.

Theorem 4.3 *Let* $\mu : \mathcal{B}(X) \to E$ *be a non-additive Borel measure which is strongly order continuous and has property* (S). *We assume that* E *has property* (EP). *Let* $\{f_n\}$ *be a sequence of Borel measurable real valued functions on* X *and* f *also such a function. If* $\{f_n\}$ *converges* μ*-a.e. to* f, *then there exists an increasing sequence* $\{A_m\} \subset \mathcal{B}(X)$ *such that* $\mu(X \setminus \cup_{m=1}^{\infty} A_m) = 0$ *and* $\{f_n\}$ *converges to* f *uniformly on* A_m *for each* $m \in N$.

Proof. Since $\{f_n\}$ converges μ-a.e. to f, by Theorem 4.2, there exists a decreasing net $\{B_\gamma \mid \gamma \in \Gamma\}$ such that $\mu(B_\gamma) \to 0$ and $\{f_n\}$ converges to f uniformly on each set $X \setminus B_\gamma$. Since E is Hausdorff and satisfies the first axiom of countability, there exists $\{V_m\} \subset \mathcal{B}_0$ such that $\cap_{m=1}^{\infty} V_m = \{0\}$ and V_m is full. Then there exists a $\{\gamma_m\}$ such that $\mu(B_{\gamma_m}) \in V_m$. Put $A_m = X \setminus \cap_{i=1}^{m} B_{\gamma_i}$ for each $m \in N$. The proof is complete. □

Theorem 4.4 *Let X be a metric space and $\mu : \mathcal{B}(X) \to E$ a non-additive Borel measure which is weakly null additive, continuous from above and has property* (S)*. We assume that E has property* (EP)*. Let $\{f_n\}$ be a sequence of Borel measurable real valued functions on X and f also such a function. If $\{f_n\}$ converges μ-a.e. to f, then for any $U \in \mathcal{B}_0$, there exists a closed set F_U such that $\mu(X \setminus F_U) \in U$ and $\{f_n\}$ converges to f uniformly on each F_U.*

Proof. Since $\{f_n\}$ converges μ-a.e. to f, by Theorem 4.3, there exists an increasing sequence $\{A_m\} \subset \mathcal{B}(X)$ such that $\{f_n\}$ converges to f uniformly on A_m for each $m = 1, 2, \ldots$ and $\mu(X \setminus \cup_{m=1}^{\infty} A_m) = 0$. Since E is Hausdorff and satisfies the first axiom of countability, there exists $\{V_n\} \subset \mathcal{B}_0$ such that $\cap_{n=1}^{\infty} V_n = \{0\}$ and V_n is full. By Theorem 4.1, μ is regular. Then for each $m \in N$, there exists an increasing sequence $\{F_{m,n}\}$ of closed sets such that $F_{m,n} \subset A_m$ and $\mu(A_m \setminus F_{m,n}) \in V_n$ for any $n \in N$. Without loss of generality, we can assume that for each $m \in N$, $\{A_m \setminus F_{m,n}\}$ is decreasing as $n \to \infty$. Then we have

$$A_m \setminus F_{m,n} \downarrow \bigcap_{n=1}^{\infty} (A_m \setminus F_{m,n}) \text{ as } n \to \infty.$$

Put $X_{m,n} = (X \setminus \cup_{m=1}^{\infty} A_m) \cup (A_m \setminus F_{m,n})$ and $D_m = \cap_{n=1}^{\infty} X_{m,n}$. Then for each $m \in N$, $X_{m,n} \downarrow D_m$ as $n \to \infty$. Since V_n is full and

$$\mu\left(\bigcap_{n=1}^{\infty} (A_m \setminus F_{m,n})\right) \le \mu\left(A_m \setminus F_{m,n}\right),$$

we have

$$\mu\left(\bigcap_{n=1}^{\infty} (A_m \setminus F_{m,n})\right) \in V_n.$$

Since $\cap_{n=1}^{\infty} V_n = \{0\}$, we have

$$\mu\left(\bigcap_{n=1}^{\infty} (A_m \setminus F_{m,n})\right) = 0.$$

By the weakly null-additivity of μ, we have $\mu(D_m) = 0$ for any $m \in N$. For any $U \in \mathcal{B}_0$, take V, $W \in \mathcal{B}_0$ with $V + W \subset U$ and V is full. By Lemma 4.1, there exists a $\theta \in \Theta$ such that $\mu\left(\cup_{m=1}^{\infty} X_{m,\theta(m)}\right) \in V$. Since $X \setminus \cup_{m=1}^{\infty} F_{m,\theta(m)} \subset \cup_{m=1}^{\infty} X_{m,\theta(m)}$ and V is full, we have

$$\mu\left(X \setminus \bigcup_{m=1}^{\infty} F_{m,\theta(m)}\right) \in V.$$

On the other hand, since $X \setminus \cup_{m=1}^{N} F_{m,\theta(m)} \downarrow X \setminus \cup_{m=1}^{\infty} F_{m,\theta(m)}$ as $N \to \infty$ and μ is continuous from above, there exists an $N_0 \in N$ such that

$$\mu\left(X \setminus \cup_{m=1}^{N_0} F_{m,\theta(m)}\right) - \mu\left(X \setminus \cup_{m=1}^{\infty} F_{m,\theta(m)}\right) \in W.$$

Then we have

$$\mu\left(X \setminus \bigcup_{m=1}^{N_0} F_{m,\theta(m)}\right) \in V + W \subset U.$$

Denote $F_U = \cup_{m=1}^{N_0} F_{m,\theta(m)}$, then F_U is a closed set, $\mu(X \setminus F_U) \in U$ and $F_U \subset \cup_{m=1}^{N} A_m$. It is easy to see that $\{f_n\}$ converges to f uniformly on F_U. □

4.2.3 Lusin's theorem

In this section, we shall further generalize well-known Lusin's theorem in classical measure theory to non-additive measure spaces in the case where the range space is an ordered topological vector space by using the results obtained in Sections 2-3. For the real valued fuzzy measure case, see [27, Theorem 4], and the Riesz space-valued fuzzy measure case, see [18, Theorem 3].

Theorem 4.5 *Let X be a metric space and $\mu : \mathcal{B}(X) \to E$ a non-additive Borel measure on X which is weakly null-additive, continuous from above and has property*

(S). *We assume that E has property* (EP). *If f is a Borel measurable real valued function on X, then for any $U \in \mathcal{B}_0$, there exists a closed set F_U such that $\mu(X \setminus F_U) \in U$ and f is continuous on each F_U.*

Proof. We prove the theorem stepwise in the following two situations.

(a) Suppose that f is a simple function, that is, $f(x) = \sum_{m=1}^{s} a_m \chi_{A_m}(x)$ $(x \in X)$, where $a_m \in R$ $(m = 1, 2, \ldots, s)$, $\chi_{A_m}(x)$ is the characteristic function of the Borel set A_m and $X = \sum_{m=1}^{s} A_m$ (a disjoint finite union). Since E is Hausdorff and satisfies the first axiom of countability, there exists $\{V_n\} \subset \mathcal{B}_0$ such that $\cap_{n=1}^{\infty} V_n = \{0\}$ and V_n is full. By Theorem 4.1, μ is regular. Then for each $m \in N$, there exists a sequence $\{F_{m,n}\}$ of closed sets such that $F_{m,n} \subset A_m$ and $\mu(A_m \setminus F_{m,n}) \in V_n$ for any $n \in N$. We may assume that $\{F_{m,n}\}$ is increasing in n for each m, without any loss of generality. Put $B_{m,n} = A_m \setminus F_{m,n}$ if $m = 1, \ldots, s$ and $B_{m,n} = \emptyset$ if $m > s$, and put $D_m = \cap_{n=1}^{\infty} B_{m,n}$. Since $\cap_{n=1}^{\infty} V_n = \{0\}$, we have $\mu(D_m) = 0$. For any $U \in \mathcal{B}_0$, there exists a $U_1 \in \mathcal{B}_0$ such that $U_1 \subset U$ and U_1 is full. By Lemma 4.1, there exists a $\theta \in \Theta$ such that

$$\mu\left(\bigcup_{m=1}^{\infty} (A_m \setminus F_{m,\theta(m)})\right) \in U_1.$$

Put $F_U = \cup_{m=1}^{s} F_{m,\theta(m)}$, then f is continuous on the closed subset F_U of X and we have

$$\mu(X \setminus F_U) = \mu\left(\bigcup_{m=1}^{s} A_m \setminus \bigcup_{m=1}^{s} F_{m,\theta(m)}\right) \leq \mu\left(\bigcup_{m=1}^{s} (A_m \setminus F_{m,\theta(m)})\right).$$

Since U_1 is full, we have $\mu(X \setminus F_U) \in U_1 \subset U$.

(b) Let f be a Borel measurable real-valued function. Then there exists a sequence $\{\phi_m\}$ of simple functions such that $\phi_m \to f$ as $m \to \infty$ on X. Since E is Hausdorff and satisfies the first axiom of countability, there exists $\{V_n\} \subset \mathcal{B}_0$ such that $\cap_{n=1}^{\infty} V_n = \{0\}$ and V_n is full. By the result obtained in (a), for each simple function ϕ_m and every $n \in N$, there exists a closed set $X_{m,n} \subset X$ such that ϕ_m is continuous on $X_{m,n}$ and $\mu(X \setminus X_{m,n}) \in V_n$. Without loss of generality, we can assume that the sequence $\{X_{m,n}\}$ of closed sets is increasing with respect to n for each m (otherwise, we can take $\cup_{i=1}^{n} X_{m,i}$ instead of $X_{m,n}$ and noting that ϕ_m is a simple function, it remains continuous on $\cup_{i=1}^{n} X_{m,i}$). Since

$$X \setminus X_{m,n} \downarrow \bigcap_{n=1}^{\infty} (X \setminus X_{m,n}) \text{ as } n \to \infty$$

and V_n is full, we have

$$\mu\left(\bigcap_{n=1}^{\infty} (X \setminus X_{m,n})\right) \in V_n.$$

Since $\cap_{n=1}^{\infty} V_n = \{0\}$, we have $\mu\left(\cap_{n=1}^{\infty} (X \setminus X_{m,n})\right) = 0$. By using Lemma 4.1, for any V_n, there exists a sequence $\{\tau_n\} \subset \Theta$ such that

$$\mu\left(\bigcup_{m=1}^{\infty} (X \setminus X_{m,\tau_n(m)})\right) \in V_n,$$

that is, $\mu\left(X \setminus \cap_{m=1}^{\infty} X_{m,\tau_n(m)}\right) \in V_n$. Since the double sequence $\{X \setminus X_{m,n}\}$ is decreasing in $n \in N$ for each $m \in N$, without any loss of generality, we may assume that for fixed $m \in N$, $\tau_1(m) < \tau_2(m) < \cdots < \tau_n(m) < \cdots$. Put $H_n = \cap_{m=1}^{\infty} X_{m,\tau_n(m)}$, then we have a sequence $\{H_n\}$ of closed sets satisfying $H_1 \subset H_2 \subset \cdots$. Since

$$X \setminus H_n \downarrow X \setminus \bigcup_{n=1}^{\infty} H_n \text{ as } n \to \infty$$

and V_n is full, we have

$$\mu\left(X \setminus \bigcup_{n=1}^{\infty} H_n\right) \in V_n.$$

Since $\cap_{n=1}^{\infty} V_n = \{0\}$, we have $\mu(X \setminus \cup_{n=1}^{\infty} H_n) = 0$. Noting that ϕ_m is continuous on $X_{m,n}$ and $H_n \subset X_{m,\tau_n(m)}$, ϕ_m is continuous on H_n for every $m \in N$.

On the other hand, since $\phi_m \to f$ as $m \to \infty$ on X, by Theorem 4.4, there exists a sequence $\{K_n\}$ of closed sets such that $\mu(X \setminus K_n) \in V_n$ and $\{\phi_m\}$ converges to f uniformly on K_n for every $n \in N$. We may assume that $\{K_n\}$ is increasing in n for each m, without any loss of generality. Since $X \setminus K_n \downarrow X \setminus \cup_{n=1}^{\infty} K_n$ as $n \to \infty$ and V_n is full, we have $\mu(X \setminus \cup_{n=1}^{\infty} K_n) \in V_n$. Since $\cap_{i=1}^{\infty} V_i = \{0\}$, we have $\mu(X \setminus \cup_{n=1}^{\infty} K_n) = 0$

and $\{\phi_m\}$ converges to f uniformly on K_n for every $n \in N$. Considering the sequence $\{(X \setminus H_n) \cup (X \setminus K_n)\}$, then we have

$$(X \setminus H_n) \cup (X \setminus K_n) \downarrow \left(X \setminus \bigcup_{n=1}^{\infty} H_n\right) \cup \left(X \setminus \bigcup_{n=1}^{\infty} K_n\right) \quad \text{as } n \to \infty.$$

Since μ is weakly null-additive, we have

$$\mu\left(\left(X \setminus \bigcup_{n=1}^{\infty} H_n\right) \cup \left(X \setminus \bigcup_{n=1}^{\infty} K_n\right)\right) = 0.$$

Moreover, since μ is continuous from above, for any $U \in \mathcal{B}_0$, there exists n_0 such that

$$\mu\left((X \setminus H_{n_0}) \cup (X \setminus K_{n_0})\right) \in U.$$

Put $F_U = H_{n_0} \cap X_{n_0}$, then F_U is a closed set and $\mu(X \setminus F_U) \in U$. We show that f is continuous on F_U. In fact, $F_U \subset H_{n_0}$ and ϕ_m is continuous on H_{n_0}, therefore ϕ_m is continuous on F_U for each $m \in N$. Noting that $\{\phi_m\}$ converges to f uniformly on F_U, then f is continuous on F_U. □

4.3 The case where μ is a continuous from above and below

In what follows, we assume that the topology on E is locally convex. Then E is called an ordered locally convex space; see [5]. The following lemma is an ordered locally convex space-valued counterpart of [27, Lemma 1] and [18, Lemma 1].

Lemma 4.2 *Let $\mu : \mathcal{F} \to E$ be a fuzzy measure. Then the following two conditions are equivalent:*

(i) *μ is weakly null-additive.*

(ii) *For any $U \in \mathcal{B}_0$ and double sequence $\{A_{m,n}\} \subset \mathcal{F}$ satisfying that for each $m \in N$, $A_{m,n} \downarrow D_m$ as $n \to \infty$ and $\mu(D_m) = 0$, there exists a $\theta \in \Theta$ such that $\mu\left(\cup_{m=1}^{\infty} A_{m,\theta(m)}\right) \in U$.*

Proof. (i)$\to$(ii): For any $U \in \mathcal{B}_0$, there exist V, $W \in \mathcal{B}_0$ such that $V + W \subset U$ and V is convex and full. For any $k \in N$, there exists a $V_k \in \mathcal{B}_0$ such that $2^k V_k \subset V$. Let $\{A_{m,n}\}$ be a double sequence such that $A_{m,n} \downarrow D_m$ as $n \to \infty$ and $\mu(D_m) = 0$ for each $m \in N$. Put $D = \cup_{m=1}^{\infty} D_m$. Then we have $\mu(D) = 0$ by the weakly null-additivity and the continuity from below of μ. For any $m \in N$ and $(n_1, \ldots, n_m) \in N^m$, $A_{1,n} \cup D \downarrow D$, $A_{1,n_1} \cup A_{2,n} \cup D \downarrow A_{1,n_1} \cup D$, $\ldots$, and $\cup_{j=1}^{m} A_{j,n_j} \cup A_{m+1,n} \cup D \downarrow \cup_{j=1}^{m} A_{j,n_j} \cup D$ hold as $n \to \infty$. Since μ is continuous from above, there exists an $n_1 \in N$ such that $\mu(A_{1,n} \cup D) - \mu(D) \in V_1$ for any $n \geq n_1$. Since $\mu(D) = 0$, $\mu(A_{1,n_1} \cup D) \in V_1$. For this n_1, $A_{1,n_1} \cup A_{2,n} \cup D \downarrow A_{1,n_1} \cup D$ as $n \to \infty$. For V_2, there exists an $n_2 \geq n_1$ such that $\mu(A_{1,n_1} \cup A_{2,n} \cup D) - \mu(A_{1,n_1} \cup D) \in V_2$ for any $n \geq n_2$. We have $\mu(A_{1,n_1} \cup A_{2,n_2} \cup D) \in \mu(A_{1,n_1} \cup D) + V_2 \subset V_1 + V_2$. Since V is convex, repeating the argument, for any $m \in N$, we have $\mu(\cup_{j=1}^{m} A_{j,n_j} \cup D) \in \sum_{j=1}^{m} V_j \subset \sum_{j=1}^{m} 2^{-j} V \subset V$. Since V is full, we have $\mu(\cup_{j=1}^{m} A_{j,n_j}) \in V$. Let $\theta \in \Theta$ satisfy $\theta(j) = n_j$ $(j = 1, 2, \ldots)$. We have $\cup_{j=1}^{m} A_{j,\theta(j)} \uparrow \cup_{j=1}^{\infty} A_{j,\theta(j)}$ as $m \to \infty$. Since μ is continuous from below, for $W \in \mathcal{B}_0$, there exists an $m_0 \in N$ such that $\mu(\cup_{j=1}^{\infty} A_{j,\theta(j)}) - \mu(\cup_{j=1}^{m} A_{j,\theta(j)}) \in W$ for any $m \geq m_0$. Then we have $\mu(\cup_{j=1}^{\infty} A_{j,\theta(j)}) \in \mu(\cup_{j=1}^{m} A_{j,\theta(j)}) + W \subset V + W \subset U$.

(ii)$\to$(i): The proof is similar to that of Lemma 4.1. Thus μ is weakly null-additive.

□

4.3.1 Regularity of measure

Theorem 4.6 *Let X be a metric space and $\mathcal{B}(X)$ a σ-field of all Borel subsets of X. Let $\mu : \mathcal{B}(X) \to E$ be a fuzzy Borel measure on X. Then μ is regular.*

Proof. By Lemma 4.2, the proof is similar to that of Theorem 4.1. □

4.3.2 Egoroff's theorem

In this subsection we show a version of the Egoroff's theorem for ordered locally convex space-valued fuzzy measures defined on a metric space. For a fuzzy measure, we have obtained the following result; see [42]:

Theorem 4.7 *Let $\mu : \mathcal{F} \to E$ be a fuzzy measure. Let $\{f_n\}$ be a sequence of $\mathcal{F}$-measurable real valued functions on X and f also such a function. If $\{f_n\}$ converges μ-a.e. to f, then $\{f_n\}$ converges μ-almost uniformly to f.*

Theorem 4.8 *Let $\mu : \mathcal{B}(X) \to E$ be a fuzzy Borel measure. Let $\{f_n\}$ be a sequence of Borel measurable real valued functions on X and f also such a function. If $\{f_n\}$ converges μ-a.e. to f, then there exists an increasing sequence $\{A_m\} \subset \mathcal{B}(X)$ such that $\mu(X \setminus \cup_{m=1}^{\infty} A_m) = 0$ and $\{f_n\}$ converges to f uniformly on A_m for each $m \in N$.*

Proof. By Theorem 4.7, the proof is similar to that of Theorem 4.3. □

Theorem 4.9 *Let X be a metric space and $\mu : \mathcal{B}(X) \to E$ a weakly null additive fuzzy Borel measure. Let $\{f_n\}$ be a sequence of Borel measurable real valued functions on X and f also such a function. If $\{f_n\}$ converges μ-a.e. to f, then for any $U \in \mathcal{B}_0$, there exists a closed set F_U such that $\mu(X \setminus F_U) \in U$ and $\{f_n\}$ converges to f uniformly on each F_U.*

Proof. By Theorem 4.8, the proof is similar to that of Theorem 4.4. □

4.3.3 Lusin's theorem

In this subsection, we give Lusin's theorem to ordered locally convex space-valued fuzzy measure spaces. For the real valued case, see [27, Theorem 4], and the Riesz space-valued case, see [18, Theorem 3].

Theorem 4.10 *Let X be a metric space and $\mu : \mathcal{B}(X) \to E$ a weakly null-additive fuzzy Borel measure on X. Let f be a Borel measurable real valued function on X. For any $U \in \mathcal{B}_0$, then there exists a closed set F_U such that $\mu(X \setminus F_U) \in U$ and f is continuous on each F_U.*

Proof. By Theorem 4.6, Lemma 4.2 and Theorem 4.9, the proof is similar to that of Theorem 4.5. □

4.4 Applications

In this section, we mention that our results are applicable to the following ordered topological vector spaces.

(i) Let T be a Hausdorff space and $C(T)$ the space of all real continuous functions defined on T endowed with a pointwise order. Then $C(T)$ is an ordered vector space. The topology of compact convergence on $C(T)$ is a locally full topology; see [5, page 159]. Moreover assume that T is a locally compact space, countable at infinity. Then its topology is defined by the sequence of semi-norms. Thus it is metrizable. For any metrizable topological vector space, there exists a countable neighborhoods of the origin in it; see [5, page 40]. Thus the first axiom of countability holds. Clearly $C(T)$ is Hausdorff. Therefore our results in Section 4 are applicable to $C(T)$.

(ii) Let $C^{\infty}(R^n)$ be the space of all real functions having continuous derivatives of any order. $C^{\infty}(R^n)$ is an ordered locally convex space but not a Riesz space; see [5, page 159]. Similar to (i), the first axiom of countability holds. Clearly $C^{\infty}(R^n)$ is Hausdorff. Therefore our results in Section 4 are applicable to $C^{\infty}(R^n)$.

(iii) Let $L_p([0,1])$ be the space of real valued Lebesgue measurable functions f defined on $[0,1]$ such that $\int_0^1 |f(x)|^p d\mu < \infty (0 < p < \infty)$ endowed with the pointwise order. Then $L_p([0,1])$ is a Riesz space. Moreover $L_p([0,1])$ is a locally solid space; see [1, example 8.6]. If the topology on a Riesz space is locally solid, then it is locally full, see [5, Chapter 7, Proposition 1]. Therefore $L_p([0,1])$ is an ordered topological vector space. However, its topology is not locally convex if $0 < p < 1$. $L_p([0,1])$ satisfies the σ-Lebesgue property, that is, the sequence $\{u_n\}$ converges to 0 with respect to the topology whenever it converges to 0 with respect to the order; see [1, example 8.6]. Moreover $L_p([0,1])$ has the Egoroff property; see [30, Section 71]. Therefore $L_p([0,1])$ has property (EP). Since its topology is defined by a unique quasi-norm, it is metrizable. Thus the first axiom of countability holds. Clearly $L_p([0,1])$ is Hausdorff. Then our results in Section 3 are applicable to $L_p([0,1])$.

Chapter 5

The Hahn-Banach theorem and separation theorem

5.1 Preliminaries

In this chapter, we treated Hahn-Banach's theorem for the mapping that take values in a partially ordered topological vector space.

First, we introduce some basic definitions for partially ordered vector space in this chapter.

Let R be the set of real numbers, N the set of natural numbers, I an indexed set, $(E, \leq)$ a partially ordered set and F a subset of E. The set F is called a *chain* if any two elements are comparable, that is, $x \leq y$ or $y \leq x$ for any x, $y \in F$. An element $x \in E$ is called a *lower bound* of F if $x \leq y$ for any $y \in F$. An element $x \in E$ is called the *minimum* of F if x is a lower bound of F and $x \in F$. If there exists a lower bound of F, then F is said to be *bounded from below*. An element $x \in E$ is called an *upper bound* of F if $y \leq x$ for any $y \in F$. An element $x \in E$ is called the *maximum* of F if x is an upper bound and $x \in F$. If there exists an upper bound of F, then F is said to be *bounded from above*. If the set of all lower bounds of F has the maximum, then the maximum is called an *infimum* of F and denoted by $\inf F$. If the set of all upper bounds of F has the minimum, then the minimum is called a *supremum* of F and denoted by $\sup F$. A partially ordered set E is said to be *complete* if every nonempty chain of E has an infimum; E is said to be *chain complete* if every nonempty chain of E which is bounded from below has an infimum; E is said to be *Dedekind complete* if every nonempty subset of E which is bounded from below has an infimum. A mapping f from E to E is called *decreasing* if $f(x) \leq x$ for every $x \in E$. For the further information of a partially ordered set; see [2, 5, 6, 30, 35].

In a complete partially ordered set, the following theorem is obtained; see [4, 20, 21].

Theorem 5.1 (Bourbaki-Kneser) *Let E be a complete partially ordered set. Let f be a decreasing mapping from E to E. Then f has a fixed point.*

Recently, T. C. Lim [31, Corollary 1] proved a following

Theorem 5.2 *Let E be a complete partially ordered set. Let f be a decreasing mapping from E to E. Then f has a minimal fixed point.*

T. C. Lim [31] proved common fixed point theorem for the family of decreasing commutative mapping, which is a generalization of Theorem 5.2.

A partially ordered set E is called a partially ordered vector space if E is a vector space and $x + z \leq y + z$ and $\alpha x \leq \alpha y$ hold whenever $x, y, z \in E$, $x \leq y$, and α is a nonnegative real number. If a partially ordered vector space E is a lattice, that is, any two elements have a supremum and an infimum, then E is called a *Riesz space.*

Let X be a vector space and E a partially ordered vector space. A mapping f from X to E is said to be *concave* if

$$f(tx + (1-t)y) \geq tf(x) + (1-t)f(y)$$

for any $x, y \in X$ and $t \in [0,1]$. A mapping f from X to E is called *sublinear* if the following conditions are satisfied.

(S1) For any $x, y \in X$, $p(x + y) \leq p(x) + p(y)$.

(S2) For any $x \in X$ and $\alpha \geq 0$ in R, $p(\alpha x) = \alpha p(x)$.

5.2 The Hahn-Banach Theorem

Lemma 5.1 *Let p be a sublinear mapping from a vector space X to a Dedekind complete partially ordered vector space E, K a nonempty convex subset of X and q a concave mapping from K to E such that $q \leq p$ on K. For any $x \in X$, let*

$$f(x) = \inf\{p(x+ty) - tq(y) \mid t \in [0,\infty) \text{ and } y \in K\}.$$

Then f is sublinear such that $f \leq p$. Moreover if g is a linear mapping from X to E, then $g \leq f$ is equivalent to $g \leq p$ on X and $q \leq g$ on K.

Proof. For any $x \in X$,

$$\{p(x+ty) - tq(y) \mid t \in [0,\infty) \text{ and } y \in K\}$$

is bounded from below. Indeed, since

$$p(x+ty) - tq(y) \geq p(ty) - p(-x) - tq(y) \geq -p(-x),$$

it is established. Since E is Dedekind complete, f is well-defined and we have $f(x) \geq -p(-x)$. By definition of f, we have $f(x) \leq p(x)$ and $f(\alpha x) = \alpha f(x)$ for any $\alpha \geq 0$. Thus (S2) is established. Let $x_1, x_2 \in X$. For any $y_1, y_2 \in K$ and $s, t > 0$, we have

$$\begin{aligned} p(x_1 + sy_1) - sq(y_1) + p(x_2 + ty_2) - tq(y_2) \\ \geq p(x_1 + x_2 + (s+t)w) - (s+t)q(w) \\ \geq f(x_1 + x_2), \end{aligned}$$

where $w = \frac{1}{s+t}(sy_1 + ty_2) \in K$. For $p(x_1 + sy_1) - sq(y_1)$, take infimum with respect to $s > 0$ and $y_1 \in K$, we have

$$f(x_1) + p(x_2 + ty_2) - tq(y_2) \geq f(x_1 + x_2)$$

and for $p(x_2 + ty_2) - tq(y_2)$, also take infimum with respect to $t > 0$ and $y_2 \in K$, we have

$$f(x_1) + f(x_2) \geq f(x_1 + x_2).$$

Thus (S1) is established. Suppose that g is a linear mapping from X to E. If $g \leq f$, then we have $g \leq p$. Moreover for any $y \in K$, since

$$-g(y) = g(-y) \leq f(-y) \leq p(-y+y) - q(y) = -q(y),$$

we have $g \geq q$ on K. To prove the converse, suppose that $g \leq p$ on X and $q \leq g$ on K. For any $x \in X$, $y \in K$ and $t > 0$, we have

$$g(x) = g(x+ty) - tg(y) \leq p(x+ty) - tq(y).$$

This implies that $g \leq f$. □

The above lemma is proved in case where the range space is a Dedekind complete Riesz space, see [35, Lemma 1.5.1].

By Theorem 5.2 and Lemma 5.1, we can give a following lemma.

Lemma 5.2 *Let f be a sublinear mapping from a vector space X to a Dedekind complete partially ordered vector space E. Then there exists a linear mapping g from X to E such that $g \leq f$.*

Proof. Let E^X be the set of mappings of X into E. Define $f \leq g$ for $f, g \in E^X$ by $f(x) \leq g(x)$ for all $x \in X$. Then $(E^X, \leq)$ is a partially ordered vector space. Put $f^*(x) = -f(-x)$ for any $x \in X$. Let

$$Y = \{h \in E^X \mid h \text{ is sublinear, } f^* \leq h \leq f\}.$$

Then Y is an ordered set. Since E is Dedekind complete, E^X is also so. Consider an arbitrary chain $Z \subset Y$. Since E^X is Dedekind complete and Z is bounded from below, there exists a $g = \inf Z$ in E^X. Then g is sublinear. Since Y is bounded from below, it holds that $g \in Y$. Thus Y is complete. Let $K = \{y\}$. Then h is also a concave mapping from K to E. We define a decreasing operator S by

$$Sh(x) = \inf\{h(x+ty) - th(y) \mid t \in [0, \infty), y \in K\}$$

for any $h \in Y$. By Lemma 5.1, Sh is sublinear and S is a mapping from Y to Y. Theorem 5.2 implies that there exists a minimal fixed point g of S. It follows from the minimality of g that for any $x, z \in X$, we have $g(x) + g(z) \leq g(x + z)$; see [10, Proposition 1]. Since g is sublinear, we also have $g(x+z) \leq g(x)+g(z)$ for any $x, z \in X$. Then we obtain that for any $x, z \in X$, $g(x + z) = g(x) + g(z)$. Let $x \in X$ and $\alpha > 0$. Since $0 = g(\alpha x - \alpha x) = \alpha g(x) + g(-\alpha x)$, we have $g(-\alpha x) = -\alpha g(x)$. Then for any real number α, we have $g(\alpha x) = \alpha g(x)$.

Since

$$0 = g(0) = g(-\alpha x + \alpha x) = g(-\alpha x) + \alpha g(x)$$

for any $\alpha > 0$ and $x \in X$, we have $g(-\alpha x) = -\alpha g(x)$. Thus $g(\alpha x) = \alpha g(x)$ for any $\alpha \in R$ and $x \in X$. Thus g is linear. □

By Lemma 5.2 and Lemma 5.1, we can prove the Hahn-Banach theorem and the Mazur-Orlicz theorem in case where the range space is a Dedekind complete partially ordered vector space.

Theorem 5.3 *Let p be a sublinear mapping from a vector space X to a Dedekind complete ordered vector space E, Y a vector subspace of X and q a linear mapping from Y to E such that $q \leq p$ on Y. Then q can be extended to a linear mapping g defined on the whole space X to E such that $g \leq p$.*

Proof. By Lemma 5.1, there exists a sublinear mapping f such that $f \leq p$. By Lemma 5.2, there exists a linear mapping g such that $g \leq f$. Then putting $K = Y$ in Lemma 5.1, we have $g \leq p$ on X and $q \leq g$ on Y. Since q is linear, for any $y \in Y$, we have

$$g(-y) \leq f(-y) \leq p(-y + y) - q(y) = -q(y) = q(-y).$$

Then we have $g \leq q$ on Y. Thus $q = g$ on Y. Therefore, the assertion holds. □

We obtain the Mazur-Orlicz theorem in a Dedekind complete partially ordered vector space.

Theorem 5.4 *Let p be a sublinear mapping from a vector space X to a Dedekind complete partially ordered vector space E. Let $\{x_j \mid j \in I\}$ be a family of elements of X and $\{y_j \mid j \in I\}$ a family of elements of E. Then the following* (1) and (2) *are equivalent.*

(1) *There exists a linear mapping f from X to E such that $f(x) \le p(x)$ for any $x \in X$ and $y_j \le f(x_j)$ for any $j \in I$.*

(2)*For any $n \in N$, $\alpha_1, \alpha_2, \ldots, \alpha_n \ge 0$ and $j_1, j_2, \ldots, j_n \in I$, we have*

$$\sum_{i=1}^{n} \alpha_i y_{j_i} \le p\left(\sum_{i=1}^{n} \alpha_i x_{j_i}\right).$$

Proof. The assertion from (1) to (2) is clear. For any $x \in X$, by (2), we have

$$-p(-x) \le p\left(x + \sum_{i=1}^{n} \alpha_i x_{j_i}\right) - \sum_{i=1}^{n} \alpha_i y_{j_i}.$$

Put

$$p_0(x) = \inf\left\{ p\left(x + \sum_{i=1}^{n} \alpha_i x_{j_i}\right) - \sum_{i=1}^{n} \alpha_i y_{j_i} \;\middle|\; \begin{array}{l} n \in N, \alpha_i \ge 0 \text{ and } j_i \in I \\ i = 1, \ldots, n \end{array} \right\}.$$

Since E is Dedekind complete, p_0 is well-defined and p_0 is sublinear. Thus by Lemma 5.2, there exists a linear mapping f from X to E such that $f(x) \le p_0(x)$ for any $x \in X$. Since $p_0(-x_j) \le -y_j$, we have

$$y_j \le -p_0(-x_j) \le f(x_j).$$

Since $p_0(x) \le p(x)$, we have $f(x) \le p(x)$. Thus the assertion holds. □

5.3 The separation theorem

Let X be a vector space, E a Dedekind complete partially ordered vector space and $X \times E$ the Cartesian product of X and E. Let A be a nonempty subset of X and $L(A)$

denotes the affine manifold spanned by A. We denotes the *algebraical relative interior* of A, that is,

$$Int(A) = \left\{ x \in X \;\middle|\; \begin{array}{l} \text{For any } x' \in L(A) \text{ there exists } \varepsilon > 0 \text{ such that} \\ x + \lambda(x' - x) \in A \text{ for any } \lambda \in [0,\, \varepsilon) \end{array} \right\}.$$

If $L(A) = X$, then we write $I(A)$ instead of $Int(A)$. Let f be a linear mapping from X to E, g a linear mapping from E to E and u_0 a point in E. Then

$$H = \{(x,\, y) \in X \times E \mid f(x) + g(y) = u_0\}$$

is empty or an affine manifold in $X \times E$. Let A, B be nonempty subsets of $X \times E$. A nonempty subset $A \subset X \times E$ is said to be *cone* (with the vertex in $x_0 \in X \times E$) if $\lambda > 0$ implies $\lambda(A - x_0) \subset (A - x_0)$. It is said that an affine manifold H separates A and B if

$$H_- \supset A \text{ and } H_+ \supset B$$

where we set

$$H_- = \{(x,\, y) \in X \times E \mid f(x) + g(y) \leq u_0\}$$

and

$$H_+ = \{(x,\, y) \in X \times E \mid f(x) + g(y) \geq u_0\}.$$

The operator P_X defined by $P_X(x,\, y) = x$ for any $(x,\, y) \in X \times E$ is called the *projection* of $X \times E$ onto X. Then P_X is a linear mapping from $X \times E$ to X. We define

$$P_X(A) = \{x \in X \mid \text{ there exists } y \in E \text{ such that } (x,\, y) \in A\}.$$

Then we have

$$P_X(A + B) = P_X(A) + P_X(A)$$

for $A \neq \emptyset$ and $B \neq \emptyset$. The subset

$$C(A) = \{\lambda z \in X \times E \mid \lambda \geq 0,\, z \in A\}$$

is called the *cone spanned by* A. If A is convex, then $C(A)$ is convex. By Lemma 5.2, we obtain the separation theorem in the Cartesian product of a vector space and a Dedekind complete partially ordered vector space.

Theorem 5.5 *Let A and B be subsets of $X \times E$ such that $C(A - B)$ is convex, and $P_X(A - B)$ satisfies the following* (i) *and* (ii) :
(i) $0 \in I(P_X(A - B))$,
(ii) *if $(x, y_1) \in A$ and $(x, y_2) \in B$, then $y_1 \geq y_2$ holds.*
Then there exists a linear mapping f from X to E and a $y_0 \in E$ such that the affine manifold

$$H = \{(x, y) \in X \times E \mid f(x) - y = y_0\}$$

separates A and B.

Proof. By assumption (i) and the definition of $I(P_X(A - B))$, for any $x \in X$ there exists $\varepsilon > 0$ and for any $\lambda \in [0, \varepsilon)$, there exists $y \in E$ such that $(\lambda x, y) \in A - B$. Then there exist $x_1, x_2 \in X$ and $y_1, y_2 \in E$ such that

$$(\lambda x, y) = (x_1 - x_2, y_1 - y_2) = (x_1, y_1) - (x_2, y_2) \in A - B.$$

Define

$$E_x = \{y \in E \mid (x, y) \in C(A - B)\}, \text{ for any } x \in X.$$

Since $\lambda^{-1}(y_1 - y_2) \in E_x$ for any $\lambda \in (0, \varepsilon)$, we have $E_x \neq \emptyset$. Let $y \in E_0$ and $y \neq 0$, then there exists $\lambda > 0$, $(x_1, y_1) \in A$ and $(x_2, y_2) \in B$ such that

$$(0, y) = \lambda\{(x_1, y_1) - (x_2, y_2)\}$$

and $x_1 = x_2$. By assumption (ii), we have $y = \lambda(y_1 - y_2) \geq 0$. We define $E_+ = \{y \in E \mid y \geq 0\}$. Then we have $y \in E_+$. Since $C(A - B)$ is a convex cone, we have $E_x + E_{x'} \subset E_{x+x'}$ for any $x, x' \in X$. We prove that for every $x \in X$ the subset E_x

possesses a lower bound in E. Since E_x is nonempty, for any $x \in X$, there exists $y' \in E$ with $-y' \in E_{-x}$. Then we have

$$y - y' \in E_x + E_{-x} \subset E_0 \subset E_+$$

for any $y \in E_x$. This implies $y' \leq y$ for any $y \in E_x$. Since E is Dedkind complete, operator $p(x) = \inf\{y \mid y \in E_x\}$ is well defined. Then $p(x)$ is sublinear. By Lemma 5.2, there exists a linear mapping f from X to E such that $f(x) \leq p(x)$ for all $x \in X$. Then for any $(x_1, y_1) \in A$, $(x_2, y_2) \in B$, take $x = x_1 - x_2$, we have

$$f(x_1 - x_2) \leq p(x_1 - x_2) \leq y_1 - y_2.$$

Therefore,

$$f(x_1) - y_1 \leq f(x_2) - y_2.$$

Since E is Dedekind complete, there exists a $y_0 \in E$ such that

$$f(x_1) - y_1 \leq y_0 \leq f(x_2) - y_2$$

for any $(x_1, y_1) \in A$, $(x_2, y_2) \in B$. Thus the affine manifold H separates A and B. $\square$

Chapter 6

Conclusion

In the thesis, the first half part, we consider Egoroff's Theorem and Lusin's Theorem for non-additive measure taking value in ordered vector spaces and ordered topological vector spaces. For the real-valued non-additive measures, there are several applications using these theorems, for the vector-valued non-additive measures, we are waiting to see how they will develop.

In the latter half part, we inspired the study of [9], we concern with the Hahn-Banach theorem and other theorems which are equivalent its theorem. Particularly, in [9], several results in optimization theory are discussing. The author will consider contributions to a further research in this area.

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