The size of the set of poles in a complete Riemannian manifold

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Acknowledgements

The author is deeply grateful to Professor Innami for his valuable advice, guidance and encouragement.

Contents

1. Introduction

It is a classical problem to investigate the existence of non-trivial pole or the behavior of geodesics on a complete surface of revolution. Let (M, p) be a pointed complete Riemannian manifold with a base point at $p \in M$ homeomorphic to the plane. We say that a pointed complete Riemannian manifold (M, p) with dimension 2 is a *surface of revolution* with the vertex at p if the Gaussian curvature $G(q)$ of M is constant on the metric t-circle

$$S_p(t) := \{q \in M \mid d(p,q) = t\}$$

around p for $t > 0$, say $G(t)$. Namely, there exists a polar coordinates (r, θ) around p such that the Riemannian metric g on a surface of revolution M is expressed as

$$g : ds^2 = dr^2 + m(r)^2 d\theta^2, \tag{1.1}$$

where the smooth function $m : [0, \infty) \to [0, \infty)$ satisfies the differential equation

$$m''(t) + G(t)m(t) = 0$$

with the initial condition $m(0) = 0, m'(0) = 1$ and is extendable to an odd function around 0. Here $2\pi m(t)$ implies the length of the parallel circle $S_p(t)$.

Let $\gamma : I \to M$ be a geodesic with unit speed in a complete Riemannian manifold M. We say that $\gamma(t_0)$ and $\gamma(t_1)$ are called a *conjugate pair* along γ if there exists a non-trivial Jacobi field along γ that vanishes at $\gamma(t_0)$ and $\gamma(t_1)$. A point $q \in M$ is called a *pole* if there exist no points conjugate to q along every geodesic $\gamma : [0, \infty) \to M$ emanating from $q = \gamma(0)$. In a surface of revolution M the vertex is a pole if M is homeomorphic to the plane. The vertex p is the unique pole in any elliptic paraboloid of revolution. On the other hand, H. von Mangoldt ([3]) proved that the set of all poles of every connected component of two-sheeted hyperboloid of revolution is a non-trivial closed ball centered at its vertex. We discuss his result under a general setting. Put

$$r_p(M) := \sup\{r \mid \text{If } d(p,q) < r, \text{ then } q \in M \text{ is a pole.}\}. \tag{1.2}$$

If M is a surface of revolution homeomorphic to the plane with the vertex at p, then $r_p(M)$ is equal to the distance between p and the farthest pole in M ([7], Lemma 1.1). Tanaka ([6]) generalized von Mangoldt's result and showed a necessary and sufficient condition for $r_p(M) > 0$, and found an equation which determines the $r_p(M)$ for a von Mangoldt's surface of revolution. Here a *von Mangoldt's surface* is by definition a surface of revolution such that the Gaussian curvature is monotone non-increasing with respect to the distance to its vertex.

We have some purposes in this article. Our first one is to give an alternative proof of Tanaka's characterization of $r_p(M) > 0$ for a surface of revolution, moreover, to make his proof much simpler. Actually, in Section 4 we prove the following theorem.

Theorem 1.1. ([6], Theorem 1.10) Let (M, p) be a surface of revolution with the vertex at p. Then $r_p(M) > 0$ if and only if M satisfies

$$\int_1^\infty \frac{1}{m(t)^2}\,dt < \infty \quad \text{and} \quad \liminf_{t\to\infty} m(t) > 0.$$

Our proof is based on the disconjugate property for the solution of the differential equation of Jacobi type, and is seemed to be simpler than the original one, whose proof is mainly based on the geodesic variation. Before the proof, we review the theory of stable Jacobi fields in Section 2. In particular, we study when we can extend a disconjugate interval for a solution of the equation of Jacobi type. In Section 3 we also review the theory of Jacobi field on a surface of revolution. We recall a lemma due to Tanaka([7], [5]).

Lemma 1.2. ([7]) Let (S, o) be a von Mangoldt's surface of revolution with the vertex at o. Let $q \in S \setminus \{o\}$. If the geodesic $\tau_q : [0, \infty) \to S$ emanating from $q = \tau_q(0)$ through o has no points conjugate to q along τ_q, then q is a pole in S.

In particular, as a result of this lemma we have

$$r_o(S) = \max\{r(q) \mid \text{There are no points conjugate to } q = \tau_q(0) \text{ along } \tau_q.\}$$

for every von Mangoldt's surface. Here $r(q)$ is the r-coordinate of the point q.

The second purpose is to prove the following theorem as an application of these theorem and lemma by an independent method. In Section 5 we will prove the following.

Theorem 1.3. ([6], Theorem 2.1) Let (S, o) be a von Mangoldt's surface of revolution such that $\displaystyle\int_1^\infty \frac{1}{m(r)^2}\,dr < \infty$. Let $\displaystyle y_\infty(t) = m(t)\int_t^\infty \frac{1}{m(r)^2}\,dr$ $(t > 0)$. Then the constant $c(m) := 2y'_\infty(0)$ exists. Set

$$\bar{F}(x) := c(m) - \int_x^\infty \frac{1}{m(r)^2}\,dr\,.$$

We then have the following.

(1) If $c(m) \le 0$, then $r_o(S) = \infty$.

(2) If $c(m) > 0$, then $r_o(S)$ is the unique zero point of the function $\bar{F}$.

Tanaka first proved Theorem 1.3, where he defined the constant $c(m)$ as follows:

$$c(m) := \int_0^\infty \frac{m(r) - rm'(r)}{m(r)^3}\,dr.$$

However, the geometrical meaning of this constant arising in the equation was not explained. We emphasize that the constant is expressed by means of the stable Jacobi field. Our method is based on the disconjugate property of Jacobi field along a ray emanating from the vertex. We will make his proof much simpler and the geometrical meaning of the equation clearer.

The third purpose is to prove Theorem 1.6 in Section 7 in order to estimate the size of the set of all poles in a complete Riemannian manifold, combining the immediate consequence of Theorem 1.4, Lemma 1.2 and Rauch's comparison theorem. Let M be a complete Riemannian manifold and T_pM the tangent space to M at a point $p \in M$. Let $\exp_p : T_pM \to M$ be the exponential map at p. Let $v \in T_pM$ be any unit vector. Then $\gamma_v(t) = \exp_p(tv)$ is the unit speed geodesic with $\gamma_v(0) = p$ and $\gamma_v'(0) = v$. Define functions i_p and c_p on the set of all unit tangent vectors at p, say S_pM, as follows:

(1) $i_p(v)$ is the least upper bound of those r such that γ_v is a minimizing geodesic in $[0, r]$,

(2) $c_p(v)$ is the least upper bound of those r such that no point is conjugate to p along γ_v in $[0, r)$.

It follows that $i_p(v) \le c_p(v)$ for all vectors $v \in S_pM$. Set

$$\begin{aligned} \widetilde{C}(p) &= \{i_p(v)v \mid v \in S_pM\}, \\ \widetilde{J}(p) &= \{c_p(v)v \mid v \in S_pM\}. \end{aligned}$$

We call $\widetilde{C}(p)$ the *tangent cut locus* at p, $C(p) = \exp_p \widetilde{C}(p)$ the *cut locus* of p and $x \in C(p)$ a *cut point* of p. We call $\widetilde{J}(p)$ the *tangent conjugate locus* at p and $x \in J(p) = \exp_p \widetilde{J}(p)$ the *conjugate point* to p.

Rauch ([2]) conjectured that $\widetilde{C}(p) \cap \widetilde{J}(p) \ne \emptyset$ for every point $p \in M$ if a Riemannian manifold M is compact and simply connected. The conjecture is valid if M is homeomorphic to the 2-sphere or isometric to a symmetric space. Weinstein ([1]) has given a negative answer to the conjecture, in general, proving that any compact differentiable manifold M not homeomorphic to the 2-sphere has a Riemannian metric on M such that there exists a point $p \in M$ whose tangent conjugate and tangent cut loci are disjoint. A well known lemma due to Klingenberg states that if $p \in M$ and $x_0 \in C(p)$ are such that $d(p, x_0) = d\big(p, C(p)\big)$, then there exists either a minimizing geodesic connecting p and x_0 along which x_0 is conjugate to p or else a geodesic loop at p through x_0 whose length is $2d\big(p, C(p)\big)$ (cf. [4]). Our contribution is a generalization of these theorems.

Theorem 1.4. Let M be a complete Riemannian manifold and $p \in M$ a point with $C(p) \ne \emptyset$. Then one of the following is true.

(1) $\widetilde{C}(p) \cap \widetilde{J}(p) \ne \emptyset$.

(2) There exist at least two geodesics connecting p and every point $q \in M$. Here we regard a constant curve as a geodesic when $q = p$.

In Section 6 we will prove this theorem. The main part of the proof is to find a geodesic which is not minimizing. It is important in the proof that the ellipsoids are star-shaped around their foci. We will pay our attention to a point in $C(p) \cap E(p,q;r_0)$ where $E(p,q;r_0)$ is the smallest ellipsoid with foci p and q intersecting $C(p)$. We will detail to the geodesics in (2) as Lemma 6.1.

We may equivalently say that a point $q \in M$ is a pole if the exponential map $\exp_q : T_qM \to M$ is a diffeomorphism. If there exists a pole $q \in M$ and the dimension of M is n, then M is diffeomorphic to the n-dimensional Euclidean space $\mathbb{E}^n$ and all geodesics emanating from q are minimizing, that is, $C(q) = \emptyset$. In particular, there exists only one geodesic connecting the pole q and every point $x \in M$. Thus, we have the following as a direct consequence of Theorem 1.4.

Corollary 1.5. Let M be a complete Riemannian manifold with a pole. We then have $\widetilde{C}(x) \cap \widetilde{J}(x) \neq \emptyset$ if a point $x \in M$ is not a pole.

We use Corollary 1.5 to estimate the size of the set of all poles in a complete Riemannian manifold with a pole. Poles are useful for the function theory on Riemannian manifolds and have been discussed in many papers. The set of poles has recently been studied in a complete surface of revolution which is homeomorphic to the plane, as stated before ([6], [7]).

Let M be a complete Riemannian manifold with a pole p and P the set of all poles in M. Let $\bar{B}(p,r)$ be the closed r-ball centered at p. Then it follows from (1.2), that

$$\bar{B}\big(p, r_p(M)\big) \subset P.$$

If M is, in addition, a surface of revolution, then

$$P = \bar{B}\big(p, r_p(M)\big).$$

Let $x \in M \setminus \{p\}$. Let $\tau_x : [0,\infty) \to M$ be the geodesic with $\tau_x(0) = x$ and $\tau_x\big(d(p,x)\big) = p$. Let $K(\pi_x)$ denote the sectional curvature of the tangent plane $\pi_x \subset T_xM$ at $x \in M$. We will prove the following theorem in Section 7, as an application of Lemma 1.2, using the Rauch comparison theorem for Jacobi vector field along minimizing geodesics passing through p and o.

Theorem 1.6. Let M be a complete Riemannian manifold with a pole p and P the set of all poles in M. Let S be a von Mangoldt's surface of revolution with the vertex at o and G its Gaussian curvature function. Then the following are true.

(1) $P \subset \bar{B}\big(p, r_o(S)\big)$ if $K(\pi_x) \geq G\big(d(p,x)\big)$ for all points $x \in M$ and all tangent planes $\pi_x \subset T_xM$.
(2) $\bar{B}\big(p, r_o(S)\big) \subset P$ if $K(\pi_x) \leq G\big(d(p,x)\big)$ for all points $x \in M$ and all tangent planes $\pi_x \subset T_xM$.

The property in Corollary 1.5, $\widetilde{C}(x) \cap \widetilde{J}(x) \neq \emptyset$ if x is not a pole, will play the most important role in the proof of Theorem 1.6. When M is simply connected and the dimension of M is two, this property is true for all points $x \in M$ with $C(x) \neq \emptyset$ ([1]). Therefore, Theorem 1.6 can be slightly changed by replacing "with a pole" by "being simply connected".

Proposition 1.7. Let M be a complete simply connected Riemannian 2-manifold with a base point at $p \in M$ and G its Gaussian curvature. If S_1 and S_2 are von Mangoldt's surfaces of revolution with the vertices at o_1, o_2 and G_1, G_2 are their Gaussian curvature functions, respectively, such that

$$G_1\big(d(p,x)\big) \leq G(x) \leq G_2\big(d(p,x)\big)$$

for all $x \in M$, then p is a pole. If P is the set of all poles in M, then

$$\bar{B}\big(p, r_{o_2}(S_2)\big) \subset P \subset \bar{B}\big(p, r_{o_1}(S_1)\big), \text{ that is, } r_{o_2}(S_2) \leq r_p(M) \leq r_{o_1}(S_1).$$

In Section 8 we will show some examples for Theorem 1.4. This article is merged with two papers, one of which have been issued as [10], the other will be issued as [8] before long.

2. Disconjugate properties for Jacobi fields

Let M be a complete Riemannian 2-manifold. Let $\gamma : [0,\infty) \to M$ be a unit speed geodesic. Let $\{\mathbf{e}_1 = \gamma', \mathbf{e}_2\}$ be an orthonormal parallel frame field along γ. We say that a vector field Y along γ is a *Jacobi field* if it satisfies the Jacobi equation

$$\nabla_{\gamma'}\nabla_{\gamma'}Y + R(Y,\gamma')\gamma' = \mathbf{0},$$

where $R : \mathcal{X}(M)^3 \to \mathcal{X}(M)$ denotes the Riemannian curvature tensor. Define a linear map

$$F_t : M_{\gamma(t)} \to M_{\gamma(t)}\ ,\ F_t(\mathbf{x}) = R\big(\mathbf{x}, \gamma'(t)\big)\gamma'(t).$$

We then have

$$\begin{aligned} &F_t(\mathbf{e}_1) = \mathbf{0}, \\ &g\big(R(\mathbf{e}_2,\gamma')\gamma', \mathbf{e}_1\big) = 0, \\ &g\big(R(\mathbf{e}_2,\gamma')\gamma', \mathbf{e}_2\big) = G\big(\gamma(t)\big). \end{aligned}$$

Let $\mathcal{J}_\gamma$ be the set of all Jacobi vector fields along γ, which forms a vector space over $\mathbb{R}$. If $Y(t) = x(t)\mathbf{e}_1(t) + y(t)\mathbf{e}_2(t) \in \mathcal{J}_\gamma$, we then have

$$x''(t) = 0 \Longleftrightarrow x(t) = c_1 t + c_2, \tag{J$_0$}$$

$$y''(t) + G\big(\gamma(t)\big)y(t) = 0. \tag{J$_\mathrm{G}$}$$

We have the following contents on the disconjugate property for later use by digesting Chapter XI in [9]. The differential equation (J$_\mathrm{G}$) is said to be *disconjugate* on I if every non-trivial solution $y : I \to \mathbb{R}$ of (J$_\mathrm{G}$) along γ vanishes at most once, where $y(t)$ means that $Y_\perp(t) = y(t)\mathbf{e}_2(t) \in \mathcal{J}_\gamma$. Then, we regard y of the solution (J$_\mathrm{G}$) as a Jacobi field along γ. The disconjugate property is stated as follows:

For each solution y_s of (J$_\mathrm{G}$) on I with $y_s(s) = 0$ and $y_s'(s) \neq 0$, we have $y_s(t) \neq 0$ for all $t \in I \setminus \{s\}$.

This property implies that the solution of (J$_\mathrm{G}$) is uniquely determined by its values at two distinct points in I.

We have a general solution y of (J$_\mathrm{G}$) from a non-trivial solution z by using the variation method of constants as following formula:

$$y(t) = z(t)\left(\int \frac{1}{z(t)^2}dtC_1 + C_2\right), \tag{2.1}$$

where C_1, C_2 are constants. This is proved as follows:

Let $y(t) = z(t)C(t)$. Then it follows that

$$\begin{aligned}
0 &= y''(t) + G(t)y(t) \\
&= z''(t)C(t) + 2z'(t)C'(t) + z(t)C''(t) + G(t)y(t) \\
&= -G(t)z(t)C(t) + 2z'(t)C'(t) + z(t)C''(t) + G(t)z(t)C(t) \\
&= 2z'(t)C'(t) + z(t)C''(t).
\end{aligned}$$

Let $C'(t) = \dfrac{u(t)}{z(t)}$. Then $C''(t) = \dfrac{u'(t)z(t) - u(t)z'(t)}{z(t)^2}$ and $\dfrac{z'(t)u(t)}{z(t)} + u'(t) = 0$. Since $\big(z(t)u(t)\big)' = 0$, we have $u(t) = \dfrac{C_1}{z(t)}$ and $C'(t) = \dfrac{C_1}{z(t)^2}$. □

Assume that $(\mathrm{J_G})$ is disconjugate on I and $c \in I$. Let y_c be the solution of $(\mathrm{J_G})$ with $y_c(c) = 0$ and $y_c'(c) = 1$. Then the solution y_s of $(\mathrm{J_G})$ with $y_s(c) = 1$ and $y_s(s) = 0$ is given by the following formula for each $s \in I \setminus \{c\}$ from (2.1)

$$y_s(t) = y_c(t) \int_t^s \frac{1}{y_c(w)^2}\, dw \tag{2.2}$$

for all t such that $c \notin (t, s)$. This is proved as follows:

We may put

$$y_s(t) = y_c(t) \left(\int_t^s \frac{1}{y_c(w)^2}\, dw C_1 + C_2 \right)$$

for all $t \in I$ such that $c \notin (t, s)$ from (2.1). Since $y_s(s) = 0$ and $y_c(s) \neq 0$, we see $C_2 = 0$ by putting $t = s$. Define

$$F : I \to \mathbb{R},\ F(t) = y_s'(t)y_c(t) - y_s(t)y_c'(t)$$

for all $t \in I$. Then

$$F'(t) = -G(t)y_s(t)y_c(t) + G(t)y_s(t)y_c(t) = 0.$$

Therefore, $F(t)$ is constant for all $t \in I$, and $F(c) = -1, F(s) = y_s'(s)y_c(s) = -1$. Since

$$y_s'(t) = y_c'(t) \int_t^s \frac{1}{y_c(w)^2}\, dw C_1 - \frac{C_1}{y_c(t)},$$

we have

$$C_1 = -y_s'(s)y_c(s) = 1$$

by putting $t = s$. □

We have from (2.2)

$$y_u(t) - y_s(t) = y_c(t) \int_s^u \frac{1}{y_c(w)^2}\, dw$$

for all $t \in [c,u]$. Differentiating it at $t = c$, we have

$$y_u'(c) - y_s'(c) = \int_s^u \frac{1}{y_c(w)^2}\, dw. \tag{2.3}$$

We get the following.

Lemma 2.1. Let $c < s$ and y_s be defined as in (2.2). Then

$$y_s'(c) \to -\infty \text{ as } s \to c+0.$$

PROOF. Let $y_c : [c,b] \to \mathbb{R}$ be the solution of $(\mathrm{J_G})$ with $y_c(c) = 0$ and $y_c'(c) = 1$. Fix $u > c$ in such a way that $(\mathrm{J_G})$ is disconjugate on $[c,u]$. By construction of y_c we find $y_c(w) = (w-c)\, y_c'(c+\theta(w-c)), 0 < \theta < 1$. Since $y_c'(c+\theta(w-c)) \to 1$ as $w \to c$, there exists for every $\varepsilon > 1$ a $\delta > c$ such that if $c < w < \delta$, then $y_c(w)^2 \le (w-c)^2\varepsilon$. Therefore,

$$\begin{aligned} y_u'(c) - y_s'(c) &= \int_s^\delta \frac{1}{y_c(w)^2}\, dw + \int_\delta^u \frac{1}{y_c(w)^2}\, dw \\ &\ge \int_s^\delta \frac{1}{y_c(w)^2}\, dw \\ &\ge \frac{1}{\varepsilon}\int_s^\delta \frac{1}{(w-c)^2}\, dw \\ &= \frac{1}{\varepsilon}\left(-\frac{1}{\delta - c} + \frac{1}{s-c}\right). \end{aligned}$$

Thus, $y_u'(c) - y_s'(c) \to \infty$, and hence, $y_s'(c) \to -\infty$ as $s \to c+0$. □

If the orientation of parameter is reversed, then we have $y_s'(c) \to +\infty$ as $s \to c-0$.

Lemma 2.2. Assume that $(\mathrm{J_G})$ is disconjugate on I. Let $c < s$ $(c, s \in I)$ and let $y_s : [c,s] \to \mathbb{R}$ be defined as in (2.2). If $y : [c,s] \to \mathbb{R}$ satisfies $(\mathrm{J_G})$ such that $y(c) = 1$ and $y(t) \ne 0$ for all $t \in [c,s]$, then $y(t) > y_s(t)$ for all $t \in (c,s]$.

PROOF. Define

$$F : [c,s] \to \mathbb{R}\,,\ F(t) = y(t) - y_s(t).$$

Then $F(t)$ satisfies $(\mathrm{J_G})$ and $F(c) = 0, F(s) = y(s) > 0$. Therefore, $F(t)$ is non-trivial. If there exists a $t_0 \in (c,s)$ such that $F(t_0) = 0$, then c and t_0 form a conjugate pair, a contradiction. □

Next, we have a condition which implies the disconjugate property.

Lemma 2.3. Assume that there exists a solution $y : I \to \mathbb{R}$ of $(\mathrm{J_G})$ with $y(t) \ne 0$ for all $t \in I$. Then $(\mathrm{J_G})$ is disconjugate on I.

PROOF. We find

$$\widetilde{y}(t) = y(t)\left(\int_{t_0}^t \frac{1}{y(w)^2}\, dw C_1 + C_2\right)$$

is a general solution of (J_G) from (2.1). Let $t_0 \in I$. If $\widetilde{y}(t_0) = 0$, we then have $C_2 = 0$. Since

$$\widetilde{y}'(t) = y'(t) \int_{t_0}^{t} \frac{1}{y(w)^2}\, dw C_1 + \frac{C_1}{y(t)},$$

it follows $C_1 = \widetilde{y}'(t_0) y(t_0)$. Let $\widetilde{y}$ be non-trivial. Then $\widetilde{y}'(t_0) \neq 0$ and

$$\widetilde{y}(t) = \widetilde{y}'(t_0) y(t_0) y(t) \int_{t_0}^{t} \frac{1}{y(w)^2}\, dw.$$

It follows that $\widetilde{y}$ vanishes only at $t = t_0$. □

Let $G : \mathbb{R} \to \mathbb{R}$ be the function as defined in (J_G).

Theorem 2.4. Assume that (J_G) is disconjugate on $(c - \varepsilon, \infty)$ for some positive ε. Let y_s, $y_{c-\varepsilon} : \mathbb{R} \to \mathbb{R}$ be the solutions of (J_G) with $y_s(c) = 1$, $y_s(s) = 0$ and with $y_{c-\varepsilon}(c) = 1$, $y_{c-\varepsilon}(c - \varepsilon) = 0$, respectively. Then $y_s(t)$ converges to $y(t)$ as $s \to \infty$ for each $t \in \mathbb{R}$. Moreover, $y : \mathbb{R} \to \mathbb{R}$ is the solution of (J_G) such that $y_{c-\varepsilon}(t) \geq y(t) > y_s(t)$ for all $t \in (c, s)$. (cf. Figure 1 in the case of $c < u < s$.)

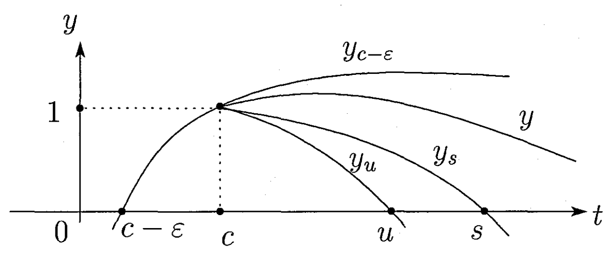

FIGURE 1. The solutions of (J_G).

PROOF. Let $y_c : \mathbb{R} \to \mathbb{R}$ be the solution of (J_G) with $y_c(c) = 0$ and $y_c'(c) = 1$. From Lemma 2.2 we have

$$y_s(t) = y_c(t) \int_{t}^{s} \frac{1}{y_c(w)^2}\, dw \ (s > t > c).$$

Let $s > u > t > c$. Then we have

$$y_s(t) - y_u(t) = y_c(t) \int_{u}^{s} \frac{1}{y_c(w)^2}\, dw > 0,$$

$$y_s'(c) - y_u'(c) = \int_{u}^{s} \frac{1}{y_c(w)^2}\, dw > 0$$

from (2.2) and (2.3). By Lemma 2.1 and the assumption, it follows

$$y_{c-\varepsilon}(t) > y_s(t) > y_u(t).$$

Therefore, there exists a function $y(t)$ such that $y_s(t) \to y(t)$ as $s \to \infty$ for each $t \in [c, \infty)$. Let $T > 0$. By Lemma 2.3 and the assumption, we see that for some $\varepsilon > 0$

$$y_{c-\varepsilon}'(t) \geq y_s'(t) > y_{T+c+1}'(t)$$

for all $s > T + c + 1$ and for all $t \in [c, T + c]$. Since

$$y_s'(t') - y_s'(t) + \int_t^{t'} G(w) y_s(w) dw = 0$$

for all $t, t' \in [-T + c, T + c]$, we find a constant C such that

$$\left| y_s'(t') - y_s'(t) \right| \leq C |t' - t| .$$

By Ascoli-Arzelà's Theorem, we have

$$\left| y'(t') - y'(t) \right| \leq C |t' - t|$$

for all $t, t' \in [-T + c, T + c]$ as $s \to \infty$. Finally, we have that y' is continuous on $\mathbb{R}$. Since

$$y'(t') - y'(t) + \int_t^{t'} G(w) y(w) dw = 0$$

for all $t, t' \in \mathbb{R}$, we have that y' is differentiable and y satisfies $(\mathrm{J_G})$. □

Combining Theorem 2.4 and (2.2), we have the following.

Corollary 2.5. Assume that $(\mathrm{J_G})$ is disconjugate on $(c - \varepsilon, \infty)$ for some positive ε. Let y_s for each $s > c$ be defined as in (2.2). Then $y_s(t)$ for each $t \in [c, \infty)$ converges to $y_\infty(t)$ as $s \to \infty$, which is the solution of $(\mathrm{J_G})$. Moreover, $y_\infty(t)$ is given by the following formula:

$$y_\infty(t) = y_c(t) \int_t^\infty \frac{1}{y_c(w)^2} dw \ (t > c).$$

Conversely, $\displaystyle\int_{c+1}^\infty \frac{1}{y_c(w)^2} dw < \infty$ shows that there exists a positive ε such that $(\mathrm{J_G})$ is disconjugate on $(c - \varepsilon, \infty)$. The following corollary will play an important role in our proof of Theorem 1.1.

Remark 2.6. In the statements in Theorem 1.3, $m(t)$ is equal to $y_0(t)$ as above, that is, $m(t)\mathbf{e}_2(t) \in \mathcal{J}_\mu$, where μ is some unit speed meridian, and so y_∞ is the solution of $(\mathrm{J_G})$ along a ray emanating from the vertex.

Corollary 2.7. Assume that $(\mathrm{J_G})$ is disconjugate on $[c, \infty)$ and

$$\int_{c+1}^\infty \frac{1}{y_c(w)^2} dw < \infty.$$

Then $[c, \infty)$ is extendable to a disconjugate interval $[c - \varepsilon, \infty)$ of $(\mathrm{J_G})$ for some positive ε.

3. Properties of Jacobi fields on a surface of revolution

Let M be a complete surface of revolution with the vertex at p homeomorphic to the plane, whose metric is expressed as (1.1). It is known that the Gaussian curvature of M at each point $q \in S_p(t)$ is given by

$$G(t) = -\frac{m''(t)}{m(t)}.$$

Let $\gamma : [0, \infty) \to M$ be a unit speed geodesic and put $\gamma(t) := \big(r(t), \theta(t)\big)$ for all $t \in [0, \infty)$. Let ν be a constant. The differential equations for a geodesic are as follows:

$$\frac{d^2u^i(t)}{dt^2} + \sum_{j,k=1}^{2} \Gamma^{\,i}_{jk} \frac{du^j(t)}{dt}\frac{du^k(t)}{dt} = 0 \ (i = 1, 2),$$

where $\Gamma^{\,i}_{jk}$ denotes Christoffel's symbol. Put $r := u^1, \theta := u^2$, then we have

$$r'' - mm'(\theta')^2 = 0,$$
$$\theta'' + 2\frac{m'}{m}r'\theta' = 0,$$

since

$$\Gamma^{1}_{11} = \Gamma^{1}_{12} = \Gamma^{2}_{11} = \Gamma^{2}_{22} = 0 \ , \ \Gamma^{1}_{22} = -mm' \ , \ \Gamma^{2}_{12} = \frac{m'}{m}.$$

From the second equation of the preceding, we have

$$\theta'(t) = \frac{\nu}{m(r(t))^2}.$$

Combining this result with (1.1), we have

$$r'(t) = \pm\frac{\sqrt{m(r(t))^2 - \nu^2}}{m(r(t))}. \tag{3.1}$$

A 1-parameter family of geodesics $\gamma_\varepsilon : [0, \infty) \times (-\varepsilon_0, \varepsilon_0) \to M, \gamma_\varepsilon(t) = \big(r(t), \theta(t) + \varepsilon\big)$ is a geodesic variation. Thus,

$$\left(\frac{\partial}{\partial \varepsilon}\right)_{\varepsilon=0} \gamma_\varepsilon(t) = \left(\frac{\partial}{\partial \theta}\right)_{\gamma(t)} \in \mathcal{J}_\gamma.$$

Put

$$\left(\frac{\partial}{\partial \theta}\right)_{\gamma(t)} =: a(t)\mathbf{e}_1(t) + b(t)\mathbf{e}_2(t),$$

where $\{\mathbf{e}_1 = \gamma', \mathbf{e}_2\}$ is an orthonormal parallel frame field along γ. Since

$$g_{\gamma(t)}\left(\frac{\partial}{\partial \theta}, \frac{\partial}{\partial \theta}\right) = m(r(t))^2$$

for all $\gamma(t) \in M \setminus \{p\}$, we have the following from (J_0)

$$a(t) = g_{\gamma(t)}\left(\frac{\partial}{\partial\theta}, \mathbf{e}_1\right) = m(r(t))\cos\xi(t) = \nu\,,$$
$$b(t) = g_{\gamma(t)}\left(\frac{\partial}{\partial\theta}, \mathbf{e}_2\right) = m(r(t))\sin\xi(t) = \pm\sqrt{m(r(t))^2 - \nu^2}\,,$$

where $\xi(t)$ denotes the angle between $\gamma'(t)$ and $\left(\frac{\partial}{\partial\theta}\right)_{\gamma(t)}$. The first formula is called Clairaut's relation.

Let $\tau_q : [0,\infty) \to M$ for each $q \in M\setminus\{p\}$ be the geodesic emanating from $q = \tau_q(0)$ through p and let $\mu_q : [0,\infty) \to M$ denote the meridian emanating from $p = \mu_q(0)$ through q. With these notation, we state the following lemmas and proposition.

Lemma 3.1. (Compare Lemma 1.1 in [6].) Let $\gamma : [0,\infty) \to M$ be a geodesic. If $r'(t) = 0$ at two distinct parameter values, then γ is not a ray.

PROOF. Let the first zero point of $r' : (0,\infty) \to \mathbb{R}$ be t_0 and the second t_1. From (3.1) and that

$$y(t) = \sqrt{m(r(t))^2 - \nu^2}$$

is the solution of (J_G), $\gamma(t_0)$ and $\gamma(t_1)$ is a conjugate pair along γ. □

Lemma 3.2. (See Lemma 1.2 in [6].) Let $\gamma : [0,\infty) \to M$ be a geodesic. If $r_0 := \lim_{t\to\infty} d(p,\gamma(t)) < \infty$, then $m'(r_0) = 0$, that is, the parallel circle $S_p(r_0)$ is a geodesic.

For simplicity, put $\rho := d(p,q)$.

Lemma 3.3. (See Lemma 1.3 in [6].) If $\liminf_{t\to\infty} m(t) = 0$, then $\mu_q|[\rho,\infty)$ for every $q \in M \setminus \{p\}$ is a unique ray emanating from q.

We give an alternative proof for the following lemma.

Lemma 3.4. (Compare Lemma 1.4 in [6].) If $\int_1^\infty \frac{1}{m(r)^2}\,dr = \infty$, then τ_q is not a ray for any $q \in M \setminus \{p\}$.

PROOF. Let $y_\rho(t) = m(t-\rho)$ for all $t \geq 0$. Then y_ρ is the solution of (J_G) along τ_q with $y_\rho(\rho) = m(0) = 0$, $y'_\rho(\rho) = m'(0) = 1$. From (2.2), the solution y_s of (J_G) with $y_s(\rho) = 1$ and $y_s(s) = 0$ is written as follows:

$$y_s(t) = m(t-\rho)\int_t^s \frac{1}{m(w-\rho)^2}\,dw = m(t-\rho)\int_{t-\rho}^{s-\rho} \frac{1}{m(r)^2}\,dr$$

for all $t > \rho$. If τ_q is a ray, then there exists no conjugate pair along τ_q. By Corollary 2.5, we have $\int_{t-\rho}^{\infty} \frac{1}{m(r)^2}\, dr < \infty$, a contradiction. $\square$

For a point $q \in M \setminus \{p\}$ and for each $\nu \in [-m(\rho), m(\rho)]$ we define two geodesics $\beta_\nu, \gamma_\nu : [0, \infty) \to M$ emanating from q, whose velocity vectors at $t = 0$ are given by

$$\beta_\nu'(0) = \sqrt{1 - \left(\frac{\nu}{m(\rho)}\right)^2} \left(\frac{\partial}{\partial r}\right)_{\beta_\nu(0)} + \frac{\nu}{m(\rho)^2} \left(\frac{\partial}{\partial \theta}\right)_{\beta_\nu(0)}, \tag{3.2}$$

$$\gamma_\nu'(0) = -\sqrt{1 - \left(\frac{\nu}{m(\rho)}\right)^2} \left(\frac{\partial}{\partial r}\right)_{\gamma_\nu(0)} + \frac{\nu}{m(\rho)^2} \left(\frac{\partial}{\partial \theta}\right)_{\gamma_\nu(0)}, \tag{3.3}$$

respectively. Thus, we have smooth 1-parameter families of geodesics whose variation vector fields are Jacobi fields

$$X_\nu(t) := \frac{\partial}{\partial \nu}\big(\beta_\nu(t)\big) \quad \text{and} \quad Y_\nu(t) := \frac{\partial}{\partial \nu}\big(\gamma_\nu(t)\big)$$

along β_ν and γ_ν, respectively. We denote by $\gamma := \gamma_c$ and $Y := Y_c$ for an arbitrary fixed $c \in \big(-m(\rho), m(\rho)\big)$. With this notation, we have the following.

Proposition 3.5. (Compare Lemma 1.6 in [6].) Let γ be the geodesic defined as above. Assume that $t_0, t_1 \in [0, \infty)$ $(t_0 < t_1)$ are the first and second zeros of $r' : [0, \infty) \to \mathbb{R}$. Then $\gamma(s)$ for $s \in (t_0, t_1)$ is a point conjugate to $\gamma(0)$ along γ if and only if

$$\left(\frac{\partial}{\partial \nu}\right)_{\nu = c} \theta\big(\gamma_\nu(s)\big) = 0.$$

Proof. Let $\gamma_\nu(t) = \big(r(t, \nu), \theta(t, \nu)\big)$. Then

$$Y_\nu(t) = \frac{\partial}{\partial \nu}\big(\gamma_\nu(t)\big) = \left(\frac{\partial}{\partial \nu}\big(r(t, \nu)\big), \frac{\partial}{\partial \nu}\big(\theta(t, \nu)\big)\right).$$

The point $\gamma(s)$ is conjugate to $\gamma(0)$ along γ if and only if

$$\left(\frac{\partial}{\partial \nu}\right)_{\nu = c} r(s, \nu) = 0 \quad \text{and} \quad \left(\frac{\partial}{\partial \nu}\right)_{\nu = c} \theta(s, \nu) = 0.$$

Thus, $\left(\frac{\partial}{\partial \nu}\right)_{\nu = c} \theta(s, \nu) = 0$ follows.

Next, we have only to prove that $\left(\frac{\partial}{\partial \nu}\right)_{\nu = c} r(s, \nu) = 0$ when $\left(\frac{\partial}{\partial \nu}\right)_{\nu = c} \theta\big(\gamma_\nu(s)\big) = 0$

holds. That $g\big(\gamma'(s), Y(s)\big) = 0$ follows by Gauss' Lemma. Then

$$\begin{aligned}
&g\big(\gamma'(s), Y(s)\big) \\
&\quad = g_{\gamma(s)}\left(r'(s)\frac{\partial}{\partial r} + \theta'(s)\frac{\partial}{\partial \theta}, \left(\frac{\partial}{\partial \nu}\right)_{\nu=c} r(s,\nu)\frac{\partial}{\partial r} + \left(\frac{\partial}{\partial \nu}\right)_{\nu=c} \theta(s,\nu)\frac{\partial}{\partial \theta}\right) \\
&\quad = r'(s)\left(\frac{\partial}{\partial \nu}\right)_{\nu=c} r(s,\nu) + m(r(s))^2\theta'(s)\left(\frac{\partial}{\partial \nu}\right)_{\nu=c} \theta(s,\nu) \\
&\quad = 0.
\end{aligned}$$

Since $\left(\frac{\partial}{\partial \nu}\right)_{\nu=c} \theta(s,\nu) = 0$ by assumption and $r'(s) \neq 0$, it follows that

$$\left(\frac{\partial}{\partial \nu}\right)_{\nu=c} r(s,\nu) = 0.$$

Namely, $Y(s) = \mathbf{0}$. □

4. Proof of Theorem 1.1

In this section we give a proof for Theorem 1.1 which is different from Tanaka's. Let M be a complete surface of revolution with the vertex at p homeomorphic to the plane. Combining Lemma 3.3 and 3.4, we have the following. We give a necessary condition that there exists a pole $q \in M \setminus \{p\}$.

Corollary 4.1. If $\liminf_{t\to\infty} m(t) = 0$ or $\displaystyle\int_1^\infty \frac{1}{m(r)^2}\,dr = \infty$, then the vertex p is the unique pole on M.

We next prove the converse of Corollary 4.1. The following proposition contains Lemma 3.4 as its special case.

Proposition 4.2. If $\displaystyle\int_1^\infty \frac{1}{m(r)^2}\,dr = \infty$, then for any point $q \in M\setminus\{p\}$ the geodesic $\gamma_\nu|[0,\infty)$ is not a ray emanating from $q = \gamma_\nu(0)$ for any $\nu \in \big(-m(\rho), m(\rho)\big)$.

Proof. When $\nu \neq 0$, if $\lim_{t\to\infty} r(t) = r_0 < \infty$, then γ_ν is not a ray by Lemma 3.2. Let $\lim_{t\to\infty} r(t) = \infty$. In the case there exist more than one zero points of r', Lemma 3.1 implies that γ_ν is not a ray. In the case where r' has a zero only at t_0, we observe that

$$y_{t_0}(t) = \frac{\sqrt{m(r(t))^2 - \nu^2}}{m'(r(t_0))}$$

is the solution of $(\mathrm{J_G})$ along γ_ν with $y_{t_0}(t_0) = 0$ and $y'_{t_0}(t_0) = 1$. If y_s is the solution of $(\mathrm{J_G})$ with $y_s(s) = 0$ and $y_s(t_0) = 1$, we then have from (2.2) that

$$\begin{aligned} y_s(t) &= m'(r(t_0))\sqrt{m(r(t))^2 - \nu^2}\int_t^s \frac{1}{m(r(w))^2 - \nu^2}\,dw \\ &= m'(r(t_0))\sqrt{m(r(t))^2 - \nu^2}\int_{r(t)}^{r(s)} \frac{m(r)}{\big(m(r)^2 - \nu^2\big)^{3/2}}\,dr \\ &\geq m'(r(t_0))\sqrt{m(r(t))^2 - \nu^2}\int_{r(t)}^{r(s)} \frac{1}{m(r)^2}\,dr \end{aligned}$$

for all $t \in (t_0, s)$. By assumption, $y_s(t)$ does not converge as $s \to \infty$. Therefore, $(\mathrm{J_G})$ is not disconjugate on $(t_0 - \varepsilon, \infty)$ for any positive ε. Thus, γ_ν is not a ray. When $\nu = 0$, τ_q is not a ray by Lemma 3.4. □

Recall that $\beta, \gamma : [0,\infty) \to M, \beta(t), \gamma(t) = \big(r(t), \theta(t)\big)$ are geodesics whose velocity vectors at $t = 0$ are given in (3.2) and (3.3), respectively.

Lemma 4.3. (Compare Lemma 1.5 in [2].) If a geodesic $\beta : [0,\infty) \to M$ does not pass through p, and if $r'(t) \neq 0$ for all $t \in (0,\infty)$, then β contains no conjugate pair.

PROOF. Clearly, $y(t) = \sqrt{m(r(t))^2 - \nu^2}$ is the solution of ($\mathrm{J_G}$) along β. If $r'(t) \neq 0$ for all $t \in (0, \infty)$, then $y(t) \neq 0$ on $(0, \infty)$ from (3.1). By Lemma 2.3, ($\mathrm{J_G}$) is disconjugate on $(0, \infty)$. □

From now on, let $\liminf_{t \to \infty} m(t) := m_0 > 0$ and β be a geodesic with

$$r(\beta(0)) = r_1 \quad \text{and} \quad \beta'(0) = \left(0\,,\, \frac{1}{m(r_1)}\right).$$

Fix a k with $0 < k < 1$. Then there exists a number $a_1 > 0$ such that if $0 \leq r_1 \leq a_1$, then $m(r_1) < k m_0$ and $m(r_1) < m(r)$ for all $r > r_1$. (cf. Figure 2.) We have the following.

Lemma 4.4. If $0 \leq r_1 \leq a_1 < r_2$ and $r_2 := r(t_2)$, then

$$\int_{t_2}^{\infty} \frac{1}{m(r(t))^2 - m(r_1)^2}\, dt < \infty \quad \text{if and only if} \quad \int_{r_2}^{\infty} \frac{1}{m(r)^2}\, dr < \infty.$$

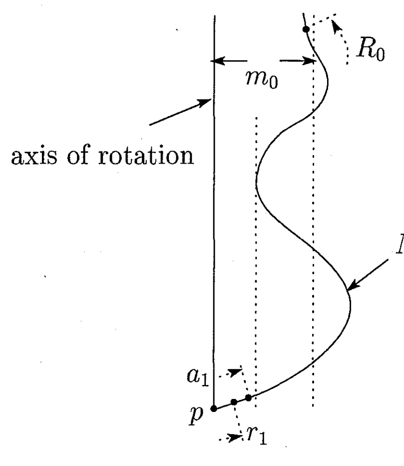

FIGURE 2. The number r_1, a_1 and R_0.

PROOF. (cf. Figure 2.) Since $r'(t) = \dfrac{\sqrt{m(r(t))^2 - m(r_1)^2}}{m(r(t))}$ from (3.1),

$$\int_u^v \frac{1}{m(r(t))^2 - m(r_1)^2}\, dt = \int_{r(u)}^{r(v)} \frac{1}{m(r)^2 - m(r_1)^2} \frac{m(r)}{\sqrt{m(r)^2 - m(r_1)^2}}\, dr.$$

It follows

$$\int_{t_2}^{v} \frac{1}{m(r(t))^2 - m(r_1)^2}\, dt \geq \int_{r_2}^{r(v)} \frac{1}{m(r)^2}\, dr.$$

Therefore, if the right hand side diverges, then the left hand side diverges.

There exists an $R_0 > 0$ such that if $R_0 < r$, then $m(r_1) < k m(r)$. If $R_0 < r(u) < r(v)$, then

$$\int_{r(u)}^{r(v)} \frac{m(r)}{\left(m(r)^2 - m(r_1)^2\right)^{3/2}}\, dr \leq \frac{1}{(1-k^2)^{3/2}} \int_{r(u)}^{r(v)} \frac{1}{m(r)^2}\, dr.$$

Therefore, if the right hand side converges, then the left hand side converges. □

Recall that $y(t) = \dfrac{\sqrt{m(r(t))^2 - m(r_1)^2}}{m'(r_1)}$ is the solution of $(\mathrm{J_G})$ along β with $y(0) = 0$ and $y'(0) = 1$. From (2.2) the solution y_s of $(\mathrm{J_G})$ with $y_s(0) = 1$ and $y_s(s) = 0$ can be written as follows for each $s > 0$:

$$y_s(t) = m'(r_1)\sqrt{m(r(t))^2 - m(r_1)^2} \int_t^s \frac{1}{m(r(w))^2 - m(r_1)^2}\, dw \ (s > t > 0).$$

By putting $c = 0$, we have the following from (2.3).

Lemma 4.5. Let $u > s > 0$. Then it follows

$$y_u'(0) - y_s'(0) = \int_s^u \frac{1}{y(w)^2}\, dw = \int_s^u \frac{m'(r_1)^2}{m(r(w))^2 - m(r_1)^2}\, dw. \tag{4.1}$$

In particular, if $\displaystyle\int_s^\infty \frac{1}{y(w)^2}\, dw < \infty$, then $\displaystyle y_\infty'(0) = \int_s^\infty \frac{1}{y(w)^2}\, dw + y_s'(0)$.

Here $y_s(t)$ and $y_\infty(t)$ are defined as in (2.2), Corollary 2.5, respectively. The values $y_s(t), y_\infty(t)$ and $y_s'(0), y_\infty'(0)$ depend on r_1. In order to show that these values, especially, $y_s'(0), y_\infty'(0)$ are continuous on r_1 in some neighborhood of p, we use the following notations:

$$y_{r_1,\infty} := y_\infty \quad \text{and} \quad y_{r_1,s} := y_s.$$

Let $0 \leq r_1 < a_1$ and $\displaystyle\int_s^\infty \frac{1}{m(r)^2}\, dr < \infty$. Then

$$h(r_1) := \int_s^\infty \frac{1}{m(r(w))^2 - m(r_1)^2}\, dw < \infty$$

by Lemma 4.4. The function $y_{r_1,\infty}$ is the solution of $(\mathrm{J_G})$ along β as stated in Remark 2.6.

Lemma 4.6. Assume that $\displaystyle\int_s^\infty \frac{1}{m(r)^2}\, dr < \infty$. Then there exists a neighborhood U of the vertex p such that $h(r(q))$ is continuous in $U \ni q$.

PROOF. Set $U = \{q \in M \mid r(q) < a_1\}$. For any $\varepsilon > 0$ there exists an $R_2 > 0$ such that if $0 < r_1 < a_1$, then

$$\int_{R_2}^\infty \frac{m(r)}{\left(m(r)^2 - m(r_1)^2\right)^{3/2}}\, dr \leq \frac{1}{(1-k^2)^{3/2}} \int_{R_2}^\infty \frac{1}{m(r)^2}\, dr < \frac{\varepsilon}{3}.$$

We have

$$h(r_1) = \int_{r(\beta(s))}^{\infty} \frac{m(r)}{\left(m(r)^2 - m(r_1)^2\right)^{3/2}}\, dr$$
$$= \int_{r(\beta(s))}^{R_2} \frac{m(r)}{\left(m(r)^2 - m(r_1)^2\right)^{3/2}}\, dr + \int_{R_2}^{\infty} \frac{m(r)}{\left(m(r)^2 - m(r_1)^2\right)^{3/2}}\, dr.$$

Let $\bar{\beta} : [0, \infty) \to M$ be the geodesic with $r\left(\bar{\beta}(0)\right) = \bar{r}_1, \bar{\beta}'(0) = \left(0, \dfrac{1}{m(\bar{r}_1)}\right), \bar{r}_1 \fallingdotseq r_1$. Then

$$h(r_1) - h(\bar{r}_1) = \int_{r(\beta(s))}^{R_2} \frac{m(r)}{\left(m(r)^2 - m(r_1)^2\right)^{3/2}}\, dr - \int_{r(\bar{\beta}(s))}^{R_2} \frac{m(r)}{\left(m(r)^2 - m(\bar{r}_1)^2\right)^{3/2}}\, dr$$
$$+ \int_{R_2}^{\infty} \frac{m(r)}{\left(m(r)^2 - m(r_1)^2\right)^{3/2}}\, dr - \int_{R_2}^{\infty} \frac{m(r)}{\left(m(r)^2 - m(\bar{r}_1)^2\right)^{3/2}}\, dr$$

and

$$\left| h(r_1) - h(\bar{r}_1) \right|$$
$$< \left| \int_{r(\beta(s))}^{R_2} \frac{m(r)}{\left(m(r)^2 - m(r_1)^2\right)^{3/2}}\, dr - \int_{r(\bar{\beta}(s))}^{R_2} \frac{m(r)}{\left(m(r)^2 - m(\bar{r}_1)^2\right)^{3/2}}\, dr \right| + \frac{2\varepsilon}{3}.$$

There exists a $\delta > 0$ such that if $\left| r_1 - \bar{r}_1 \right| < \delta$, then $\left| h(r_1) - h(\bar{r}_1) \right| < \dfrac{\varepsilon}{3} + \dfrac{2\varepsilon}{3} = \varepsilon$. Thus, $h \circ r$ is continuous in U. □

As $u \to \infty$ in (4.1), we have

$$y'_{r_1,\infty}(0) = m'(r_1)^2 h(r_1) + y'_{r_1,s}(0).$$

In this consequence, $y'_{r_1,\infty}(0)$ is continuous at $r_1 \in [0, a_1]$, where $y_{r_1,s}$ is the solution of $(\mathrm{J_G})$ along β with $y_{r_1,s}(s) = 0$ and $y_{r_1,s}(0) = 1$ for each $s > 0$. From Corollary 2.5 we have

$$y_{r_1,\infty}(t) = m'(r_1)\sqrt{m(r(t))^2 - m(r_1)^2} \int_t^{\infty} \frac{1}{m(r(w))^2 - m(r_1)^2}\, dw \ (t > 0).$$

We remark that the right hand side of the above equation is an expression of a Jacobi field on the interval $(0, \infty)$ and the expression is not available in any interval containing 0. We think that it is the restriction of a Jacobi vector field $y_{r_1,\infty}$ defined along a whole geodesic $\beta : (-\infty, \infty) \to M$. We can extend an interval with no conjugate pair as follows:

Lemma 4.7. Assume that $\displaystyle\int_1^{\infty} \frac{1}{m(r)^2}\, dr < \infty$. If a geodesic $\beta : (-\infty, \infty) \to M$ through $q = \beta(0) \in U$ is tangent to the parallel circle around p at q, that is, $\beta'(0) = \left(0, \dfrac{1}{m(r_1)}\right)$, then there exists a $\delta_{r_1} > 0$ such that there is no conjugate pair on $(-\delta_{r_1}, \infty)$ along the geodesic β where $r_1 = r(\beta(0))$. Furthermore, δ_{r_1} is continuous on r_1.

PROOF. We observe from Lemma 4.5 and Lemma 4.6 that $y'_{r_1,\infty}(0)$ exists and that $h(r_1)$ is continuous on $r_1 \in [0, a_1)$. Since $y_{r_1,\infty}(0) = 1$ and $y'_{r_1,\infty}(0)$ exists, we can extend the disconjugate interval of $y_{r_1,\infty}$ as follows:

If there are zeros of $y_{r_1,\infty}$, we then put $\delta_{r_1} := -t(r_1)$, where $t(r_1)$ is the maximum zero of zeros of $y_{r_1,\infty}$. Clearly, $t(r_1) < 0$. If there are no zeros, we put $\delta_{r_1} = \infty$. In this consequence, the interval which has no conjugate pairs extends from $[0, \infty)$ to $(-\delta_{r_1}, \infty)$ as showed in Corollary 2.7 and this is the maximal disconjugate interval. Since the solution of (J_G) depends continuously on the initial condition, the function $\delta_{r_1}$is continuous on r_1. □ (cf. Figure 3.)

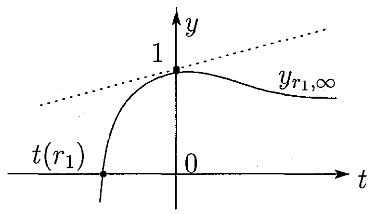

FIGURE 3. The maximum zero of zeros of $y_{r_1,\infty}$.

We enter our final stage to the proof of Theorem 1.1.

Lemma 4.8. Assume that

$$\liminf_{t \to \infty} m(t) > 0 \quad \text{and} \quad \int_1^\infty \frac{1}{m(r)^2}\, dr < \infty.$$

Then there exists a positive b such that any point q with $d(p, q) \leq b$ is a pole.

PROOF. By assumption that $\int_1^\infty \frac{1}{m(r)^2}\, dr < \infty$, we have a $\delta_0 > 0$, where δ_0 is given by putting $r_1 = 0$ for δ_{r_1} in Lemma 4.7. There exists an $a_2 > 0$ such that if $0 \leq r_1 < a_2 < a_1$, then $|\delta_{r_1} - \delta_0| \leq \frac{\delta_0}{2}$, that is, $\delta_{r_1} \geq \frac{\delta_0}{2}$. Put $b := \min\left(a_2, \frac{\delta_0}{2}\right)$. For any point q in the b-neighborhood of p, there is no conjugate pair along any geodesic emanating from q.

For a geodesic $\beta : [0, \infty) \to M$ with $r(\beta(0)) = r_1 < b$ whose velocity vector at $t = 0$ is defined as (3.2), we have

$$y(t) = \sqrt{m(r(t))^2 - c^2} \neq 0$$

on $(0, \infty)$ for any fixed $c \in [0, m(r_1)]$. Therefore, (J_G) is disconjugate on $(0, \infty)$ along β by Lemma 2.3.

For a geodesic $\gamma : [0, \infty) \to M$ whose velocity vector at $t = 0$ is defined as (3.3), the following is true. Let q_0 be a point such that $r'(q_0) = 0$, that is, $d(p, q_0) = d(p, \gamma([0, \infty)))$ with $r(q_0) < r_1$. Let q_1 be a point such that $d(p, q_1) = d(p, q)$, $q_1 \neq q$

and $q_1 \in \gamma([0,\infty))$. Since

$$d(q_0, q) = \frac{d(q, q_1)}{2} \leq \frac{d(q,p) + d(p, q_1)}{2} \leq b \leq \frac{\delta_0}{2} \leq \delta_{r_1},$$

there also exist no points conjugate to q along γ by Lemma 4.7. (cf. Figure 4.) Therefore, every point q in the b-neighborhood of p is a pole. □

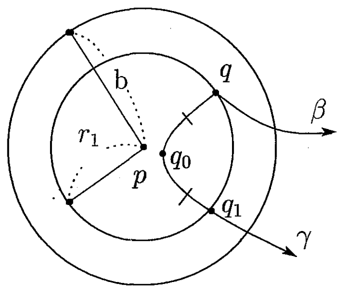

FIGURE 4. The geodesics β, γ emanate from q.

By Corollary 4.1 and Lemma 4.8, we have Theorem 1.1.

5. Proof of Theorem 1.3

In this section we prove Theorem 1.3. Let (S, o) be a von Mangoldt's surface of revolution with the vertex at o. We determine the number $r_o(S)$. The proof is based on Lemma 1.2. We find the equation whose solution is $r_o(S)$. Since $m(0) = 0, m'(0) = 1$ and from (2.3) we have

$$y_u'(0) - y_s'(0) = \int_s^u \frac{1}{m(r)^2}\,dr = \int_1^u \frac{1}{m(r)^2}dr - \int_1^s \frac{1}{m(r)^2}\,dr.$$

Thus,

$$y_u'(0) - \int_1^u \frac{1}{m(r)^2}\,dr = y_s'(0) - \int_1^s \frac{1}{m(r)^2}\,dr.$$

This shows that these values do not depend on parameter s. Then we can set

$$C = y_s'(0) - \int_1^s \frac{1}{m(r)^2}\,dr = y_1'(0)$$

where C is a constant. From Corollary 2.5 and the assumption, both

$$y_\infty(t) = m(t)\int_t^\infty \frac{1}{m(r)^2}\,dr \ (t > 0)$$

and

$$y_\infty'(0) = \int_1^\infty \frac{1}{m(r)^2}\,dr + C$$

exist. Let an $x > 0$ be a number such that the maximal disconjugate interval of $(\mathrm{J_G})$ along τ_q is $(-x, \infty)$. Then

$$y_\infty'(0) = \int_1^\infty \frac{1}{m(r)^2}\,dr + y_x'(0) - \int_1^x \frac{1}{m(r)^2}\,dr = \int_x^\infty \frac{1}{m(r)^2}\,dr + y_x'(0).$$

Since the Gaussian curvature $G(\tau_q(t))$ along τ_q is symmetric with respect to the vertex p, the x satisfies $y_\infty'(0) = -y_x'(0)$. (cf. Figure 5.) Since $y_s'(0)$ is monotone increasing on s, we have $y_\infty'(0) > y_x'(0)$.

In the case where $c(m) \leq 0$, we have $-y_x'(0) \leq 0$, a contradiction. Namely, $(-\infty, \infty)$ is the disconjugate interval of $(\mathrm{J_G})$. We then have $r_o(S) = \infty$.

In the case where $c(m) > 0$, it follows that

$$y_\infty'(0) = \int_x^\infty \frac{1}{m(r)^2}\,dr + y_x'(0) = -y_x'(0).$$

Therefore,

$$\begin{aligned} 0 &= 2y_x'(0) + \int_x^\infty \frac{1}{m(r)^2}\,dr \\ &= 2\left(y_\infty'(0) - \int_x^\infty \frac{1}{m(r)^2}\,dr\right) + \int_x^\infty \frac{1}{m(r)^2}\,dr \\ &= c(m) - \int_x^\infty \frac{1}{m(r)^2}\,dr. \end{aligned}$$

Thus, we have the equation $\bar{F}(x) = 0$ and the results. □

The geometrical meaning of the constant $c(m)$ is $2y'_\infty(0)$ as above.

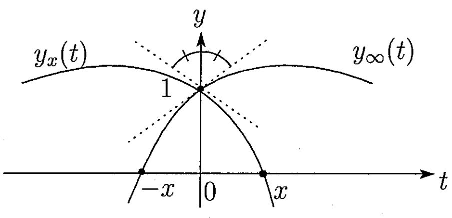

FIGURE 5. The relation of y_x to y_∞.

Remark 5.1. Furthermore, put $c(m, r_1) := 2y'_{r_1,\infty}(0)$, and

$$\bar{F}(r_1, x) := c(m, r_1) - \int_x^\infty \frac{m'(r_1)^2}{m(r(t))^2 - m(r_1)^2}\, dt.$$

If $c(m, r_1) > 0$, then there exists an $x = x(r_1)$ such that

$$\bar{F}\big(r_1, x(r_1)\big) = 0 \quad \text{and} \quad \delta_{r_1} = x(r_1).$$

Then, $\big(-x(r_1), \infty\big)$ is the maximal disconjugate interval along a geodesic β such that $r\big(\beta(0)\big) = r_1$ and $r'\big(\beta(0)\big) = 0$.

6. Proof of Theorem 1.4

In this section we will prove Theorem 1.4. We assume that (1) of Theorem 1.4 is not true, that is, $\widetilde{C}(p) \cap \widetilde{J}(p) = \emptyset$. Then there are at least two minimizing geodesics connecting p and every point $x \in C(p)$ (cf. [4]). Thus, we may assume that $q \notin C(p)$.

Let U denote the set of all $v \in T_pM$ such that $\gamma_v(t) = \exp_p tv$ is a minimizing geodesic in $t \in [0, 1]$. Let φ be the restriction of $\exp_p$ to U. Notice that φ is injective in the interior of U and the boundary of U is $\widetilde{C}(p)$. The map $\varphi : \text{Int}\, U \to M$ is bi-Lipschitz diffeomorphism on any bounded set where Int U is the interior of U. Let Z_k be a sequence of minimizing geodesics from a point $q \in M$ and contained in $M \setminus C(p)$. Let $\widetilde{Z}_k$ be a sequence of curves in T_pM one of whose endpoints is a point $\widetilde{q}$ such that $\varphi(\widetilde{Z}_k) = Z_k$ and $\varphi(\widetilde{q}) = q$. It follows from bi-Lipschiz continuity of φ that if Z_k converges to a minimizing geodesic Z, then $\widetilde{Z}_k$ converges to a curve $\widetilde{Z} \subset T_pM$ such that $\varphi(\widetilde{Z}) = Z$. This fact will be used later.

Let F be a function on M given by

$$F(x) := d(p, x) + d(q, x)$$

for all $x \in M$. Then $F^{-1}\big((d(p, q), r]\big), r > d(p, q)$, is star-shaped around both p and q, that is, all minimizing geodesics $T(p, x)$ and $T(q, x)$ are contained in $F^{-1}\big((d(p, q), r]\big)$ for every point $x \in F^{-1}\big((d(p, q), r]\big)$. Since $C(p)$ is closed, there exists a point $x_0 \in C(p)$ such that

$$F(x_0) = \min\{F(x) \mid x \in C(p)\}.$$

The following lemma shows the details of Theorem 1.4 (2).

Lemma 6.1. Let M be a complete Riemannian manifold and $p, q \in M$ with $C(p) \neq \emptyset$, $\widetilde{C}(p) \cap \widetilde{J}(p) = \emptyset$ and $q \notin C(p)$. If $x_0 \in C(p)$ is the point given as above, then the following hold.

(1) If $x_0 \notin C(q)$, then the number of minimizing geodesics from p to x_0 is exactly two, say $T_1(p, x_0)$ and $T_2(p, x_0)$. Moreover, one of $T_1(p, x_0) \cup T(x_0, q)$ and $T_2(p, x_0) \cup T(x_0, q)$ is a geodesic crossing $C(p)$ and the other is a geodesic reflecting against $C(p)$ at x_0.

(2) If $x_0 \in C(q)$, then the numbers of minimizing geodesics from p to x_0 and x_0 to q are exactly two, respectively. Moreover, two of $T_1(p, x_0) \cup T_1(x_0, q), T_1(p, x_0) \cup T_2(x_0, q), T_2(p, x_0) \cup T_1(x_0, q)$ and $T_2(p, x_0) \cup T_2(x_0, q)$ are geodesics crossing $C(p)$ and the others are geodesics reflecting against $C(p)$ at x_0.

Here we say that a unit speed and broken geodesic $\gamma : [0, a] \to M$ reflects against a hypersurface $H \subset M$ at $x = \gamma(b) \in H$ if

$$\gamma'(b + 0) \neq \gamma'(b - 0) \ \text{ and } \ g\big(\gamma'(b - 0), v\big) = g\big(\gamma'(b + 0), v\big)$$

for all tangent vectors $v \in T_xH$ where $\gamma'(b \pm 0) = \lim_{t \to +0} \gamma'(b \pm t)$.

PROOF. In order to prove (1) we treat the case $x_0 \notin C(q)$. Let $T = T(x_0, q)$ be the unique minimizing geodesic connecting x_0 and q. Suppose in addition that there exist at least three minimizing geodesics connecting p and x_0. Choose two of them, T_1 and T_2, such that neither of $T \cup T_1$ and $T \cup T_2$ is a geodesic. Namely, $T \cup T_1$ and $T \cup T_2$ are broken at x_0. Since $d(q, y) \leq d(q, x_0) + d(x_0, y)$ for every point $y \in T_1 \cup T_2 \setminus \{p, x_0\}$, we find points $y_i \in T_i$ sufficiently close to x_0 such that $F(y_i) < F(x_0)$ for $i = 1, 2$, meaning that $T(q, y_i) \cap C(p) = \emptyset$. Let the curves $\widetilde{T_i} \subset U, i = 1, 2$, be such that $\widetilde{T_i}$ joins the origin O of T_pM and a point $\widetilde{x}_{0i} \in \varphi^{-1}(x_0)$ with $\varphi(\widetilde{T_i}) = T_i$. We then have new curves $\varphi^{-1}(T(q, y_i))$ connecting $\widetilde{q}$ and $\widetilde{y}_i = \varphi^{-1}(y_i) \in \widetilde{T_i}$. Since x_0 is not conjugate to p, the points $\widetilde{y}_i$ are close to $\widetilde{x}_{0i}$. Letting $y_i \to x_0$ we have two curves $\widetilde{W_i}, i = 1, 2$, connecting $\widetilde{q}$ and $\widetilde{x}_{0i}$, respectively, such that $\varphi(\widetilde{W_i}) = T$. This is impossible. In fact, let $\gamma(t), \widetilde{\omega}_1(t)$ and $\widetilde{\omega}_2(t), t \in [0, 1]$, be parameterizations of $T, \widetilde{W_1}$ and $\widetilde{W_2}$, respectively, such that $\gamma(0) = q$ and $\varphi(\widetilde{\omega}_1(t)) = \varphi(\widetilde{\omega}_2(t)) = \gamma(t)$ for all $t \in [0, 1]$. Let $t_0 = \max\{t \in [0, 1] \mid \widetilde{\omega}_1(s) = \widetilde{\omega}_2(s) \text{ for all } s \in [0, t]\}$. Then $t_0 > 0$ because φ is injective in the interior of U. Since φ is diffeomorphic on some neighborhood around $\widetilde{\omega}_1(t_0) = \widetilde{\omega}_2(t_0)$ because of $\widetilde{C}(p) \cap \widetilde{J}(p) = \emptyset$, we have $t_0 = 1$, contradicting that $\widetilde{\omega}_1(1) = \widetilde{x}_{01} \neq \widetilde{x}_{02} = \widetilde{\omega}_2(1)$. Thus there are exactly two minimizing geodesics L_1 and L_2 connecting p and x_0. From above argument, we may assume that $L_1 \cup T$ is a geodesic L connecting p and q. Let $\gamma_1 : [0, a] \to M$ and $\gamma_2 : [0, b] \to M$ be the parameterizations of geodesics L and L_2, respectively, where $a = F(x_0)$ and $b = d(p, x_0)$. The cut locus $C(p)$ is smooth in some neighborhood of x_0 because x_0 is not conjugate to p along both γ_1 and γ_2. Therefore, we have

$$g(\gamma_2'(b-0), v) = g(\gamma_1'(b-0), v) = g(\gamma_1'(b+0), v)$$

for all tangent vectors $v \in T_{x_0}C(p)$. Where the first equality follows from $F(x_0) = \min\{F(x) \mid x \in C(p)\}$ and the second equality follows from $\gamma_1'(b-0) = \gamma_1'(b+0)$. This proves that (1) is true.

In order to prove (2) we treat the case $x_0 \in C(q)$. Let T be a minimizing geodesic connecting q and x_0. Let $q_1 \in T \setminus \{x_0, q\}$ be such that $q_1 \notin C(p)$. We can choose such a point q_1 because of $q \notin C(p)$. Moreover, the point q_1 satisfies $x_0 \notin C(q_1)$. Let F_1 be a function defined by

$$F_1(x) := d(p, x) + d(q_1, x)$$

for all $x \in M$. We will prove

$$F_1(x_0) = \min\{F_1(x) \mid x \in C(p)\}.$$

In order prove this, we suppose there exists a point $x_1 \in C(p)$ with $F_1(x_1) < F_1(x_0)$. We then have

$$\begin{aligned} F(x_1) &= d(p, x_1) + d(q, x_1) \\ &\leq d(p, x_1) + d(q, q_1) + d(q_1, x_1) = F_1(x_1) + d(q, q_1) \\ &< F_1(x_0) + d(q, q_1) = d(p, x_0) + d(q_1, x_0) + d(q, q_1) \\ &= d(p, x_0) + d(q, x_0) = F(x_0). \end{aligned}$$

This contradicts the choice of x_0. If we use q_1 anf F_1 instead of q and F, respectively, then (1) is true. Letting $q_1 \to q$, we have one geodesic stated in (2). To complete the proof we have to show the number of minimizing geodesics connecting q and x_0 is exactly two. Suppose first that the number is one. Then the argument for (1) is valid to get a contradiction, since any minimizing geodesic connecting q and $x \in T_i(p, x_0) \setminus \{x_0\}$ is not contained in any $T_i(p, x_0) \cup T(x_0, q)$ because of $x_0 \in C(q)$, meaning that $F(x) < F(x_0)$. Suppose there exist at least three minimizing geodesics connecting q and x_0. We then find at least two broken geodesics with break point at x_0 which are a union of $T_i(p, x_0)$ and some $T = T(x_0, q)$. Thus the argument for (1) is valid to get a contradiction again when we use a point $q_1 \in T \setminus \{q\}$ instead of q as before. This completes the proof of (2). □

7. Proof of Theorem 1.6

In this section we prove Theorem 1.6.

PROOF of (1). Let $q \in M \backslash \{p\}$ be a pole. Take a point $s \in S$ with $d(o,s) = d(p,q)$. Then $\tau_q : [0,\infty) \to M$ is a ray with $\tau_q(0) = q, \tau_q\big(d(p,q)\big) = p$, and $\tau_s : [0,\infty) \to S$ is a geodesic with $\tau_s(0) = s$ and $\tau_s\big(d(o,s)\big) = o$. Since τ_s lies in a union of two meridians, we have

$$d\big(p, \tau_q(t)\big) = d\big(o, \tau_s(t)\big)$$

for all $t \in [0, \infty)$. It follows from the assumption,

$$K\big(\pi_{\tau_q(t)}\big) \geq G\big(d(p,\tau_q(t))\big) = G\big(d(o,\tau_s(t))\big) = G\big(\tau_s(t)\big)$$

for all $t \in [0,\infty)$, where

$$K(\pi_x) = \frac{\langle R(u,v)v, u\rangle}{\|u\|^2\|v\|^2 - \langle u,v\rangle^2}$$

denotes the sectional curvature of the tangent plane π_x at $x \in M$, being spanned by two independent tangent vectors $u, v \in T_xM$, by putting $\langle\cdot,\cdot\rangle := g(\cdot,\cdot)$. It follows that $K\big(\pi_{\tau_q(t)}\big)$ at $\tau_q(t) \in M$ is greater than or equal to the Gaussian curvature $G\big(\tau_s(t)\big)$ at $\tau_s(t) \in S$. Since there is no point conjugate to q along τ_q, the Rauch comparison theorem for Jacobi vector fields ([4]) shows that τ_s has no point conjugate to s. Lemma 1.2 proves that s is a pole in S and

$$r_p(M) \leq r_o(S), \text{ that is, } P \subset \bar{B}\big(p, r_o(S)\big).$$

This completes the proof of (1).

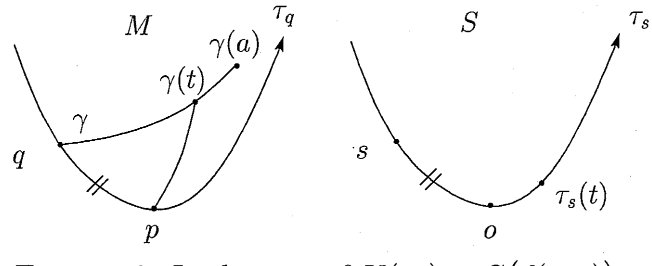

FIGURE 6. In the case of $K(\pi_x) \leq G\big(d(p,x)\big)$.

PROOF of (2). (cf. Figure 6.) Let $q \in M$ be a point with $d(p,q) \leq r_o(S)$. We will prove $C(q) = \emptyset$ which implies that q is a pole in M. Suppose for indirect proof that $C(q) \neq \emptyset$. Then it follows from Corollary 1.5 that $\widetilde{C}(q) \cap \widetilde{J}(q) \neq \emptyset$, say $\widetilde{x} \in \widetilde{C}(q) \cap \widetilde{J}(q)$ and $x = \exp_q \widetilde{x}$. Therefore, we have a minimizing geodesic $\gamma : [0,a] \to M$ with $\gamma(0) = q$ and $\gamma(a) = x$ such that x is conjugate to q along γ where $a = d(q,x)$. Let $s \in S$ be a point with $d(o,s) = d(p,q)$. Then s is a pole and $\tau_s : [0,\infty) \to S$ is a ray with $\tau_s(0) = s$ and $\tau_s\big(d(o,s)\big) = o$. We then have from the triangle inequality

$$d\big(p, \gamma(t)\big) \geq |\, t - d(p,q)\,| = |\, t - d(o,s)\,| = d\big(o, \tau_s(t)\big)$$

for all $t \in [0, a]$. Thus, we have

$$K(\pi_{\gamma(t)}) \le G\big(d(p, \gamma(t))\big) \le G\big(d(o, \tau_s(t))\big) = G\big(\tau_s(t)\big)$$

for all $t \in [0, a]$, since S is a von Mangoldt's surface of revolution. It follows that $K(\pi_{\gamma(t)})$ at $\gamma(t) \in M$ is less than or equal to $G(\tau_s(t))$ at $\tau_s(t) \in S$ for all $t \in [0, a]$. Since s is a pole, it follows from the Rauch comparison theorem for Jacobi vector fields that γ has no point conjugate to q, a contradiction. We then have $C(q) = \emptyset$. Namely, q is a pole. Hence,

$$r_p(M) \ge r_o(S), \text{ that is, } P \supset \bar{B}\big(p, r_o(S)\big).$$

This completes the proof of (2). □

8. Some examples

The examples in this section are helpful to understand the role of the assumption in the argument in Section 6.

(1) Lift of curves meeting a conjugate point.

Here we discuss what happens when $\widetilde{C}(p) \cap \widetilde{J}(p) \neq \emptyset$. (cf. Figure 7.) Let

$$M = \{(x, y, z) \in \mathbb{E}^3 \mid x^2 + y^2 + x^2 = 1\} \text{ and } p = (0, 0, 1).$$

Let (r, θ) be the polar coordinates in T_pM such that

$$\exp_p(r, \theta) = (\sin r \cos\theta, \sin r \sin\theta, \cos r).$$

Then $\widetilde{C}(p) = \widetilde{J}(p) = \{(r, \theta) \mid r = \pi,\ 0 \leq \theta < 2\pi\}$ and $C(p) = \{(0, 0, -1)\}$. Let $U = \{(r, \theta) \mid 0 \leq r \leq \pi\} \subset T_pM$. Set $x_0 = (0, 0, -1)$. Let $q = \exp_p(r_0, 0)$ for $r_0 \in (0, \pi)$. Then it is of course that $F(x_0) = \min\{d(p, x) + d(q, x) \mid x \in C(p)\}$. Take a point $\widetilde{x}_0 = (\pi, \theta_0) \in T_pM$ where $0 < \theta_0 < \pi$ or $\pi < \theta_0 < 2\pi$ such that $\exp_p \widetilde{x}_0 = x_0$. Let $y(r) = \exp_p(r, \theta_0), 0 < r < \pi$. Then $y(r) \in T = \exp_p\left([0, \pi] \times \{\theta_0\}\right)$. We have a curve $\widetilde{W}(r) = \exp_p^{-1}\left(T(q, y(r))\right) \subset U$ which can be parameterized by θ, connecting $\widetilde{q} = \exp_p^{-1} q$ and $\widetilde{y}(r) = \exp_p^{-1} y(r)$. As $y(r) \to x_0$, the sequence of curves $\widetilde{W}(r)$ converges to the union of curves $\{(r, 0) \mid r_0 \leq r \leq \pi\}$ and $\{(\pi, \theta) \mid 0 \leq \theta \leq \theta_0\} \subset \widetilde{C}(p)$ or $\{(\pi, \theta) \mid \theta_0 \leq \theta \leq 2\pi\} \subset \widetilde{C}(p)$. Those curves for $0 < \theta_0 < \pi$ and $\pi < \theta_0 < 2\pi$ branch at $(\pi, 0)$. This does not happen when $\widetilde{C}(p) \cap \widetilde{J}(p) = \emptyset$, as was seen in the

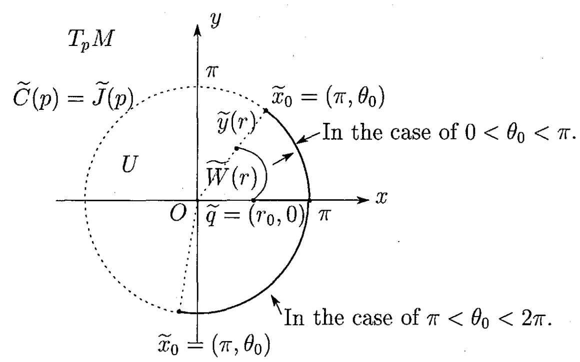

FIGURE 7. The curve to which $\widetilde{W}(r)$ converges in T_pM as $y(r) \to x_0$ in M.

proof of Theorem 1.4.

(2) Lift of ellipses in a flat cylinder.

Here we discuss what happens when ellipses meet $\widetilde{C}(p)$ as being larger in T_pM. (cf. Figure 8.) Let

$$M = \{(x, y, z) \in \mathbb{E}^3 \mid x^2 + z^2 = 1\} \text{ and } p = (1, 0, 0), q = (0, 2, -1).$$

Then $C(p) = \{(-1, y, 0) \mid y \in \mathbb{R}\}$ and $C(q) = \{(0, y, 1) \mid y \in \mathbb{R}\}$. Since the Gaussian curvature $G(p)$ at every point $p \in M$ equals 0, there are no points conjugate to p along a minimizing geodesic, that is, $\widetilde{C}(p) \cap \widetilde{J}(p) = \emptyset$. We identify M with $\mathbb{E}^2/\Gamma$ where $\mathbb{E}^2 = \{(x, y) \mid x, y \in \mathbb{R}\}$ and Γ is the isometry group generated by a translation $(x, y) \mapsto (x, y+2\pi)$, that is, the identification is given by $M \ni (\cos\theta, y, \sin\theta) \mapsto (y, \theta \bmod 2\pi) \in \mathbb{E}^2$. The tangent plane T_pM is identified with $\mathbb{E}^2$ also. Then $\widetilde{C}(p) = \{(x, \pm\pi) \mid x \in \mathbb{R}\}$ and $U = \{(x, y) \mid x \in \mathbb{R}, -\pi \leq y \leq \pi\}$. If $\varphi = \exp_p|U$, then $\varphi^{-1}(p) = (0, 0) =: \widetilde{p}_0$ and $\varphi^{-1}(q) = (2, -\pi/2) =: \widetilde{q}_0$ by this identification. Set $\widetilde{q}_1 = (2, 3\pi/2)$, meaning $\varphi(\widetilde{q}_1) = q$. Furthermore, $\varphi^{-1}(C(q)) = \{(x, \pi/2) \mid x \in \mathbb{R}\}$. Let $E(p, q; r) = \{w \mid F(w) := d(p, w) + d(q, w) = r\}$ and $D(p, q; r) = \{w \mid F(w) \leq r\}$ for each $r > d(p, q)$. Then $\varphi^{-1}\big(E(p, q; r)\big)$ changes for r as follows:

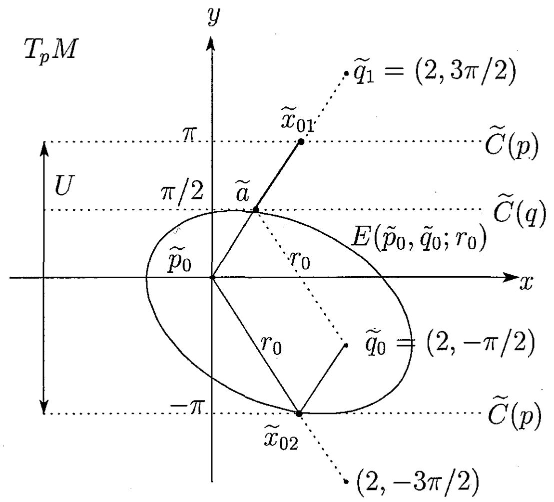

FIGURE 8. At the moment when the ellipse just meets $C(p)$ on M.

(1) $\varphi^{-1}\big(E(p, q; r)\big) = E(\widetilde{p}_0, \widetilde{q}_0; r)$ if r satisfies $d(p, q) < r < r_0$, where $r_0 = \min\{F(w) \mid w \in C(p)\} = \sqrt{4 + 9\pi^2/4}$.
(2) $\varphi^{-1}\big(E(p, q; r_0)\big) = E(\widetilde{p}_0, \widetilde{q}_0; r_0) \cup T(\widetilde{a}, \widetilde{x}_{01})$ where $\widetilde{a} = E(\widetilde{p}_0, \widetilde{q}_0; r_0) \cap T(\widetilde{p}_0, \widetilde{q}_1)$ and $\{\widetilde{x}_{01}\} = \widetilde{C}(p) \cap T(\widetilde{p}_0, \widetilde{q}_1)$. (cf. Figure 8.)
(3) $\varphi^{-1}\big(E(p, q; r)\big) = \partial\big(D(\widetilde{p}_0, \widetilde{q}_0; r) \cup D(\widetilde{p}_0, \widetilde{q}_1; r)\big) \cap U$ if r satisfies $r > r_0$ where ∂X is the boundary of X.

Let $\widetilde{x}_{02} = \widetilde{x}_{01} - (0, 2\pi)$. If $x_0 \in C(p)$ satisfies $F(x_0) = \min\{F(x) \mid x \in C(p)\}$, then $\varphi^{-1}(x_0) = \{\widetilde{x}_{01}, \widetilde{x}_{02}\}$. Moreover, $\varphi\big(T(\widetilde{p}_0, \widetilde{x}_{01}) \cup T(\widetilde{x}_{02}, \widetilde{q}_0)\big)$ is a geodesic connecting p and q in M. The geodesic reflecting against $C(p)$ at x_0 is identified with $\varphi\big(T(\widetilde{p}_0, \widetilde{x}_{02}) \cup T(\widetilde{x}_{02}, \widetilde{q}_0)\big)$. It is remarkable that any sequence of points y_j such that $y_j \in E(p, q; r_j)$ for $r_j < r_0$ with $r_j \to r_0$ cannot converge to any point in $T(a, x_0) \setminus \{a, x_0\}$.

Let $p=(1,0,0), q=(1,2,0)$. Then $C(p)=C(q)=\{(-1,y,0 \mid y\in\mathbb{R})\}$ and $x_0\in C(q)$. The geodesics crossing $C(p)$ are identified with $\varphi\big(T(\widetilde{p}_0,\widetilde{x}_{01})\cup T(\widetilde{x}_{02},\widetilde{q}_0)\big)$ and $\varphi\big(T(\widetilde{p}_0,\widetilde{x}_{02})\cup T(\widetilde{x}_{01},\widetilde{q}_0)\big)$. The geodesics reflecting against $C(p)$ at x_0 are identified with $\varphi\big(T(\widetilde{p}_0,\widetilde{x}_{01})\cup T(\widetilde{x}_{01},\widetilde{q}_0)\big)$ and $\varphi\big(T(\widetilde{p}_0,\widetilde{x}_{02})\cup T(\widetilde{x}_{02},\widetilde{q}_0)\big)$. (cf. Figure 9.)

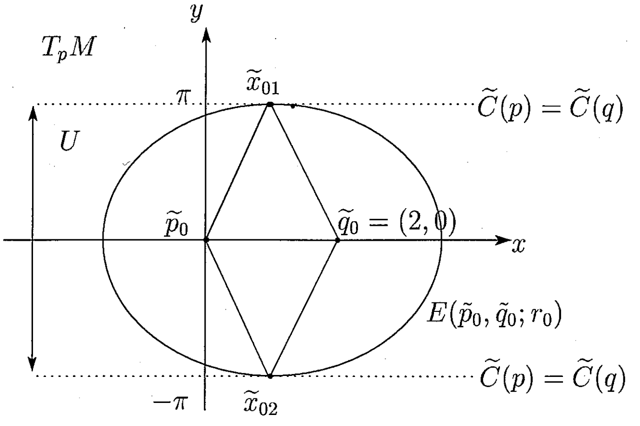

FIGURE 9. In the case of $x_0\in C(q)$.

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