Study on Scalarization Methods for Sets in Optimization Theory

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Preface

Set-valued analysis is one of the important tools in nonlinear analysis and optimization theory. Especially, continuity of set-valued maps plays very important roles in many areas of optimization theory. Recently, many researchers have been investigated vector optimization problems with set-valued maps which are called "set-valued optimization" or "set optimization." This research has been developed as a generalization of vector optimization for around thirty years.

On the other hand, scalarization methods have been widely developed by many researchers. One of the purpose of scalarization methods is the replacement of an optimization problem (especially, vector optimization problem) by a suitable scalar optimization problem which is an optimization problem with a real-valued objective functional where the objective functional is the composite function of a scalarizing function and the objective function of the original optimization problem. Moreover, by using the scalar optimization theory which is applied to the above composite function, the vector optimization theory is obtained. In recent years, some researchers propose scalarizing functions for sets and investigate those properties and some applications. However, the usefulness of them in set-valued analysis and optimization is not well known because such study has not always been systematically nor it is not emphasized in those areas by using scalarization methods for sets.

In the thesis, based on the results of our study listed in the last of the thesis, we establish two types of scalarizing functions for sets and those properties. Moreover, by using these properties we present several results for set-valued analysis and optimization.

The results in the thesis show the usefulness of scalarization methods for sets in several problems of set-valued analysis and optimization. I hope that the results obtained in the thesis will help further progress in those areas.

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Contents

1 Introduction **1**

2 Basic concepts **7**

2.1 Ordered topological vector spaces 7

2.2 Set-relation in an ordered topological vector space 9

3 Convexity and continuity of set-valued map **15**

3.1 Convexity of set-valued map 15

3.2 Continuity of set-valued map 21

4 Scalarization methods for sets **25**

4.1 Scalarizing functions for sets 25

4.2 Inherited properties of scalarizing functions 38

5 Applications **53**

5.1 Optimality conditions for set-valued optimization 53

5.2 Fan's inequality for set-valued maps 63

5.3 Continuity of cone-convex function 67

5.4 Saddle point theorems for set-valued maps 72

5.5 Minimax theorems for set-valued maps 86

6 Conclusion **97**

Bibliography **99**

A list of the Author's work **107**

Chapter 1

Introduction

Optimization theory is one of the important tools to study economics, engineering, and other areas. In general, optimization problems are defined by $\min_{x \in S} f(x)$ or $\max_{x \in S} f(x)$ where f is a real-valued function on a nonempty set S. If f is a vector-valued function then it is called "multiobjective optimization" or "vector optimization." It is well known that the convexity and continuity of f play very important roles in those optimization problems as well as variational problems, saddle point problems, minimax problems and complementarity problems. Especially, convex lower semicontinuous functions have many convenient properties and applications in those problems (see [1, 8, 9, 14, 32, 35, 58]). Furthermore, many researchers consider some types of convexity for vector-valued or set-valued maps, and investigate properties and applications of those generalized convex functions (see [13, 24, 33, 36, 42, 51, 57]).

Set-valued analysis has been widely developed and produced many applications in nonlinear analysis and optimization theory (see [5, 6, 7, 9, 33, 51, 59] and references therein). For example, it is well known that the subdifferential mapping of a convex function is set-valued map, and it is one of the important tools in convex analysis and optimization. Moreover, the continuity of set-valued map is very important in the area of optimization as well as in other fields of mathematics, such as fixed point theory, stability theory, analysis of algorithms, and differential inclusions. In recent years, set-valued optimization problems which are vector optimization problems with set-valued maps have been investigated by many researchers (see [2, 3, 4, 12, 28, 29, 30, 33, 34, 43, 44, 45, 46, 57] and references therein). These optimization problems are closely related to problems in fuzzy programming and optimal control. In fact, differential inclusions play very important roles in those problems. It means that set-valued analysis and optimization seem to have the potential to become a bridge of different areas in optimization.

Scalarization is one of the important tools in vector optimization, control theory, and various fields of applied mathematics. (see [24, 25, 33, 37, 40, 51, 61, 73] and references therein). For example, L_2-norm and L_∞-norm play very important roles in control theory. In vector optimization problems, weighted sum approach and weighted Chebyshev norm approach are often used. It is well known that a solution of scalarized vector optimization problems by those approaches is also a solution of parent vector optimization problems. Moreover, the sublinear scalarization method for vectors introduced by Gerstewitz (Tammer) has many useful properties to study vector optimization problems when the image space of a function f is not convex (see [21, 22, 24]). Also, some researchers consider certain generalizations of this scalarization and apply to several problems in set-valued optimization (see [19, 20, 54, 62]). In [27], Hamel and Löhne define different types of scalarizing functions for sets, which evaluate a nonempty set as a real number or $\pm\infty$ by a certain comparison with a given reference set along a given direction based on set-relations introduced in [47]. By using these functions, they show generalized results on Ekeland variational principle in an abstract space like topological vector space without such strong assumption as convexity. In [63], Shimizu and Tanaka give a similar result to a minimal element theorem in [27] under different assumptions. Hernández and Marín [28] show two existence theorems of solutions for set-valued optimization problems.

In the thesis, based on the approach of Hamel and Löhne [27], and six kinds of set-relations introduced in [47], we propose twelve types of scalarizing functions for sets and investigate some properties of them. Moreover, by using these properties we present the following five applications:

First, we introduce some optimality conditions for set-valued optimization problems with respect to set-relations introduced in [47]. Optimality conditions for set-valued optimization problems based on scalarization are given in [29]. Based on these results, we obtain optimality conditions for more general set-valued optimization problems.

Second, we give four types of Ky Fan minimax inequality for set-valued maps. Ky Fan minimax inequality in [18] is one of the important results in convex analysis as well as nonlinear analysis and it has many applications in those areas (see [5, 7, 9, 58]). Recently, Georgiev and Tanaka [19, 20] generalize an equivalent form of scalar Ky Fan minimax inequality into set-valued four cases by using two types of nonlinear scalarizing functions for sets which are extensions of sublinear scalarizing functions for vectors used in [21]. Based on these results, we give more simple proofs of similar results.

Third, we demonstrate a notion of continuity for set-valued maps under some convexity assumptions. It is well known that each real-valued convex function f is locally Lipschitz continuous on $\mathbb{R}^n$. This result is useful in convex analysis and optimization (see [9, 14, 58]). By using this result and scalarizing functions for sets, we prove the continuity of cone-convex set-valued map.

Fourth, we present certain saddle point theorems for set-valued maps. Saddle point problems for real-valued or vector-valued functions are closely connected with game theory, Lagrangian duality, and several equilibrium problems (see [5, 6, 9, 14, 33, 51, 58, 59] and references therein). Moreover, saddle point problems for set-valued maps play important roles in Lagrangian duality of vector optimization problems, multiobjective games, and various fields of applied mathematics. In 1992, Luc and Vergas [52] propose a concept of saddle point for compact set-valued maps (that is, the image of the map at each point is a compact set) based on a vector criterion which is a generalization of vector-valued saddle point, and show necessary and sufficient conditions for the existence of those saddle points. On the other hand, some researchers show other types of existence theorems for those saddle point (see [11, 38]). Moreover, in [18, 75], they consider a generalization of those saddle points and show several existence theorems. In the thesis, we define a saddle point of set-valued maps by using two types of set-relations, and show some saddle point theorems with noncompact set-valued maps.

Fifth, we give some minimax theorems for set-valued maps. Minimax theorems including saddle point problems for real-valued or vector-valued functions have been investigated by many researchers (see [15, 16, 17, 50, 55, 64, 66, 67, 68, 69, 71] and references therein). In recent years, some researchers consider concepts of minimax and maximin values for set-valued maps by using minimal and maximal elements for vectors, and show some types of minimax theorems for set-valued maps (see [11, 38, 48, 49, 52, 75]). Also, to show their results they use scalarization methods for vectors. In the thesis, we consider minimax and maximin values of set-valued maps based on set-relations, and show two types of minimax theorems for set-valued maps based on Tanaka's vector-valued minimax theorems (see [69]).

The organization of the thesis is as follows. In Chapter 2, we introduce some basic definitions and mathematical methodology on comparison between two sets in an ordered topological vector space. In Chapter 3, we introduce several definitions of convexity and continuity of set-valued map. In Chapter 4, we give two types of scalarizing functions for sets and investigate several properties of them including the monotonicity and the inheritance on cone-convexity and cone-continuity of parent set-valued map. In Chapter 5, as applications of scalarization methods for sets introduced in Chapter 4, we present several results in set-valued analysis and optimization. In Section 5.1, we show optimality conditions for set-valued optimization problems. In particular, we propose new solution concepts of those problems. In Section 5.2, we give four types generalizations of Fan's inequality for set-valued maps. In Section 5.3, we show the continuity of cone-convex function. Especially, we introduce a concept of local Lipschitz continuity of set-valued map, and show that each set-valued map has this property under some convexity assumptions. In Section 5.4, we consider four types of saddle point for set-valued maps, and show sufficient conditions for the existence of them under weaker conditions than ordinary saddle point theorems for set-valued maps. In Section 5.5, we propose two types of minimax and maximin values of set-valued map, respectively, and show new minimax theorems for set-valued maps.

Chapter 2

Basic concepts

In this chapter, we recall some basic definitions and properties which will be used in the thesis.

2.1 Ordered topological vector spaces

Firstly, we give the preliminary terminology and notation. Let Y be a real topological vector space, A, B nonempty subsets of Y. We denote the topological interior, topological closure, complement of A by $\operatorname{int} A$, $\operatorname{cl} A$ and A^c, respectively; the conical hull of A by $\operatorname{cone}(A)$; the algebraic sum, algebraic difference of A and B by $A + B := \{a + b | a \in A, b \in B\}$, $A - B := \{a - b | a \in A, b \in B\}$, respectively; the composite function of two functions f and g by $g \circ f$. Moreover, we denote the set of all real numbers by $\mathbb{R}$; the set of all nonnegative real numbers by $\mathbb{R}_+$; the set of all extended real numbers by $\bar{\mathbb{R}}$, that is, $\bar{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$; n-dimensional Euclidean space by $\mathbb{R}^n$. Furthermore, we denote the origin of Y by θ_Y; all of nonempty subsets of Y by $\wp(Y)$.

Let us recall some concepts and properties in a real topological vector space with order structure. Let (Y, τ_Y) be a real topological vector space (Y, for short) and C a nontrivial convex cone (that is, $C \neq \{\theta_Y\}$, $C \neq Y$, $C + C = C$ and $\lambda C \subset C$ for all

$\lambda \geq 0$) in Y. Then we define the partial order on Y by C as follows:

$$x \leq_C y \text{ if } y - x \in C \text{ for } x, y \in Y.$$

We say that Y is ordered by C if Y is equipped with $\leq_C$. It is well known that if C is pointed (that is, $C \cap (-C) = \{\theta_Y\}$) then $\leq_C$ is antisymmetric. When $x \leq_C y$ for $x, y \in Y$, we define the order interval with respect to C between x and y by $[x, y] := \{z \in Y | x \leq_C z \,\text{and}\, z \leq_C y\}$. We say that $B \subset C$ is a *base* of C if B is convex and for each $c \in C \setminus \{\theta_Y\}$ has a unique representation of the form $c = \lambda b$ for some $\lambda > 0$ and $b \in B$. Let Y^* be the topological dual space of Y, that is, the vector space of all linear continuous functionals from Y into $\mathbb{R}$. Then, we define the topological dual cone of C and its quasi interior by

$$C^* := \{y^* \in Y^* | y^*(x) \geq 0, \,\text{for all}\, x \in C\},$$

$$\text{qi}C^* := \{y^* \in Y^* | y^*(x) > 0, \,\text{for all}\, x \in C \setminus \{\theta_Y\}\},$$

respectively. It is easy to check that if $\text{int}\, C \neq \emptyset$ and C has the base B then there exists $y_0^* \in \text{qi}C^*$ such that

$$B = \{x \in C | y_0^*(x) = 1\}. \tag{2.1}$$

The following property is an important relationship between the topology and the ordering.

Definition 2.1 (normal, [56]) Let Y be a real topological vector space ordered by a convex cone C. We say that C is *normal for the topology* τ_Y (normal, for short) if there is a base of neighborhoods of $\{\theta_Y\}$ consisting of sets A with the property $A = (A + C) \cap (A - C)$.

It is well known that a normal cone has many useful properties in topological vector spaces (see [24, 31, 33, 56]). For example, if C is normal then $\text{cl}\, C$ is pointed, and then C is pointed, too (see [24]). This concept will be used in Chapter 5.

Throughout this chapter, X is a real topological vector space, Y is a real topological vector space ordered by a nontrivial closed convex cone C.

Let us recall the following cone-properties of sets which will be used in the thesis.

Definition 2.2 (cone-property, [51]) A set $A \in \wp(Y)$ is said to be

(i) C-*convex* if $A + C$ is convex;

(ii) C-*closed* if $A + C$ is closed;

(iii) C-*proper* if $A + C \neq Y$;

(iv) C-*bounded* if for any U which is an open neighborhood of θ_Y, there exists $t \in \mathbb{R}_+$ such that $A \subset tU + C$.

(v) f_C-*convex* (resp., *closed, proper, bounded*) if A is C and $(-C)$-convex (resp., closed, proper, bounded).

Moreover, we say that a set-valued map $F : X \to \wp(Y)$ is C-*property valued on* X if $F(x)$ has the C-property for each $x \in X$.

2.2 Set-relation in an ordered topological vector space

In Section 2.1, we introduce the ordering between two vectors $a, b \in Y$ as follows:

$$(1)\ a \in b - C \text{ (that is, } a \leq_C b), \quad (2)\ a \in b + C \text{ (that is, } b \leq_C a).$$

When we replace a vector $a \in Y$ with a set $A \in \wp(Y)$, that is, we consider comparisons between a vector and a set with respect to C, there are four types of comparable cases and in-comparable case. Comparable cases are as follows: for $A \in \wp(Y)$, $b \in Y$,

$$\begin{array}{ll} (1)\ A \subset (b - C), & (2)\ A \cap (b - C) \neq \emptyset, \\ (3)\ A \cap (b + C) \neq \emptyset, & (4)\ A \subset (b + C). \end{array}$$

In the same way, when we replace a vector $b \in Y$ with a set $B \in \wp(Y)$, that is, we consider comparisons between two sets with respect to C, there are twelve types of comparable cases and one in-comparable case. For sets $A, B \in \wp(Y)$, A would be inferior to B if we have one of the following situations:

$$\begin{array}{ll} (1)\ A \subset (\cap_{b\in B}(b-C)), & (2)\ A \cap (\cap_{b\in B}(b-C) \neq \emptyset, \\ (3)\ (\cup_{a\in A}(a+C)) \supset B, & (4)\ ((\cup_{a\in A}(a+C)) \cap B) \neq \emptyset, \\ (5)\ (\cap_{a\in A}(a+C)) \supset B, & (6)\ ((\cap_{a\in A}(a+C)) \cap B) \neq \emptyset, \\ (7)\ A \subset (\cup_{b\in B}(b-C)), & (8)\ (A \cap (\cup_{b\in B}(b-C))) \neq \emptyset. \end{array}$$

Also, there are eight converse situations in which B would be inferior to A. Actually relationships (1) and (4) coincide with relationships (5) and (8), respectively. Therefore, we define the following six kinds of classification for set-relationships; see Figure 3.1. These relationships are proposed by Kuroiwa, Tanaka, and Ha [47].

Definition 2.3 (set-relation, [47]) For sets $A, B \in \wp(Y)$, we write

$A \leq_C^{(1)} B$ by $A \subset \bigcap_{b\in B}(b-C)$, equivalently $B \subset \bigcap_{a\in A}(a+C)$;

$A \leq_C^{(2)} B$ by $A \cap \left(\bigcap_{b\in B}(b-C)\right) \neq \emptyset$;

$A \leq_C^{(3)} B$ by $B \subset (A+C)$;

$A \leq_C^{(4)} B$ by $\left(\bigcap_{a\in A}(a+C)\right) \cap B \neq \emptyset$;

$A \leq_C^{(5)} B$ by $A \subset (B-C)$;

$A \leq_C^{(6)} B$ by $A \cap (B-C) \neq \emptyset$, equivalently $(A+C) \cap B \neq \emptyset$.

Proposition 2.1 ([47]) *For sets $A, B \in \wp(Y)$, the following statements hold.*

$$\begin{array}{ll} A \leq_C^{(1)} B \text{ implies } A \leq_C^{(2)} B; & A \leq_C^{(1)} B \text{ implies } A \leq_C^{(4)} B; \\ A \leq_C^{(2)} B \text{ implies } A \leq_C^{(3)} B; & A \leq_C^{(4)} B \text{ implies } A \leq_C^{(5)} B; \\ A \leq_C^{(3)} B \text{ implies } A \leq_C^{(6)} B; & A \leq_C^{(5)} B \text{ implies } A \leq_C^{(6)} B. \end{array}$$

Remark 2.1 Since C is a convex cone including the zero vector of Y, the following elementary properties hold for sets $A, B \in \wp(Y)$:

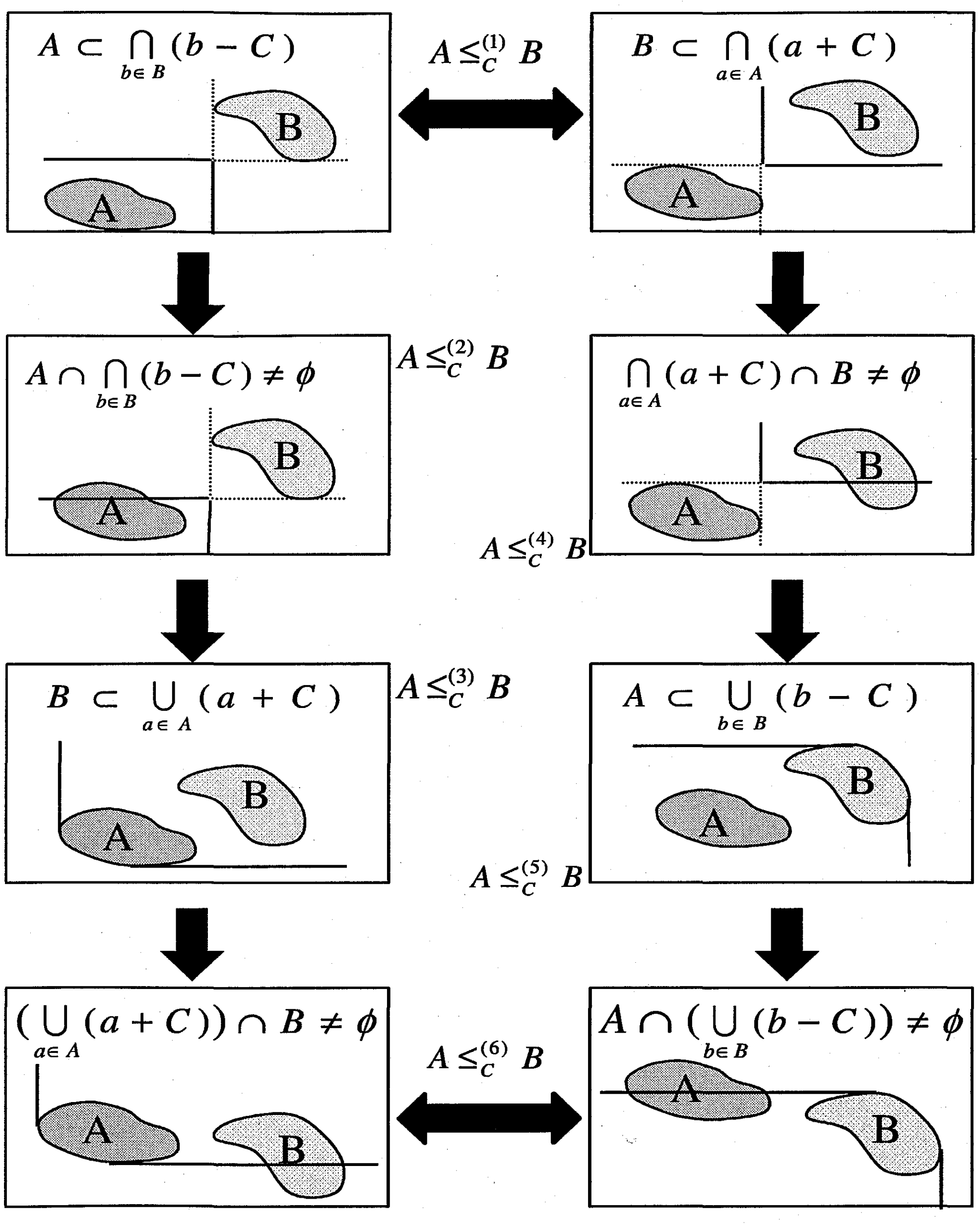

Figure 2.1: Six kinds of classification for set-relationship.

(i) $\bigcap_{a\in A}(a+C) = \bigcap_{y\in A-C}(y+C)$ and $\bigcap_{b\in B}(b-C) = \bigcap_{y\in B+C}(y-C)$;

(ii) $A+C = A+C+C$ and $B-C = B-C-C$;

(iii) $B \subset (A+C)$ if and only if $(B+C) \subset (A+C)$;

(iv) $A \subset (B-C)$ if and only if $(A-C) \subset (B-C)$.

Moreover, conditions $A \subset B$ and $A \cap B \neq \emptyset$ are invariant under translation and scalar multiplication, that is, for $y \in Y$ and $\alpha > 0$,

(v) $A \subset B$ implies $(A+y) \subset (B+y)$ and $(\alpha A) \subset (\alpha B)$;

(vi) $A \cap B \neq \emptyset$ implies $(A+y) \cap (B+y) \neq \emptyset$ and $(\alpha A) \cap (\alpha B) \neq \emptyset$.

Hence, for sets $A, B, D, E \in \wp(Y)$ and $\alpha \in \mathbb{R}$, the following property holds:

(vii) $A \cap B \neq \emptyset$ and $D \cap E \neq \emptyset$ implies $(A+D) \cap (B+E) \neq \emptyset$.

Proposition 2.2 *For sets $A, B \subset Y$, the following equivalences hold.*

(i) $A \leq_C^{(1)} B \Leftrightarrow A \leq_C^{(1)} (B+C) \Leftrightarrow (A-C) \leq_C^{(1)} B \Leftrightarrow (A-C) \leq_C^{(1)} (B+C)$

(ii) $A \leq_C^{(2)} B \Leftrightarrow A \leq_C^{(2)} (B+C) \Leftrightarrow (A+C) \leq_C^{(2)} B \Leftrightarrow (A+C) \leq_C^{(2)} (B+C)$

(iii) $A \leq_C^{(3)} B \Leftrightarrow A \leq_C^{(3)} (B+C) \Leftrightarrow (A+C) \leq_C^{(3)} B \Leftrightarrow (A+C) \leq_C^{(3)} (B+C)$

(iv) $A \leq_C^{(4)} B \Leftrightarrow A \leq_C^{(4)} (B-C) \Leftrightarrow (A-C) \leq_C^{(4)} B \Leftrightarrow (A-C) \leq_C^{(4)} (B-C)$

(v) $A \leq_C^{(5)} B \Leftrightarrow A \leq_C^{(5)} (B-C) \Leftrightarrow (A-C) \leq_C^{(5)} B \Leftrightarrow (A-C) \leq_C^{(5)} (B-C)$

(vi) $A \leq_C^{(6)} B \Leftrightarrow A \leq_C^{(6)} (B-C) \Leftrightarrow (A+C) \leq_C^{(6)} B \Leftrightarrow (A+C) \leq_C^{(6)} (B-C)$

Proof. Since C is a convex cone, the statements are clear. $\square$

Proposition 2.3 *For sets $A, B \in \wp(Y)$, the following statements hold.*

(i) *For each* $j = 1, \ldots, 6$,

$$A \leq_C^{(j)} B \text{ implies } (A + y) \leq_C^{(j)} (B + y) \text{ for } x \in Y, \text{ and}$$

$$A \leq_C^{(j)} B \text{ implies } \alpha A \leq_C^{(j)} \alpha B \text{ for } \alpha > 0;$$

(ii) *For each* $j = 1, \ldots, 5$, $\leq_C^{(j)}$ *is transitive;*

(iii) *For each* $j = 3, 5, 6$, $\leq_C^{(j)}$ *is reflexive.*

Proof. By Remark 2.1, statement (i) is clear, and also by the definition of each set-relation and $\theta_Y \in C$, statement (iii) is clear. We show the statement (ii). First we prove the case of $j = 2$. For sets $A, B, D \in \wp(Y)$, let $A \leq_C^{(2)} B$ and $B \leq_C^{(2)} D$. Then, by the definition of $\leq_C^{(2)}$,

$$A \cap \left\{ \bigcap_{b \in B} (b - C) \right\} \neq \emptyset; \tag{2.2}$$

$$B \cap \left\{ \bigcap_{d \in D} (d - C) \right\} \neq \emptyset. \tag{2.3}$$

By (2.2), there exists $a \in A$ such that $a \in b - C$ for any $b \in B$. By (2.3), there exists $b' \in B$ such that $b' \in d - C$ for any $d \in D$. Hence, $a \in d - C$ for any $d \in D$, that is, $A \leq_C^{(2)} B$. Consequently, $\leq_C^{(2)}$ are transitive. The case of $j = 4$ is proved in the same way and the other cases are obvious. □

Chapter 3

Convexity and continuity of set-valued map

In this chapter, we introduce some definitions of convexity and continuity of set-valued map, and their properties.

Throughout this chapter, X is a real topological vector space, Y is a real topological vector space ordered by a nontrivial convex cone C, f is a real-valued function from X into $\bar{\mathbb{R}}$, and F is a set-valued map from X into $\wp(Y)$.

3.1 Convexity of set-valued map

In this section, we introduce some definitions of convexity for set-valued map, and we define concavity as the dual notion of convexity for set-valued map in the sense of set-relations.

We do not need a topology to mention results on convexities, and hence X is a real vector space and Y is a real vector space ordered by C in this section.

At first, we recall some definitions of convexity for real-valued function.

Definition 3.1 (convex) A real-valued function f is called *convex* if the set

$$\operatorname{epi}(f) := \{(x, r) \in X \times \mathbb{R} | f(x) \leq r\}$$

is convex or empty.

It is easy to check that the following property holds.

Proposition 3.1 *A function $g : X \to \mathbb{R}$ is convex on X if and only if for any $x, y \in X$ and $\lambda \in [0,1]$,*

$$g(\lambda x + (1-\lambda)y) \leq \lambda g(x) + (1-\lambda)g(y).$$

Definition 3.2 (quasiconvex) A real-valued function f is called *quasiconvex* if for each $x, y \in X$ and $\lambda \in [0,1]$,

$$f(\lambda x + (1-\lambda)y) \leq \max\{f(x), f(y)\}.$$

If $-f$ is a quasiconvex function, then f is called a *quasiconcave function.*

In [68, 69], Tanaka proposes several convexity concepts for vector-valued function based on the above convexity concepts and the partial ordering $\leq_C$. By using these concepts and six kinds of set-relations with respect to C introduced in Definition 2.3, we consider natural extensions of some concepts of convexity of real-valued function to set-valued map.

Definition 3.3 (C-convex, [47]) For each $j = 1, \ldots, 6$, a set-valued map F is called *type (j) C-convex* if for each $x, y \in X$ and $\lambda \in (0,1)$,

$$F(\lambda x + (1-\lambda)y) \leq_C^{(j)} \lambda F(x) + (1-\lambda)F(y).$$

Let A be a nonempty subset of X and $\mathcal{V}$ a family of nonempty subsets of Y. We say that $\mathcal{V}$ is *convex* if for any $V_1, V_2 \in \mathcal{V}$ and $\lambda \in (0,1)$, $\lambda V_1 + (1-\lambda)V_2 \in \mathcal{V}$. Moreover, $A \times \mathcal{V} \subset X \times \wp Y$ is *convex* if for any $(a_1, V_1), (a_2, V_2) \in A \times \mathcal{V}$ and $\lambda \in (0,1)$, $\lambda(a_1, V_1) + (1-\lambda)(a_2, V_2) \in A \times \mathcal{V}$. By this convexity concept, (i) of Proposition 2.3 and Remark 2.1, we obtain the following results.

Proposition 3.2 *For each $j = 1, \ldots, 6$, the following statements hold:*

(i) *If a set-valued map F is type (j) C-convex on X then the set*

$$\mathrm{gr}_l\left(F;\leq_C^{(j)}\right) := \{(x,V) \in X \times \wp(Y) | F(x) \leq_C^{(j)} V\}$$

is convex.

(ii) *If a set-valued map F is type (j) C-concave on X then the set*

$$\mathrm{gr}_u\left(F;\leq_C^{(j)}\right) := \{(x,V) \in X \times \wp(Y) | V \leq_C^{(j)} F(x)\}$$

is convex.

Definition 3.4 (properly quasi C-convex, [47]) For each $j = 1,\ldots,6$, a set-valued map F is called *type (j) properly quasi C-convex* if for each $x, y \in X$ and $\lambda \in (0,1)$,

$$F(\lambda x + (1-\lambda)y) \leq_C^{(j)} F(x) \quad \text{or} \quad F(\lambda x + (1-\lambda)y) \leq_C^{(j)} F(y).$$

Definition 3.5 (natural quasi C-convex, [47]) For each $j = 1,\ldots,6$, a set-valued map F is called *type (j) natural quasi C-convex* if for each $x, y \in X$ and $\lambda \in (0,1)$, there exists $\mu \in [0,1]$ such that

$$F(\lambda x + (1-\lambda)y) \leq_C^{(j)} \mu F(x) + (1-\mu)F(y).$$

If C is replaced by $\mathrm{int}\, C$ then we call it *type (j) strictly natural quasi C-convex.*

The relationships between these cone-convexities with $j = 1,\ldots,6$ are as follows:

Proposition 3.3 ([47]) *For each $j = 1,\ldots,6$,*

(i) *if a set-valued map F is type (j) C-convex, then F is type (j) natural quasi C-convex, and*

(ii) *if a set-valued map F is type (j) properly quasi C-convex, then F is type (j) natural quasi C-convex.*

To illustrate relationships among these convexities, we give the following simple examples.

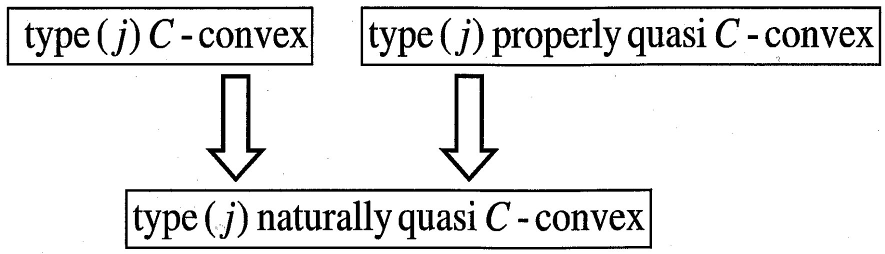

Figure 3.1: Relationships among cone-convexities for set-valued maps.

Example 3.1 Let $X := \mathbb{R}$, $Y := \mathbb{R}^2$, $C := \mathbb{R}^2_+$. We consider a set-valued map $F : X \to \wp(Y)$

$$F(x) := \left[\begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} x \\ x^2 \end{pmatrix} \right].$$

Then, for each $j = 2, \ldots, 6$, F is type (j) C-convex, but F is not type (5) properly quasi C-convex.

Example 3.2 Let $X := [0,1]$, $Y := \mathbb{R}^2$, $C := \mathbb{R}^2_+$. We consider a set-valued map $F : X \to \wp(Y)$

$$F(x) := \begin{cases} \left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right] & (x = 0, 1), \\ \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\} & (0 < x < 1). \end{cases}$$

Then, for each $j = 1, \ldots, 6$, F is type (j) C-convex and moreover F is type (j) properly quasi C-convex.

Example 3.3 Let $X := \mathbb{R}_+$, $Y := \mathbb{R}^2$, $C := \mathbb{R}^2_+$. We consider a set-valued map

$F : X \to \wp(Y)$

$$F(x) := \begin{cases} \left[\begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} x \\ x \end{pmatrix}\right] & (x \leq 1), \\ \left[\begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} x \\ 1 \end{pmatrix}\right] & (1 \leq x \leq 2), \\ \left[\begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} x \\ x-1 \end{pmatrix}\right] & (2 \leq x). \end{cases}$$

Then, for each $j = 2, \dots, 6$, F is type (j) properly quasi C-convex, but neither $j = 4$ nor $j = 5$ type (j) C-convex.

Example 3.4 Let $X := \mathbb{R}$, $Y := \mathbb{R}^2$, $C := \mathbb{R}^2_+$. We consider a set-valued map $F : X \to \wp(Y)$

$$F(x) := \begin{cases} \left[\begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} x \\ -x+1 \end{pmatrix}\right] & (x \leq 0), \\ \left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] & (0 \leq x < 1), \\ \left[\begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} x \\ x-1 \end{pmatrix}\right] & (1 \leq x). \end{cases}$$

Then, for each $j = 2, \dots, 6$, F is type (j) natural quasi C-convex, but neither $j = 4$ nor $j = 5$ type (j) C-convex and type (j) properly quasi C-convex.

Moreover, we propose cone-concavity as the dual notion of cone-convexity for set-valued map in the sense of set-relations. Usually, the dual notion of convexity (resp. cone-convexity) for a real-valued (resp. vector-valued) function F is defined by the convexity (resp. cone-convexity) of $-F$, and then F is called concave (resp. cone-concave). However in the case of set-valued maps, it is not necessarily formed. For example, if F is a type (3) C-concave function in the usual definition (e.g., [47]), then $-F$ is a type (3) C-convex function and hence F satisfies

$$\lambda F(x) + (1-\lambda)F(y) \leq_C^{(5)} F(\lambda x + (1-\lambda)y)$$

for each $x, y \in X$ and $\lambda \in (0,1)$, but this condition seems to be the cone-concavity of F in the set-relation $\leq_C^{(5)}$. Hence, we define the dual notion of cone-convexity for set-valued map directly in a different way.

Definition 3.6 (C-concave) For each $j = 1, \ldots, 6$, a set-valued map F is called *type* (j) *C-concave* if for each $x, y \in X$ and $\lambda \in (0,1)$,

$$\lambda F(x) + (1-\lambda)F(y) \leq_C^{(j)} F(\lambda x + (1-\lambda)y).$$

Definition 3.7 (properly quasi C-concave) For each $j = 1, \ldots, 6$, a set-valued map F is called *type* (j) *properly quasi C-concave* if for each $x, y \in X$ and $\lambda \in (0,1)$,

$$F(x) \leq_C^{(j)} F(\lambda x + (1-\lambda)y) \quad \text{or} \quad F(y) \leq_C^{(j)} F(\lambda x + (1-\lambda)y).$$

Definition 3.8 (natural quasi C-concave) For each $j = 1, \ldots, 6$, a set-valued map F is called *type* (j) *natural quasi C-concave* if for each $x, y \in X$ and $\lambda \in (0,1)$, there exists $\mu \in [0,1]$ such that

$$\mu F(x) + (1-\mu)F(y) \leq_C^{(j)} F(\lambda x + (1-\lambda)y).$$

If C is replaced by $\operatorname{int} C$ then we call it *type* (j) *strictly natural quasi C-concave.*

The relationships among these cone-concavities with $j = 1, \ldots, 6$ are as follows:

Proposition 3.4 *For each $j = 1, \ldots, 6$,*

(i) *if a set-valued map F is type (j) C-concave, then F is type (j) natural quasi C-concave, and*

(ii) *if a set-valued map F is type (j) properly quasi concave, then F is type (j) natural quasi C-concave.*

Proof. In the same way in [47], the statements are proved straightforward. □

3.2 Continuity of set-valued map

In this section, we introduce some continuity concepts for set-valued map. Firstly, we recall definitions of continuity and semicontinuity of real-valued function, and its property.

Definition 3.9 (continuity for real-valued function) Let $x \in X$.

(i) A real-valued function f is called *lower semicontinuous at* x if for any $\epsilon > 0$ there exists U which is an open neighborhood of x such that $f(x) - \epsilon < f(y)$ for all $y \in U$. We shall say that f is lower semicontinuous on X if f is lower semicontinuous at every $x \in X$.

(ii) A real-valued function f is called *upper semicontinuous at* x if for any $\epsilon > 0$ there exists U which is an open neighborhood of x such that $f(y) < f(x) + \epsilon$ for all $y \in U$. We shall say that f is upper semicontinuous on X if f is upper semicontinuous at every $x \in X$.

(iii) A real-valued function f is called *continuous at* x if f is both of lower and upper semicontinuous at x. We shall say that f is continuous on X if f is continuous at every $x \in X$.

It is well known that the following proposition holds.

Proposition 3.5 *The following statements hold:*

(i) *If a real-valued function f is lower semicontinuous on X then the set*

$$\mathrm{lev}_r^l(f) := \{x \in X | f(x) \leq r\}$$

is closed or empty for all $r \in \mathbb{R}$.

(ii) *If a real-valued function f is upper semicontinuous on X then the set*

$$\mathrm{lev}_r^u(f) := \{x \in X | r \leq f(x)\}$$

is closed or empty for all $r \in \mathbb{R}$.

Next, we recall usual definitions of continuity for set-valued maps.

Definition 3.10 (continuity for set-valued map, [24]) Let $x \in X$.

(i) A set-valued map F is called *lower continuous at* x if for every open set V with $F(x) \cap V \neq \emptyset$, there exists an open neighborhood U of x such that $F(y) \cap V \neq \emptyset$ for all $y \in U$. We shall say that F is *lower continuous on* X if F is lower continuous at every point $x \in X$.

(ii) A set-valued map F is called *upper continuous at* x if for every open set V with $F(x) \subset V$, there exists an open neighborhood U of x such that $F(y) \subset V$ for all $y \in U$. We shall say that F is *upper continuous on* X if F is upper continuous at every point $x \in X$.

(iii) A set-valued map F is called *continuous on* X if F is both lower and upper continuous on X.

Classically, we find the terms "lower semicontinuous" and "upper semicontinuous" for these notions. Instead, in the thesis, we use the terms "lower continuous" and "upper continuous" along the lines of [24], because both notions above coincide with the usual continuity of single-valued functions when the set-valued map is singleton, that is, $F(x) = \{f(x)\}$ for some function $f : X \to Y$.

For cone-continuity of set-valued maps, there are many concepts; see [39] in 1999, [19] in 2000 and [24] in 2003. In the thesis, we use the following typical definitions of cone-continuity for set-valued maps based on [24].

Definition 3.11 (C-continuity for set-valued maps, [24]) Let $x \in X$.

(i) A set-valued map F is called *lower C-continuous at* x if for every open set $V \subset Y$ with $F(x) \cap V \neq \emptyset$, there exists an open neighborhood U of x such that $F(y) \cap (V + C) \neq \emptyset$ for all $y \in U$. We shall say that F is *lower C-continuous on* X if F is lower C-continuous at every point $x \in X$.

(ii) A set-valued map F is called *upper C-continuous at x* if for every open set V with $F(x) \subset V$, there exists an open neighborhood U of x such that $F(y) \subset V + C$ for all $y \in U$. We shall say that F is *upper C-continuous on X* if F is upper C-continuous at every point $x \in X$.

(iii) A set-valued map F is called *C-continuous on X* if F is both lower and upper C-continuous on X.

Remark 3.1 It is clear that $f(x) - \epsilon < f(y)$ (resp., $f(x) + \epsilon > f(y)$) is equivalent to $f(y) \in f(x) + (-\epsilon, \epsilon) + \mathbb{R}_+$ (resp., $f(y) \in f(x) + (-\epsilon, \epsilon) - \mathbb{R}_+$). Hence, when $Y = \mathbb{R}$ and $C = \mathbb{R}_+$, lower C-continuity and upper C-continuity for singleton set-valued maps coincide with the usual lower semicontinuity for real-valued functions. Also, lower $(-C)$-continuity and upper $(-C)$-continuity for singleton set-valued maps coincide with the usual upper semicontinuity for real-valued functions. By symbolic interpretation, C and $-C$ correspond to "lower" and "upper," respectively.

By Theorem 7.2 in [51], we obtain the following lemma. This lemma plays important roles in Sections 5.4 and 5.5.

Lemma 3.1 *Let A be a compact subset of X and $F : A \to \wp(Y)$. Assume that F is upper C-continuous on A. If F is C-proper (resp., closed, bounded) valued on A, then* $\bigcup_{x \in A} F(x)$ *is C-proper (resp., closed, bounded).*

Chapter 4

Scalarization methods for sets

In this chapter, we introduce some scalarization methods for sets and some properties of them which are very useful when we consider the replacement of a set-valued optimization problem by a suitable scalar optimization problem.

Throughout this chapter, X is a real topological vector space, Y is a finite dimensional real Hausdorff topological vector space, C is a nontrivial, closed, pointed convex cone with $\operatorname{int} C \neq \emptyset$.

4.1 Scalarizing functions for sets

At first, we recall linear scalarization methods for vectors. The most representative example of linear scalarizing functions is an inner product. Let $y, k \in Y$ and $A \in \wp(Y)$. In the case of vector, it is defined by

$$h(y; k) := \langle y, k \rangle.$$

Based on this scalarization, we can consider the following scalarizing functions for sets defined by

$$\varphi(A; k) := \inf_{y \in A} \langle y, k \rangle, \quad \psi(A; k) := \sup_{y \in A} \langle y, k \rangle.$$

Next, we introduce sublinear scalarizing functions for vectors which have very useful properties on nonlinear vector optimization. By this result, sublinear scalarizing

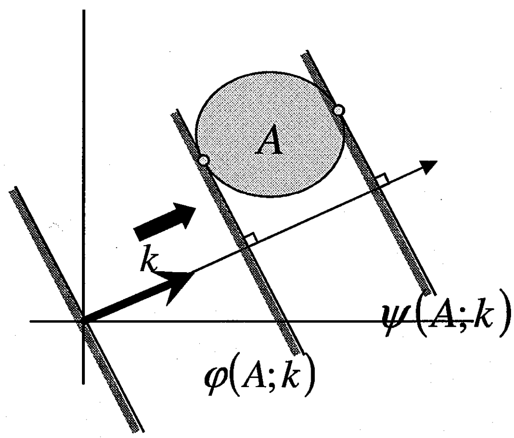

Figure 4.1: An image of scalarizing functions for sets based on an inner product.

functions for vectors defined as follows:

$$(1)\quad h_C(y;k) := \inf\{t \in \mathbb{R} : y \in tk - C\}$$

$$(2)\quad -h_C(-y;k) = \sup\{t \in \mathbb{R} : y \in tk + C\}$$

where $k \in \operatorname{int} C$. This function $h_C(y;k)$ has been treated in some papers; see, [21] and [60], and it is regarded as a generalization of the Chebyshev scalarization. Essentially, $h_C(y;k)$ is equivalent to the smallest strictly monotonic function with respect to $\operatorname{int} C$ defined by Luc in [51]. Note that $h_C(\cdot;k)$ is positively homogeneous and subadditive for every fixed $k \in \operatorname{int} C$, and hence it is sublinear and continuous.

By using these sublinear functions, we define the following four types of nonlinear scalarizing functions for sets proposed by Nishizawa, Tanaka, and Georgiev [54].

$$(1)\quad \psi_C^{(1)}(A;k) := \sup\{h_C(y;k) : y \in A\},$$

$$(2)\quad \varphi_C^{(1)}(A;k) := \inf\{h_C(y;k) : y \in A\},$$

$$(3)\quad \varphi_C^{(2)}(A;k) = \sup\{-h_C(-y;k) : y \in A\},$$

$$(4)\quad \psi_C^{(2)}(A;k) = \inf\{-h_C(-y;k) : y \in A\}.$$

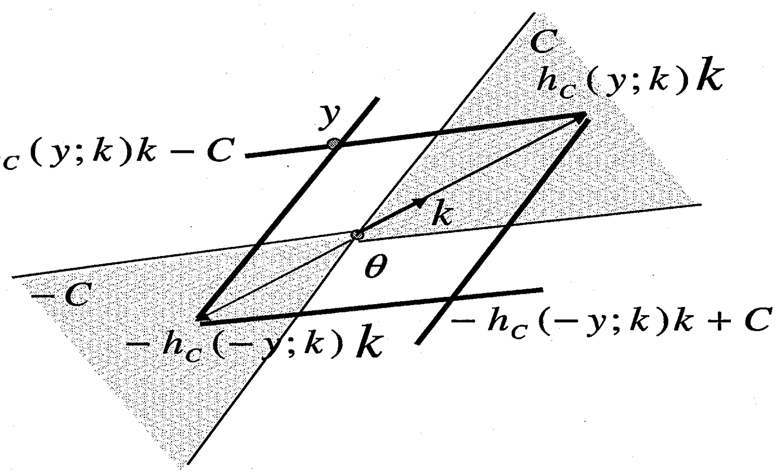

Figure 4.2: An image of sublinear scalarizing functions for vectors.

Functions (1) and (4) have symmetric properties and then results for function (4) $\psi_C^{(2)}$ can be easily proved by those for function (1) $\psi_C^{(1)}$. Similarly, the results for function (3) $\varphi_C^{(2)}$ can be deduced by those for function (2) $\varphi_C^{(1)}$. By using these four functions we measure each image of set-valued map F with respect to its 4-tuple of scalars, which can be regarded as standpoints for the evaluation of the image with respect to convex cone C.

Next, we introduce another scalarization method proposed in [27]. In [27], they propose the four types of nonlinear scalarizing functions for sets by the approach different from the above. Let $k \in \operatorname{int} C$ and $\mathcal{V} \subset \wp(Y)$.

(i) We assume that $\mathcal{V}$ is $\leq_C^{(3)}$-bounded, that is, there is a topological bounded set $V' \in \wp(Y)$ and a nonempty set $V'' \in \wp(Y)$ such that

$$V' \leq_C^{(3)} V \leq_C^{(3)} V'' \text{ for all } V \in \mathcal{V}.$$ 1

Then, the functional $z^l : \wp(Y) \to \mathbb{R} \cup \{\pm\infty\}$, defined by

$$z^l(V) := \inf\{t \in \mathbb{R} : tk + V'' \subset V + \operatorname{cl} C\}.$$

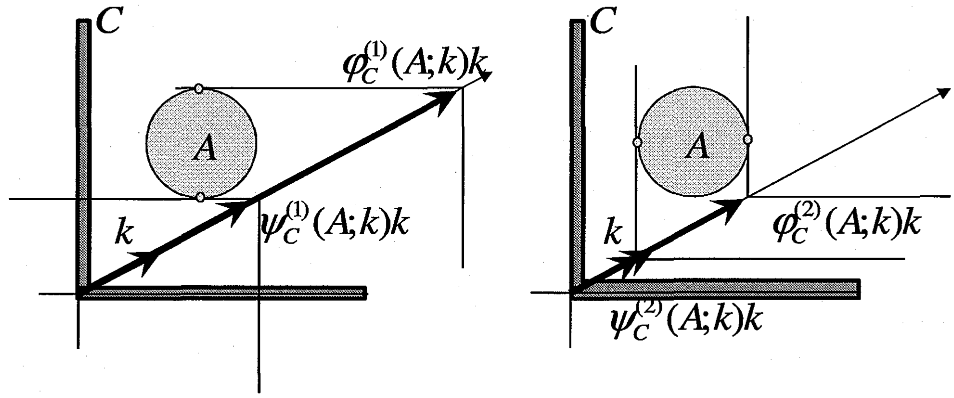

Figure 4.3: An image of four types of nonlinear scalarizing functions for sets in [54].

(ii) We assume that $\mathcal{V}$ is $\leq_C^{(5)}$-bounded, that is, there is a nonempty set $W' \in \wp(Y)$ and a topological bounded set $W'' \in \wp(Y)$ such that

$$W' \leq_C^{(5)} V \leq_C^{(5)} W'' \text{ for all } V \in \mathcal{V}.$$

Then, the functional $z^u : \wp(Y) \to \mathbb{R} \cup \{\pm\infty\}$, defined by

$$z^u(V) := -\inf\{t \in \mathbb{R} : t(-k) + W' \subseteq V - \operatorname{cl} C\}.$$

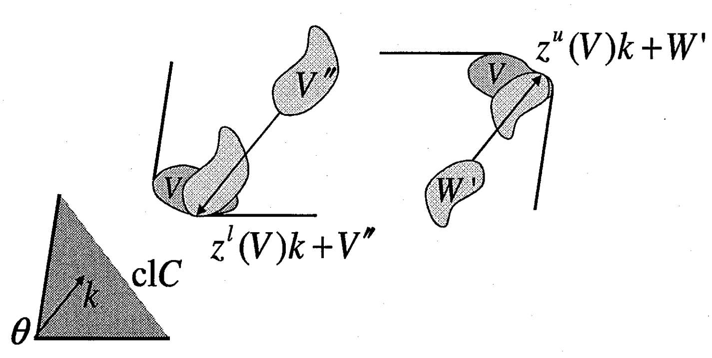

Figure 4.4: An image of Hamel-Löhne type scalarizing functions for sets.

Based on these scalarization methods for sets and six kinds of set-relations, we propose the following nonlinear scalarizing functions for sets. Let $V, V' \in \wp(Y)$ and

$k \in \operatorname{int} C$. For each $j = 1, \ldots, 6$, $I^{(j)}_{(k,V')} : \wp(Y) \to \mathbb{R} \cup \{\pm\infty\}$ and $S^{(j)}_{(k,V')} : \wp(Y) \to \mathbb{R} \cup \{\pm\infty\}$ are defined by

$$I^{(j)}_{(k,V')}(V) := \inf \left\{ t \in \mathbb{R} \,\middle|\, V \leq^{(j)}_C (tk + V') \right\},$$

$$S^{(j)}_{(k,V')}(V) := \sup \left\{ t \in \mathbb{R} \,\middle|\, (tk + V') \leq^{(j)}_C V \right\},$$

respectively. They are regarded as extensions of scalarizing functions for vectors and for sets. The original idea on the sublinear scalarization for vectors was dealt by Krasnosel'skij [41] in 1962 and by Rubinov [60] in 1977, and then it was applied to vector optimization with its concrete definition by Tammer (Gerstewitz) [22] in 1983, and to separation theorems for not necessary convex sets by Tammer (Gerstewitz) and Iwanow [23] in 1985. In recent years, several scalarization ideas for sets are proposed in [54, 27, 63], and all of them are special cases of unified types of scalarizing functions above.

Proposition 4.1 *Let $A, B, V, V' \in \wp(Y)$ and $k \in \operatorname{int} C$. Then, the following statements hold:*

(i) *For each $j = 1, \ldots, 6$,*

$$V \leq^{(j)}_C (tk + V') \quad \textit{implies} \quad V \leq^{(j)}_C (sk + V') \quad \textit{for any } s \geq t,$$

$$(tk + V') \leq^{(j)}_C V \quad \textit{implies} \quad (sk + V') \leq^{(j)}_C V \quad \textit{for any } s \leq t.$$

(ii) *For each $j = 1, \ldots, 6$ and $\alpha \in \mathbb{R}$,*

$$I^{(j)}_{(k,V')}(V + \alpha k) = I^{(j)}_{(k,V')}(V) + \alpha,$$

$$S^{(j)}_{(k,V')}(V + \alpha k) = S^{(j)}_{(k,V')}(V) + \alpha.$$

(iii) *For each $j = 1, \ldots, 5$,*

$$A \leq^{(j)}_C B \quad \textit{implies} \quad I^{(j)}_{(k,V')}(A) \leq I^{(j)}_{(k,V')}(B) \quad \textit{and} \quad S^{(j)}_{(k,V')}(A) \leq S^{(j)}_{(k,V')}(B).$$

Proof. Statement (i) is clear. We show statement (ii). Let $s := I^{(j)}_{(k,V')}(V)$. Then, for any $\varepsilon > 0$, there exists $s(\varepsilon)$ such that

$$s \leq s(\varepsilon) < s + \varepsilon \quad \text{and} \quad V \leq^{(j)}_C (s(\varepsilon)k + V).$$

By Proposition 4.3, we obtain $(V + \alpha k) \leq^{(j)}_C ((s(\varepsilon) + \alpha)k + V')$ and so

$$I^{(j)}_{(k,V')}(V + \alpha k) \leq s(\varepsilon) + \alpha < s + \varepsilon + \alpha.$$

Since ε is an arbitrary, we obtain $I^{(j)}_{(k,V')}(V + \alpha k) \leq s + \alpha = I^{(j)}_{(k,V')}(V) + \alpha$. Conversely, we can prove $I^{(j)}_{(k,V')}(V + \alpha k) \geq I^{(j)}_{(k,V')}(V) + \alpha$ in the same way. Consequently, we obtain $I^{(j)}_{(k,V')}(V + \alpha k) = I^{(j)}_{(k,V')}(V) + \alpha$. In the same way as the case of $I^{(j)}_{(k,V')}$, the equality $S^{(j)}_{(k,V')}(V + \alpha k) = S^{(j)}_{(k,V')}(V) + \alpha$ is shown.

Finally, we show statement (iii). Let A and B be nonempty subsets of Y satisfying $A \leq^{(j)}_C B$. The case of and $I^{(j)}_{(k,V')}(B) = +\infty$ is trivial. In the case of $I^{(j)}_{(k,V')}(B) = -\infty$, $B \leq^{(j)}_C (tk + V')$ for all $t \in \mathbb{R}$, and then it follows from the transitivity of $\leq^{(j)}_C$ that $I^{(j)}_{(k,V')}(A) = -\infty$. Hence we consider the case of $s := I^{(j)}_{(k,V')}(B) \in \mathbb{R}$. By the definition of infimum, for any $\varepsilon > 0$, there exists $s(\varepsilon)$ such that

$$s \leq s(\varepsilon) < s + \varepsilon \quad \text{and} \quad B \leq^{(j)}_C (s(\varepsilon)k + V').$$

Since $A \leq^{(j)}_C B$ and $\leq^{(j)}_C$ is transitive, we obtain $A \leq^{(j)}_C (s(\varepsilon)k + V')$, that is, $I^{(j)}_{(k,V')}(A) \leq s(\varepsilon) < s + \varepsilon$. Since ε is arbitrary, $I^{(j)}_{(k,V')}(A) \leq s = I^{(j)}_{(k,V')}(B)$. Consequently, $I^{(j)}_{(k,V')}(\cdot)$ is monotone with respect to $\leq^{(j)}_C$. In the same way, the monotonicity with respect to each set-relation of $S^{(j)}_{(k,V')}$ is proved, and the proof is completed. □

Proposition 4.2 *Let $A \in \wp(Y)$. Then, the following statements hold:*

(i) *For any $k \in \operatorname{int} C$ and $V' \in \wp(Y)$,*

$$I^{(3)}_{(k,V')}(A) \leq I^{(2)}_{(k,V')}(A) \leq I^{(1)}_{(k,V')}(A),$$

(ii) *For any* $k \in \operatorname{int} C$ *and* $V' \in \wp(Y)$,

$$I^{(5)}_{(k,V')}(A) \leq I^{(4)}_{(k,V')}(A) \leq I^{(1)}_{(k,V')}(A).$$

Proof. Assume that there exist k_0 and V_0 such that $I^{(1)}_{(k_0,V_0)}(A) < I^{(2)}_{(k_0,V_0)}(A)$. Let $t_1 := I^{(1)}_{(k_0,V_0)}(A)$. Then, there exists $\epsilon > 0$ such that

$$t_1 < t_1 + \epsilon < I^{(2)}_{(k_0,V_0)}(A) \quad \text{and} \quad A \leq^{(1)}_C (t_1 + \epsilon)k_0 + V_0.$$

By Proposition 2.1, we obtain $A \leq^{(2)}_C (t_1 + \epsilon)k_0 + V_0$ and then $I^{(2)}_{(k_0,V_0)}(A) \leq t_1 + \epsilon$. This is a contradiction. Hence, we obtain $I^{(2)}_{(k,V')}(A) \leq I^{(1)}_{(k,V')}(A)$ for each $k \in \operatorname{int} C$ and $V' \in \wp(Y)$. Similarly, we can prove $I^{(3)}_{(k,V')}(A) \leq I^{(2)}_{(k,V')}(A)$. Thus, we have

$$I^{(3)}_{(k,V')}(A) \leq I^{(2)}_{(k,V')}(A) \leq I^{(1)}_{(k,V')}(A).$$

Statement (ii) can be proved in the same way. □

Proposition 4.3 *Let* $A \in \wp(Y)$. *Then, the following statements hold:*

(i) *For each fixed* $k \in \operatorname{int} C$ *and* $V' \in \wp(Y)$,

$$S^{(1)}_{(k,V')}(A) \leq S^{(2)}_{(k,V')}(A) \leq S^{(3)}_{(k,V')}(A).$$

(ii) *For each fixed* $k \in \operatorname{int} C$ *and* $V' \in \wp(Y)$,

$$S^{(1)}_{(k,V')}(A) \leq S^{(4)}_{(k,V')}(A) \leq S^{(5)}_{(k,V')}(A).$$

Proof. We can prove the statements in a similar way to the proof of Proposition 4.2. □

Proposition 4.4 *For each* $j = 1, \ldots, 5$, $k \in \operatorname{int} C$ *and* $V' \in \wp(Y)$, *if* $I^{(j)}_{(k,V')}(V'), S^{(j)}_{(k,V')}(V') \in \mathbb{R}$ *then* $I^{(j)}_{(k,V')}(V') \geq 0$ *and* $S^{(j)}_{(k,V')}(V') \leq 0$. *In particular,*

$$V' \leq^{(j)}_C V' \quad \textit{implies} \quad I^{(j)}_{(k,V')}(V') = S^{(j)}_{(k,V')}(V') = 0.$$

Proof. Firstly, we show $I^{(j)}_{(k,V')}(V') \geq 0$ and $S^{(j)}_{(k,V')}(V') \leq 0$. Let $t_j = I^{(j)}_{(k,V')}(V')$ and assume that $t_j < 0$. Then, there exists $\epsilon > 0$ and $t(\epsilon) \in \mathbb{R}$ such that

$$t_j < t(\epsilon) < t_j + \epsilon < 0 \quad \text{and} \quad V' \leq^{(j)}_C t(\epsilon)k + V'. \tag{4.1}$$

By (iii) of Proposition 4.1,

$$I^{(j)}_{(k,V')}(V') \leq I^{(j)}_{(k,V')}(t(\epsilon)k + V').$$

Moreover, it follows from (ii) of Proposition 4.1 that

$$I^{(j)}_{(k,V')}(t(\epsilon)k + V') = I^{(j)}_{(k,V')}(V') + t(\epsilon).$$

Hence, we obtain $t_j \leq t_j + t(\epsilon)$ and so $t(\epsilon) \geq 0$. This contradicts (4.1). Consequently, we have $I^{(j)}_{(k,V')}(V') \geq 0$. The case of $S^{(j)}_{(k,V')}(V')$ are proved in a similar way. Next, we assume that $V' \leq^{(j)}_C V'$. By the definitions of $I^{(j)}_{(k,V')}$ and $S^{(j)}_{(k,V')}$, it is clear that $I^{(j)}_{(k,V')}(V') = S^{(j)}_{(k,V')}(V') = 0$, and the proof is completed. $\square$

Proposition 4.5 *Let $A \in \wp(Y)$. For each $k \in \operatorname{int} C$ and $V' \in \wp(Y)$, the following statements hold:*

(i) *If A is $(-C)$-bounded and V' is C-bounded then $I^{(1)}_{(k,V')}(A) \in \mathbb{R}$.*

(ii) *For each $j = 2, 3$, if A is C-proper and V' is C-bounded then $I^{(j)}_{(k,V')}(A) \in \mathbb{R}$.*

(iii) *For each $j = 4, 5$, if A is $(-C)$-bounded and V' is $(-C)$-proper then $I^{(j)}_{(k,V')}(A) \in \mathbb{R}$.*

(iv) *If A is C-proper and V' is $(-C)$-bounded then $I^{(6)}_{(k,V')}(A) \in \mathbb{R}$.*

Proof. At first, we prove statement (i). Assume that A is $(-C)$-bounded and V' is C-bounded. Then, by the definition of C-boundedness and $\operatorname{int} C \neq \emptyset$ there exist $\bar{a}, \bar{v} \in Y$ such that

$$A \subset \bar{a} - C \quad \text{and} \quad V' \subset \bar{v} + C.$$

Now, we consider the set $[-k, k]$. Since $k \in \operatorname{int} C$, we obtain $\operatorname{int}([-k, k]) \neq \emptyset$ and $\operatorname{int}([-k, k])$ contains the origin θ_Y. As a result, $\operatorname{int}([-k, k])$ is absorbing. Let $U := \operatorname{int}([-k, k])$. Then, there exists $\bar{t} \geq 0$ such that

$$\bar{a} - \bar{v} \in \bar{t}U \subset \bar{t}k - C.$$

Moreover, by the definition of $\leq_C^{(1)}$ and (i) of Proposition 2.3,

$$A \leq_C^{(1)} \{\bar{a}\} \leq_C^{(1)} \{\bar{t}k + \bar{v}\} \leq_C^{(1)} \bar{t}k + V'.$$

Thus, from (ii) of Proposition 2.3 and the definition of $I^{(1)}_{(k,V')}$, we have

$$I^{(1)}_{(k,V')}(A) \leq \bar{t} < \infty.$$

On the other hand, for any $a \in A - C$, there exists $t_a \geq 0$ such that

$$a - \bar{v} \in t_a U \subset -t_a k + C.$$

Hence, by (iii) of Proposition 4.1 we obtain

$$I^{(1)}_{(k,V')}(\{-t_a k + \bar{v}\}) \leq I^{(1)}_{(k,V')}(\{a\}). \tag{4.2}$$

Furthermore, since U is absorbing, for any $v \in V'$ there exists $t_{\bar{v}} \geq 0$ such that

$$v - \bar{v} \in t_{\bar{v}} U \subset t_{\bar{v}} k - C.$$

Therefore, we have $v - t_{\bar{v}} k \in \bar{v} - C$ and so

$$-t_{\bar{v}} = I^{(3)}_{(k,V')}(-t_{\bar{v}} k + V') \leq I^{(3)}_{(k,V')}(\{-t_{\bar{v}} k + v\}) \leq I^{(3)}_{(k,V')}(\{\bar{v}\}).$$

From (i) of Proposition 4.2, we obtain

$$-t_{\bar{v}} \leq I^{(3)}_{(k,V')}(\{\bar{v}\}) \leq I^{(1)}_{(k,V')}(\{\bar{v}\}). \tag{4.3}$$

By (4.2), (4.3) and (ii) of Proposition 4.1,

$$-\infty < -t_{\bar{v}} - t_a \leq -t_a + I^{(1)}_{(k,V')}(\{\bar{v}\}) \leq I^{(1)}_{(k,V')}(\{a\}). \tag{4.4}$$

Moreover, since $a \in A - C$ and C is a convex cone, $I^{(1)}_{(k,V')}(\{a\}) \leq I^{(1)}_{(k,V')}(A)$. For this result and (4.4), we obtain $-\infty < I^{(1)}_{(k,V')}(A)$. Consequently, $I^{(1)}_{(k,V')}(A) \in \mathbb{R}$.

Next, we prove statement (ii). We consider the case $j = 3$. Assume that A is C-proper and V' is C-bounded. Since A is C-proper, there exists $\bar{a} \in Y$ such that $\bar{a} \notin A + C$. It follows from the C-boundedness of V' that there exists $\bar{v} \in Y$ such that $V' \subset \bar{v} + C$. Let $U := \operatorname{int}([-k,\, k])$. Then, U is an absorbing neighborhood of the origin θ_Y, and so for any $a \in A$, there exists $t_a \geq 0$ such that

$$a - \bar{v} \in t_a U - C \subset t_a k - C.$$

Thus, we obtain

$$t_a k + V' \subset t_a k + \bar{v} + C \subset a + C \subset A + C,$$

and hence $I^{(3)}_{(k,V')}(A) \leq t_a < \infty$. Next, we prove $-\infty < I^{(3)}_{(k,V')}(A)$. Since U is absorbing, for any $v \in V'$, there exists $\hat{t} \geq 0$ such that

$$v - \bar{a} \in \hat{t} U \subset \hat{t} k - C.$$

Thus, we obtain $-\hat{t}k + v \in \bar{a} - C$ and so

$$(-\hat{t}k + V') \cap (\bar{a} - C) \neq \emptyset.$$

For this result and $\bar{a} \notin A + C$, it is easy to check that $-\hat{t}k + V' \not\subset A + C$. Therefore, we have

$$-\infty < -\hat{t} \leq I^{(3)}_{(k,V')}(A).$$

Consequently, $I^{(3)}_{(k,V')}(A) \in \mathbb{R}$. The remainder cases of $j = 2$, (iii), and (iv) can be proved similarly. $\square$

Proposition 4.6 *Let $A \in \wp(Y)$. For each $k \in \operatorname{int} C$ and $V' \in \wp(Y)$, the following statements hold:*

(i) *If A is C-bounded and V' is $(-C)$-bounded then $S^{(1)}_{(k,V')}(A) \in \mathbb{R}$.*

(ii) *For each* $j = 2, 3$, *if* A *is* C*-bounded and* V' *is* C*-proper then* $S^{(j)}_{(k,V')}(A) \in \mathbb{R}$.

(iii) *For each* $j = 4, 5$, *if* A *is* $(-C)$*-proper and* V' *is* $(-C)$*-bounded then* $S^{(j)}_{(k,V')}(A) \in \mathbb{R}$.

(iv) *If* A *is* $(-C)$*-proper and* V' *is* C*-bounded then* $S^{(6)}_{(k,V')}(A) \in \mathbb{R}$.

Proof. We can prove the statements in a similar way to the proof of Proposition 4.5. □

Next, we give several useful properties of $I^{(*)}_{(k,V')}$ and $S^{(*)}_{(k,V')}$ with $* = 3, 5$ which will be used in Chapter 5.

Proposition 4.7 *Let* $A \in \wp(Y)$ *and* $r \in \mathbb{R}$. *For each* $k \in \operatorname{int} C$ *and* $V' \in \wp(Y)$, *the following statements hold:*

(i) *If* A *is* C*-closed, then*

$$I^{(3)}_{(k,V')}(A) \leq r \quad \textit{implies} \quad A \leq^{(3)}_C rk + V'.$$

(ii) *If* V' *is* $(-C)$*-closed, then*

$$I^{(5)}_{(k,V')}(A) \leq r \quad \text{implies} \quad A \leq^{(5)}_C rk + V'.$$

(iii) *If* V' *is* C*-closed, then*

$$S^{(3)}_{(k,V')}(A) \geq r \quad \textit{implies} \quad rk + V' \leq^{(3)}_C A.$$

(iv) *If* A *is* $(-C)$*-closed, then*

$$S^{(5)}_{(k,V')}(A) \geq r \quad \textit{implies} \quad rk + V' \leq^{(5)}_C A.$$

Proof. We show statement (i) only since the others can be proved similarly. Let A be C-closed. Assume that there exists $(k_0, V_0') \in \operatorname{int} C \times \wp(Y)$ such that $I^{(3)}_{(k_0,V_0')}(A) \leq r$

and $A \not\leq_C^{(3)} rk_0 + V_0'$. Then, there exists $v' \in rk_0 + V_0'$ such that $v' \in (A+C)^c$. Since A is C-closed, $(A+C)^c$ is open. Hence, there exists $\delta > 0$ such that

$$v' + \delta k_0 \notin A + C.$$

Thus, $(r+\delta)k_0 + V_0' \not\subset A + C$ and then

$$r < r + \delta < I^{(3)}_{(k_0,V_0')}(A).$$

This contradicts $I^{(3)}_{(k_0,V_0')}(A) \leq r$. Consequently, $A \leq_C^{(3)} rk_0 + V_0'$. □

Proposition 4.8 *Let $A, B \in \wp(Y)$. Then the following statements hold:*

(i) *If the following two conditions are satisfied:*

  (a) $I^{(3)}_{(k,v)}(B)$ *is finite for each $k \in \operatorname{int} C$ and $v \in Y$,*

  (b) *B is C-closed and $B \subset A + \operatorname{int} C$,*

  then

$$I^{(3)}_{(k,v)}(A) < I^{(3)}_{(k,v)}(B)$$

  where $I^{(3)}_{(k,v)}(A) := \{t \in \mathbb{R} | \{tk + v\} \leq_C^{(3)} A\}$.

(ii) *If the following two conditions are satisfied:*

  (a) $S^{(5)}_{(k,v)}(B)$ *is finite for each $k \in \operatorname{int} C$ and $v \in Y$,*

  (b) *A is $(-C)$-closed and $A \subset B - \operatorname{int} C$,*

  then

$$S^{(5)}_{(k,v)}(A) < S^{(5)}_{(k,v)}(B)$$

  where $S^{(5)}_{(k,v)}(A) := \{t \in \mathbb{R} | A \leq_C^{(5)} \{tk + v\}\}$.

Proof. We prove statement (i) only; statement (ii) can be proved in a similar way. Let $t_A := I^{(3)}_{(k,v)}(A)$ and $t_B := I^{(3)}_{(k,v)}(B)$. By the assumption and (i) of Proposition 4.7, we obtain

$$(t_B k + v) \in B + C \subset A + \operatorname{int} C = \operatorname{int}(A + C).$$

Hence, there exists U which is a neighborhood of the origin θ_Y such that

$$t_B k + v + U \subset A + C.$$

Moreover, there exists $t_0 > 0$ such that $-t_0 k \in U$ and so we obtain

$$(t_B - t_0)k + v \in t_B k + v + U \subset A + C.$$

Thus, we have $I^{(3)}_{(k,v)}(A) \leq t_B - t_0 < t_B$. □

The following proposition will be used in Section 5.2.

Proposition 4.9 *Let $A, B \in \wp(Y)$. Then the following statements hold:*

(i) *If A and B are C-bounded and there is an open neighborhood G of θ such that $A \leq^{(3)}_C B + G$ then for each $k \in \operatorname{int} C$,*

$$S^{(3)}_{(k,\theta_Y)}(A) < S^{(3)}_{(k,\theta_Y)}(B).$$

(ii) *If A and B are $(-C)$-bounded and there is an open neighborhood G of θ such that $A + G \leq^{(5)}_C B$ then for each $k \in \operatorname{int} C$,*

$$I^{(5)}_{(k,\theta_Y)}(A) < I^{(5)}_{(k,\theta_Y)}(B).$$

Proof. We prove statement (i) only; statement (ii) can be proved in a similar way. Next, let $s_A := S^{(3)}_{(k,\theta_Y)}(A)$ and $s_B := S^{(3)}_{(k,\theta_Y)}(B)$. By (ii) of Proposition 4.6 and the definition of s_A, for any $\epsilon > 0$ there exists $s(\epsilon) \in \mathbb{R}$ such that

$$s_A - \epsilon < s(\epsilon) \leq s_A \quad \text{and} \quad \{s(\epsilon)k\} \leq^{(3)}_C A,$$

that is, $A \subset s(\epsilon)k + C$. From G is absorbing, there exists $s_0 > 0$ such that $-s_0k \in G$ and so we obtain

$$B - s_0k \subset B + G \subset A + C,$$

Thus we have

$$B \subset A + s_0k + C \subset (s(\epsilon) + s_0)\, k + C,$$

that is, $((s(\epsilon) + s_0)\, k + \{\theta\}) \leq_C^{(3)} B$. By the definition of s_B, we get $s(\epsilon) + s_0 \leq s_B$, and hence $s_A - \epsilon + s_0 \leq s_B$. Since ϵ is an arbitrary positive real number, we obtain $s_A < s_A + s_0 \leq s_B$. Consequently, $S^{(3)}_{(k,\theta_Y)}(A) < S^{(3)}_{(k,\theta_Y)}(B)$. □

Remark 4.1 In Proposition 4.8, the conditions of C-closeness (resp., $(-C)$-closeness) are necessary. Let $Z := \mathbb{R}^2$, $C := \mathbb{R}^2_+$, $A := \left[\binom{0}{0}, \binom{1}{1}\right]$, $B := \operatorname{int} A$ and $v := \theta_Y$. Then, $I^{(3)}_{(k,v)}(A) = I^{(3)}_{(k,v)}(B)$ for any $k \in \operatorname{int} C$ although $B \subset A + \operatorname{int} C$.

4.2 Inherited properties of scalarizing functions

In Section 4.1, we prove the monotonicity of unified types of scalarizing functions with respect to set-relations. In this section, we use this property to prove that some kinds of cone-convexity and cone-continuity for set-valued map are inherited to unified types of scalarizing functions. At first, we remark that the unified types of scalarizing functions have an important merit on the inheritance properties in contrast with the approach of [54].

Let $k \in \operatorname{int} C$, $V' \in \wp(Y)$, and $F : X \to \wp(Y)$ a set-valued map. For any $x \in X$ and for each $j = 1, \ldots, 6$, we consider the following composite functions:

$$(I^{(j)}_{(k,V')} \circ F)(x) := I^{(j)}_{(k,V')}(F(x)),$$

$$(S^{(j)}_{(k,V')} \circ F)(x) := S^{(j)}_{(k,V')}(F(x)).$$

Then, we can directly discuss inherited properties on cone-convexity and cone-continuity of parent set-valued map F to $I^{(j)}_{(k,V')} \circ F$ and $S^{(j)}_{(k,V')} \circ F$ in an analogous fashion to linear scalarizing function like inner product. Thus, we show how some kinds of cone-convexity and cone-continuity of parent set-valued map are inherited to unified types of scalarizing functions.

Proposition 4.10 *Let $F : X \to \wp(Y)$. For each $k \in \operatorname{int} C$ and $V' \in \wp(Y)$, the following statements hold:*

(i) *For each $j = 1, \ldots, 5$, if F is type (j) properly quasi C-convex then $I^{(j)}_{(k,V')} \circ F$ and $S^{(j)}_{(k,V')} \circ F$ are quasiconvex.*

(ii) *For each $j = 1, \ldots, 5$, if F is type (j) properly quasi C-concave then $I^{(j)}_{(k,V')} \circ F$ and $S^{(j)}_{(k,V')} \circ F$ are quasiconcave.*

Proof. By (iii) of Proposition 4.1 and the definition of properly quasiconvexity, it is clear. □

Proposition 4.11 *Let $F : X \to \wp(Y)$. For each $k \in \operatorname{int} C$ and $V' \in \wp(Y)$, the following statements hold:*

(i) *For each $j = 1, 2, 3$, if F is type (j) natural quasi C-convex then $I^{(j)}_{(k,V')} \circ F$ is quasiconvex.*

(ii) *For each $j = 4, 5$, if F is type (j) natural quasi C-convex and V' is $(-C)$-convex then $I^{(j)}_{(k,V')} \circ F$ is quasiconvex.*

Proof. Since F is type (j) natural quasi C-convex, for any $x, y \in X$ and $\lambda \in (0, 1)$, there exists $\mu \in [0, 1]$ such that

$$F(\lambda x + (1 - \lambda)y) \leq^{(j)}_C \mu F(x) + (1 - \mu)F(y). \tag{4.5}$$

Let $t_x := I^{(j)}_{(k,V')}(F(x))$, $t_y := I^{(j)}_{(k,V')}(F(y))$, and $t := I^{(j)}_{(k,V')}(F(\lambda x + (1 - \lambda)y))$. Then, we would like to prove $t \leq t_x$ or $t \leq t_y$. The case of $t = -\infty$ is obvious. Hence we consider the case of $t > -\infty$. Assume that there exists $(k_0, V_0 \in \operatorname{int} C \times \wp(Y)$ such that $I^{(j)}_{(k_0,V_0)}$ is not quasiconvex on X, that is, there exist $x, y \in X$ and $\lambda \in (0,1)$ such that

$$\max\{I^{(j)}_{(k_0,V_0)}(F(x)), I^{(j)}_{(k_0,V_0)}(F(y))\} < I^{(j)}_{(k_0,V_0)}(F(\lambda x + (1-\lambda)y)).$$

Let $t_x := I^{(j)}_{(k_0,V_0)}(F(x))$, $t_y := I^{(j)}_{(k_0,V_0)}(F(y))$, and $t := I^{(j)}_{(k_0,V_0)}(F(\lambda x + (1-\lambda)y))$ with $t_x \leq t_y$. Then there exists t^* such that

$$t_y < t^* < t \quad \text{and} \quad F(y) \leq^{(j)}_C (t^* k_0 + V_0).$$

It follows from (ii) of Proposition 2.3, we obtain $F(x) \leq^{(j)}_C (t^* k_0 + V_0)$.

First we prove statement (i). At first we consider the case $j = 1$. By the definition of $\leq^{(1)}_C$, we obtain

$$F(x) \subset \bigcap_{z \in t^* k_0 + V_0} (z - C) \quad \text{and} \quad F(y) \subset \bigcap_{z \in t^* k_0 + V_0} (z - C).$$

Since $\bigcap_{z \in t^* k_0 + V_0} (z - C)$ is a convex set, for any $s \in [0,1]$,

$$sF(x) + (1-s)F(y) \subset \bigcap_{z \in t^* k_0 + V_0} (z - C),$$

and so, we obtain $\mu F(x) + (1-\mu)F(y) \subset \bigcap_{z \in t^* k_0 + V_0} (z - C)$, that is,

$$\mu F(x) + (1-\mu)F(y) \leq^{(1)}_C (t^* k_0 + V_0). \tag{4.6}$$

By (4.5), (4.6), and (iii) of Proposition 4.1, we have

$$I^{(1)}_{(k_0,V_0)}(F(\lambda x + (1-\lambda)y)) \leq t^*,$$

which contradicts $t^* < t$. Next, we consider the case $j = 2$. By the definition of $\leq^{(2)}_C$, we obtain

$$F(x) \cap \left(\bigcap_{z \in t^* k_0 + V_0} (z - C) \right) \neq \emptyset \quad \text{and} \quad F(y) \cap \left(\bigcap_{z \in t^* k_0 + V_0} (z - C) \right) \neq \emptyset.$$

Since $\bigcap_{z \in t^* k_0 + V_0} (z - C)$ is convex, by (vi) and (vii) of Remark 2.1, for any $s \in [0,1]$, we obtain

$$(sF(x) + (1-s)F(y)) \cap \left(\bigcap_{z \in t^* k_0 + V_0} (z - C) \right) \neq \emptyset,$$

and so, $(\mu F(x)+(1-\mu)F(y))\cap\left(\bigcap_{z\in t^*k_0+V_0}(z-C)\right)\neq\emptyset$, that is,

$$\mu F(x)+(1-\mu)F(y)\leq_C^{(2)}(t^*k_0+V_0).$$

Hence we have

$$I^{(2)}_{(k_0,V_0)}(F(\lambda x+(1-\lambda)y))\leq t^*,$$

which contradicts $t^*<t$. The case $j=3$ is proved in the same way.

Next, we prove statement (ii). At first, we consider the case of $j=4$. By the definition of $\leq_C^{(4)}$ and convexity of C, we obtain

$$\left(\bigcap_{\bar{x}\in F(x)}(\bar{x}+C)\right)\cap(t^*k_0+V_0-C)\neq\emptyset,$$

and

$$\left(\bigcap_{\bar{y}\in F(y)}(\bar{y}+C)\right)\cap(t^*k_0+V_0-C)\neq\emptyset.$$

Since V_0 is a $(-C)$-convex set, $t^*k_0+V_0-C$ is convex and so, by (vi) and (vii) of Remark 2.1, for any $s\in[0,1]$, we obtain

$$\left(s\bigcap_{\bar{x}\in F(x)}(\bar{x}+C)+(1-s)\bigcap_{\bar{y}\in F(y)}(\bar{y}+C)\right)\cap(t^*k_0+V_0-C)\neq\emptyset. \tag{4.7}$$

Here, for sets $A,B\in\wp(Y)$, the following inclusion is formed:

$$\left(s\bigcap_{a\in A}(a+C)+(1-s)\bigcap_{b\in B}(b+C)\right)\subset\bigcap_{z\in sA+(1-s)B}(z+C).$$

Hence

$$\left(s\bigcap_{\bar{x}\in F(x)}(\bar{x}+C)+(1-s)\bigcap_{\bar{y}\in F(y)}(\bar{y}+C)\right)\subset\bigcap_{\bar{z}\in sF(x)+(1-s)F(y)}(\bar{z}+C),$$

and then it follows from (4.7) that we obtain

$$\left(\bigcap_{\bar{z}\in sF(x)+(1-s)F(y)}(\bar{z}+C)\right)\cap(t^*k_0+V_0-C)\neq\emptyset,$$

that is, $sF(x) + (1-s)F(y) \leq_C^{(4)} (t^*k_0 + V_0 - C)$. By (iv) of Proposition 2.2, we have $sF(x) + (1-s)F(y) \leq_C^{(4)} (t^*k_0 + V_0)$ for any $s \in [0,1]$, and hence

$$\mu F(x) + (1-\mu)F(y) \leq_C^{(4)} (t^*k_0 + V_0). \tag{4.8}$$

By (4.5), (4.8), and (ii) of Proposition 2.3, we have

$$I^{(4)}_{(k_0,V_0)}(F(\lambda x + (1-\lambda)y)) \leq t^*.$$

This is a contradiction to $t^* < t$. The case $j = 5$ is proved in the same way, and the proof is completed. □

Proposition 4.12 *Let $F : X \to \wp(Y)$. For each $k \in \operatorname{int} C$ and $V' \in \wp(Y)$, the following statements hold:*

(i) *For each $j = 1, 4, 5$, if F is type (j) natural quasi C-concave then $S^{(j)}_{(k,V')} \circ F$ is quasiconcave.*

(ii) *For each $j = 2, 3$, if F is type (j) natural quasi C-concave and V' is C-convex then $S^{(j)}_{(k,V')} \circ F$ is quasiconcave.*

Proof. By the same way in Proposition 4.11, the statements are proved. □

Next, we introduce some continuous properties are inherited to composite functions of set-valued maps and scalarizing functions. To show these results, we give the following lemma.

Lemma 4.1 *Let $A \in \wp(Y)$. Then, the following statements hold:*

(i) $\{\operatorname{cl}(A+C)\}^c = \{\operatorname{cl}(A+C)\}^c - C$ *and* $\{\operatorname{cl}(A-C)\}^c = \{\operatorname{cl}(A-C)\}^c + C$;

(ii) $\operatorname{int}(A+C) = \operatorname{int}(A+C) + C$ *and* $\operatorname{int}(A-C) = \operatorname{int}(A-C) - C$.

Proof. Since C is a convex cone, we can prove easily by the definitions of the closure and interior. $\square$

Proposition 4.13 *Let $F : X \to \wp(Y)$. For each $k \in \operatorname{int} C$ and $V' \in \wp(Y)$, the following statements hold:*

(i) *For each $j = 1, 4, 5$,*

  (a) *if F is lower C-continuous on X, then $I^{(j)}_{(k,V')} \circ F$ is lower semicontinuous on X,*

  (b) *if F is upper $(-C)$-continuous on X, then $I^{(j)}_{(k,V')} \circ F$ is upper semicontinuous on X.*

(ii) *For each $j = 2, 3, 6$,*

  (c) *if F is lower $(-C)$-continuous on X, then $I^{(j)}_{(k,V')} \circ F$ is upper semicontinuous on X,*

  (d) *if F is upper C-continuous on X, then $I^{(j)}_{(k,V')} \circ F$ is lower semicontinuous on X.*

Proof. The proof throughout the whole of the theorem is given by the same method, and so we shall prove in cases $j = 3, 5$.

First, we prove (a) and (d). For $j = 3, 5$, we show that

$$\operatorname{lev}^l_r(I) := \left\{x \in X \middle| \left(I^{(j)}_{(k,V')} \circ F\right)(x) \leq r\right\}$$

is closed for any $r \in \mathbb{R}$, that is, for any net $\{x_\alpha\}_{\alpha \in J} \subset \operatorname{lev}^l_r(I)$,

$$x_\alpha \to \bar{x} \Rightarrow \bar{x} \in \operatorname{lev}^l_r(I),$$

where J is a directed set. Assume that there exist $\bar{r} \in \mathbb{R}$, $\{x_\beta\}_{\beta \in J} \subset \operatorname{lev}^l_{\bar{r}}(I)$, and $\bar{x} \in X$ such that

$$x_\beta \to \bar{x} \quad \text{and} \quad \bar{x} \notin \operatorname{lev}^l_{\bar{r}}(I).$$

Let $t_{\bar{x}} := \left(I^{(j)}_{(k,V')} \circ F\right)(\bar{x})$. Then there exist $\epsilon > 0$ and $\delta > 0$ such that $\bar{r} < \bar{r} + \epsilon < \bar{r} + \epsilon + \delta < t_{\bar{x}}$ because $\bar{x} \notin \mathrm{lev}^{l}_{\bar{r}}(I)$. Let $t_{\beta} := \left(I^{(j)}_{(k,V')} \circ F\right)(x_{\beta})$ for any $\beta \in J$. Then $t_{\beta} \leq \bar{r}$. Therefore, $t_{\beta} \leq \bar{r} < \bar{r} + \epsilon < \bar{r} + \epsilon + \delta < t_{\bar{x}}$ and so we obtain

$$F(\bar{x}) \not\leq^{(j)}_{C} (\bar{r} + \epsilon + \delta)k + V' \quad \text{and} \quad F(x_{\beta}) \leq^{(j)}_{C} (\bar{r} + \epsilon)k + V'. \tag{4.9}$$

(a): We consider the case of $j = 5$. By (4.9) and the definition of type (5) set-relation, we have

$$F(\bar{x}) \not\subset (\bar{r} + \epsilon + \delta)k + V' - C \quad \text{and} \quad F(x_{\beta}) \subset (\bar{r} + \epsilon)k + V' - C. \tag{4.10}$$

Since C is a convex cone, $k \in \operatorname{int} C$ and $\delta > 0$,

$$\mathrm{cl}\,((\bar{r} + \epsilon)k + V' - C) \subset (\bar{r} + \epsilon + \delta)k + V' - C,$$

and then

$$\{(\bar{r} + \epsilon + \delta)k + V' - C\}^{c} \subset \{\mathrm{cl}\,((\bar{r} + \epsilon)k + V' - C)\}^{c}. \tag{4.11}$$

Hence, by (4.10) and (4.11), we have

$$F(\bar{x}) \cap (\{\mathrm{cl}\,((\bar{r} + \epsilon)k + V' - C)\}^{c}) \neq \emptyset,$$

and

$$F(x_{\beta}) \cap (\{\mathrm{cl}\,((\bar{r} + \epsilon)k + V' - C)\}^{c}) = \emptyset.$$

By (i) of Lemma 4.1, we obtain

$$\{\mathrm{cl}\,((\bar{r} + \epsilon)k + V' - C)\}^{c} = \{\mathrm{cl}\,((\bar{r} + \epsilon)k + V' - C)\}^{c} + C.$$

Consequently, we have

$$F(\bar{x}) \cap (\{\mathrm{cl}\,((\bar{r} + \epsilon)k + V' - C)\}^{c}) \neq \emptyset,$$

and

$$F(x_{\beta}) \cap (\{\mathrm{cl}\,((\bar{r} + \epsilon)k + V' - C)\}^{c} + C) = \emptyset.$$

This is a contradiction to the lower C-continuity of F on X. Consequently, $I^{(5)}_{(k,V')} \circ F$ is lower semicontinuous on X for each $(k, V') \in \operatorname{int} C \times \wp(Y)$.

(d): We consider the case $j = 3$. By (4.9) and the definition of type (3) set-relation, we obtain

$$(\bar{r} + \epsilon + \delta)k + V' \not\subset F(\bar{x}) + C \quad \text{and} \quad (\bar{r} + \epsilon)k + V' \subset F(x_\beta) + C. \tag{4.12}$$

Assume that $F(x_\beta) \subset F(\bar{x}) - \delta k + C$, then we obtain $(\bar{r} + \epsilon)k + V' \subset F(x_\beta) + C \subset F(\bar{x}) - \delta k + C$, hence, $(\bar{r} + \epsilon + \delta)k + V' \subset F(\bar{x}) + C$. This is a contradiction to (4.12), and so we have

$$F(x_\beta) \not\subset F(\bar{x}) - \delta k + C. \tag{4.13}$$

Moreover, since C is a convex cone, $k \in \operatorname{int} C$ and $\delta > 0$,

$$F(\bar{x}) \subset F(\bar{x}) + C \subset \operatorname{int}(F(\bar{x}) - \delta k + C). \tag{4.14}$$

Hence, by (4.13) and (4.14), we have

$$F(\bar{x}) \subset \operatorname{int}(F(\bar{x}) - \delta k + C) \quad \text{and} \quad F(x_\beta) \not\subset \operatorname{int}(F(\bar{x}) - \delta k + C).$$

By (ii) of Lemma 4.1, we obtain

$$\operatorname{int}(F(\bar{x}) - \delta k + C) = \operatorname{int}(F(\bar{x}) - \delta k + C) + C.$$

Consequently, we have

$$F(\bar{x}) \subset \operatorname{int}(F(\bar{x}) - \delta k + C)$$

and

$$F(x_\beta) \not\subset \operatorname{int}(F(\bar{x}) - \delta k + C) + C.$$

This is a contradiction to the upper C-continuity of F on X. Consequently, $I^{(3)}_{(k,V')} \circ F$ is lower semicontinuous on X for each $(k, V') \in \operatorname{int} C \times \wp(Y)$.

Second, we prove (b) and (c). For each $j = 3, 5$, we show that

$$\operatorname{lev}^u_r(I) := \left\{x \in X \,|\, r \leq \left(I^{(j)}_{(k,V')} \circ F\right)(x)\right\}$$

is closed for any $r \in \mathbb{R}$, that is, for any $\{x_\alpha\}_{\alpha \in J} \subset \mathrm{lev}^u_r(I)$,

$$x_\alpha \to \bar{x} \Rightarrow \bar{x} \in \mathrm{lev}^u_r(I).$$

Assume that there exist $\bar{r} \in \mathbb{R}$, $\{x_\beta\}_{\beta \in J} \subset \mathrm{lev}^u_{\bar{r}}(I)$, and $\bar{x} \in X$ such that

$$x_\beta \to \bar{x} \quad \text{and} \quad \bar{x} \notin \mathrm{lev}^u_{\bar{r}}(I).$$

Let $t_{\bar{x}} := \left(I^{(j)}_{(k,V')} \circ F\right)(\bar{x})$. Then there exist $\epsilon > 0$ and $\delta > 0$ such that $t_{\bar{x}} < \bar{r} - \epsilon < \bar{r} - \epsilon + \delta < \bar{r}$ because $\bar{x} \notin \mathrm{lev}^u_{\bar{r}}(I)$. Let $t_\beta := \left(I^{(j)}_{(k,V')} \circ F\right)(x_\beta)$ for any $\beta \in J$. Then $\bar{r} \leq t_\beta$. Therefore, $t_{\bar{x}} < \bar{r} - \epsilon < \bar{r} - \epsilon + \delta < \bar{r} \leq t_\beta$ and so we obtain

$$F(\bar{x}) \leq^{(j)}_C (\bar{r} - \epsilon)k + V' \quad \text{and} \quad F(x_\beta) \not\leq^{(j)}_C (\bar{r} - \epsilon + \delta)k + V'. \tag{4.15}$$

(b): We consider the case $j = 5$. By (4.15) and the definition of type (5) set-relation, we have

$$F(\bar{x}) \subset (\bar{r} - \epsilon)k + V' - C \quad \text{and} \quad F(x_\beta) \not\subset (\bar{r} - \epsilon + \delta)k + V' - C. \tag{4.16}$$

Since C is a convex cone, $k \in \operatorname{int} C$ and $\delta > 0$,

$$(\bar{r} - \epsilon)k + V' - C \subset \operatorname{int}((\bar{r} - \epsilon + \delta)k + V' - C). \tag{4.17}$$

Hence, by (4.16) and (4.17), we have

$$F(\bar{x}) \subset \operatorname{int}((\bar{r} - \epsilon + \delta)k + V' - C)$$

and

$$F(x_\beta) \not\subset \operatorname{int}((\bar{r} - \epsilon + \delta)k + V' - C).$$

By (ii) of Lemma 4.1, we obtain

$$\operatorname{int}((\bar{r} - \epsilon + \delta)k + V' - C) = \operatorname{int}((\bar{r} - \epsilon + \delta)k + V' - C) - C.$$

Consequently, we have

$$F(\bar{x}) \subset \operatorname{int}((\bar{r} - \epsilon + \delta)k + V' - C)$$

and

$$F(x_\beta) \not\subset \operatorname{int}((\bar{r} - \epsilon + \delta)k + V' - C) - C.$$

This is a contradiction to the upper $(-C)$-continuity of F on X. Consequently, $I^{(5)}_{(k,V')} \circ F$ is upper semicontinuous on X for each $(k, V') \in \operatorname{int} C \times \wp(Y)$.

(c): We consider the case $j = 3$. By (4.15) and the definition of type (3) set-relation, we obtain

$$(\bar{r} - \epsilon)k + V' \subset F(\bar{x}) + C \quad \text{and} \quad (\bar{r} - \epsilon + \delta)k + V' \not\subset F(x_\beta) + C. \tag{4.18}$$

Assume that $F(\bar{x}) \subset F(x_\beta) - \delta k + C$, then we obtain $(\bar{r} - \epsilon)k + V' \subset F(\bar{x}) + C \subset F(x_\beta) - \delta k + C$, hence, we have $(\bar{r} - \epsilon + \delta)k + V' \subset F(x_\beta) + C$. This is a contradiction to (4.18), and so we have

$$F(\bar{x}) \not\subset F(x_\beta) - \delta k + C. \tag{4.19}$$

Moreover, since C is a convex cone, $k \in \operatorname{int} C$ and $\delta > 0$,

$$F(x_\beta) \subset \operatorname{cl}(F(x_\beta) + C) \subset F(x_\beta) - \delta k + C. \tag{4.20}$$

Hence, by (4.19) and (4.20), we have $F(\bar{x}) \cap (\{\operatorname{cl}(F(x_\beta) + C)\}^c) \neq \emptyset$ and $F(x_\beta) \cap (\{\operatorname{cl}(F(x_\beta) + C)\}^c) = \emptyset$. It follows from (i) of Lemma 4.1 that we obtain

$$\{\operatorname{cl}(F(x_\beta) + C)\}^c = \{\operatorname{cl}(F(x_\beta) + C)\}^c - C.$$

Consequently, we have

$$F(\bar{x}) \cap (\{\operatorname{cl}(F(x_\beta) + C)\}^c) \neq \emptyset$$

and

$$F(x_\beta) \cap (\{\operatorname{cl}(F(x_\beta) + C)\}^c - C) = \emptyset.$$

This is a contradiction to the lower $(-C)$-continuity of F on X. Consequently, $I^{(3)}_{(k,V')} \circ F$ is upper semicontinuous on X for each $(k, V') \in \operatorname{int} C \times \wp(Y)$. □

Proposition 4.14 *Let $F : X \to \wp(Y)$. For each $k \in \operatorname{int} C$ and $V' \in \wp(Y)$, the following statements hold:*

(i) *For each* $j = 1, 2, 3$,

(a) *if* F *is lower* $(-C)$*-continuous on* X*, then* $S^{(j)}_{(k,V')} \circ F$ *is upper semicontinuous on* X*,*

(b) *if* F *is upper* C*-continuous on* X*, then* $S^{(j)}_{(k,V')} \circ F$ *is lower semicontinuous on* X*.*

(ii) *For each* $j = 4, 5, 6$,

(c) *if* F *is lower* C*-continuous on* X*, then* $S^{(j)}_{(k,V')} \circ F$ *is lower semicontinuous on* X*,*

(d) *if* F *is upper* $(-C)$*-continuous on* X*, then* $S^{(j)}_{(k,V')} \circ F$ *is upper semicontinuous on* X*.*

Proof. By the same way as the proof of Proposition 4.13, the statements are proved. □

By Propositions 4.13 and 4.14, we obtain the following corollaries.

Corollary 4.1 *Let* $F : X \to \wp(Y)$*. For each* $k \in \operatorname{int} C$ *and* $V' \in \wp(Y)$*, the following statements hold:*

(i) *For each* $j = 1, 4, 5$,

(a) *if* F *is lower continuous on* X*, then* $I^{(j)}_{(k,V')} \circ F$ *is lower semicontinuous on* X*,*

(b) *if* F *is upper continuous on* X*, then* $I^{(j)}_{(k,V')} \circ F$ *is upper semicontinuous on* X*.*

(ii) *For each* $j = 2, 3, 6$,

(c) *if* F *is lower continuous on* X*, then* $I^{(j)}_{(k,V')} \circ F$ *is upper semicontinuous on* X*,*

(d) *if F is upper continuous on X, then $I^{(j)}_{(k,V')} \circ F$ is lower semicontinuous on X.*

Corollary 4.2 *Let $F : X \to \wp(Y)$. For each $k \in \operatorname{int} C$ and $V' \in \wp(Y)$, the following statements hold:*

(i) *For each $j = 1, 2, 3$,*

(a) *if F is lower continuous on X, then $S^{(j)}_{(k,V')} \circ F$ is upper semicontinuous on X,*

(b) *if F is upper continuous on X, then $S^{(j)}_{(k,V')} \circ F$ is lower semicontinuous on X.*

(ii) *For each $j = 4, 5, 6$,*

(c) *if F is lower continuous on X, then $S^{(j)}_{(k,V')} \circ F$ is lower semicontinuous on X,*

(d) *if F is upper continuous on X, then $S^{(j)}_{(k,V')} \circ F$ is upper semicontinuous on X.*

By Propositions 4.13, 4.14 and Corollaries 4.1, 4.2, we summarize the inheritance properties on continuity and cone-continuity of parent set-valued maps via the unified types of scalarizing functions in Table 4.1. By symbolic interpretation, (semi-)continuity notions with prefixes C and $-C$ are inherited to the semicontinuity with "lower" and "upper," respectively.

Example 4.1 Let $X := \mathbb{R}$, $Y := \mathbb{R}^2$ and $C := \mathbb{R}^2_+$. We consider a set-valued map

Table 4.1: Inherited properties on semicontinuity of set-valued maps via scalarization.

F	$I^{(j)}_{(k,V')} \circ F$		$S^{(j)}_{(k,V')} \circ F$	
	$j = 1,4,5$	$j = 2,3,6$	$j = 4,5,6$	$j = 1,2,3$
l.c. on X	l.s.c. on X	u.s.c. on X	l.s.c. on X	u.s.c. on X
u.c. on X	u.s.c. on X	l.s.c. on X	u.s.c. on X	l.s.c. on X
C-l.c. on X	l.s.c. on X	$(*)$	l.s.c. on X	$(*)$
C-u.c. on X	$(*)$	l.s.c. on X	$(*)$	l.s.c. on X
$(-C)$-l.c. on X	$(*)$	u.s.c. on X	$(*)$	u.s.c. on X
$(-C)$-u.c. on X	u.s.c. on X	$(*)$	u.s.c. on X	$(*)$

$F : X \to \wp(Y)$ defined by

$$F(x) := \begin{cases} \left[\begin{pmatrix} x \\ x \end{pmatrix}, \begin{pmatrix} x \\ 0 \end{pmatrix} \right] & (x \leq -1), \\ \\ \left[\begin{pmatrix} x \\ x+2 \end{pmatrix}, \begin{pmatrix} x \\ 3 \end{pmatrix} \right] & (-1 < x < 1), \\ \\ \left[\begin{pmatrix} x-1 \\ 0 \end{pmatrix}, \begin{pmatrix} x-1 \\ x \end{pmatrix} \right] & (1 \leq x). \end{cases}$$

It is easy to check that F is upper C-continuous on X. Let $k := \binom{1}{1}$ and $V' := \left[\binom{0}{0}, \binom{1}{1}\right]$, then we have

$$\left(I^{(3)}_{(k,V')} \circ F \right)(x) = \begin{cases} x & (x \leq -1), \\ \\ x+2 & (-1 < x < 1), \\ \\ x-1 & (1 \leq x). \end{cases}$$

Hence $I^{(3)}_{(k,V')} \circ F$ is lower semicontinuous on X.

Remark 4.2 If F is neither lower continuous on X nor upper continuous on X, we can not apply the results in Corollaries 4.1 and 4.2 to the composite functions of F and each of the unified types of scalarizing functions. However, by Propositions 4.13 and 4.14, we get a clue to confirm cone-continuity of a parent set-valued map F by checking semicontinuity of the scalarizing functions.

Example 4.2 Let $X := \mathbb{R}$, $Y := \mathbb{R}^2$ and $C := \mathbb{R}^2_+$. We consider a set-valued map $G : X \to \wp(Y)$ defined by

$$G(x) := \begin{cases} \left[\begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} x \\ x^2 \end{pmatrix}\right] & (x \leq -1), \\ \left[\begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} x \\ x+3 \end{pmatrix}\right] & (-1 < x < 1), \\ \left[\begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} x \\ 5x \end{pmatrix}\right] & (1 \leq x). \end{cases}$$

It is easy to check that G is upper C-continuous on X. Let $k := \binom{1}{1}$ and $V' := \left[\binom{0}{0}, \binom{1}{1}\right]$, then we have

$$\left(I^{(5)}_{(k,V')} \circ G\right)(x) = \begin{cases} x^2 - 1 & (x \leq -1), \\ x + 2 & (-1 < x < 1), \\ 5x - 1 & (1 \leq x). \end{cases}$$

Hence $I^{(5)}_{(k,V')} \circ G$ is neither lower semicontinuous nor upper semicontinuous on X.

Example 4.3 Let $X := \mathbb{R}$, $Y := \mathbb{R}^2$ and $C := \mathbb{R}^2_+$. We consider a set-valued map

$H : X \to \wp(Y)$ defined by

$$H(x) := \begin{cases} \left[\begin{pmatrix} x-1 \\ 0 \end{pmatrix}, \begin{pmatrix} x-1 \\ -x \end{pmatrix} \right] & (x < -1), \\ \left[\begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right] & (-1 \le x \le 0), \\ \left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} x+2 \\ x+2 \end{pmatrix} \right] & (0 < x). \end{cases}$$

It is easy to check that H is lower $(-C)$-continuous on X. Let $k := \binom{1}{1}$ and $V' := \left[\binom{0}{0}, \binom{1}{1}\right]$, then we have

$$\left(S^{(5)}_{(k,V')} \circ H \right)(x) = \begin{cases} x-2 & (x < -1), \\ -1 & (-1 \le x \le 0), \\ x+1 & (0 < x). \end{cases}$$

Hence $S^{(5)}_{(k,V')} \circ H$ is neither lower semicontinuous nor upper semicontinuous on X.

Remark 4.3 Each cell with $(*)$ in Table 4.1 is undetermined on semicontinuity for the scalarizing functions. By Examples 4.2 and 4.3, $I^{(5)}_{(k,V')} \circ G$ and $S^{(5)}_{(k,V')} \circ H$ are neither lower semicontinuous on X nor upper semicontinuous on X.

Chapter 5

Applications

In this chapter, by using several properties of scalarizing functions for sets introduced in Chapter 4, we present several results in set-valued analysis and optimization.

Throughout this chapter, we use mainly $\leq_C^{(j)}$ with $j = 1, \ldots, 5$ (especially, $\leq_C^{(3)}$ and $\leq_C^{(5)}$), because almost of results in optimization research areas are based on ordering with transitivity (in particular, partial ordering and total ordering), however, from (ii) of Proposition 2.3, $\leq_C^{(6)}$ does not satisfy the transitivity.

5.1 Optimality conditions for set-valued optimization

Let Y be a finite dimensional real topological vector space ordered by a pointed closed convex cone C with $\operatorname{int} C \neq \emptyset$. Firstly, we recall a concept of minimal (resp., maximal, weak minimal, weak maximal) element of a set $A \in \wp(Y)$. A point $a_0 \in A$ is said to be *minimal* (resp., *maximal*) *element of* A if

$$(a_0 - C) \cap A = \{a_0\} \quad (\text{resp.}, (a_0 + C) \cap A = \{a_0\}).$$

If C is replaced by $\operatorname{int} C \cup \{\theta_Y\}$ then a_0 is called *weak minimal* (resp., *weak maximal*) *element of* A. We denote the family of all minimal (resp., maximal) elements of A and the family of all weak minimal (resp., weak maximal) elements of A by $\mathrm{Min}_{\mathrm{v}}A$ (resp., $\mathrm{Max}_{\mathrm{v}}A$) and $\mathrm{Min}^{\mathrm{w}}_{\mathrm{v}}A$ (resp., $\mathrm{Max}^{\mathrm{w}}_{\mathrm{v}}A$), respectively.

Based on these definitions and $\leq_C^{(j)}$ with $j = 1, \ldots, 5$, we consider natural extensions of these concepts to a family of sets. Let $\mathcal{A} \subset \wp(\mathcal{Y})$. We say that a set $A_0 \in \mathcal{A}$ is *type* (j) *minimal element of* $\mathcal{A}$ if for any $A \in \mathcal{A}$,

$$A \leq_C^{(j)} A_0 \quad \text{implies} \quad A_0 \leq_C^{(j)} A;$$

type (j) *maximal element of* $\mathcal{A}$ if for any $A \in \mathcal{A}$,

$$A_0 \leq_C^{(j)} A \quad \text{implies} \quad A \leq_C^{(j)} A_0.$$

If C is replaced by $\operatorname{int} C$ then A_0 is called *type* (j) *weak minimal* (resp., *weak maximal*) *element of* $\mathcal{A}$. We denote the family of all type (j) minimal elements (resp., maximal elements) of $\mathcal{A}$ by $\mathrm{Min}_{(j)}\mathcal{A}$ (resp., $\mathrm{Max}_{(j)}\mathcal{A}$). Also, we denote the family of all type (j) weak minimal elements (resp., weak maximal elements) of $\mathcal{A}$ by $\mathrm{Min}^{\mathrm{w}}_{(j)}\mathcal{A}$ (resp., $\mathrm{Max}^{\mathrm{w}}_{(j)}\mathcal{A}$).

Throughout this section, X is a real Hausdorff topological vector space, Y is a finite dimensional real Hausdorff topological vector space ordered by a closed pointed convex cone C with $\operatorname{int} C \neq \emptyset$.

Let $F : X \to \wp(Y)$ and $v \in Y$. We consider the following set-valued optimization problems:

$$(\mathrm{SP}) \begin{cases} \text{Optimize} & F(x) \\ \text{Subject to} & x \in X. \end{cases}$$

Then we define minimal (resp., weak minimal) and maximal (resp., weak maximal) efficient solutions of (SP) with respect to $\leq_C^{(j)}$ with $j = 1, \ldots, 5$. A point $x_0 \in X$ is said to be *type* (j) *minimal* (resp., *maximal, weak minimal, weak maximal*) *efficient solution* of (SP) if $F(x_0)$ is a type (j) minimal (resp., maximal, weak minimal, weak maximal) element of $F(X) := \{F(x) : x \in X\}$. Firstly, we introduce necessary and sufficient conditions for the existence of type $(*)$ minimal (resp., maximal) efficient solutions for (SP) where $* = 3, 5$.

Theorem 5.1 ([29]) *Let $F : X \to \wp(Y)$. Assume that the following two conditions hold:*

(i) *F is C-closed, C-bounded valued on X and $x_0 \in X$.*

(ii) *For each $k \in \operatorname{int} C$ and $x \in X$, $\left(I^{(3)}_{(k,F(x_0))} \circ F\right)(x) = 0$ if and only if $F(x_0) \leq^{(3)}_C F(x)$ and $F(x) \leq^{(3)}_C F(x_0)$.*

Then, x_0 is a type (3) minimal efficient solution of (SP) if and only if x_0 is a solution of the scalar optimization problem $\inf\limits_{x \in X} \left(I^{(3)}_{(k,F(x_0))} \circ F\right)(x)$.

Based on Theorem 5.1, we obtain the following theorem.

Theorem 5.2 *Let $F : X \to \wp(Y)$. Assume that the following conditions hold:*

(i) *F is $(-C)$-closed, $(-C)$-bounded valued on X and $x_0 \in X$.*

(ii) *For each $k \in \operatorname{int} C$ and $x \in X$, $\left(I^{(5)}_{(k,F(x_0))} \circ F\right)(x) = 0$ if and only if $F(x_0) \leq^{(5)}_C F(x)$ and $F(x) \leq^{(5)}_C F(x_0)$.*

Then x_0 is a type (5) minimal efficient solution of (SP) if and only if x_0 is a solution of the scalar optimization problem $\inf\limits_{x \in X} \left(I^{(5)}_{(k,F(x_0))} \circ F\right)(x)$.

Proof. We assume that x_0 is a type (5) minimal efficient solution of (SP). Then, for any $x \in X \setminus \{x_0\}$

$$F(x) \leq^{(5)}_C F(x_0) \quad \text{implies} \quad F(x_0) \leq^{(5)}_C F(x).$$

By Propositions 4.1 and 4.4, we obtain

$$(I^{(5)}_{(k,F(x_0))} \circ F)(x_0) = 0 \quad \text{and} \quad F(x) \not\leq^{(5)}_C F(x_0).$$

Hence,

$$\left(I^{(5)}_{(k,F(x_0))} \circ F\right)(x) \geq 0 = \left(I^{(5)}_{(k,F(x_0))} \circ F\right)(x_0). \tag{5.1}$$

Moreover, by (iii) of Proposition 4.1 we have $F(x_0) \leq^{(5)}_C F(x)$ and then

$$\left(I^{(5)}_{(k,F(x_0))} \circ F\right)(x_0) \leq \left(I^{(5)}_{(k,F(x_0))} \circ F\right)(x). \tag{5.2}$$

Hence, by (5.1) and (5.2) we obtain

$$\left(I^{(5)}_{(k,F(x_0))} \circ F\right)(x_0) \leq \left(I^{(5)}_{(k,F(x_0))} \circ F\right)(x)$$

for any $x \in X \setminus \{x_0\}$. Consequently, x_0 is a solution of $\inf\limits_{x\in X} \left(I^{(5)}_{(k,F(x_0))} \circ F\right)(x)$.

Conversely, we assume that x_0 is a solution of $\inf\limits_{x\in X} \left(I^{(5)}_{(k,F(x_0))} \circ F\right)(x)$. Then,

$$\left(I^{(5)}_{(k,F(x_0))} \circ F\right)(x_0) \leq \left(I^{(5)}_{(k,F(x_0))} \circ F\right)(x)$$

for any $x \in X \setminus \{x_0\}$. Suppose that x_0 is not a type (5) minimal efficient solution of (SP). Then, there exists $\bar{x} \in X \setminus \{x_0\}$ such that

$$F(\bar{x}) \leq^{(5)}_C F(x_0) \quad \text{and} \quad F(x_0) \not\leq^{(5)}_C F(\bar{x}). \tag{5.3}$$

By Proposition 4.4, $\left(I^{(5)}_{(k,F(x_0))} \circ F\right)(x_0) = 0$ and so we obtain

$$0 = \left(I^{(5)}_{(k,F(x_0))} \circ F\right)(x_0) \leq \left(I^{(5)}_{(k,F(x_0))} \circ F\right)(\bar{x}). \tag{5.4}$$

Moreover, by (iii) of Proposition 4.1

$$\left(I^{(5)}_{(k,F(x_0))} \circ F\right)(\bar{x}) \leq \left(I^{(5)}_{(k,F(x_0))} \circ F\right)(x_0) = 0. \tag{5.5}$$

Hence, it follows from (5.4) and (5.5) that we obtain

$$\left(I^{(5)}_{(k,F(x_0))} \circ F\right)(\bar{x}) = \left(I^{(5)}_{(k,F(x_0))} \circ F\right)(x_0) = 0$$

and then

$$F(\bar{x}) \leq^{(5)}_C F(x_0) \quad \text{and} \quad F(x_0) \leq^{(5)}_C F(\bar{x}).$$

This contradicts (5.3). Consequently, x_0 is a type (5) minimal efficient solution of (SP).

□

By Theorems 5.1 and 5.2, we obtain the following corollaries.

Corollary 5.1 *Let $F : X \to \wp(Y)$. Assume that the following conditions hold:*

(i) *F is C-closed, C-bounded valued on X and $x_0 \in X$.*

(ii) *For each $k \in \operatorname{int} C$ and $x \in X$, $\left(I^{(3)}_{(k,F(x_0))} \circ F\right)(x; k) = 0$ if and only if $F(x_0) \leq^{(3)}_C F(x)$ and $F(x) \leq^{(3)}_C F(x_0)$.*

Then, x_0 is a type (3) maximal efficient solution of (SP) if and only if x_0 is a solution of the scalar optimization problem $\sup\limits_{x \in X} \left(I^{(3)}_{(k,F(x_0))} \circ F\right)(x)$.

Corollary 5.2 *Let $F : X \to \wp(Y)$. Assume that the following two conditions hold:*

(i) *F is $(-C)$-closed, $(-C)$-bounded valued on X and $x_0 \in X$.*

(ii) *For each $k \in \operatorname{int} C$ and $x \in X$, $\left(I^{(5)}_{(k,F(x_0))} \circ F\right)(x) = 0$ if and only if $F(x_0) \leq^{(5)}_C F(x)$ and $F(x) \leq^{(5)}_C F(x_0)$.*

Then x_0 is a type (5) maximal efficient solution of (SP) if and only if x_0 is a solution of the scalar optimization problem $\sup\limits_{x \in X} \left(I^{(5)}_{(k,F(x_0))} \circ F\right)(x)$.

Next, we show sufficient conditions for the existence of type (j) weak minimal (resp, weak maximal) efficient solutions of (SP) with $j = 1, \dots, 5$.

Theorem 5.3 *Let $F : X \to \wp(Y)$, $k \in \operatorname{int} C$ and $v \in Y$. Assume that F is C-closed valued on X. For each $j = 1, 2, 3$, the following statements hold:*

(i) *If $x(k; v) \in X$ is a solution of $\inf\limits_{x \in X} \left(I^{(3)}_{(k,v)} \circ F\right)(x)$, then $x(k; v)$ is a type (j) weak minimal efficient solution of (SP).*

(ii) *If $x(k; v) \in X$ is a solution of $\inf\limits_{x \in X} (I^{(3)}_{(k,v)} \circ F)(x)$, then $x(k; v)$ is a type (j) minimal efficient solution of (SP).*

(iii) *If $x(k; v) \in X$ is a solution of $\sup\limits_{x \in X} \left(I^{(3)}_{(k,v)} \circ F\right)(x)$, then $x(k; v)$ is a type (j) weak maximal efficient solution of (SP).*

(iv) *If $x(k; v) \in X$ is a solution of $\sup\limits_{x \in X} (I^{(3)}_{(k,v)} \circ F)(x;)$, then $x(k; v)$ is a type (j) maximal efficient solution of (SP).*

Proof. We show that statement (i) only; the others can be proved in a similar way. Assume that $x(k;v)$ is a solution of $\inf_{x\in X}\left(I^{(3)}_{(k,v)}\circ F\right)(x)$. Then, for any $x\in X\setminus\{x(k;v)\}$

$$\left(I^{(3)}_{(k,v)}\circ F\right)(x(k;v))\leq\left(I^{(3)}_{(k,v)}\circ F\right)(x). \tag{5.6}$$

By (ii) of Proposition 4.5, if $\left(I^{(3)}_{(k,v)}\circ F\right)(x(k;v))$ is not finite then $F(x(k;v))+C=Y$ and so $x(k;v)$ is a type (j) weak minimal solution of (SP). Hence we may assume that $\left(I^{(3)}_{(k,v)}\circ F\right)(x(k;v))$ is finite and $x(k;v)$ is not a type (j) weak minimal solution of (SP). Then there exists $\bar{x}\in X\setminus\{x(k;v)\}$ such that

$$F(\bar{x})\leq^{(j)}_{\operatorname{int}C}F(x(k;v))\quad\text{and}\quad F(x(k;v))\not\leq^{(j)}_{\operatorname{int}C}F(\bar{x}).$$

It follows from Proposition 2.1 that $F(\bar{x})\leq^{(3)}_{\operatorname{int}C}F(x(k;v))$. Hence, by (i) of Proposition 4.4 we obtain $\left(I^{(3)}_{(k,v)}\circ F\right)(\bar{x})<\left(I^{(3)}_{(k,v)}\circ F\right)(x(k;v))$. This contradicts (5.6). Consequently, $x(k;v)$ is a type (j) weak minimal efficient solution of (SP). □

Similarly, we obtain the following result:

Theorem 5.4 *Let $F:X\to\wp(Y)$, $k\in\operatorname{int}C$ and $v\in Y$. Assume that F is $(-C)$-closed valued on X. For each $j=1,4,5$, the following statements hold:*

(i) *If $x(k;v)\in X$ is a solution of $\inf_{x\in X}\left(S^{(5)}_{(k,v)}\circ F\right)(x)$, then $x(k;v)$ is a type (j) weak minimal efficient solution of* (SP).

(ii) *If $x(k;v)\in X$ is a solution of $\inf_{x\in X}\left(S^{(5)}_{(k,v)}\circ F\right)(x)$, then $x(k;v)$ is a type (j) minimal solution of* (SP).

(iii) *If $x(k;v)\in X$ is a solution of $\sup_{x\in X}\left(S^{(5)}_{(k,v)}\circ F\right)(x)$, then $x(k;v)$ is a type (j) weak maximal efficient solution of* (SP).

(iv) *If $x(k;v)\in X$ is a solution of $\sup_{x\in X}\left(S^{(5)}_{(k,v)}\circ F\right)(x)$, then $x(k;v)$ is a type (j) maximal solution of* (SP).

By Theorems 5.3 and 5.4, we can define another solution concepts for (SP). Let $F : X \to \wp(Y)$. Then, $x_0 \in X$ is said to be *proper minimal* (resp., *proper maximal*) *efficient solution of* (SP) if there exists $k \in \operatorname{int} C$ and $v \in Y$ such that $x(k; v)$ is a unique solution of $\inf_{x\in X} \left(I^{(3)}_{(k,v)} \circ F\right)(x)$ (resp., $\sup_{x\in X} \left(S^{(5)}_{(k,v)} \circ F\right)(x)$); *proper weak minimal* (resp., *proper weak maximal*) *efficient solution of* (SP) if there exists $k \in \operatorname{int} C$ and $v \in Z$ such that $x(k; v)$ is a solution of $\inf_{x\in X} \left(I^{(3)}_{(k,v)} \circ F\right)(x)$ (resp., $\sup_{x\in X} \left(S^{(5)}_{(k,v)} \circ F\right)(x)$). We denote the sets that are image of a proper minimal efficient solution (resp., proper maximal efficient solution) and a proper weak minimal efficient solution (resp., proper weak maximal efficient solution) of (SP) by $\mathrm{Min_p}F(X)$ (resp., $\mathrm{Max_p}F(X)$) and $\mathrm{Min^w_p}F(x)$ (resp., $\mathrm{Max^w_p}F(X)$), respectively.

To illustrate the concept of proper efficient solution, we give the following simple example.

Example 5.1 Let $X = (0, +\infty)$, $Y = \mathbb{R}^2$, $C := \left\{ \binom{x}{y} \in Y \,\middle|\, x, y \geq 0 \right\}$. Then we consider a set-valued map $F : X \to \wp(Y)$

$$F(x) := \begin{cases} \left[\begin{pmatrix} 1 \\ \frac{2}{x} \end{pmatrix}, \begin{pmatrix} 1 \\ \frac{2}{x} + 1 \end{pmatrix} \right] & (0 < x \leq 1), \\ \left[\begin{pmatrix} x \\ \frac{2}{x} \end{pmatrix}, \begin{pmatrix} x \\ \frac{2}{x} + 1 \end{pmatrix} \right] & (1 \leq x \leq 2), \\ \left[\begin{pmatrix} x \\ 1 \end{pmatrix}, \begin{pmatrix} x \\ 2 \end{pmatrix} \right] & (2 \leq x). \end{cases}$$

Let $v = \theta_Y$, and take $x \in [1, 2]$. If $k(x) := \begin{pmatrix} x \\ \frac{2}{x} \end{pmatrix}$ then it is clear that

$$\left(I^{(3)}_{(k(x),v)} \circ F\right)(x) < \left(I^{(3)}_{(k(x),v)} \circ F\right)(y)$$

for any $y \in X \setminus \{x\}$. If $x < 1$ then it is easy to check that $F(1) \leq^{(3)}_C F(x)$, and then it follows from (iii) of Proposition 4.1 that $F(x) \notin \mathrm{Min_p}F(X)$. However, if $k(x) = \begin{pmatrix} 1 \\ \frac{2}{x} \end{pmatrix}$ then

$$\left(I^{(3)}_{(k(x),v)} \circ F\right)(x) \leq \left(I^{(3)}_{(k(x),v)} \circ F\right)(y)$$

for any $y \in X$, and hence we have $F(x) \in \mathrm{Min}_{\mathrm{p}}^{\mathrm{w}} F(X)$. Similarly, we obtain $F(x) \notin \mathrm{Min}_{\mathrm{p}} F(X)$ and $F(x) \in \mathrm{Min}_{\mathrm{p}}^{\mathrm{w}} F(X)$ when $x > 2$. Consequently, we have $\mathrm{Min}_{\mathrm{p}} F(X) = \{F(x) | 1 \leq x \leq 2\}$ and $\mathrm{Min}_{\mathrm{p}}^{\mathrm{w}} F(X) = F(X)$.

It is clear that if $x_0 \in X$ is a proper minimal efficient solution (resp., proper maximal efficient solution) then x_0 is a proper weak minimal efficient solution (resp., proper weak maximal efficient solution). Moreover, by Proposition 4.8, Theorems 5.3 and 5.4 we obtain the following relationships.

(i) If F is C-closed valued on X and $\mathrm{Min}_{\mathrm{p}} F(X) \neq \emptyset$ then $\mathrm{Min}_{\mathrm{p}} F(X) \subset \mathrm{Min}_{(j)} F(X)$ where $j = 1, 2, 3$.

(ii) If F is $(-C)$-closed valued on X and $\mathrm{Max}_{\mathrm{p}} F(X) \neq \emptyset$ then $\mathrm{Max}_{\mathrm{p}} F(X) \subset \mathrm{Max}_{(j)} F(X)$ where $j = 1, 4, 5$.

(iii) If F is C-closed valued on X and $\mathrm{Min}_{\mathrm{p}}^{\mathrm{w}} F(X) \neq \emptyset$ then $\mathrm{Min}_{\mathrm{p}}^{\mathrm{w}} F(X) \subset \mathrm{Min}_{(j)}^{\mathrm{w}} F(X)$ where $j = 1, 2, 3$.

(iv) If F is $(-C)$-closed valued on X $\mathrm{Max}_{\mathrm{p}}^{\mathrm{w}} F(X) \neq \emptyset$ then $\mathrm{Max}_{\mathrm{p}}^{\mathrm{w}} F(X) \subset \mathrm{Max}_{(j)}^{\mathrm{w}} F(X)$ where $j = 1, 4, 5$.

However, the converse is not true in general. Let $X := \mathbb{R}$, $Y := \mathbb{R}^2$, $C := \mathbb{R}^2_+$. We consider the following set-valued map defined by

$$F(x) := \begin{cases} \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right] & (x < 0), \\[2ex] \left[\begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}, \begin{pmatrix} \frac{4}{3} \\ \frac{4}{3} \end{pmatrix}\right] & (0 \leq x < \frac{1}{2}), \\[2ex] \left\{ \lambda \begin{pmatrix} \frac{1}{3} \\ \frac{4}{3} \end{pmatrix} + (1 - \lambda) \begin{pmatrix} \frac{5}{3} \\ \frac{1}{4} \end{pmatrix} \middle| \lambda \in [0, 1] \right\} & (\frac{1}{2} \leq x < 1), \\[2ex] \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right] & (1 \leq x). \end{cases}$$

Then, $\mathrm{Min}_{\mathrm{p}}F(X) = \mathrm{Min}_{\mathrm{p}}^{\mathrm{w}}F(X) = X \setminus [\frac{1}{2}, 1)$ although $\mathrm{Min}_{(3)}F(X) = \mathrm{Min}_{(3)}^{\mathrm{w}}F(X) = X$.

By using some properties of $I^{(3)}_{(k,v)}$ and $S^{(5)}_{(k,v)}$, we can obtain the following results which will be used in Section 5.5.

Theorem 5.5 *Let $F : X \to \wp(Y)$ and $\bar{x} \in X$. Then the following statements hold:*

(i) *If the following conditions are satisfied:*

(a) *F is C-closed valued on X,*

(b) *$\bigcup_{x\in X} F(x)$ is C-closed and C-proper,*

then $F(\bar{x}) \in \mathrm{Min}_{\mathrm{p}}^{\mathrm{w}}F(X)$ if and only if $F(\bar{x}) \not\subset \bigcup_{x\in X}(F(x) + \mathrm{int}\, C)$.

(ii) *If the following conditions are satisfied:*

(a) *F is $(-C)$-closed valued on X,*

(b) *$\bigcup_{x\in X} F(x)$ is $(-C)$-closed and $(-C)$-proper,*

then $F(\bar{x}) \in \mathrm{Max}_{\mathrm{p}}^{\mathrm{w}}F(X)$ if and only if $F(\bar{x}) \not\subset \bigcup_{x\in X}(F(x) - \mathrm{int}\, C)$.

Proof. We only prove statement (i); statement (ii) can be proved in a similar way. Firstly, we show that if $F(\bar{x}) \in \mathrm{Min}_{\mathrm{p}}^{\mathrm{w}}F(X)$ then $F(\bar{x}) \not\subset \bigcup_{x\in X}(F(x) + \mathrm{int}\, C)$. To use contradiction, we assume that $F(\bar{x}) \subset \bigcup_{x\in X}(F(x) + \mathrm{int}\, C)$. Let $k \in \mathrm{int}\, C$ and $v \in Z$. By condition (b), it is clear that $F(\bar{x})$ is C-proper. Then it follows from (i) of Proposition 4.8 that we obtain

$$I^{(3)}_{(k,v)}\left(\bigcup_{x\in X} F(x)\right) < \left(I^{(3)}_{(k,v)} \circ F\right)(\bar{x}).$$

Let $\bar{t} := \left(I^{(3)}_{(k,v)} \circ F\right)(\bar{x})$. By (i) of Proposition 4.7, $\bar{t}k + v \in F(\bar{x}) + C$. Since $F(\bar{x}) \subset \bigcup_{x\in X}(F(x) + \mathrm{int}\, C)$, there exists $x(k;v) \in X$ such that $\bar{t}k+v \in F(x(k;v))+\mathrm{int}\, C$. Hence we have

$$\left(I^{(3)}_{(k,v)} \circ F\right)(x(k;v)) < \left(I^{(3)}_{(k,v)} \circ F\right)(\bar{x}).$$

Since k and v are arbitrary, this contradicts $F(\bar{x}) \in \mathrm{Min}_{\mathrm{p}}^{\mathrm{w}} F(X)$.

Next, we assume that $F(\bar{x}) \notin \mathrm{Min}_{\mathrm{p}}^{\mathrm{w}} F(X)$ and $F(\bar{x}) \not\subset \bigcup_{x \in X} (F(x) + \operatorname{int} C)$. Then, there exists $\bar{v} \in F(\bar{x})$ such that $\bar{v} \notin \bigcup_{x \in X} (F(x) + \operatorname{int} C)$. Hence, for any $k \in \operatorname{int} C$, we have

$$\left(I^{(3)}_{(k,\bar{v})} \circ F\right)(\bar{x}) \leq 0 \leq I^{(3)}_{(k,\bar{v})}\left(\bigcup_{x \in X} F(x)\right).$$

Then for any $k \in \operatorname{int} C$ and $v \in Z$ there exists $x(k; v) \in X$ such that

$$\left(I^{(3)}_{(k,v)} \circ F\right)(x(k; v)) < \left(I^{(3)}_{(k,v)} \circ F\right)(\bar{x}). \tag{5.7}$$

Since $F(\bar{x}) \not\subset \bigcup_{x \in X} (F(x) + \operatorname{int} C)$, there exists $\bar{v} \in F(\bar{x})$ such that

$$\bar{v} \notin \bigcup_{x \in X} (F(x) + \operatorname{int} C). \tag{5.8}$$

From (5.7) and (iii) of Proposition 4.1, there exists $x(k; \bar{v}) \in X$ such that

$$\left(I^{(3)}_{(k,\bar{v})} \circ F\right)(x(k; \bar{v})) < \left(I^{(3)}_{(k,\bar{v})} \circ F\right)(\bar{x}) \leq I^{(3)}_{(k,\bar{v})}(\{\bar{v}\}).$$

Let $t_1 := \left(I^{(3)}_{(k,\bar{v})} \circ F\right)(x(k; \bar{v}))$. Since $I^{(3)}_{(k,\bar{v})}(\{\bar{v}\}) = 0$, there exists $\epsilon > 0$ such that

$$t_1 < -\epsilon \quad \text{and} \quad -\epsilon k + \bar{v} \in F(x(k; \bar{v})) + C.$$

Since C is a convex cone and $k \in \operatorname{int} C$, we obtain $\bar{v} + \operatorname{int}[-\epsilon k, \epsilon k] \subset F(x(k; \bar{v})) + C$. Hence

$$\bar{v} \in F(x(k; \bar{v})) + \operatorname{int} C \subset \bigcup_{x \in X} (F(x) + \operatorname{int} C)$$

which contradicts (5.8). Consequently, statement (i) holds. □

Theorem 5.6 *Let $F : X \to \wp(Y)$ and $\bar{x} \in X$. Then the following statements hold:*

(i) *If the following conditions are satisfied:*

(a) *F is C-closed valued on X,*

(b) $\bigcup_{x\in X} F(x)$ *is* C*-closed and* C*-proper,*

then $F(\bar{x}) \in \mathrm{Min_p} F(X)$ *if and only if* $F(\bar{x}) \not\subset \bigcup_{x\in X\setminus\{\bar{x}\}} (F(x)+C)$.

(ii) *If the following conditions are satisfied:*

(a) F *is* $(-C)$*-closed valued on* X*,*

(b) $\bigcup_{x\in X} F(x)$ *is* $(-C)$*-closed and* $(-C)$*-proper,*

then $F(\bar{x}) \in \mathrm{Max_p} F(X)$ *if and only if* $F(\bar{x}) \not\subset \bigcup_{x\in X\setminus\{\bar{x}\}} (F(x)-C)$.

Proof. We can prove this lemma in a similar way to the proof of Theorem 5.5. □

5.2 Fan's inequality for set-valued maps

The following theorem is equivalent to Theorem 1 in [18] of Ky Fan minimax inequality; this equivalence was proved by Takahashi [65] firstly in 1976.

Theorem 5.7 *Let* A *be a nonempty compact convex subset of a Hausdorff topological vector space and* $f : A \times A \to \mathbb{R}$*. If* f *satisfies the following conditions:*

(i) *for each fixed* $y \in A$*,* $f(\cdot, y)$ *is lower semicontinuous,*

(ii) *for each fixed* $x \in A$*,* $f(x, \cdot)$ *is quasi concave,*

(iii) *for all* $x \in A$*,* $f(x,x) \le 0$*,*

then there exists $\bar{x} \in A$ *such that* $f(\bar{x}, y) \le 0$ *for all* $y \in A$*.*

In set-valued case, there are four types generalizations of $f(x, y) \le 0$ (see Figure 5.1). Hence, based on Theorem 5.7 we shall show four kinds of Ky Fan minimax inequality for set-valued maps.

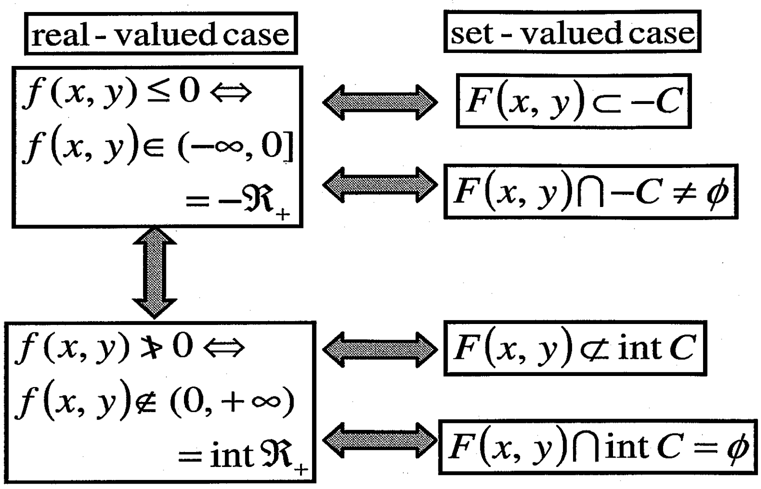

Figure 5.1: Relationships between real-valued inequality and set-relation.

Throughout this section, X is a real Hausdorff topological vector space, $Y := \mathbb{R}^n$ ordered by a nontrivial closed convex cone C with $\operatorname{int} C \neq \emptyset$.

Theorem 5.8 *Let A be a nonempty compact convex subset of X and $F : A \times A \to \wp Y$. If F satisfies the following conditions:*

(i) *F is $(-C)$-bounded valued on $A \times A$,*

(ii) *for each fixed $y \in A$, $F(\cdot, y)$ is lower C-continuous on A,*

(iii) *for each fixed $x \in A$, $F(x, \cdot)$ is type (5) properly quasi C-concave on A,*

(iv) *for all $x \in A$, $F(x,x) \subset -C$,*

then there exists $\bar{x} \in A$ such that $F(\bar{x}, y) \subset -C$ for all $y \in A$.

Proof. For this end, we consider the function $I^{(5)}_{(k,\theta_Y)} \circ F$ where $k \in \operatorname{int} C$. From (iii) of Proposition 4.5, $\left(I^{(5)}_{(k,\theta_Y)} \circ F\right)(x,y) \in \mathbb{R}$ for any $x, y \in A$. Moreover, by Propositions 4.10 and 4.13, we obtain

(a) for each fixed $y \in A$, $\left(I^{(5)}_{(k,\theta_Y)} \circ F\right)(\cdot, y)$ is lower semicontinuous,

(b) for each fixed $x \in A$, $\left(I^{(5)}_{(k,\theta_Y)} \circ F\right)(x, \cdot)$ is quasi concave.

Also, by the definition of $I^{(5)}_{(k,\theta_Y)}$, $\left(I^{(5)}_{(k,\theta_Y)} \circ F\right)(x, x) \leq 0$ for all $x \in A$. Hence, we can apply the result of Theorem 5.7 to $\left(I^{(5)}_{(k,\theta_Y)} \circ F\right)$, that is, there exists $\bar{x} \in A$ such that $\left(I^{(5)}_{(k,\theta_Y)} \circ F\right)(\bar{x}, y) \leq 0$ for all $y \in A$. Clearly, $\{\theta_Y\}$ is $(-C)$-closed and then from (ii) of Proposition 4.7, we have $F(\bar{x}, y) \subset -C$. □

Theorem 5.9 *Let A be a nonempty compact convex subset of X and $F : A \times A \to \wp(Y)$. If F satisfies the following conditions:*

(i) *F is C-proper and C-closed valued on $A \times A$,*

(ii) *for each fixed $y \in A$, $F(\cdot, y)$ is upper C-continuous,*

(iii) *for each fixed $x \in A$, $F(x, \cdot)$ is type* (3) *properly quasi C-concave,*

(iv) *for all $x \in A$, $F(x, x) \cap (-C) \neq \emptyset$,*

then there exists $\bar{x} \in A$ such that $F(\bar{x}, y) \cap (-C) \neq \emptyset$ for all $y \in A$.

Proof. In a similar way to the proof of Theorem 5.8, by Propositions 4.10, 4.13 and Theorem 5.7, there exists $\bar{x} \in A$ such that

$$\left(I^{(3)}_{(k,\{\theta\})} \circ F\right)(\bar{x}, y) \leq 0$$

for all $y \in A$. By (i) of Proposition 4.7, it is equivalent to $\{\theta_Y\} \subset F(\bar{x}, y) + C$ and then $F(\bar{x}, y) \cap (-C) \neq \emptyset$ for all $y \in A$. □

Theorem 5.10 *Let A be a nonempty compact convex subset of X and $F : A \times A \to \wp(Y)$. If F satisfies the following conditions:*

(i) *F is $(-C)$-proper valued on $A \times A$,*

(ii) *for each fixed* $y \in A$, $F(\cdot, y)$ *is lower* C*-continuous,*

(iii) *for each fixed* $x \in A$, $F(x, \cdot)$ *is type* (5) *natural quasi* C*-concave,*

(iv) *for all* $x \in A$, $F(x, x) \cap \operatorname{int} C = \emptyset$,

then there exists $\bar{x} \in A$ *such that* $F(\bar{x}, y) \cap \operatorname{int} C = \emptyset$ *for all* $y \in A$.

Proof. In a similar way to the proof of Theorem 5.8, by Propositions 4.11, 4.14 and Theorem 5.7, there exists $\bar{x} \in A$ such that

$$\left(S^{(5)}_{(k,\theta_Y)} \circ F\right)(\bar{x}, y) \leq 0$$

for all $y \in Y$. It is equivalent to $\left(S^{(5)}_{(k,\theta_Y)} \circ F\right)(\bar{x}, y) \not> 0$. Hence, by (ii) of Proposition 4.8 we have $\{\theta_Y\} \not\leq^{(5)}_{\operatorname{int} C} F(\bar{x}, y)$. Therefore we get $F(\bar{x}, y) \cap \operatorname{int} C = \emptyset$ for all $y \in A$. □

Theorem 5.11 *Let* A *be a nonempty compact convex subset of* X *and* $F : A \times A \to \wp(Y)$. *If* F *satisfies the following conditions:*

(i) F *is compact-valued on* $A \times A$,

(ii) *for each fixed* $y \in A$, $F(\cdot, y)$ *is upper* C*-continuous,*

(iii) *for each fixed* $x \in A$, $F(x, \cdot)$ *is type* (3) *natural quasi* C*-concave,*

(iv) *for all* $x \in A$, $F(x, x) \not\subset \operatorname{int} C$,

then there exists $\bar{x} \in A$ *such that* $F(\bar{x}, y) \not\subset \operatorname{int} C$ *for all* $y \in A$.

Proof. In a similar way to the proof of Theorem 5.8, by Propositions 4.12, 4.14 and Theorem 5.7, there exists $\bar{x} \in A$ such that

$$\left(S^{(3)}_{(k,\theta_Y)} \circ F\right)(\bar{x}, y) \leq 0$$

for all $y \in A$. It is equivalent to $\left(S^{(3)}_{(k,\theta_Y)} \circ F\right)(\bar{x}, y) \not> 0$. Hence, by (i) of Proposition 4.9, we have $\{\theta_Y\} \not\leq^{(3)}_C F(\bar{x}, y) + G$ for any open neighborhood G of θ_Y. Therefore we get $F(\bar{x}, y) \not\subset \operatorname{int} C$ for all $y \in A$. $\square$

5.3 Continuity of cone-convex function

In this section, we prove the continuity of cone-convex set-valued map. In particular, we introduce a concept of local Lipschitz continuity for set-valued map and show each set-valued map has this property on $\mathbb{R}^n$ under some convexity assumptions.

Throughout this section, $X = \mathbb{R}^n$, $Y := \mathbb{R}^m$ ordered by a normal closed convex cone C with $\operatorname{int} C \neq \emptyset$, $F : X \to \wp(Y)$.

Firstly, let us recall the concept of local Lipschitz continuity of real-valued functions.

Definition 5.1 (locally Lipschitz continuous) Let $f : X \to \mathbb{R}$, $x \in X$ and $N_\epsilon(x) := \{y \in X : \|y - x\| < \epsilon\}$. Then f is said to be *locally Lipschitz continuous at x* if for any $\epsilon > 0$ there exists $M > 0$ such that for any $y \in N_\epsilon(x)$,

$$f(y) - f(x) \leq M\|y - x\|.$$

We shall say that f is locally Lipschitz on X if f is locally Lipschitz continuous at every $x \in X$.

By using a convex cone C, we consider a natural extension of local Lipschitz continuity of real-valued functions to set-valued maps.

Definition 5.2 (locally C-Lipschitz continuous) Let $x \in X$ and $N_\epsilon(x) := \{y \in X : \|y - x\| < \epsilon\}$. Then F is said to be *locally C-Lipschitz continuous at x* if for any $\epsilon > 0$ there exist $k \in \operatorname{int} C$ and $M > 0$ such that for any $y \in N_\epsilon(x)$,

$$F(y) \subset F(x) + [-M\|y - x\|k, M\|y - x\|k].$$

We shall say that F is locally C-Lipschitz on X if F is locally C-Lipschitz continuous at every $x \in X$.

Firstly, we give the following inherited properties of $I^{(3)}_{(k,V')}$ and $S^{(5)}_{(k,V')}$ as special cases of Propositions 4.11 and 4.12.

Lemma 5.1 *Let $k \in \operatorname{int} C$ and $V' \in \wp(Y)$. Then the following statements hold:*

(i) *If the following two conditions hold:*

(a) *V' is C-convex,*

(b) *F is type (3) C-convex,*

then $I^{(3)}_{(k,V')} \circ F$ is convex on X.

(ii) *If the following two conditions hold:*

(a) *V' is $(-C)$-convex,*

(b) *F is type (5) C-convex,*

then $I^{(5)}_{(k,V')} \circ F$ is convex on X.

Proof. We show statement (i) only; statement (ii) can be proved in the same way. Take $\left(x_1, \left(I^{(3)}_{(k,V')} \circ F\right)(x_1)\right), \left(x_2, \left(I^{(3)}_{(k,V')} \circ F\right)(x_2)\right) \in \operatorname{epi}\left(I^{(3)}_{(k,V')} \circ F\right)$. Let $t_1 := \left(I^{(3)}_{(k,V')} \circ F\right)(x_1)$ and $t_2 := \left(I^{(3)}_{(k,V')} \circ F\right)(x_2)$. By the definition of $I^{(3)}_{(k,V')} \circ F$, it is easy to check that for any $\epsilon > 0$ we obtain

$$F(x_1) \leq^{(3)}_C (t_1 + \epsilon)k + V \quad \text{and} \quad F(x_2) \leq^{(l)}_C (t_2 + \epsilon)k + V.$$

Since V' is C-convex, for any $\lambda \in [0,1]$ we have

$$\lambda F(x_1) + (1-\lambda)F(x_2) \leq^{(3)}_C (\lambda t_1 + (1-\lambda)t_2 + \epsilon)k + V'.$$

Hence, by (iii) of Proposition 4.1,

$$I^{(3)}_{(k,V')}(\lambda F(x_1) + (1-\lambda)F(x_2)) \leq \lambda t_1 + (1-\lambda)t_2 + \epsilon.$$

Since ϵ is an arbitrary positive real number,

$$I^{(3)}_{(k,V)}(\lambda F(x_1) + (1-\lambda)F(x_2)) \leq \lambda t_1 + (1-\lambda)t_2.$$

On the other hand, it follows from the type (3) C-convexity of F that we obtain

$$F(\lambda x_1 + (1-\lambda)x_2) \leq_C^{(3)} \lambda F(x_1) + (1-\lambda)F(x_2).$$

Hence, we obtain

$$\left(I^{(3)}_{(k,V')} \circ F\right)(\lambda x_1 + (1-\lambda)x_2) \leq \lambda t_1 + (1-\lambda)t_2$$

and then the set epi $\left(I^{(3)}_{(k,V')} \circ F\right)$ is convex. Consequently, $I^{(3)}_{(k,V')} \circ F$ is convex on X, and the proof is completed. □

By the same way in the proof of Lemma 5.1, we can obtain the following lemma.

Lemma 5.2 *Let $k \in \operatorname{int} C$ and $V' \in \wp(Y)$. Then the following statements hold:*

(i) *If the following two conditions hold:*

(a) *V' is C-convex,*

(b) *F is type (3) C-concave,*

then $S^{(3)}_{(k,V')} \circ F$ is concave on X.

(ii) *If the following two conditions hold:*

(a) *V' is $(-C)$-convex,*

(b) *F is type (5) C-concave,*

then $S^{(5)}_{(k,V')} \circ F$ is concave on X.

Theorem 5.12 *Let $\mathcal{S}$ be the family of all nonempty f_C-bounded and f_C-convex subsets of Y. If F satisfies the following conditions:*

(i) *F is f_C-bounded and $(-C)$-convex valued on X,*

(ii) *F is type (3) C-convex and type (5) C-convex on X,* *then F is $(-C)$-continuous on X.*

Proof. By the convexity assumption and Lemma 5.1, for each $* = 3, 5$ and fixed $(k, V') \in (\operatorname{int} C) \times \mathcal{S}$, $I^{(*)}_{(k,V')} \circ F$ is convex on X. Moreover, by (ii) and (iii) of Proposition 4.5, $I^{(*)}_{(k,V')} \circ F$ is finite on X. Thus $I^{(*)}_{(k,V')} \circ F$ is continuous on X. We shall prove the lower and upper $(-C)$-continuity of F on X.

At first, assume that there exists $\bar{x} \in X$ such that F is not lower $(-C)$-continuous at $\bar{x}$, that is, there exists an open set U with $F(\bar{x}) \cap U \neq \emptyset$ such that for any W which is an open neighborhood of $\bar{x}$, there exists $x_W \in W$ such that $F(x_W) \cap (U - C) = \emptyset$. Since U is open, $(F(\bar{x}) + \operatorname{int} C) \cap U \neq \emptyset$. Let $v \in (F(\bar{x}) + \operatorname{int} C) \cap U$ and consider the function $I^{(3)}_{(k,V')} \circ F$. Then it is clear that $v \notin F(x_W) + C$ and $I^{(3)}_{(k,V')} \circ F$ is continuous on X. Moreover, since C is normal there exists $\bar{t} > 0$ such that $v + [-\bar{t}k, \bar{t}k] \subset (F(\bar{x}) + \operatorname{int} C) \cap U$. Hence, we obtain

$$\left(I^{(3)}_{(k,v)} \circ F\right)(\bar{x}) \leq -\bar{t} < 0 \leq \left(I^{(3)}_{(k,v)} \circ F\right)(x_W).$$

Since W is an arbitrary, this is a contradiction to the upper semicontinuity of $I^{(3)}_{(k,v)} \circ F$.

Next, we assume that there exists $\bar{x} \in X$ such that F is not upper $(-C)$-continuous at $\bar{x}$, that is, there exists an open set U with $F(\bar{x}) \subset U$ such that for any W which is an open neighborhood of $\bar{x}$, there exists $x_W \in W$ such that $F(x_W) \not\subset U - C$. Since C is normal, there exists $\bar{t} > 0$ such that $F(\bar{x}) \subset F(\bar{x}) + \operatorname{int}[-\bar{t}k, \bar{t}k] \subset U$. By the $(-C)$-convexity of $F(\bar{x})$ and the convexity of $\operatorname{int}[-\bar{t}k, \bar{t}k]$, $V := F(\bar{x}) + \operatorname{int}[-\bar{t}k, \bar{t}k]$ is $(-C)$-convex. Also, V is f_C-bounded. Hence, $I^{(5)}_{(k,V')} \circ F$ is continuous on X. However, since $\bar{t} > 0$ and $F(x_W) \not\subset V - C$ we obtain

$$\left(I^{(5)}_{(k,V')} \circ F\right)(\bar{x}) < 0 \leq \left(I^{(5)}_{(k,V')} \circ F\right)(x_W).$$

Since W is an arbitrary, this is a contradiction. Consequently, F is $(-C)$-continuous on X. □

Theorem 5.13 *Let $\mathcal{S}$ be the family of all nonempty f_C-bounded and f_C-convex subsets of Y. If F satisfies the following conditions:*

(i) *F is f_C-bounded and C-convex valued on X,*

(ii) *F is type (3) C-concave and type (5) C-concave on X,*

then F is C-continuous on X.

Proof. We can prove this theorem by a similar way to the proof of Theorem 5.12. □

Next, we give the local C-Lipschitz continuity of set-valued map under some convexity assumptions.

Theorem 5.14 *If F satisfies the following conditions:*

(i) *F is f_C-convex, f_C-closed and f_C-bounded valued on X,*

(ii) *F is type (5) C-convex and type (3) C-concave on X,*

then F is locally C-Lipschitz on X.

Proof. Let $x \in X$ and $k \in \operatorname{int} C$. By Lemma 5.1 and Proposition 4.5, $I^{(5)}_{(k,F(x))} \circ F$ is a finite convex function on X. Hence, $I^{(5)}_{(k,F(x))} \circ F$ is locally Lipschitz continuous on X, that is, for any $\epsilon > 0$ there exists $M_1 > 0$ such that for any $y \in N_\epsilon(x)$,

$$\left| \left(I^{(5)}_{(k,F(x))} \circ F \right)(y) - \left(I^{(5)}_{(k,F(x))} \circ F \right)(x) \right| \leq M_1 \parallel y - x \parallel .$$

By the definition of $I^{(5)}_{(k,F(x))} \circ F$, (i) of Proposition 4.1 and (i) of Lemma 5.2,

$$F(y) \subset F(x) + M_1 \parallel y - x \parallel k - C. \tag{5.9}$$

Similarly, $S^{(3)}_{(k,F(x))} \circ F$ is locally Lipschitz continuous on X and then there exists $M_2 > 0$ such that for any $y \in N_\epsilon(x)$,

$$\left| \left(S^{(3)}_{(k,F(x))} \circ F \right)(y) - \left(S^{(3)}_{(k,F(x))} \circ F \right)(x) \right| \leq M_2 \parallel y - x \parallel .$$

Hence, we have

$$F(y) \subset F(x) - M_2 \parallel y - x \parallel k + C. \tag{5.10}$$

Let $M' := \max\{M_1, M_2\}$. By (5.9) and (5.10), for any $y \in N_\epsilon(x)$,

$$F(y) \subset F(x) + [-M' \parallel y - x \parallel k, M' \parallel y - x \parallel k].$$

Since x is an arbitrary, F is locally C-Lipschitz on X, and the proof is completed. $\square$

To illustrate Theorem 5.14, we present the following simple example.

Example 5.2 Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $C := \mathbb{R}^2_+$. A set-valued map F is defined by

$$F(x) = \left[\begin{pmatrix} x \\ -x^2 - 1 \end{pmatrix}, \begin{pmatrix} x \\ x^2 + 1 \end{pmatrix}\right].$$

Then it is clear that F satisfies conditions (i) and (ii) in Theorem 5.14. Hence, F is locally C-Lipschitz on X. In fact, for any $x \in X$, $\epsilon > 0$, and for any $y \in N_\epsilon(x)$, we obtain

$$F(y) \subset F(x) + \left[-\|y - x\| \begin{pmatrix} 1 \\ (x+\epsilon)^2 + 1 \end{pmatrix}, \|y - x\| \begin{pmatrix} 1 \\ (x+\epsilon)^2 + 1 \end{pmatrix}\right] \quad (x \geq 0),$$

$$F(y) \subset F(x) + \left[-\|y - x\| \begin{pmatrix} 1 \\ (x-\epsilon)^2 + 1 \end{pmatrix}, \|y - x\| \begin{pmatrix} 1 \\ (x-\epsilon)^2 + 1 \end{pmatrix}\right] \quad (x \leq 0).$$

5.4 Saddle point theorems for set-valued maps

In this section, we introduce four types of saddle point of set-valued map and investigate existence theorems for those saddle points with noncompact set-valued maps.

Throughout this section, X and Y are real Hausdorff topological vector spaces, $Z := \mathbb{R}^n$ ordered by a closed convex cone C with $\operatorname{int} C \neq \emptyset$, B is a compact base of C.

Firstly, we give the following lemma.

Lemma 5.3 *Let $F : X \to \wp(Z)$, $k \in \operatorname{int} C$ and $v \in Z$. Assume that $I^{(3)}_{(k,v)} \circ F$ and $S^{(5)}_{(k,v)} \circ F$ are finite on X. Then the following statements hold:*

(i) *If F is C-closed valued on X then $I^{(3)}_{(k,v)} \circ F$ is continuous on $(\operatorname{int} C) \times Z$. In particular, if $I^{(3)}_{(k,v)} \circ F$ is continuous on X for each $(k, v) \in (\operatorname{int} C) \times Z$ then it is continuous on $(\operatorname{int} C) \times Z \times X$;*

(ii) *If F is $(-C)$-closed valued on X then $S^{(5)}_{(k,v)} \circ F$ is continuous on $(\operatorname{int} C) \times Z$. In particular, if $S^{(5)}_{(k,v)} \circ F$ is continuous on X for each $(k,v) \in (\operatorname{int} C) \times Z$ then it is continuous on $(\operatorname{int} C) \times Z \times X$.*

Proof. At first, we show $\left(I^{(3)}_{(k,v)} \circ F\right)(x)$ is continuous on $(\operatorname{int} C) \times Z$. Take $(\bar{k}, \bar{v}) \in (\operatorname{int} C) \times Z$ and $\epsilon > 0$. Let $t := \left(I^{(3)}_{(\bar{k},\bar{v})} \circ F\right)(x)$. By the assumption, $t \in \mathbb{R}$. We consider the following two sets:

$$U_{\bar{k}}(\epsilon) := \bar{k} + \operatorname{int}\left[-\frac{\epsilon}{4(\mid t \mid +\epsilon)}\bar{k},\ \frac{\epsilon}{4(\mid t \mid +\epsilon)}\bar{k}\right],$$
$$U_{\bar{v}}(\epsilon) := \bar{v} + \operatorname{int}\left[-\frac{\epsilon}{4}\bar{k},\ \frac{\epsilon}{4}\bar{k}\right].$$

Since C is normal, $U_{\bar{k}}(\epsilon)$ and $U_{\bar{v}}(\epsilon)$ are open neighborhoods of $\bar{k}$ and $\bar{v}$, respectively. Hence, by the definition of $I^{(3)}_{(k,v)} \circ F$, we show that for any $k \in U_{\bar{k}}(\epsilon)$ and $v \in U_{\bar{v}}(\epsilon)$,

$$(t-\epsilon)k + v \notin F(x) + C \quad \text{and} \quad (t+\epsilon)k + v \in F(x) + \operatorname{int} C.$$

We take $k \in U_{\bar{k}}(\epsilon)$ and $v \in U_{\bar{v}}(\epsilon)$. Then, there exists $r_1, r_2 \in \operatorname{int}\left[-\frac{\epsilon}{4}\bar{k}, \frac{\epsilon}{4}\bar{k}\right]$ such that

$$k = \bar{k} + \frac{r_1}{\mid t \mid +\epsilon} \quad \text{and} \quad v = r_2 + \bar{v}.$$

We consider $(t-\epsilon)k + v$. Then,

$$\begin{aligned}
(t-\epsilon)k + v &= (t-\epsilon)\left(\bar{k} + \frac{r_1}{\mid t \mid +\epsilon}\right) + r_2 + \bar{v} \\
&= (t-\epsilon)\bar{k} + \frac{t-\epsilon}{\mid t \mid +\epsilon} r_1 + r_2 + \bar{v} \\
&\in (t-\epsilon)\bar{k} + \frac{\epsilon}{4}\bar{k} + \frac{\epsilon}{4}\bar{k} + \bar{v} - C \\
&= (t - \frac{\epsilon}{2})\bar{k} + \bar{v} - C.
\end{aligned}$$

Since $(t - \frac{\epsilon}{2})\bar{k} + \bar{v} \notin F(x) + C$, we obtain $\left(I^{(3)}_{(k,v)} \circ F\right)(x) > t - \epsilon$. Similarly, we consider $(t+\epsilon)k + v$. Then,

$$\begin{aligned}
(t+\epsilon)k + v &= (t+\epsilon)\left(\bar{k} + \frac{r_1}{\mid t \mid +\epsilon}\right) + r_2 + \bar{v} \\
&= (t+\epsilon)\bar{k} + \frac{t+\epsilon}{\mid t \mid +\epsilon} r_1 + r_2 + \bar{v} \\
&\in (t+\epsilon)\bar{k} - \frac{\epsilon}{4}\bar{k} - \frac{\epsilon}{4}\bar{k} + \bar{v} + C \\
&= (t + \frac{\epsilon}{2})\bar{k} + \bar{v} + C.
\end{aligned}$$

Thus, $(t+\epsilon)k+v \in F(x)+\operatorname{int} C$ and then $\left(I^{(3)}_{(k,v)} \circ F\right)(x) < t+\epsilon$. Accordingly, $\left(I^{(3)}_{(\cdot,\cdot)} \circ F\right)(x)$ is continuous on $(\operatorname{int} C)\times Z$. Next, we assume that $I^{(3)}_{(k,v)} \circ F$ is continuous on X. Take any $(k',v',x') \in (\operatorname{int} C)\times Z \times X$ and $\epsilon > 0$. Then, by the continuity of $I^{(3)}_{(k',v')} \circ F$ on X, there exists a neighborhood $U_{x'}$ of x' such that for all $x \in U_{x'}$,

$$\left(I^{(3)}_{(k',v')} \circ F\right)(x') - \epsilon < \left(I^{(3)}_{(k',v')} \circ F\right)(x) < \left(I^{(3)}_{(k',v')} \circ F\right)(x') + \epsilon.$$

Let $t' := \left(I^{(3)}_{(k',v')} \circ F\right)(x')$. By the definition of $I^{(3)}_{(k,v)} \circ F$,

$$(t'-\epsilon)k'+v' \notin F(x)+C \quad \text{and} \quad (t'+\epsilon)k'+v' \in F(x)+\operatorname{int} C.$$

Hence, by the normality of C and C-closeness of F, there exists an open neighborhood $U_{k',v'}$ of (k',v') such that for any $(k,v) \in U_{k',v'}$,

$$(t'-\epsilon)k+v \notin F(x)+C \quad \text{and} \quad (t'+\epsilon)k+v \in F(x)+\operatorname{int} C,$$

and then

$$\left(I^{(3)}_{(k',v')} \circ F\right)(x') - \epsilon < \left(I^{(3)}_{(k',v')} \circ F\right)(x) < \left(I^{(3)}_{(k',v')} \circ F\right)(x') + \epsilon,$$

for any $(k,v,x) \in U_{k',v'} \times U_{x'}$. Similarly, we can prove statement (ii). This completes the proof. □

Now we define two types of scalarizing fucntions for sets based on $I^{(3)}_{(k,v)}$ and $S^{(5)}_{(k,v)}$. Let $A \in \wp(Z)$ and $v \in Z$. Then, $\varphi_v : \wp(Z) \to \mathbb{R} \cup \{\pm\infty\}$ and $\psi_v : \wp(Z) \to \mathbb{R} \cup \{\pm\infty\}$ are defined by

$$\varphi_v(A) := \inf_{k\in B} I^{(3)}_{(k,v)}(A),$$

$$\psi_v(A) := \sup_{k\in B} S^{(5)}_{(k,v)}(A),$$

respectively. Then, we introduce some properties of these functions.

Lemma 5.4 *Let $A \in \wp(Z)$ and $v \in Z$. Then, the following statements hold:*

(i) $\varphi_v(A) < \infty$ *and* $\psi_v(A) > -\infty$.

(ii) *If A is C-proper then there exists $v_0 \in Z$ such that $\varphi_{v_0}(A) > -\infty$.*

(iii) *If A is $(-C)$-proper then there exists $v_0 \in Z$ such that $\psi_{v_0}(A) < \infty$.*

(iv) *If A is C-closed then there exists $k' \in B$ such that*

$$\varphi_v(A) = \min\{t \in \mathbb{R} | tk' + v \in A + C\}.$$

(v) *If A is $(-C)$-closed then there exists $k' \in B$ such that*

$$\psi_v(A) = \max\{t \in \mathbb{R} | tk' + v \in A - C\}.$$

Proof. By Lemma 5.3, we obtain

$$\varphi_v(A) = \min_{k \in B} I^{(3)}_{(k,v)}(A) \quad \text{and} \quad \psi_v(A) = \max_{k \in B} S^{(5)}_{(k,v)}(A). \tag{5.11}$$

Hence, it follows from (ii) of Proposition 4.5 that statement (i) is clear. We show statement (ii). Since A is C-proper, there exists $v_0 \in Z$ such that $v_0 \notin A + C$. We consider the function φ_{v_0}. Then it is easy to check that $-\infty < 0 \leq \varphi_{v_0}(A)$. Similarly, we can prove statement (iii).

Next, we show statement (iv). By (5.11), there exists $k' \in B$ such that

$$\varphi_v(A) = I^{(3)}_{(k',v)}(A).$$

Let $t_A := I^{(3)}_{(k',v)}(A)$ and assume that $t_A k' + v \notin A + C$. Then $t_A k' + v \in (A + C)^c$. Since A is C-closed, $(A + C)^c$ is open and then there exist $k_0 \in \operatorname{int} C$ and $t_0 \in \mathbb{R}_+$ such that for any $0 < t < t_0$,

$$t_A k' + v + [-tk_0, tk_0] \subset (A + C)^c.$$

Since $k' \in B$ and $k_0 \in \operatorname{int} C$, it is easy to check that there exists $\hat{t} > 0$ such that $\hat{t}k' \in [-tk_0, tk_0]$. Hence, we obtain

$$(t_A + \hat{t})k' + v \notin A + C.$$

This is a contradiction. Moreover, statement (v) can be proved in the same way, and the proof is completed. $\square$

Lemma 5.5 *Let $A_1, A_2 \in \wp(Z)$ and $v \in Z$. Then, the following statements hold:*

(i) *If $A_1 \leq_C^{(3)} A_2$ then $\varphi_v(A_1) \leq \varphi_v(A_2)$.*

(ii) *If $A_1 \leq_C^{(5)} A_2$ then $\psi_v(A_1) \leq \psi_v(A_2)$.*

(iii) *If A_i is C-closed and C-proper with $i = 1, 2$ then there exists $v_0 \in Z$ such that for any $\alpha \in [0, 1]$,*

$$\varphi_{v_0}(\alpha A_1 + (1-\alpha)A_2) = \alpha\varphi_{v_0}(A_1) + (1-\alpha)\varphi_{v_0}(A_2).$$

(iv) *If A_i is $(-C)$-closed and $(-C)$-proper then there exists $v_0 \in Z$ such that for any $\alpha \in [0, 1]$,*

$$\psi_{v_0}(\alpha A_1 + (1-\alpha)A_2) = \alpha\psi_{v_0}(A_1) + (1-\alpha)\psi_{v_0}(A_2).$$

Proof. By the definitions of φ_v and ψ_v, statements (i) and (ii) are clear. We show statement (iii). Assume that A_i is C-closed and C-proper with $i = 1, 2$. Since $\operatorname{int} C \neq \emptyset$, there exists $v_0 \in Z$ such that

$$v_0 \notin (A_1 + C) \cup (A_2 + C).$$

Let us consider the function φ_{v_0}. By (i) and (ii) of Lemma 5.4,

$$\varphi_{v_0}(A_1), \varphi_{v_0}(A_2) \in \mathbb{R}_+.$$

Moreover, it follows from (iv) of Lemma 5.4 that we obtain

$$\varphi_{v_0}(A_i) = \min\{t \in \mathbb{R}_+ | (tB + v) \cap (A_i + C) \neq \emptyset\}, \tag{5.12}$$

with $i = 1, 2$. Let $t_{A_1} := \varphi_{v_0}(A_1)$, $t_{A_2} := \varphi_{v_0}(A_2)$. By (5.12),

$$(t_{A_1}B + v_0) \cap (A_1 + C) \neq \emptyset \quad \text{and} \quad (t_{A_2}B + v_0) \cap (A_2 + C) \neq \emptyset.$$

Hence, for any $\alpha \in [0, 1]$, we obtain

$$((\alpha t_{A_1} + (1-\alpha)t_{A_2})B + v_0) \cap (\alpha A_1 + (1-\alpha)A_2 + C) \neq \emptyset.$$

It is equivalent to

$$\varphi_{v_0}(\alpha A_1 + (1-\alpha)A_2) \leq \alpha t_{A_1} + (1-\alpha)t_{A_2}.$$

Next, we show the converse. Let $\hat{A}_i := (A_i + C) \cap (v_0 + C)$. Firstly we show $\varphi_{v_0}(A_i) = \varphi_{v_0}(\hat{A}_i)$. Let $t_{\hat{A}_i} := \varphi_{v_0}(\hat{A}_i)$ and assume that $t_{A_i} < t_{\hat{A}_i}$. By the definition of φ_{v_0}, for any $k \in B$,

$$t_{A_i}k + v_0 \notin \hat{A}_i + C.$$

It is clear that $\hat{A}_i + C = \hat{A}_i$ and $t_{A_i}k + v_0 \in v_0 + C$. Thus,

$$t_{A_i}k + v_0 \notin A_i + C.$$

This is a contradiction. Hence we obtain $\varphi_{v_0}(A_i) = \varphi_{v_0}(\hat{A}_i)$, and then it follows from (2.1) and (5.12) that we obtain

$$\varphi_{v_0}(A_i) = \inf_{a_i \in \hat{A}_i} x_0^*(a_i - v_0). \tag{5.13}$$

Assume that there exists $\alpha_0 \in (0,1)$ such that

$$\varphi_{v_0}(\alpha_0 A_1 + (1-\alpha_0)A_2) < \alpha_0 t_{A_1} + (1-\alpha_0)t_{A_2}. \tag{5.14}$$

By (iv) of Lemma 5.4 and (5.13), there exists $a_1' \in \hat{A}_1$ and $a_2' \in \hat{A}_2$ such that for any $a_1 \in \hat{A}_1$ and $a_2 \in \hat{A}_2$,

$$x_0^*(a_1') \leq x_0^*(a_1) \quad \text{and} \quad x_0^*(a_2') \leq x_0^*(a_2). \tag{5.15}$$

Moreover, it is easy to check that $\alpha_0\hat{A}_1 + (1-\alpha_0)\hat{A}_2$ is C-closed, and then there exists $a' \in \alpha_0\hat{A}_1 + (1-\alpha_0)\hat{A}_2$ such that for any $a \in \alpha_0\hat{A}_1 + (1-\alpha_0)\hat{A}_2$,

$$x_0^*(a') \leq x_0^*(a).$$

Then there exist $\hat{a_1} \in \hat{A}_1$ and $\hat{a_2} \in \hat{A}_2$ such that $a' = \alpha_0\hat{a_1} + (1-\alpha_0)\hat{a_2}$. It is easy to check that

$$\alpha_0\hat{A}_1 + (1-\alpha_0)\hat{A}_2 = (\alpha_0 A_1 + (1-\alpha_0)A_2 + C) \cap (v_0 + C).$$

Hence we obtain

$$\alpha_0 x_0^*(\hat{a}_1) + (1-\alpha_0)x_0^*(\hat{a}_2) = \varphi_{v_0}(\alpha_0 A_1 + (1-\alpha_0)A_2).$$

It follows from (5.13) and (5.14) that

$$\alpha_0 x_0^*(\hat{a}_1) + (1-\alpha_0)x_0^*(\hat{a}_2) < \alpha_0 x_0^*(a_1') + (1-\alpha_0)x_0^*(a_2').$$

This contradicts (5.15). Consequently, we have

$$\varphi_{v_0}(\alpha A_1 + (1-\alpha)A_2) = \alpha\varphi_v(A_1) + (1-\alpha)\varphi_v(A_2),$$

for any $\alpha \in [0,1]$. Statement (iv) can be proved in a similar way, and the proof is completed. $\square$

Lemma 5.6 *Let $A_1, A_2 \in \wp(Z)$. Then, the following statements hold:*

(i) *If the following two conditions are satisfied:*

(a) *A_1 is C-proper and A_2 is C-closed,*

(b) $A_1 \leq^{(3)}_{\mathrm{int}\, C} A_2$,

then $\varphi_v(A_1) < \varphi_v(A_2)$ for each fixed $v \in Z$.

(ii) *If the following two conditions are satisfied:*

(c) *A_1 is $(-C)$-closed and A_2 is $(-C)$-proper,*

(d) $A_1 \leq^{(5)}_{\mathrm{int}\, C} A_2$,

then $\psi_v(A_1) < \psi_v(A_2)$ for each fixed $v \in Z$.

Proof. At first, we show statement (i). Assume that conditions (a) and (b) are satisfied. By (i) and (ii) of Lemma 5.4, $\varphi_v(A_i) \in \mathbb{R}$ with $i = 1,2$ for any $v \in Z$. Let $t_{A_2} := \varphi_v(A_2)$. Then there exists $k' \in B$ such that $t_{A_2}k' + v \in A_2 + C$. Moreover, by condition (b) there exists $\tilde{a_1} \in A_1$ such that $t_{A_2}k' + v \in \tilde{a_1} + \operatorname{int} C$. Then there exists $\epsilon > 0$ such that

$$(t_{A_2} - \epsilon)k' + v \in \tilde{a_1} + \operatorname{int} C \subset A_1 + C.$$

Hence we obtain $\varphi v(A_1) \leq t_{A_2} - \epsilon < \varphi_v(A_2)$. Statement (ii) can be proved similarly. □

Lemma 5.7 *Let $F : X \to \wp(Z)$. Then the following statements hold:*

(i) *If F is C-closed valued on X then for any $v \in Z$, the solution x_0 of $\inf_{x \in X} (\varphi_v \circ F)(x)$ (resp., $\sup_{x \in X} (\varphi_v \circ F)(x)$) is also the type* (3) *weak minimal (resp., type* (3) *weak maximal) efficient solution of* (SP).

(ii) *If F is $(-C)$-closed valued on X then for any $v \in Z$, the solution x_0 of $\inf_{x \in X} (\psi_v \circ F)(x)$ (resp., $\sup_{x \in X} (\psi_v \circ F)(x)$) is also the type* (5) *weak minimal (resp., type* (5) *weak maximal) efficient solution of* (SP).

Proof. By Theorems 5.3, 5.4 and Lemma 5.6, it is clear. □

Similarly, we can obtain the following lemma.

Lemma 5.8 *Let $F : X \to \wp(Z)$. Then the following statements hold:*

(i) *If F is C-closed valued on X then for any $v \in Z$, the unique solution of $\inf_{x \in X} (\varphi_v \circ F)(x)$ (resp., $\sup_{x \in X} (\varphi_v \circ F)(x)$) is also the type* (3) *minimal (resp., type* (3) *maximal) efficient solution of* (SP).

(ii) *If F is $(-C)$-closed valued on X then for any $v \in Z$, the unique solution of $\inf_{x \in X} (\psi_v \circ F)(x)$ (resp., $\sup_{x \in X} (\psi_v \circ F)(x)$) is also the type* (5) *minimal (resp., type* (5) *maximal) efficient solution of* (SP).

Let $F : X \times Y \to \wp(Z)$. Now, we define four types of saddle point of F based on set-relations. For each $* = 3, 5$, a point $(x_0, y_0) \in X \times Y$ is called *type* $(*)$ *C-saddle point* if

$$F(x_0, y_0) \in \operatorname{Min}_{(*)} F(X, y_0) \cap \operatorname{Max}_{(*)} F(x_0, Y);$$

type $(3,5)$ *C-saddle point* if

$$F(x_0,y_0) \in \mathrm{Min}_{(3)}F(X,y_0) \cap \mathrm{Max}_{(5)}F(x_0,Y);$$

type $(5,3)$ *C-saddle point* if

$$F(x_0,y_0) \in \mathrm{Min}_{(5)}F(X,y_0) \cap \mathrm{Max}_{(3)}F(x_0,Y).$$

If C is replaced by $\operatorname{int} C$ then we call it type $(*)$ (resp., type $(3,5)$, type $(5,3)$) weak C-saddle point. It is easy to check that type (3) (resp., type $(3,5)$) C-saddle point (resp., weak C-saddle point) is type (5) (resp., type $(5,3)$) $(-C)$-saddle point (resp., weak $(-C)$-saddle point). Moreover, if F is single-valued then these concepts are equivalent. Hence, we consider only type (3) and type $(3,5)$ cone-saddle point.

Firstly, we show sufficient conditions for the existence of type (3) cone-saddle points.

Theorem 5.15 *Let S and T be nonempty compact convex subsets of X and Y, respectively, and $v \in Z$. If $F : S \times T \to \wp(Z)$ satisfies that*

(i) *F is C-closed and C-proper valued on $S \times T$,*

(ii) *$F(\cdot, y)$ is type* (3) *natural quasi C-convex and upper C-continuous on S for every $y \in T$,*

(iii) *$F(x, \cdot)$ is type* (3) *natural quasi C-concave, upper C-continuous and lower $(-C)$-continuous on T for every $x \in S$,*

then, F has at least one type (3) *weak C-saddle point.*

Proof. At first, we show there exists $v_0 \in Z$ such that

$$v_0 \notin \bigcup_{(x,y)\in S\times T} F(x,y) + C.$$

By Lemma 3.1, for each $y \in T$ there exists $v(y) \in Z$ such that

$$v(y) \notin \bigcup_{x\in S} F(x,y) + C,$$

and then we obtain

$$\bigcup_{x\in S} F(x,y) \subset (v(y)-C)^c$$

Since F is upper C-continuous on T, there exists U_y which is an open neighborhood of y such that

$$\bigcup_{(x,w)\in S\times U_y} F(x,w) \subset (v(y)-C)^c.$$

Moreover, since T is compact there exists U_{y_i} with $i=1,\cdots,n$ such that $T\subset \bigcup_{i=1}^{n} U_{y_i}$, and then we obtain

$$\bigcup_{(x,y)\in S\times T} F(x,y) \subset \bigcup_{i=1}^{n}(v(y_i)-C)^c.$$

Clearly, $\bigcup_{i=1}^{n} v(y_i)$ is C-proper. Hence there exists $v_0\in Z$ such that

$$v_0 \notin \bigcup_{(x,y)\in S\times T} F(x,y)+C.$$

Let us consider the function $\varphi_{v_0}\circ F$. By Lemmas 5.5 and 5.6, it is easy to check that

(a) $(\varphi_{v_0}\circ F)(x,y)\in\mathbb{R}$ for any $(x,y)\in S\times T$,

(b) $(\varphi_{v_0}\circ F)(\cdot,y)$ is quasiconvex on S for every $y\in T$,

(c) $(\varphi_{v_0}\circ F)(x,\cdot)$ is quasiconcave on T for every $x\in S$.

We show the continuity of $\varphi_{v_0}\circ F$. Assume that there exists $(x',y')\in S\times T$ such that $(\varphi_{v_0}\circ F)(\cdot,y')$ is not lower semicontinuous at x', that is, there exists $\epsilon_1>0$ such that for any $U_\alpha(x')$ which is an open neighborhood of x', there exists $x_\alpha\in U_\alpha(x')$ such that

$$(\varphi_{v_0}\circ F)(x_\alpha,y') \le (\varphi_{v_0}\circ F)(x',y')-\epsilon_1.$$

Hence, there exists $k'\in B$ such that we obtain

$$(\varphi_v\circ F)(x',y') = I^{(3)}_{(k',v_0)}(F(x',y')) \quad\text{and}\quad F(x_\alpha,y')\not\subset F(x',y')-\frac{1}{2}\epsilon_1 k'+\operatorname{int}C.$$

It is clear that $F(x',y') \subset F(x',y') - \frac{1}{2}\epsilon_1 k' + \operatorname{int} C$ and $F(x',y') - \frac{1}{2}\epsilon_1 k' + \operatorname{int} C$ is open. Since α is an arbitrary, this contradicts the upper C-continuity of F on S for every $y \in T$. Similarly, we assume that $(\varphi_{v_0} \circ F)(x',\cdot)$ is not upper semicontinuous at y', that is, there exists $\epsilon_2 > 0$ such that for any $U_\alpha(y')$ which is an open neighborhood of y', there exists $y_\alpha \in U_\alpha(y')$ such that

$$(\varphi_{v_0} \circ F)(x',y') + \epsilon_2 \leq (\varphi_{v_0} \circ F)(x',y_\alpha).$$

Hence, we obatin

$$F(x',y_\alpha) \cap \left\{ \left((\varphi_{v_0} \circ F)(x',y') + \frac{1}{2}\epsilon_2 \right) k' + v - \operatorname{int} C \right\} = \emptyset.$$

It is clear that

$$F(x',y') \cap \left\{ \left((\varphi_{v_0} \circ F)(x',y') + \frac{1}{2}\epsilon_2 \right) k' + v - \operatorname{int} C \right\} \neq \emptyset,$$

and $\left((\varphi_{v_0} \circ F)(x',y') + \frac{1}{2}\epsilon_2\right) k' + v - \operatorname{int} C$ is open. This contradicts the lower $(-C)$-continuity of F on T for every $x \in S$. Hence we obtain

(d) $(\varphi_{v_0} \circ F)(\cdot,y)$ is lower semicontinuous on S for every $y \in T$,

(e) $(\varphi_{v_0} \circ F)(x,\cdot)$ is upper semicontinuous on T for every $x \in S$.

By (a)–(e), we can apply Sion's minimax theorem [64] to $\varphi_{v_0} \circ F$, that is, there exists $(x_0,y_0) \in S \times T$ such that

$$(\varphi_{v_0} \circ F)(x_0,y) \leq (\varphi_{v_0} \circ F)(x_0,y_0) \leq (\varphi_{v_0} \circ F)(x,y_0),$$

for any $x \in S$ and $y \in T$. Hence it follows from Lemma 5.7 that

$$F(x_0,y_0) \in \mathrm{Min}^{\mathrm{w}}_{(3)} F(X,y_0) \cap \mathrm{Max}^{\mathrm{w}}_{(3)} F(x_0,Y).$$

The proof is completed. □

Theorem 5.16 *Let S and T be nonempty compact convex subsets of X and Y, respectively, and $v \in Z$. If $F : S \times T \to \wp(Z)$ satisfies that*

(i) *F is C-closed and C-proper valued on $S \times T$,*

(ii) *$F(\cdot, y)$ is type (3) strictly natural quasi C-convex and upper C-continuous on S for every $y \in T$,*

(iii) *$F(x, \cdot)$ is type (3) strictly natural quasi C-concave, upper C-continuous and lower $(-C)$-continuous on T for every $x \in S$,*

then, F has at least one type (3) C-saddle point.

Proof. It follows from the proof of Theorem 5.15 that there exists $v_0 \in Z$ such that

$$v_0 \notin \bigcup_{(x,y) \in S \times T} F(x, y).$$

Let us consider the function $\varphi_{v_0} \circ F$. Then, we obtain

(a) $(\varphi_{v_0} \circ F)(x, y) \in \mathbb{R}$ for any $(x, y) \in S \times T$,

(b) $(\varphi_{v_0} \circ F)(\cdot, y)$ is lower semicontinuous on S for every $y \in T$,

(c) $(\varphi_{v_0} \circ F)(x, \cdot)$ is upper semicontinuous on T for every $x \in S$.

Moreover, by (i) and (iii) of Lemma 5.5,

(d) $(\varphi_{v_0} \circ F)(\cdot, y)$ is quasiconvex on S for every $y \in T$,

(e) $(\varphi_{v_0} \circ F)(\cdot, y)$ is quasiconcave on T for every $x \in S$.

Now we consider the following two set-valued maps:

$$M(y) := \{\bar{x} \in S |\, (\varphi_{v_0} \circ F)(\bar{x}, y) = \min_{x \in S} (\varphi_{v_0} \circ F)(x, y)\},$$

$$N(x) := \{\bar{y} \in T |\, (\varphi_{v_0} \circ F)(x, \bar{y}) = \max_{y \in T} (\varphi_{v_0} \circ F)(x, y)\}.$$

We show M is singleton on T. By (d), it is easy to check that M is nonempty and convex-valued on T. Assume that there exists $x_1, x_2 \in M(y)$ with $x_1 \neq x_2$. Then, for any $\lambda \in (0, 1)$, $\lambda x_1 + (1 - \lambda)x_2 \in M(y)$. Hence, for any $x \in S$,

$$(\varphi_{v_0} \circ F)(\lambda x_1 + (1 - \lambda)x_2, y) \leq (\varphi_{v_0} \circ F)(x, y).$$

However, by the definition of type (3) strictly natural quasi C-convex, there exists $\mu \in [0,1]$ such that

$$F(\lambda x_1 + (1-\lambda)x_2) \leq^{(3)}_{\operatorname{int} C} \mu F(x_1) + (1-\mu)F(x_2)$$

and then,

$$\begin{aligned} & (\varphi_{v_0} \circ F)(\lambda x_1 + (1-\lambda)x_2, y) < \varphi_{v_0}(\mu F(x_1, y) + (1-\mu)F(x_2, y)) \\ \leq \; & \mu (\varphi_{v_0} \circ F)(x_1, y) + (1-\mu)(\varphi_{v_0} \circ F)(x_2, y) \\ = \; & \min_{x \in X} (\varphi_{v_0} \circ F)(x, y), \end{aligned}$$

which is a contradiction. Hence M is singleton on T. Similarly, N is singleton on S. Accordingly, by (a)–(e) and (i) of Lemma 5.8, F has at least one type (3) C-saddle point, and the proof is completed. □

Next, we show existence theorems for type $(3,5)$ cone-saddle points. For this end, we recall the following lemma.

Lemma 5.9 ([67]) *Let S and T be nonempty compact convex subsets of X and Y, respectively, $f, g : S \times T \to \mathbb{R}$. If the following two conditions are satisfied:*

(i) *$f(\cdot, y)$ is quasiconvex and continuous on S for every $y \in T$,*

(ii) *$g(x, \cdot)$ is quasiconcave and continuous on T for every $x \in S$,*

then there exists $(x_0, y_0) \in S \times T$ such that

$$f(x_0, y_0) \leq f(x, y_0) \quad \text{and} \quad g(x_0, y) \leq g(x_0, y_0),$$

for any $x \in S$ and $y \in T$.

Theorem 5.17 *Let S and T be nonempty compact convex subsets of X and Y, respectively. If $F : S \times T \to \wp(Z)$ satisfies that*

(i) *F is f_C-closed and f_C-proper valued on $S \times T$,*

(ii) *$F(\cdot, y)$ is type* (3) *natural quasi C-convex and continuous on S for every $y \in T$,*

(iii) *$F(x, \cdot)$ is type* (5) *natural quasi C-concave and continuous on T for every $x \in S$,*

then, F has at least one type $(3, 5)$ *weak C-saddle point.*

Proof. By Theorem 5.15, we obtain $\bigcup_{(x,y)\in S\times T} F(x, y)$ is f_C-closed and f_C-proper. Then there exists $u_0, v_0 \in Z$ such that

$$u_0 \notin \bigcup_{(x,y)\in S\times T} F(x, y) + C \quad \text{and} \quad v_0 \notin \bigcup_{(x,y)\in S\times T} F(x, y) - C.$$

Let us consider the composite functions $\varphi_{u_0} \circ F$ and $\psi_{v_0} \circ F$. It follows from the proof of Theorem 5.15 that $(\varphi_{u_0} \circ F)(\cdot, y)$ is quasiconvex and continuous on S for every $y \in T$. By a similar way, we can prove the quasiconcavity and continuity of $\psi_{v_0} \circ F$. Hence, by Lemma 5.9 there exists $(x_0, y_0) \in S \times T$ such that

$$(\varphi_{u_0} \circ F)(x_0, y_0) \leq (\varphi_{u_0} \circ F)(x, y_0),$$

and

$$(\psi_{v_0} \circ F)(x_0, y) \leq (\psi_{v_0} \circ F)(x_0, y_0),$$

for any $x \in S$ and $y \in T$, respectively. Thus, it follows from Lemma 5.7 that

$$F(x_0, y_0) \in \mathrm{Min}^{\mathrm{w}}_{(3)} F(X, y_0) \cap \mathrm{Max}^{\mathrm{w}}_{(5)} F(x_0, Y),$$

and the proof is completed. □

By Theorems 5.15, 5.16 and 5.17, we obtain the following existence theorem for type $(3, 5)$ C-saddle point.

Theorem 5.18 *Let S and T be nonempty compact convex subsets of X and Y, respectively. If $F : S \times T \to \wp(Z)$ satisfies that*

(i) *F is f_C-closed and f_C-proper valued on $S \times T$,*

(ii) $F(\cdot, y)$ *is type* (3) *strictly natural quasi* C*-convex and continuous on* S *for every* $y \in T$,

(iii) $F(x, \cdot)$ *is type* (5) *strictly natural quasi* C*-concave and continuous on* T *for every* $x \in S$,

then, F *has at least one type* $(3, 5)$ C*-saddle point.*

5.5 Minimax theorems for set-valued maps

Throughout this section, X and Y are real Hausdorff topological vector space, $Z := \mathbb{R}^n$ with the Euclidean norm $\|\cdot\|$, and C is a nontrivial closed convex cone with $\operatorname{int} C \neq \emptyset$.

Let $F : X \times Y \to \wp(Z)$. Based on several solution concepts of (SP) introduced in Section 5.1, we consider the following two types of minimax and maximin values of F, respectively:

$$\begin{aligned}
\mathrm{Min}_{(3)}\mathrm{Max}_{\mathrm{p}}F(x,y) &:= \mathrm{Min}_{(3)}\{F(x,y) | F(x,y) \in \mathrm{Max}_{\mathrm{p}}F(x,Y),\ x \in X\},\\
\mathrm{Max}_{(5)}\mathrm{Min}_{\mathrm{p}}F(x,y) &:= \mathrm{Max}_{(5)}\{F(x,y) | F(x,y) \in \mathrm{Min}_{\mathrm{p}}F(X,y),\ y \in Y\},\\
\mathrm{Min}_{(3)}\mathrm{Max}_{\mathrm{p}}^{\mathrm{w}}F(x,y) &:= \mathrm{Min}_{(3)}\{F(x,y) | F(x,y) \in \mathrm{Max}_{\mathrm{p}}^{\mathrm{w}}F(x,Y),\ x \in X\},\\
\mathrm{Max}_{(5)}\mathrm{Min}_{\mathrm{p}}^{\mathrm{w}}F(x,y) &:= \mathrm{Max}_{(5)}\{F(x,y) | F(x,y) \in \mathrm{Min}_{\mathrm{p}}^{\mathrm{w}}F(X,y),\ y \in Y\}.
\end{aligned}$$

In this section, we show some minimax theorems for set-valued maps. For this end, we consider two set-valued maps $S : (\operatorname{int} C) \times Z \times Y \to \wp(X)$ and $T : (\operatorname{int} C) \times Z \times X \to \wp(Y)$ defined by

$$S(k; v; y) := \left\{ x \in X \,\middle|\, \left(I^{(3)}_{(k,v)} \circ F\right)(x,y) = \min_{x \in X} \left(I^{(3)}_{(k,v)} \circ F\right)(x,y) \right\}, \tag{5.16}$$

$$T(k; v; x) := \left\{ y \in Y \,\middle|\, \left(S^{(5)}_{(k,v)} \circ F\right)(x,y) = \max_{y \in Y} \left(S^{(5)}_{(k,v)} \circ F\right)(x,y) \right\}, \tag{5.17}$$

respectively.

At first, we give the following lemmas.

Lemma 5.10 *Let $F : X \times Y \to \wp(Z)$. Then the following statements hold:*

(i) *If the following two conditions hold:*

(a) *F is C-closed valued on X for every $y \in Y$,*

(b) *$\bigcup_{x\in X} F(x,y)$ is compact for every $y \in Y$,*

then for each $y \in Y$, there exists a compact subset K of $(\operatorname{int} C) \times Z$ such that

$$\mathrm{Min}_{\mathrm{p}}^{\mathrm{w}} F(X,y) = F\left(\left\{\bigcup_{(k,v)\in K} S(k;v;y)\right\}, y\right).$$

In particular, if $S(k;v;y)$ is singleton for any $(k,v,y) \in (\operatorname{int} C) \times Z \times Y$, then

$$\mathrm{Min}_{\mathrm{p}} F(X,y) = F\left(\left\{\bigcup_{(k,v)\in K} S(k;v;y)\right\}, y\right).$$

(ii) *If the following two conditions hold:*

(a) *F is $(-C)$-closed valued on Y for every $x \in X$,*

(b) *$\bigcup_{y\in X} F(x,y)$ is compact for every $y \in Y$,*

then for each $y \in Y$, there exists a compact subset K of $(\operatorname{int} C) \times Z$ such that

$$\mathrm{Max}_{\mathrm{p}}^{\mathrm{w}} F(x,Y) = F\left(x, \left\{\bigcup_{(k,v)\in K} T(k;v;x)\right\}\right).$$

In particular, if $T(k;v;x)$ is singleton for any $(k,v,y) \in (\operatorname{int} C) \times Z \times Y$, then

$$\mathrm{Max}_{\mathrm{p}} F(x,Y) = F\left(x, \left\{\bigcup_{(k,v)\in K} T(k;v;x)\right\}\right).$$

Proof. We only prove statement (i); statement (ii) can be proved in a similar way. Let $U := \{z \in Z \mid \|z\| \leq 1\}$ and $y \in Y$. Since $\bigcup_{x\in X} F(x,y)$ is compact, there exists $\{z_1,\ldots,z_n\} \subset \bigcup_{x\in X} F(x,y)$ such that

$$\bigcup_{x\in X} F(x,y) \subset \bigcup_{i=1}^{n}(z_i + \operatorname{int} U) \subset \bigcup_{i=1}^{n}(z_i + U).$$

Clearly, $\bigcup_{i=1}^{n}(z_i+U)$ is compact. Hence, there exists $a \in Z$ such that

$$\bigcup_{i=1}^{n}(z_i+U) \subset a + \operatorname{int} C.$$

Let $V' := \bigcup_{i=1}^{n}(z_i+U) - \{a\}$. Then it is easy to check that $\operatorname{cone}(V') \subset (\operatorname{int} C) \cup \{\theta_Z\}$ and $\operatorname{cone}(V')$ is a closed cone with nonempty topological interior. Now we consider the set $V^* := U \cap \operatorname{cone}(V')$. Let $K := (V^* \setminus \{\theta_Z\}) \times \{a\}$. Then, it is clear that K is a nonempty compact set and the relation

$$F\left(\left\{\bigcup_{(k,v)\in K} S(k;v;y)\right\}, y\right) \subset \operatorname{Min}_{\mathrm{p}}^{\mathrm{w}} F(X,y)$$

holds.

Next, we prove the converse. To use contradiction, we assume that there exist $\bar{y} \in Y$ and $F(\bar{x},\bar{y}) \in \operatorname{Min}_{\mathrm{p}}^{\mathrm{w}} F(X,\bar{y})$ such that

$$F(\bar{x},\bar{y}) \not\subset F\left(\left\{\bigcup_{(k,v)\in K} S(k;v;\bar{y})\right\}, \bar{y}\right).$$

Then, for any $(k,v) \in K$, there exists $x(k;v) \in X$ such that

$$\left(I^{(k)}_{(v,\circ)}F\right)(x(k;v),\bar{y}) < \left(I^{(k)}_{(v,\circ)}F\right)(\bar{x},\bar{y}). \tag{5.18}$$

By (i) of Theorem 5.5, $F(\bar{x},y) \not\subset \bigcup_{x\in X}(F(x,y)+\operatorname{int} C)$. Then there exists $b \in F(\bar{x},\bar{y})$ such that

$$b \notin \bigcup_{x\in X}(F(x,\bar{y})+\operatorname{int} C). \tag{5.19}$$

Obviously, $b \in \{a\} + \operatorname{cone}(V')$ and then there exists $\hat{k} \in V^*$ and $\hat{t} \in \mathbb{R}_+$ such that $b = \hat{t}\hat{k} + a$. From (5.18) and (iii) of Proposition 4.1, there exists $x(\hat{k};a) \in X$ such that

$$\left(I^{(3)}_{(\hat{k},a)} \circ F\right)(x(\hat{k};a),\bar{y}) < \left(I^{(3)}_{(\hat{k},a)} \circ F\right)(\bar{x},\bar{y}) \le I^{(3)}_{(\hat{k},a)}(\{b\}) = \hat{t}.$$

Let $t_1 := \left(I^{(3)}_{(\hat{k},a)} \circ F\right)(x(\hat{k};a),\bar{y})$. Then there exists $\epsilon > 0$ such that

$$t_1 < \hat{t} - \epsilon < \hat{t} \quad \text{and} \quad (\hat{t}-\epsilon)\hat{k} + a \in F(x(\hat{k};a),\bar{y}) + C.$$

Since $\hat{k} \in \operatorname{int} C$, we obtain

$$\hat{t}\hat{k} + a + \operatorname{int}[-\epsilon\hat{k}, \epsilon\hat{k}] = b + \operatorname{int}[-\epsilon\hat{k}, \epsilon\hat{k}] \subset F(x(\hat{k}; a), \bar{y}) + C$$

and then $b \in F(x(\hat{k}; a), \bar{y}) + \operatorname{int} C$. This contradicts (5.19). Similarly, if (5.16) is singleton valued on $(\operatorname{int} C) \times Z \times Y$, it follows from (i) of Theorem 5.6 that we obtain

$$\mathrm{Min_p} F(X, y) = F\left(\bigcup_{(k,v) \in K} S(k; v; y),\, y \right).$$

This completes the proof. □

Lemma 5.11 **([29])** *Let A be a compact subset of X. Then the following statements hold:*

(i) *If F is C-closed valued and upper C-continuous on A then*

$$\mathrm{Min}_{(3)} F(A) \neq \emptyset,$$

where $F(A) := \{F(x) : x \in A\}$. In particular, $F(A) \subset \mathrm{Min}_{(3)} F(A) + C$.

(ii) *If F is $(-C)$-closed valued and upper $(-C)$-continuous on A then*

$$\mathrm{Max}_{(5)} F(A) \neq \emptyset.$$

In particular, $F(A) \subset \mathrm{Max}_{(5)} F(A) - C$.

Theorem 5.19 *Let A and B be nonempty compact convex subsets of X and Y, respectively. If $F : A \times B \to \wp(Z)$ satisfies that*

(i) *F is continuous and compact valued on $A \times B$,*

(ii) *for any $y \in B$, $F(\cdot, y)$ is type (3) natural quasi C-convex on A,*

(iii) *for any $x \in A$, $F(x, \cdot)$ is type (5) natural quasi C-concave on B,* *then,*

$$(\mathrm{Min}_{(3)}\mathrm{Max}_{\mathrm{p}}^{\mathrm{w}} F(x,y) + C) \cap (\mathrm{Max}_{(5)}\mathrm{Min}_{\mathrm{p}}^{\mathrm{w}} F(x,y) - C) \neq \emptyset.$$

Proof. For the proof of this theorem, we consider the following two set-valued maps:

$$S^*(y) := \{x \in A | F(x,y) \in \mathrm{Min}_{\mathrm{p}}^{\mathrm{w}} F(X,y)\},$$

$$T^*(x) := \{y \in B | F(x,y) \in \mathrm{Max}_{\mathrm{p}}^{\mathrm{w}} F(x,Y)\}.$$

We show these set-valued maps satisfy the following three conditions:

(a) S^* and T^* are nonempty valued on B and A, respectively,

(b) S^* is compact valued and upper continuous on B,

(c) T^* is compact valued and upper continuous on A.

(a): By Propositions 4.13 and 4.14, we obtain $I^{(3)}_{(k,v)} \circ F$ and $S^{(5)}_{(k,v)} \circ F$ are continuous on $A \times B$ for every $k \in \mathrm{int}\, C$ and $v \in Z$. Thus, (5.16) and (5.17) are nonempty on B and A, respectively. Hence, S^* and T^* are nonempty valued on B and A, respectively.

(b): By (i) of Lemma 5.10, for each $y \in B$ there exists a compact set $K \subset (\mathrm{int}\, C) \times Z$ such that

$$S^*(y) = \bigcup_{(k,v) \in K} S(k,v,y).$$

At first, we show that S^* is compact valued on B. Since B is compact, we prove S^* is closed valued on B. Assume that there exists $\bar{y} \in B$ such that $S^*(\bar{y})$ is not closed, that is, there exist a net $\{x_\alpha : \alpha \in J\} \subset S^*(\bar{y})$ and $\bar{x} \in A$ such that

$$x_\alpha \to \bar{x} \notin S^*(\bar{y})$$

where J is a directed set. By the definition of $S^*(\bar{y})$, for each x_α there exists $(k_\alpha, v_\alpha) \in K$ such that for any $x \in A$,

$$\left(I^{(3)}_{(k_\alpha,v_\alpha)} \circ F\right)(x_\alpha, \bar{y}) \leq \left(I^{(3)}_{(k_\alpha,v_\alpha)} \circ F\right)(x, \bar{y}).$$

Since K is compact, there exists $(\bar{k}, \bar{v}) \in K$ such that

$$(k_\alpha, v_\alpha) \to (\bar{k}, \bar{v}).$$

By the continuity of $I^{(3)}_{(k,v)} \circ F$ on A for every $y \in B$ and (i) of Lemma 5.3, we obtain

$$\left(I^{(3)}_{(\bar{k},\bar{v})} \circ F\right)(\bar{x}, \bar{y}) \leq \left(I^{(3)}_{(\bar{k},\bar{v})} \circ F\right)(x, \bar{y}),$$

for any $x \in A$. This contradicts $\bar{x} \notin S^*(\bar{y})$. Thus, S^* is compact valued on B. Next, we show S^* is upper continuous on B. Assume that there exists $y' \in B$ such that S^* is not upper continuous at y', that is, there exists an open set V' with $S^*(y') \subset V'$ such that for any open neighborhood U_α of y', there exists $y_\alpha \in U_\alpha$ such that

$$S^*(y_\alpha) \not\subset V'.$$

Hence, there exists $x_\alpha \in S^*(y_\alpha)$ such that $x_\alpha \in A \cap (V')^c$. Moreover, by the definition of $S^*(y_\alpha)$, for each x_α there exists $(k_\alpha, v_\alpha) \in K$ such that

$$\left(I^{(3)}_{(k_\alpha,v_\alpha)} \circ F\right)(x_\alpha, y_\alpha) = \min_{x \in A} \left(I^{(3)}_{(k_\alpha,v_\alpha)} \circ F\right)(x, y_\alpha). \tag{5.20}$$

Let $N := A \cap (V')^c$ and consider the net $\{(k_\alpha, v_\alpha, x_\alpha) : \alpha \in I\}$. Since $K \times N$ is compact, there exists $(\hat{k}, \hat{v}, \hat{x}) \in K \times N$ such that

$$(k_\alpha, v_\alpha, x_\alpha) \to (\hat{k}, \hat{v}, \hat{x}).$$

By (i) of Lemma 5.3, $I^{(3)}_{(k,v)} \circ F$ is continuous on $K \times N \times B$, and then it follows from (5.20) that

$$\left(I^{(3)}_{(\hat{k},\hat{v})} \circ F\right)(\hat{x}, y') = \min_{x \in A} \left(I^{(3)}_{(\hat{k},\hat{v})} \circ F\right)(x, y').$$

This is a contradiction to $\hat{x} \in N$. Hence, S^* is upper continuous on B.

(c): We can prove in a similar way to the proof of statement (b).

Let

$$S_0(y) := \{y\} \times \{x \in A | F(x,y) \in \mathrm{Min}_{\mathrm{p}}^{\mathrm{w}} F(A,y)\},$$

and

$$T_0(x) := \{x\} \times \{y \in B | F(x,y) \in \mathrm{Max}_{\mathrm{p}}^{\mathrm{w}} F(x,B)\}.$$

By (a), (b) and (c), S_0 and T_0 are nonempty compact valued, and upper continuous on B and A, respectively. Hence it follows from Lemmas 3.1 and 5.11 that we obtain

$$\mathrm{Min}_{(3)} \mathrm{Max}_{\mathrm{p}}^{\mathrm{w}} F(x,y) \neq \emptyset \quad \text{and} \quad \mathrm{Max}_{(5)} \mathrm{Min}_{\mathrm{p}}^{\mathrm{w}} F(x,y) \neq \emptyset. \tag{5.21}$$

On the other hand, by the continuity of $I^{(3)}_{(k,v)} \circ F$ and $S^{(5)}_{(k,v)} \circ F$ for every $k \in \operatorname{int} C$ and $v \in Z$, the following two statements hold:

(a)' (5.16) is nonempty closed valued and upper continuous on B,

(b)' (5.17) is nonempty closed valued and upper continuous on A.

Moreover, by Propositions 4.11 and 4.12, for each $k \in \operatorname{int} C$ and $v \in Z$, $I^{(3)}_{(k,v)} \circ F$ is quasiconvex on A for every $y \in B$ and $S^{(5)}_{(k,v)} \circ F$ is quasiconcave on B for every $x \in A$.

(c)' (5.16) is convex valued on B for every $k \in \operatorname{int} C$ and $v \in Z$,

(d)' (5.17) is convex valued on A for every $k \in \operatorname{int} C$ and $v \in Z$.

Therefore, it follows from (a)', (b)', (c)', and (d)' that we can apply Browder's coincidence theorem [10] to (5.16) and (5.17), that is, for every $k \in \operatorname{int} C$ and $v \in Z$ there exists $(x_0, y_0) \in A \times B$ such that

$$x_0 \in S(k; v; y_0) \quad \text{and} \quad y_0 \in T(k; v; x_0).$$

Thus,

$$F(x_0, y_0) \in \mathrm{Max}_{\mathrm{p}}^{\mathrm{w}} F(x_0, B) \cap \mathrm{Min}_{\mathrm{p}}^{\mathrm{w}} F(A, y_0).$$

By Lemma 5.11 and (5.21), we obtain

$$F(x_0, y_0) \subset (\mathrm{Min}_{(3)} \mathrm{Max}_{\mathrm{p}}^{\mathrm{w}} F(x,y) + C) \cap (\mathrm{Max}_{(5)} \mathrm{Min}_{\mathrm{p}}^{\mathrm{w}} F(x,y) - C).$$

This completes the proof. □

As a special case of Theorem 5.19, we obtain the following vector-valued minimax theorem.

Corollary 5.3 *Let A and B be nonempty compact convex subsets of X and Y, respectively. If $F : A \times B \to \wp(Z)$ satisfies that*

(i) *F is continuous and singleton valued on $A \times B$,*

(ii) *for any $y \in B$, $F(\cdot, y)$ is type (3) natural quasi C-convex on A,*

(iii) *for any $x \in A$, $F(x, \cdot)$ is type (5) natural quasi C-concave on B,*

then,

$$(\mathrm{Min}_{(3)}\mathrm{Max}^{\mathrm{w}}_{\mathrm{v}} F(x,y) + C) \cap (\mathrm{Max}_{(5)}\mathrm{Min}^{\mathrm{w}}_{\mathrm{v}} F(x,y) - C) \neq \emptyset.$$

Proof. By Theorem 5.19, we may show the following equalities hold:

$$\mathrm{Min}^{\mathrm{w}}_{\mathrm{v}} F(A,y) = \mathrm{Min}^{\mathrm{w}}_{\mathrm{p}} F(A,y) \quad \text{and} \quad \mathrm{Max}^{\mathrm{w}}_{\mathrm{v}} F(x,B) = \mathrm{Max}^{\mathrm{w}}_{\mathrm{p}} F(x,B).$$

$\mathrm{Min}^{\mathrm{w}}_{\mathrm{p}} F(A,y) \subset \mathrm{Min}^{\mathrm{w}}_{\mathrm{v}} F(A,y)$ is clear. We prove the converse. Take $F(\bar{x},y) \in \mathrm{Min}^{\mathrm{w}}_{\mathrm{v}} F(A,y)$. Then

$$(F(\bar{x},y) - (\mathrm{int}\, C \cup \{\theta_Z\})) \cap \left(\bigcup_{x \in X} F(x,y) \right) = \{F(\bar{x},y)\}.$$

By (i) of Theorem 5.5, $F(\bar{x},y) \in \mathrm{Min}^{\mathrm{w}}_{\mathrm{p}} F(A,y)$. Similarly, we obtain $\mathrm{Max}^{\mathrm{w}}_{\mathrm{v}} F(x,B) = \mathrm{Max}^{\mathrm{w}}_{\mathrm{p}} F(x,B)$, and the proof is completed. □

Remark 5.1 Let $f : X \times Y \to Z$. In Corollary 5.3, we present a vector-valued minimax theorem based on $\mathrm{Min}_{(3)}\mathrm{Max}^{\mathrm{w}}_{\mathrm{v}} f(x,y)$ and $\mathrm{Max}_{(5)}\mathrm{Min}^{\mathrm{w}}_{\mathrm{v}} f(x,y)$. In general, minimax and maximin values of f are defined as follows (see [16, 17, 67, 69]):

$$\mathrm{Min}_{\mathrm{v}} \bigcup_{x \in X} \mathrm{Max}^{\mathrm{w}}_{\mathrm{v}} f(x,y) \quad \text{and} \quad \mathrm{Max}_{\mathrm{v}} \bigcup_{y \in Y} \mathrm{Min}^{\mathrm{w}}_{\mathrm{v}} f(x,y).$$

The following simple example shows that

$$\mathrm{Max}_{(5)}\mathrm{Min}_{\mathrm{v}}^{\mathrm{w}} f(x,y) \neq \mathrm{Max}_{\mathrm{v}} \bigcup_{x\in X} \mathrm{Min}_{\mathrm{v}}^{\mathrm{w}} f(x,y).$$

Let $X := [0,1]$, $Y := [-2,2]$, $C := \left\{ \binom{x}{y} \in \mathbb{R}^2 \middle| 0 \le y \le x \right\}$. Then we define a vector-valued function $f : X \times Y \to \mathbb{R}^2$ by

$$f(x,y) := \begin{pmatrix} 0 \\ x \end{pmatrix} + \begin{pmatrix} y \\ y^2 \end{pmatrix}.$$

Then $\mathrm{Max}_{(5)}\mathrm{Min}_{\mathrm{v}}^{\mathrm{w}} f(x,y) = \{\mathrm{Min}_{\mathrm{v}}^{\mathrm{w}} f(x,y) | y \ge 1\}$ and

$$\mathrm{Max}_{\mathrm{v}} \bigcup_{x\in X} \mathrm{Min}_{\mathrm{v}}^{\mathrm{w}} f(x,y) = \{f(x,y)|y=-2\} \cup \{f(x,y)|1 \le y,\, x=0\}.$$

Hence $\mathrm{Max}_{(5)}\mathrm{Min}_{\mathrm{v}}^{\mathrm{w}} f(x,y) \neq \mathrm{Max}_{\mathrm{v}} \bigcup_{x\in X} \mathrm{Min}_{\mathrm{v}}^{\mathrm{w}} f(x,y)$. By a similar simple example, we can check that $\mathrm{Min}_{(3)}\mathrm{Max}_{\mathrm{v}}^{\mathrm{w}} f(x,y) \neq \mathrm{Min}_{\mathrm{v}} \bigcup_{y\in Y} \mathrm{Max}_{\mathrm{v}}^{\mathrm{w}} f(x,y)$.

Next, we introduce a strong minimax theorem for set-valued maps.

Theorem 5.20 *Let A and B be nonempty compact convex subsets of X and Y, respectively. If $F : A \times B \to \wp(Z)$ satisfies that*

(i) *F is continuous and compact valued on $A \times B$,*

(ii) *for any $y \in B$, $F(\cdot, y)$ is type (3) strictly natural quasi C-convex on A,*

(iii) *for any $x \in A$, $F(x, \cdot)$ is type (5) strictly natural quasi C-concave on B,*

then,

$$(\mathrm{Min}_{(3)}\mathrm{Max}_{\mathrm{p}} F(x,y) + C) \cap (\mathrm{Max}_{(5)}\mathrm{Min}_{\mathrm{p}} F(x,y) - C) \neq \emptyset.$$

Proof. By Theorem 5.16, it is easy to check that (5.16) is singleton valued on $(\mathrm{int}\, C) \times Z \times Y$. Accordingly, it follows from the proof of Theorem 5.19 that we obtain this theorem. □

By Corollary 5.3 and Theorem 5.20, we obtain the following corollary.

Corollary 5.4 *Let A and B be nonempty compact convex subsets of X and Y, respectively. If $F : A \times B \to \wp(Z)$ satisfies that*

(i) *F is continuous and singleton valued on $A \times B$,*

(ii) *for any $y \in B$, $F(\cdot, y)$ is type (3) strictly natural quasi C-convex on A,*

(iii) *for any $x \in A$, $F(x, \cdot)$ is type (5) strictly natural quasi C-concave on B,*

then,

$$(\mathrm{Min}_{(3)}\mathrm{Max}_{\mathrm{v}}F(x,y) + C) \cap (\mathrm{Max}_{(5)}\mathrm{Min}_{\mathrm{v}}F(x,y) - C) \neq \emptyset.$$

Chapter 6

Conclusion

In the thesis, we introduce some scalarization methods for sets containing with several ordinary scalarization methods for sets, and investigate those properties. Especially, the monotonicity and inheritance properties are very useful. In fact, almost of results in Chapter 5 can be proved by using these properties.

In Section 5.1, we present several optimality conditions for (SP) which are more generalizations of ordinary optimality conditions for (SP) based on scalarization. In particular, we propose new solution concepts for (SP) and investigate those optimality conditions. In Section 5.2, we show four types of Ky Fan inequality for set-valued maps with more simple proof than Georgiev and Tanaka type Ky Fan inequality for set-valued maps (see [19, 20]). In Section 5.3, we give several types of continuity of cone-convex set-valued maps. Especially, we propose a concept of local Lipschitz continuity of set-valued maps which is a natural extension of real-valued case, and show that any set-valued map has this property on $\mathbb{R}^n$ under some convexity assumptions. In Section 5.4, we propose four types of saddle point for set-valued map, and show sufficient conditions for the existence of them by using special scalarizing functions for sets. Our results do not necessary need to the continuity and the compactness of set-valued map on a product space. This means that our results are more generalizations of ordinary saddle point theorems for set-valued maps (see [11, 38, 52]). In Section 5.5, we consider two types of minimax and maximin values of set-valued map, respectively, and show minimax theorems for set-valued maps which are natural extensions of some vector-valued minimax theorems. In addition, our minimax theorems lead to more natural form of vector-valued minimax theorems than ordinary vector-valued minimax theorems.

In the thesis, we consider several scalarizing functions for sets and investigate some properties of them. Moreover, by using these properties we prove several results in set-valued analysis and optimization. These results show that scalarization methods are very useful to study set-valued analysis and optimization. The author believes that the study in the thesis will contribute to a further research in set-valued analysis and optimization theory.

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