

STUDY ON ALGEBRAIC AND TOPOLOGICAL STRUCTURES OF COMMUTATIVE BANACH ALGEBRAS

Dai Honma

Doctoral Program in Information Science and Engineering
Graduate School of Science and Technology

Niigata University

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Abstract

In Banach algebra theory, it is well-known that the algebraic structure of a commutative Banach algebra is closely related with the topological structure of the maximal ideal space of the algebra. Let $C(X)$ be the Banach algebra of all complex-valued continuous functions on a compact Hausdorff space X . In Chapter 1, we investigate solvability of the algebraic equations over $C(X)$ via topological structure of the maximal ideal space X . In Section 1, we consider the following property that $C(X)$ may have:

- (*) For every $f \in C(X)$ there exist $p, q \in \mathbb{N}$ such that $q/p \notin \mathbb{N}$ and the equation $z^p - f^q = 0$ has a solution in $C(X)$.

The condition (*) means that for every $f \in C(X)$ there exist $g \in C(X)$ and $p, q \in \mathbb{N}$ such that $q/p \notin \mathbb{N}$ and $g^p = f^q$. Our purpose is to characterize the algebraic structure (*) of $C(X)$ in terms of topological structure of the maximal ideal space X . We shall give such characterizations for the case that X is locally connected or first countable. We also give some non-trivial relations between (*) and algebraic closedness (and square-root closedness). In Section 2, we study n -th root closedness for $C(X)$.

In Chapter 2, we investigate sufficient conditions for a mapping between commutative Banach algebras to be an algebra isomorphism. Molnár [31] characterized algebra isomorphisms between Banach algebras of the type $C(X)$ by a spectrum-preserving property. The property is that the spectrum of the product of the images of any two elements is equal to the spectrum of the product of those two elements. Molnár also gave a similar characterization of algebra *-isomorphisms between Banach *-algebras of the type $C(X)$. The studies in this chapter are inspired by this result of Molnár. Let $C_0(X)$ be the Banach *-algebra of all complex-valued continuous functions on a locally compact Hausdorff space X which vanish at infinity.

In Section 1, we study peripheral spectrum version of Molnár Theorem and give a characterization of algebra $*$ -isomorphisms between algebras of the type $C_0(X)$. It is an extension of the result of Molnár and characterization for Banach algebras without unit. In Molnár's proof of the results mentioned above, it is crucial that the mappings preserve the order of the functions. The ideas of our proof is different of that of Molnár. In Section 2, we give a norm-preserving property as a sufficient condition for mappings between algebras of the type $C(X)$ to be algebra $*$ -isomorphisms. The characterization is also an extension of the result of Molnár.

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CHAPTER 1

On algebraic equations in algebras of continuous functions

1. On a characterization of compact Hausdorff space X for which certain algebraic equations are solvable in $C(X)$

1.1. Introduction. Let X be a compact Hausdorff space. We will denote by $C(X)$ the commutative Banach algebra of all complex-valued continuous functions on X with respect to the supremum norm $\|\cdot\|$. Let $P(x, z)$ be a monic polynomial over $C(X)$ with respect to z , that is, for a positive integer n and $f_0, f_1, \dots, f_{n-1} \in C(X)$,

$$P(x, z) = z^n + f_{n-1}(x)z^{n-1} + \dots + f_1(x)z + f_0(x)$$

for $x \in X$. When X is a singleton, say $\{x_0\}$, we may regard $C(\{x_0\})$ as the complex number field \mathbb{C} . It is well-known as the fundamental theorem of algebra that every monic polynomial equation with complex coefficients has a solution in \mathbb{C} . So, the complex number field \mathbb{C} is algebraically closed. We say that $C(X)$ is algebraically closed if and only if $P(x, z) = 0$ has a solution in $C(X)$ for every monic polynomial $P(x, z)$ over $C(X)$. That is, $C(X)$ is algebraically closed if and only if for every monic polynomial $P(x, z)$ over $C(X)$ there exists $g \in C(X)$ such that $P(x, g(x)) = 0$ for every $x \in X$. A natural question arises as whether $C(X)$ is algebraically closed.

Let X be a locally connected compact connected metric space. In 1959, Fort, Jr. ([10, Result 2]) essentially proved that if $C(X)$ is square-root closed, then X never contains simple closed arc. Note that $C(X)$ is said to be square-root closed if and only if for every $f \in C(X)$ there exists $g \in C(X)$ such that $g^2 = f$. In other words, $C(X)$ is square-root closed if and only if the quadratic equation $z^2 - f = 0$

has a solution in $C(X)$ for every $f \in C(X)$. By the definition, $C(X)$ is square-root closed whenever $C(X)$ is algebraically closed. From a result of Fort, Jr., we see that $C(S^1)$ is not square-root closed. Here S^1 denotes the unit circle.

Deckard and Percy ([7, 8]) proved that $C(X)$ is algebraically closed if X is a Stonian space, that is, a compact Hausdorff space such that the closure of every open set is open, or a totally disconnected compact Hausdorff space, or a linearly ordered and order complete topological space. To prove these, the fundamental tool is that for every $x_0 \in X$ the solutions of $P(x, z) = 0$ vary continuously with respect to x in some open neighbourhood of x_0 (see [7, Lemma 2.2]). Deckard and Percy ([8, Theorem 2]) also proved that $C([0, 1])$ is algebraically closed. Recall that $C(S^1)$ is not algebraically closed. So, it is interesting to give a characterization of X in order that $C(X)$ is algebraically closed. Deckard and Percy also mentioned that if X is the closure of the graph of the function $y = \sin 1/x$, $0 < x \leq 2\pi$, then $C(X)$ is not algebraically closed.

In 1966, Čirka ([4]) considered this problem from the other viewpoint. Let A be a uniform algebra on a locally connected compact Hausdorff space X . Čirka proved that if for every $f \in A$ there exists $g \in A$ such that $g^2 = f$, then $A = C(X)$. In 1967, Countryman, Jr. ([5]) gave a necessary and sufficient condition for a first countable compact Hausdorff space X in order that $C(X)$ is algebraically closed. For the precise statement, see Theorem B of this section. Roughly speaking, $C(X)$ is algebraically closed if and only if X never contains the unit circle S^1 nor the graph of the function $y = \sin 1/x$, $0 < x \leq 2\pi$. Moreover, this result states that $C(X)$ is algebraically closed if $C(X)$ is square-root closed. For locally connected X , Hatori and Miura ([13, Theorem 2.2]) gave a characterization of $C(X)$ to be square-root closed in terms of the covering dimension $\dim X$ and the first Čech cohomology

group $\check{H}^1(X; \mathbb{Z})$ of X with the integer coefficients. More explicitly, when X is locally connected the algebra $C(X)$ is square-root closed if and only if $\dim X \leq 1$ and $\check{H}^1(X; \mathbb{Z}) = 0$. Miura and Nijima ([29, Theorem 3.3]) proved that if X is locally connected, then square-root closedness implies algebraic closedness. Note, by definition, that the converse is valid for every compact Hausdorff space. So, algebraic closedness is equivalent to weaker property of square-root closedness whenever X is first countable or locally connected. Recently, Feinstein and Oliver ([9, Theorem 9]) characterized algebraic closedness of $C(X)$ in terms of the extendability of endomorphisms to Cole or Arens-Hoffman extensions (cf. [6]).

Miura ([28]) introduced a weaker property of “approximate” square-root closedness. To be more explicit, the algebra $C(X)$ is approximately square-root closed if and only if for every $f \in C(X)$ with $\|f\| \leq 1$ there exist $g, h \in C(X)$ such that $f = gh$ and $\|g - h\| \leq \varepsilon$, where $\varepsilon \geq 0$ is a given constant. Moreover, he proved that approximate square-root closedness was equivalent to square-root closedness when X is locally connected. Kawamura and Miura ([23, Theorem 1.3]) studied approximate n -th root closedness of $C(X)$ and proved that this property of $C(X)$ is equivalent to the n -divisibility of $\check{H}^1(X; \mathbb{Z})$ when $\dim X \leq 1$. Chigogidze, Karasev, Kawamura and Valov ([3]) investigated (commutative) C^* -algebras with approximate n -th root property.

Gorin and Karahanjan ([12]) studied k -th root version of Čirka Theorem. In 1979, Karahanjan ([22]) generalized a result of Čirka ([4]) in the following way. Let A be a uniform algebra on a locally connected compact Hausdorff space X . If, for every $f \in A$, there exist $g \in A$ and $p, q \in \mathbb{N}$, the set of all natural numbers, such that $q/p \notin \mathbb{N}$ and $g^p = f^q$, then $A = C(X)$. Here, we note that if we replace $q/p \notin \mathbb{N}$ with $q/p \in \mathbb{N}$ in the above, then the condition is obviously true for all A .

In this section, we are concerned with the following property that $C(X)$ may have:

- (*) For every $f \in C(X)$ there exist $g \in C(X)$ and $p, q \in \mathbb{N}$ such that $q/p \notin \mathbb{N}$ and $g^p = f^q$.

If $C(X)$ is square-root closed, then $C(X)$ has this property. In fact, for every $f \in C(X)$ there exists $g \in C(X)$ such that $g^2 = f$, that is, p and q are independent of a particular choice of $f \in C(X)$. Since $C(S^1)$ is not square-root closed, a natural question arises. Does the condition (*) holds for $C(S^1)$? We give a negative answer (Lemma 1.4) to this question. Moreover, when X is locally connected we give a necessary and sufficient condition for X in order that $C(X)$ have the property (*). As a corollary, we also prove that if X is locally connected, or first countable, then the condition (*) holds for $C(X)$ if and only if $C(X)$ is algebraically closed; In this case, (*) for $C(X)$ is equivalent to the square-root closedness of $C(X)$.

1.2. Results. Before stating our results, we need some terminologies and symbols.

We say that a topological space T is *hereditarily unicoherent* if $M \cap N$ is connected for every pair of closed and connected subsets M and N of T . For example, the unit circle S^1 is *not* hereditarily unicoherent.

We say that a topological space T is *almost locally connected* if T contains no mutually disjoint connected closed subsets C_n ($n \in \mathbb{N}$), which are open in the closure of $\cup_{n \in \mathbb{N}} C_n$ in T , with the following property: There exist $x_n, y_n \in C_n$ such that $\{x_n\}_{n \in \mathbb{N}}$ and $\{y_n\}_{n \in \mathbb{N}}$ converge to distinct points. For example, the closure of the graph of the function $y = \sin 1/x$, $0 < x \leq 2\pi$ is *not* almost locally connected.

Let Y be a normal space and n a non-negative integer. The covering dimension $\dim Y$ of Y is less than or equal to n if for every finite open covering \mathfrak{A} of Y there exists a refinement \mathfrak{B} of \mathfrak{A} such that each $y \in Y$ belongs to at most $(n+1)$ elements

of \mathfrak{B} . It is well-known that $\dim Y \leq n$ if and only if for every closed subset F of Y and every S^n -valued continuous function f on F , there exists an S^n -valued continuous function \tilde{f} on Y such that $\tilde{f}|_F = f$, where S^n is the n -sphere (cf. [32]).

Let X be a compact Hausdorff space. Then $\check{H}^1(X; \mathbb{Z})$ denotes the first Čech cohomology group of X with integer coefficients. Let $C(X)^{-1}$ be the multiplicative group of all invertible elements of $C(X)$ and $\exp C(X) = \{e^f : f \in C(X)\}$. It is well-known that $\check{H}^1(X; \mathbb{Z})$ is isomorphic to the quotient group $C(X)^{-1}/\exp C(X)$, by a theorem of Arens and Royden [11]. In particular, $\check{H}^1(X; \mathbb{Z})$ is *trivial* if and only if $C(X)^{-1} = \exp C(X)$.

Now we are ready to state our main result. The main result of this section is as follows:

THEOREM 1.1. *Let X be a locally connected compact Hausdorff space. Then the following conditions are equivalent.*

- (a) *For every $f \in C(X)$ there exist $g \in C(X)$ and $p, q \in \mathbb{N}$ such that $q/p \notin \mathbb{N}$ and $g^p = f^q$.*
- (b) *X is hereditarily unicoherent.*
- (c) *$\dim X \leq 1$ and $\check{H}^1(X; \mathbb{Z})$ is trivial.*
- (d) *$\{g^p : g \in C(X)\}$ is uniformly dense in $C(X)$ for every $p \in \mathbb{N}$.*
- (e) *For each $f \in C(X)$ and $p \in \mathbb{N}$ there exists $g \in C(X)$ such that $g^p = f$.*

COROLLARY 1.2. *Let X be a locally connected compact Hausdorff space. Then the following conditions are equivalent.*

- (a) *For every $f \in C(X)$ there exist $g \in C(X)$ and $p, q \in \mathbb{N}$ such that $q/p \notin \mathbb{N}$ and $g^p = f^q$.*
- (b) *$\{g^p : g \in C(X)\}$ is uniformly dense in $C(X)$ for every $p \in \mathbb{N}$.*
- (c) *For each $f \in C(X)$ and $p \in \mathbb{N}$ there exists $g \in C(X)$ such that $g^p = f$.*
- (d) *$C(X)$ is algebraically closed.*

- (e) $C(X)$ is square-root closed.
- (f) X is hereditarily unicoherent.
- (g) $\dim X \leq 1$ and $\check{H}^1(X; \mathbb{Z})$ is trivial.

COROLLARY 1.3. *Let X be a first countable compact Hausdorff space. Then each of the following conditions implies the other.*

- (a) *For each $f \in C(X)$ there exist $g \in C(X)$ and $p, q \in \mathbb{N}$ such that $q/p \notin \mathbb{N}$ and $g^p = f^q$.*
- (b) $C(X)$ is algebraically closed.
- (c) $C(X)$ is square-root closed.
- (d) X is hereditarily unicoherent and almost locally connected.
- (e) X is almost locally connected, $\dim X \leq 1$ and $\check{H}^1(X; \mathbb{Z})$ is trivial.

1.3. Lemmata. We require some lemmata before proving Theorem 1.1. To prove Lemma 1.4 and 1.5, we use ideas by Countryman, Jr. [5, Lemma 2.1, Lemma 2.3].

LEMMA 1.4. *Let X be a compact Hausdorff space. If the condition (a) of Theorem 1.1 holds, then X is hereditarily unicoherent.*

PROOF. Assume that the condition (a) holds. We will show that X is hereditarily unicoherent. Suppose not. Then, by definition, there exist non-empty closed connected subsets M and N of X such that $M \cap N$ is disconnected. So, there are non-empty closed subsets A and B such that $M \cap N = A \cup B$ and $A \cap B = \emptyset$. Let f be a continuous mapping from X into the closed unit interval $[0, 1]$ such that $f(x) = 0$ on A and $f(x) = 1$ on B . Put

$$h(x) = \begin{cases} \exp(i\pi f(x)) & (x \in M) \\ \exp(-i\pi f(x)) & (x \in N \setminus M). \end{cases}$$

Then we see that h is continuous on $M \cup N$. Let $\tilde{h} \in C(X)$ be a mapping so that $\tilde{h}|_{M \cup N} = h$. By the condition (a), there exist positive integers p, q and an element \tilde{g}

in $C(X)$ such that p does not divide q and $\tilde{h}^q = \tilde{g}^p$. Put $q = sp + r$, where s and r are integers with $1 \leq r \leq p - 1$ (note $q/p \notin \mathbb{N}$). Since h does not vanish on $M \cup N$, the function $g = \tilde{g}|_{M \cup N}/h^s$ is a well-defined continuous mapping from $M \cup N$ into \mathbb{C} . Since $\tilde{h}^q = \tilde{g}^p$, for each $x \in M \cup N$ we obtain

$$g^p(x) = \left(\frac{\tilde{g}(x)}{h^s(x)} \right)^p = \frac{\tilde{h}^q(x)}{h^{sp}(x)} = h^{q-sp}(x) = h^r(x),$$

and so $h^r = g^p$ on $M \cup N$. Since

$$g^p(x) = h^r(x) = \exp(i\pi r f(x))$$

for $x \in M$, we get

$$g(x) = \omega(x) \exp\left(\frac{i\pi r f(x)}{p}\right)$$

for every $x \in M$, where $\omega(x)$ is one of the p -th roots of 1. The above equation and the continuity of f and g imply that $\omega(x)$ is a continuous mapping from M into the set of all p -th roots of 1. Since M is connected, ω must be constant. So there is a p -th root ω_0 of 1 such that

$$(1) \quad g(x) = \omega_0 \exp\left(\frac{i\pi r f(x)}{p}\right)$$

for each x in M . In a way similar to the above, we see that there exists a p -th root γ_0 of 1 such that

$$(2) \quad g(x) = \gamma_0 \exp\left(-\frac{i\pi r f(x)}{p}\right)$$

for each x in N .

Pick $x_0 \in A$ arbitrarily. Since $x_0 \in A \subset M \cap N$, the equations (1) and (2) imply that

$$\omega_0 \exp\left(\frac{i\pi r f(x_0)}{p}\right) = g(x_0) = \gamma_0 \exp\left(-\frac{i\pi r f(x_0)}{p}\right).$$

Recall that $f = 0$ on A and $f = 1$ on B , and so $f(x_0) = 0$. We thus obtain $\omega_0 = \gamma_0$.

For $y \in B$, it follows from (1), (2) and $\omega_0 = \gamma_0$ that

$$\omega_0 \exp\left(\frac{i\pi r}{p}\right) = g(y) = \omega_0 \exp\left(-\frac{i\pi r}{p}\right),$$

because $B \subset M \cap N$. Thus we have $r/p \in \mathbb{N}$, which contradicts $1 \leq r < p - 1$. We conclude that X is hereditarily uncoherent. \square

LEMMA 1.5. *Let X be a compact Hausdorff space. If the condition (a) of Theorem 1.1 holds, then X is almost locally connected.*

PROOF. Assume that (a) holds and suppose that X is not almost locally connected. By definition, X contains mutually disjoint connected closed subsets C_n ($n \in \mathbb{N}$), which are open in $\overline{\cup_{n \in \mathbb{N}} C_n}$, the closure of $\cup_{n \in \mathbb{N}} C_n$ in X , with the following property: to each $n \in \mathbb{N}$ there correspond $x_n, y_n \in C_n$ such that $\{x_n\}_{n \in \mathbb{N}}$ and $\{y_n\}_{n \in \mathbb{N}}$ converge to distinct points, say x_0 and y_0 . Put $F = \cup_{n \in \mathbb{N}} C_n$. Since X is a compact Hausdorff space, there exist open neighborhoods A and B of x_0 and y_0 respectively such that $\bar{A} \cap \bar{B} = \emptyset$. Let f be a continuous mapping from X into the interval $[-1, 1]$ such that $f(x) = 1$ on \bar{A} and $f(x) = -1$ on \bar{B} . We consider the following mapping h from \bar{F} into \mathbb{C} :

$$h(x) = \begin{cases} f(x) + \frac{i}{n} (1 - f^2(x)) & (x \in C_n; n \text{ is even}) \\ f(x) - \frac{i}{n} (1 - f^2(x)) & (x \in C_n; n \text{ is odd}) \\ f(x) & (x \in \bar{F} \setminus F). \end{cases}$$

We see that $h \in C(\bar{F})$. Let $\tilde{h} \in C(X)$ be a mapping with $\tilde{h}|_{\bar{F}} = h$. Since the condition (a) of Theorem 1.1 is assumed to hold, there exist a continuous mapping $\tilde{g} \in C(X)$ and $p, q \in \mathbb{N}$ with $q/p \notin \mathbb{N}$ such that $\tilde{h}^q = \tilde{g}^p$ on X . Put $q = sp + r$, where s and r are integers with $1 \leq r \leq p - 1$ (note $q/p \notin \mathbb{N}$). Now we define a mapping g from \bar{F} into \mathbb{C} as follows:

$$g(x) = \begin{cases} \frac{\tilde{g}(x)}{h^s(x)} & (x \in \bar{F}, h(x) \neq 0) \\ 0 & (x \in \bar{F}, h(x) = 0). \end{cases}$$

Recall that $\tilde{h}|_{\bar{F}} = h$. Since $\tilde{h}^q = \tilde{g}^p$ on X , for each $x \in \bar{F}$ with $h(x) \neq 0$ we obtain

$$g^p(x) = \left(\frac{\tilde{g}(x)}{h^s(x)} \right)^p = \frac{\tilde{h}^q(x)}{h^{sp}(x)} = h^{q-sp}(x) = h^r(x),$$

and so $h^r(x) = g^p(x)$ whenever $x \in \bar{F}$, $h(x) \neq 0$. It follows that $g \in C(\bar{F})$ such that $h^r = g^p$ on \bar{F} .

Pick $n \in \mathbb{N}$ arbitrarily. By the definition of h , there is a continuous mapping θ_n on C_n such that $h(x) = |h(x)| \exp(i\theta_n(x))$ for every $x \in C_n$ and that $\theta_n(C_n) \subset [0, \pi]$ if n is even and $\theta_n(C_n) \subset [-\pi, 0]$ if n is odd. Since $h^r = g^p$ on \bar{F} , for each $x \in C_n$

$$g^p(x) = |h(x)|^r \exp(ir\theta_n(x)),$$

and so there is a p -th root $\omega_n(x)$ of 1 such that

$$g(x) = \omega_n(x) |h(x)|^{r/p} \exp\left(\frac{ir\theta_n(x)}{p}\right).$$

Since h , g and θ_n are continuous, $\omega_n(x)$ is a continuous mapping from C_n into the set of all p -th roots of 1. Furthermore, since C_n is connected, $\omega_n(x)$ must be constant, say ω_n . So,

$$(3) \quad g(x) = \omega_n |h(x)|^{r/p} \exp\left(\frac{ir\theta_n(x)}{p}\right) \quad (x \in C_n).$$

Since $\{x_n\}_{n \in \mathbb{N}}$ and $\{y_n\}_{n \in \mathbb{N}}$ converge to $x_0 \in A$ and $y_0 \in B$, respectively, we may assume that $\{x_n\}_{n \in \mathbb{N}} \subset A$ and $\{y_n\}_{n \in \mathbb{N}} \subset B$. Recall that $f = 1$ on \bar{A} and $f = -1$ on \bar{B} . So, we get $h(x_n) = 1$ and $h(y_n) = -1$ for every $n \in \mathbb{N}$. Since $\theta_{2n}(C_{2n}) \subset [0, \pi]$ and $\theta_{2n-1}(C_{2n-1}) \subset [-\pi, 0]$ for every $n \in \mathbb{N}$, it follows from the equation $h(x) = |h(x)| \exp(i\theta_n(x))$ that $\theta_n(x_n) = 0$, $\theta_{2n}(y_{2n}) = \pi$ and $\theta_{2n-1}(y_{2n-1}) = -\pi$ for every $n \in \mathbb{N}$. It follows from (3) that $g(x_n) = \omega_n$ converges to $g(x_0)$. On the other hand, since $g(y_n)$ converges to $g(y_0)$, we see from (3) that both $g(y_{2n}) = \omega_{2n} \exp(ir\pi/p)$ and $g(y_{2n-1}) = \omega_{2n-1} \exp(-ir\pi/p)$ converge to $g(y_0)$. That is,

$$g(x_0) \exp\left(\frac{ir\pi}{p}\right) = g(y_0) = g(x_0) \exp\left(\frac{-ir\pi}{p}\right).$$

Since $|g(x_0)| = |h(x_0)|^{r/p} = |f(x_0)|^{r/p} = 1$, we see that $\exp(ir\pi/p) = \exp(-ir\pi/p)$. In other words, $r/p \in \mathbb{N}$, which contradicts $1 \leq r \leq p-1$. We thus conclude that X is almost locally connected. \square

The following results, Lemma 1.6 and 1.7 are deduced from [29, Theorem 3.3]; Moreover, Lemma 1.7 is well-known (cf. [25, Chap.VIII §57 Section III, Theorem 3, p.438]). Here we give a proof for the sake of completeness.

LEMMA 1.6. *Let X be a locally connected compact Hausdorff space. If X is hereditarily unicoherent, then $\dim X \leq 1$.*

PROOF. Let $\mathfrak{A} = \{O_k\}_{k=1}^n$ be a finite open covering of X . We show that there is an open refinement \mathfrak{B} for \mathfrak{A} such that every $x \in X$ is in at most two elements of \mathfrak{B} . Since X is assumed to be locally connected, it follows from [29, Lemma 3.2] that X is an A-space, that is, the class of all open sets whose boundaries are finite sets forms an open base. Without loss of generality we may assume that each O_k has at most finitely many boundary points. Put $B = \bigcup_{k=1}^n (\overline{O_k} \setminus O_k)$, where $\overline{}$ denotes the closure in X . We define a mutually disjoint open family $\{V_k\}_{k=1}^n$ as follows:

$$V_1 = O_1 \setminus B \quad \text{and} \quad V_k = O_k \setminus \left(B \cup \bigcup_{j=1}^{k-1} \overline{O_j} \right) \quad \text{for } k = 2, 3, \dots, n.$$

Since $\{O_k\}_{k=1}^n$ is an open covering of X , we see that $\bigcup_{k=1}^n V_k = X \setminus B$.

Since B consists of at most finitely many points, to each $x \in B$ there corresponds an open neighborhood U_x of x with the following property: $U_x \subset O_k$ for some k and $U_x \cap U_y = \emptyset$ whenever $x, y \in B$, $x \neq y$. Put $\mathfrak{B} = \{V_k\}_{k=1}^n \cup \{U_x : x \in B\}$. We see that \mathfrak{B} is an open covering of X . Recall that both $\{V_k\}_{k=1}^n$ and $\{U_x : x \in B\}$ are mutually disjoint. This implies that if $x \in X$, then at most two elements of \mathfrak{B} contain x . So, we get $\dim X \leq 1$. \square

LEMMA 1.7. *Let X be a locally connected compact Hausdorff space. If X is hereditarily unicoherent, then $\check{H}^1(X, \mathbb{Z})$ is trivial.*

PROOF. Assume that X is hereditarily unicoherent. By a theorem of Arens and Royden, it is enough to show that the equality $C(X)^{-1} = \exp C(X)$ holds.

Since $\exp C(X) \subset C(X)^{-1}$, it suffices to prove that $C(X)^{-1} \subset \exp C(X)$. To do this, pick $f \in C(X)^{-1}$ arbitrarily. Since X is locally connected, each connected component of X is open. It follows that X has at most finitely many connected components. Without loss of generality, we may assume that X is connected. Recall that $f \in C(X)^{-1}$, and so f vanishes nowhere. Since X is locally connected, for each x in X there exists a connected open neighborhood V_x of x and a continuous mapping g_x from the closure $\overline{V_x}$ of V_x into \mathbb{C} such that $f = e^{g_x}$ on $\overline{V_x}$. Since X is compact, there are finite number of points x_1, x_2, \dots, x_{n+1} such that $\bigcup_{k=1}^{n+1} V_{x_k} = X$. For simplicity, we denote $g_k = g_{x_k}$ and $V_k = V_{x_k}$ for $k = 1, 2, \dots, n+1$. Note that $\{\overline{V_k}\}_{k=1}^{n+1}$ is a class of non-empty connected closed sets with $\bigcup_{k=1}^{n+1} \overline{V_k} = X$. Since X is connected, $\overline{V_1}$ intersects at least one of $\overline{V_2}, \overline{V_3}, \dots, \overline{V_{n+1}}$; we may assume that $\overline{V_1}$ meets $\overline{V_2}$. Then $e^{g_1} = f = e^{g_2}$ on $\overline{V_1} \cap \overline{V_2}$, and so we have $e^{g_1 - g_2} = 1$ on $\overline{V_1} \cap \overline{V_2}$. Since X is hereditarily unicoherent, $\overline{V_1} \cap \overline{V_2}$ is connected. Hence by the continuity of $g_1 - g_2$, the equation $e^{g_1 - g_2} = 1$ implies the existence of an integer k_1 such that

$$g_1 - g_2 = 2k_1\pi i \quad \text{on} \quad \overline{V_1} \cap \overline{V_2}.$$

We define a mapping \tilde{g}_1 from $\overline{V_1} \cup \overline{V_2}$ into \mathbb{C} as follows:

$$\tilde{g}_1(x) = \begin{cases} g_1(x) & (x \in \overline{V_1}) \\ g_2(x) + 2k_1\pi i & (x \in \overline{V_2} \setminus \overline{V_1}). \end{cases}$$

It is easy to see that \tilde{g}_1 is continuous on $\overline{V_1} \cup \overline{V_2}$ and

$$f = e^{\tilde{g}_1} \quad \text{on} \quad \overline{V_1} \cup \overline{V_2}.$$

In the same way, $\overline{V_1} \cup \overline{V_2}$ intersects at least one of $\overline{V_3}, \overline{V_4}, \dots, \overline{V_{n+1}}$. We may assume that $\overline{V_1} \cup \overline{V_2}$ meets $\overline{V_3}$. The equation $e^{\tilde{g}_1} = f = e^{g_3}$ holds on $(\overline{V_1} \cup \overline{V_2}) \cap \overline{V_3}$, and so $e^{\tilde{g}_1 - g_3} = 1$ on $(\overline{V_1} \cup \overline{V_2}) \cap \overline{V_3}$. Since X is hereditarily unicoherent, $(\overline{V_1} \cup \overline{V_2}) \cap \overline{V_3}$ is connected. Hence by the continuity of $\tilde{g}_1 - g_3$, there exists an integer k_2 such that

$$\tilde{g}_1 - g_3 = 2k_2\pi i \quad \text{on} \quad (\overline{V_1} \cup \overline{V_2}) \cap \overline{V_3}.$$

We define a mapping \tilde{g}_2 from $(\overline{V_1} \cup \overline{V_2}) \cup \overline{V_3}$ into \mathbb{C} as follows: If x is in $\overline{V_1} \cup \overline{V_2}$, let $\tilde{g}_2(x) = \tilde{g}_1(x)$, and let $\tilde{g}_2(x) = g_3(x) + 2k_2\pi i$ otherwise. It is easy to see that \tilde{g}_2 is continuous on $\overline{V_1} \cup \overline{V_2} \cup \overline{V_3}$ and

$$f = e^{\tilde{g}_2} \quad \text{on} \quad \overline{V_1} \cup \overline{V_2} \cup \overline{V_3}.$$

Continuing this process, we have a continuous mapping \tilde{g}_n on $\cup_{k=1}^{n+1} \overline{V_k}$ such that

$$f = e^{\tilde{g}_n} \quad \text{on} \quad \bigcup_{k=1}^{n+1} \overline{V_k}$$

Since $\cup_{k=1}^{n+1} \overline{V_k} = X$, we have that $f \in \exp C(X)$. Since $f \in C(X)^{-1}$ was arbitrary, we conclude that $C(X)^{-1} \subset \exp C(X)$ and the proof is complete. \square

LEMMA 1.8. *Let X be a compact Hausdorff space. If $\dim X \leq 1$ and $\check{H}^1(X; \mathbb{Z})$ is trivial, then $\{g^p : g \in C(X)\}$ is uniformly dense in $C(X)$ for every $p \in \mathbb{N}$.*

PROOF. Pick $p \in \mathbb{N}$ and $f \in C(X)$ arbitrarily. We show that for every $\varepsilon > 0$ there exists $g \in C(X)$ such that $\|f - g^p\| < \varepsilon$. Without loss of generality we may assume that $\|f\| \leq 1$. Choose $k \in \mathbb{N}$ so that $2^p/\varepsilon^p < k$. Then put

$$E_k = \left\{ x \in X : |f(x)| \geq \frac{1}{k} \right\}.$$

Since $\dim X \leq 1$, there exists $u \in C(X)^{-1}$ with $|u| = 1$ on X such that $u = f/|f|$ on E_k . Then $\tilde{u}(x) = \max\{|f(x)|, 1/k\}u(x)$ is in $C(X)^{-1}$ with $\tilde{u}| = f$ on E_k . Since $\check{H}^1(X; \mathbb{Z})$ is trivial, by a theorem of Arens and Royden there exists $v \in \exp C(X)$ such that $\tilde{u} = v^p$. We define two mappings g and h as follows :

$$\begin{aligned} g(x) &= \frac{\sqrt[p]{|f(x)|} v(x)}{|v(x)|} \quad (x \in X), \\ h(x) &= \begin{cases} 0 & (f(x) = 0) \\ \frac{f(x)}{g(x)^{p-1}} & (f(x) \neq 0). \end{cases} \end{aligned}$$

Then we see that $g, h \in C(X)$, $\|g\| \leq 1$ and $f = g^{p-1}h$. Since $f(= \tilde{u}) = v^p$ on E_k , we see that $g = v = h$ on E_k . Therefore

$$\begin{aligned} \|g - h\| &= \sup\{|g(x) - h(x)| : x \in X \setminus E_k\} \\ &\leq 2 \sup\left\{\sqrt[p]{|f(x)|} : x \in X \setminus E_k\right\} \leq 2 \left(\frac{1}{k}\right)^{1/p} < \varepsilon. \end{aligned}$$

Since $f = g^{p-1}h$ and $\|g\| \leq 1$, it follows that

$$\|f - g^p\| = \|g^{p-1}h - g^p\| \leq \|g^{p-1}\| \|h - g\| < \varepsilon.$$

This completes the proof. \square

The case where $p = 2$ in Lemma 1.9 was essentially proved in [1, Corollary 5.9]. Here, we generalize the result to the case where $p \geq 2$.

LEMMA 1.9. *Let X be a locally connected compact Hausdorff space and $p \in \mathbb{N}$ with $p \geq 2$. If $\{f_n^p\}_{n \in \mathbb{N}} \subset C(X)$ converges uniformly to $f \in C(X)$, then there is a Cauchy subsequence of $\{f_n\}_{n \in \mathbb{N}}$.*

PROOF. For each $k \in \mathbb{N}$, set

$$E(k) = \left\{x \in X : |f(x)| > \frac{1}{k}\right\}.$$

Note that the closure $\overline{E(k)}$ of $E(k)$ in X is a compact subset of $E(2k)$. Since X is locally connected, each connected component of $E(2k)$ is open. So, there are finitely many connected components $C(k, 1), C(k, 2), \dots, C(k, N_k)$ such that $C(k, j) \cap E(k) \neq \emptyset$ for each j , $1 \leq j \leq N_k$ and that

$$(4) \quad E(k) \subset \bigcup_{j=1}^{N_k} C(k, j) \subset E(2k).$$

Pick $x_{k,j} \in C(k, j) \cap E(k)$ for each $k \in \mathbb{N}$ and j , $1 \leq j \leq N_k$. By a diagonal argument, we obtain a subsequence of $\{f_n\}_{n \in \mathbb{N}}$ converging at each point $x_{k,j}$, which we denote by the same letter $\{f_n\}_{n \in \mathbb{N}}$. We show that $\{f_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence

in $C(X)$. Put $\omega_l = \exp(2l\pi i/p)$ for $l = 0, 1, 2, \dots, p-1$. Fix $k \in \mathbb{N}$ arbitrarily. We define $\varepsilon(k)$ as follows:

$$(5) \quad \varepsilon(k) = \min \left\{ \frac{1}{2k} - \left(\frac{1}{2k} \right)^p, \left(\frac{1}{4k} |\omega_1 - 1| \right)^p \right\}.$$

Since $\lim_{n \rightarrow \infty} \|f_n^p - f\| = 0$ and since $\{f_n\}$ converges at each point $x_{k,j}$, we have, for a sufficiently large $n(k) \in \mathbb{N}$,

$$(6) \quad \|f_n^p - f_m^p\| < \varepsilon(k),$$

$$(7) \quad \|f_n^p - f\| < \varepsilon(k),$$

$$(8) \quad |f_n(x_{k,j}) - f_m(x_{k,j})| < \varepsilon(k)^{1/p}$$

for $n, m \geq n(k)$ and $j = 1, 2, \dots, N_k$. Fix $n, m \geq n(k)$ and $x \in E(2k)$ arbitrarily.

Since

$$f_n^p(x) - f_m^p(x) = \prod_{l=0}^{p-1} (f_n(x) - \omega_l f_m(x)),$$

it follows from (6) that there exists l with $0 \leq l \leq p-1$ such that the inequality

$$(9) \quad |f_n(x) - \omega_l f_m(x)| < \varepsilon(k)^{1/p}$$

holds. To prove the uniqueness of such l , suppose that there exists another l' , $l \neq l'$ such that the equation (9) is valid for l' in place of l . We get

$$\begin{aligned} |\omega_l - \omega_{l'}| |f_m(x)| &\leq |\omega_l f_m(x) - f_n(x)| + |f_n(x) - \omega_{l'} f_m(x)| \\ &< 2\varepsilon(k)^{1/p} \leq \frac{1}{2k} |\omega_1 - 1|, \end{aligned}$$

and so

$$(10) \quad |\omega_l - \omega_{l'}| |f_m(x)| < \frac{1}{2k} |\omega_1 - 1|.$$

On the other hand, since $x \in E(2k)$, the inequality (7) implies that

$$|f_m(x)|^p \geq |f(x)| - |f(x) - f_m^p(x)| > \frac{1}{2k} - \varepsilon(k) \geq \left(\frac{1}{2k} \right)^p.$$

It follows that

$$|\omega_l - \omega_{l'}| |f_m(x)| \geq |\omega_1 - 1| |f_m(x)| \geq \frac{1}{2k} |\omega_1 - 1|,$$

which contradicts (10). Hence the uniqueness is proved.

Since $x \in E(2k)$ was arbitrary, we have proved that to each $x \in E(2k)$ there corresponds a unique l such that (9) holds. This implies that if we define

$$G_l(k) = \{x \in E(2k) : |f_n(x) - \omega_l f_m(x)| < \varepsilon(k)^{1/p}\}$$

for $l = 0, 1, \dots, p-1$, then $\{G_l(k)\}_{l=0}^{p-1}$ is a mutually disjoint family with $E(2k) = \bigcup_{l=0}^{p-1} G_l(k)$. Since $G_l(k)$ is open for $l = 0, 1, 2, \dots, p-1$, each connected component of $E(2k)$ is contained in a unique $G_l(k)$. By the inequality (8), we get $x_{k,j} \in G_0(k)$ for $j = 1, 2, \dots, N_k$. Hence $C(k, j) \subset G_0(k)$ for $j = 1, 2, \dots, N_k$. By the definition of $G_l(k)$, it follows from (4) that

$$(11) \quad |f_n(x) - f_m(x)| < \varepsilon(k)^{1/p}$$

for every $x \in E(k)$. If $x \in X \setminus E(k)$, then we see from (7) that

$$|f_n(x)|^p \leq |f(x)| + \varepsilon(k) < \frac{1}{k} + \frac{1}{2k} < \frac{2}{k}.$$

Thus, we have that

$$(12) \quad |f_n(x) - f_m(x)| \leq |f_n(x)| + |f_m(x)| < 2 \left(\frac{2}{k} \right)^{1/p}$$

for every $x \in X \setminus E(k)$. It follows from (5), (11) and (12) that

$$\|f_n - f_m\| \leq 2 \left(\frac{2}{k} \right)^{1/p}.$$

Since $k \in \mathbb{N}$ and $n, m > n(k)$ are arbitrary, $\{f_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $C(X)$.

□

Although Lemma 1.10 and 1.11 are well-known (cf. [25, Chap.VIII §57 Section I, Theorem 8, p.435] and [25, Chap.VIII §46 Section XI, Theorem 2, p.165], respectively), for the sake of completeness we give a proof.

LEMMA 1.10. *Let X be a compact Hausdorff space. Then the following conditions are equivalent.*

- (a) $\check{H}^1(X; \mathbb{Z})$ is trivial.

(b) For each connected component X_λ of X , $\check{H}^1(X_\lambda; \mathbb{Z})$ is trivial.

For a compact Hausdorff space X , it is well-known that

(#) Every connected component X_λ of X is the intersection of all clopen sets G_μ of X such that $X_\lambda \subset G_\mu$.

We can prove the following as an application of (#).

(b) If O is open with $X_\lambda \subset O$ for some connected component X_λ of X , then there is clopen G such that $X_\lambda \subset G \subset O$.

In fact, if $G_\mu \supset X_\lambda$ is clopen with $\bigcap_{\mu \in I} G_\mu = X_\lambda$, then $\{X \setminus G_\mu\}_{\mu \in I}$ becomes an open covering of the closed subset $X \setminus O$, and so $X \setminus O \subset \bigcup_{i=1}^n (X \setminus G_{\mu_i})$ for some $\mu_1, \mu_2, \dots, \mu_n \in I$. Then the clopen $\bigcap_{i=1}^n G_{\mu_i}$ satisfies $X_\lambda \subset \bigcap_{i=1}^n G_{\mu_i} \subset O$.

PROOF OF LEMMA 1.10. First we show that (a) implies (b). Suppose that (a) is true. Let X_λ be an arbitrary connected component of X . It is enough to show that $C(X_\lambda)^{-1} = \exp C(X_\lambda)$ by a theorem of Arens-Royden. Since $\exp C(X_\lambda) \subset C(X_\lambda)^{-1}$, we show that $C(X_\lambda)^{-1} \subset \exp C(X_\lambda)$. Pick $f \in C(X_\lambda)^{-1}$ arbitrary. By the Tietze extension theorem, there exists a continuous extension \tilde{f} of f to all of X . Continuity of \tilde{f} implies that \tilde{f} does not vanish on a certain open set O that contains X_λ . Therefore, combining with the condition (b), we obtain a clopen set G which satisfies that $X_\lambda \subset G \subset O$. Now we define a mapping \mathfrak{F} from X into \mathbb{C} as follows: Let $\mathfrak{F}(x) = \tilde{f}(x)$ if $x \in G$, and $\mathfrak{F}(x) = 1$ otherwise. Then we see that $\mathfrak{F} \in C(X)^{-1}$ with $\mathfrak{F} = f$ on X_λ . Because $\check{H}^1(X; \mathbb{Z})$ is assumed to be trivial, there exists $g \in C(X)$ such that $\mathfrak{F} = \exp g$. It follows that $f = \exp(g|_{X_\lambda})$. Thus we see that $f \in C(X_\lambda)^{-1}$. Since f was arbitrary, we conclude that $C(X_\lambda)^{-1} \subset \exp C(X_\lambda)$.

Next we show that (b) implies (a). Suppose that (b) is true. It is enough to show that $C(X)^{-1} \subset \exp C(X)$. Pick $\tilde{f} \in C(X)^{-1}$ arbitrarily. Since (b) is true, to every connected component X_λ of X , the equation $C(X_\lambda)^{-1} = \exp C(X_\lambda)$ holds. Thus to

each λ , there corresponds $g_\lambda \in C(X_\lambda)$ such that $\tilde{f}|_{X_\lambda} = \exp g_\lambda$ holds. Let \tilde{g}_λ be a continuous extension of g_λ to the whole space X . If we put $\tilde{h}_\lambda = \tilde{f}/\exp \tilde{g}_\lambda$ on X , then $\tilde{h}_\lambda = 1$ on X_λ . Continuity of \tilde{h}_λ implies that there exists an open neighborhood $O_\lambda \supset X_\lambda$ such that $\tilde{h}_\lambda(O_\lambda) \subset \{z \in \mathbb{C} : |z - 1| < 1/2\}$. Therefore, combining with (‡), we obtain a clopen set G_λ which satisfies $X_\lambda \subset G_\lambda \subset O_\lambda$. Since $\tilde{h}_\lambda(G_\lambda) \subset \{z \in \mathbb{C} : |z - 1| < 1/2\}$, a continuous logarithm \log from $\{z \in \mathbb{C} : |z - 1| < 1/2\}$ into \mathbb{C} is well-defined. So, we get

$$\tilde{f} = \tilde{h}_\lambda \exp \tilde{g}_\lambda = \exp(\tilde{g}_\lambda + \log \tilde{h}_\lambda) \quad \text{on} \quad G_\lambda.$$

Since $\{G_\lambda\}_\lambda$ is an open covering of the compact space X , this covering has a finite open subcovering $\{G_{\lambda_k}\}_{k=1}^n$. The corresponding mappings to G_k are denoted by \tilde{g}_k, \tilde{h}_k ($k = 1, \dots, n$). Since every member of this covering is clopen, without loss of generality, we may assume that $G_{\lambda_{k_1}} \cap G_{\lambda_{k_2}} = \emptyset$ ($k_1 \neq k_2$). Now we define a mapping \tilde{g} from X into \mathbb{C} as follows. If $x \in X$, then there exists a unique k such that $x \in G_k$; Let $\tilde{g}(x) = \tilde{g}_k(x) + \log \tilde{h}_k(x)$. Then we see that $\tilde{g} \in C(X)$ and $\tilde{f} = \exp \tilde{g}$. Thus we conclude that $C(X)^{-1} \subset \exp C(X)$ and this completes the proof. \square

LEMMA 1.11. *Let X be a compact Hausdorff space. Then the following conditions are equivalent.*

- (a) $\dim X \leq 1$.
- (b) *For each connected component X_λ of X , $\dim X_\lambda \leq 1$.*

PROOF. A proof of (a) \Rightarrow (b) is elementary and omitted (cf. [32]).

Conversely, suppose that (b) is true. Let F be a closed subset of X and f an S^1 -valued continuous mapping on F . We show that there exists an S^1 -valued continuous mapping \tilde{f} on X such that $\tilde{f}|_F = f$. Let X_λ be a connected component of X . Since $\dim X_\lambda \leq 1$, there exists an S^1 -valued continuous extension g_λ of $f|_{F \cap X_\lambda}$ to X_λ . We define a mapping h_λ from $F \cup X_\lambda$ into \mathbb{C} as follows: Let $h_\lambda(x) = g_\lambda(x)$ if $x \in X_\lambda$,

and $h_\lambda(x) = f(x)$ if $x \in F \setminus X_\lambda$. Then we see that h_λ is an S^1 -valued continuous mapping on $F \cup X_\lambda$ satisfying $h_\lambda = f$ on F . Let \tilde{h}_λ be a continuous extension of h_λ to all of X . By definition, $|\tilde{h}_\lambda| = |h_\lambda| = 1$ on F . Continuity of \tilde{h}_λ implies that there exists an open neighborhood O_λ of X_λ such that \tilde{h}_λ never vanishes on O_λ . Therefore, combined with (‡), there exists a clopen set G_λ such that $X_\lambda \subset G_\lambda \subset O_\lambda$. Thus \tilde{h}_λ never vanishes on G_λ . Since $\{G_\lambda\}_\lambda$ is an open covering of the compact space X , $\{G_\lambda\}_\lambda$ has a finite subcovering $\{G_{\lambda_k}\}_{k=1}^n$ for X . Since every G_{λ_k} is clopen, without loss of generality, we may assume that $G_{\lambda_{k_1}} \cap G_{\lambda_{k_2}} = \emptyset$ ($k_1 \neq k_2$). Now we define a mapping \tilde{f} on X as follows. If $x \in X$, then there exists a unique k such that $x \in G_{\lambda_k}$. We put $\tilde{f}(x) = \tilde{h}_{\lambda_k}(x) / |\tilde{h}_{\lambda_k}(x)|$. Since $h_{\lambda_k} = f$ on F for every k , we see that \tilde{f} is an S^1 -valued continuous mapping on X such that $\tilde{f}|_F = f$ and this completes the proof. \square

1.4. Proof of Results.

PROOF OF THEOREM 1.1. (a) \Rightarrow (b) By Lemma 1.4. (b) \Rightarrow (c) By Lemma 1.6 and 1.7. (c) \Rightarrow (d) By Lemma 1.8. (e) \Rightarrow (a) By definition.

(d) \Rightarrow (e) Suppose that $\{g^p : g \in C(X)\}$ is uniformly dense in $C(X)$ for every $p \in \mathbb{N}$. Pick $f \in C(X)$ and $p \in \mathbb{N}$ arbitrarily. By hypothesis, there exists a sequence $\{g_n^p\}_{n \in \mathbb{N}}$ such that g_n^p converges to f as $n \rightarrow \infty$. By Lemma 1.9, there is a Cauchy subsequence $\{g_{n_j}\}_{j \in \mathbb{N}}$ of $\{g_n\}_{n \in \mathbb{N}}$. Since $C(X)$ is complete, there exists $g \in C(X)$ such that g_{n_j} converges to g as $j \rightarrow \infty$. It follows that $f = g^p$ and the proof is complete. \square

REMARK. Let us consider the following two conditions.

(d') $\{g^p : g \in C(X)\}$ is uniformly dense in $C(X)$ for some $p \in \mathbb{N}$ with $p \geq 2$.

(e') There exists $p \in \mathbb{N}$, $p \geq 2$ with the following property: For each $f \in C(X)$ there is $g \in C(X)$ such that $g^p = f$.

Then the implications (e) of Theorem 1.1 \Rightarrow (e') \Rightarrow (d') are obviously true. If, in addition, X is locally connected, then (d') with Lemma 1.8 implies that every element in $C(X)$ is the p -th power of another. So, we get (d') \Rightarrow (e'). Consequently, both (d') and (e') are also equivalent to all of the conditions from (a) to (e) of Theorem 1.1 whenever X is locally connected. Note that Kawamura and Miura [23, Theorem 1.3] proved that if X is a compact Hausdorff space with $\dim X \leq 1$, then the condition (d') above is equivalent to that $\check{H}^1(X; \mathbb{Z})$ is p -divisible.

It is well-known [29, Theorem 3.3] that if X is locally connected, then $C(X)$ is algebraically closed if and only if $C(X)$ is square root closed as is stated in the following theorem.

THEOREM A ([29]). *Let X be a locally connected compact Hausdorff space. Then the following conditions are equivalent.*

- (1) $C(X)$ is algebraically closed.
- (2) $C(X)$ is square-root closed.
- (3) $\dim X \leq 1$ and $\check{H}^1(X; \mathbb{Z})$ is trivial.
- (4) X is hereditarily unicoherent.

PROOF OF COROLLARY 1.2. This is just an application of Theorem 1.1 and Theorem A. □

If X is first countable, then we see that the condition (a) of Theorem 1.1 holds if and only if $C(X)$ is algebraically closed. To prove this, we need the following result, which was essentially proved by Countryman, Jr. [5] (see also [29]).

THEOREM B ([5, 29]). *Let X be a first countable compact Hausdorff space. Then the following conditions are equivalent.*

- (1) $C(X)$ is algebraically closed.

- (2) $C(X)$ is square-root closed.
- (3) X is almost locally connected and hereditarily unicoherent.
- (4) X is almost locally connected and for every connected component X_λ of X , X_λ is locally connected, $\dim X_\lambda \leq 1$ and $\check{H}^1(X_\lambda; \mathbb{Z})$ is trivial.

PROOF OF COROLLARY 1.3. (b) \Leftrightarrow (c) \Leftrightarrow (d): By Theorem B, each of the conditions (b), (c) and (d) implies the other.

(a) \Rightarrow (d): It follows from Lemma 1.4 and 1.5 that (a) implies (d).

(c) \Rightarrow (a): It is obvious that (c) implies (a).

Finally, we show that (e) is equivalent to the condition (4) of Theorem B. It follows from Lemma 1.10 and 1.11 that (4) of Theorem B implies (e). Conversely, we prove that (e) implies (4) of Theorem B. By [5, Proof of Lemma 2.5], we see that each connected component X_λ of X is locally connected. It follows from Lemma 1.10 and 1.11 that (e) implies (4) of Theorem B, and the proof is complete. \square

2. Remarks on n -th root closedness for commutative C^* -algebras

2.1. Introduction. Let $n \in \mathbb{N}$ with $n \geq 2$. We say that $C(X)$ is n -th root closed if for every $f \in C(X)$ there exists $g \in C(X)$ such that $g^n = f$. Clearly, $C(X)$ is n -th root closed for every $n \in \mathbb{N}$ whenever $C(X)$ is algebraically closed. In this section, we will make up some results obtained in the preceding section with respect to n -th root closedness.

2.2. Results.

THEOREM 1.12. *Let X be a first countable compact Hausdorff space. Then the following conditions are equivalent:*

- (1) $C(X)$ is n -th root closed for some $n \in \mathbb{N}$ with $n \geq 2$
- (2) $C(X)$ is n -th root closed for every $n \in \mathbb{N}$ with $n \geq 2$
- (3) $C(X)$ is algebraically closed.

COROLLARY 1.13. *Let X be a first countable compact Hausdorff space. Then the following conditions are equivalent:*

- (1) $C(X)$ is n -th root closed for some $n \in \mathbb{N}$ with $n \geq 2$
- (2) $C(X)$ is n -th root closed for every $n \in \mathbb{N}$ with $n \geq 2$
- (3) $C(X)$ is algebraically closed
- (4) $C(X)$ is square-root closed
- (5) For every $f \in C(X)$ there exist $g \in C(X)$ and $p, q \in \mathbb{N}$ such that $q/p \neq \mathbb{N}$ and $g^p = f^q$
- (6) X is almost locally connected and hereditarily unicoherent
- (7) X is almost locally connected, $\dim X \leq 1$ and $\check{H}^1(X; \mathbb{Z}) = 0$.

PROOF OF THEOREM 1.12. By definition, the implications $(3) \Rightarrow (2) \Rightarrow (1)$ are true. So, it is enough to show that $(1) \Rightarrow (3)$ holds. By definition, the implication $(1) \Rightarrow (a)$ of Corollary 1.3 is true. Since X is first countable, Corollary 1.3 shows that (a) is equivalent to the condition that $C(X)$ is algebraically closed. So we have that (1) implies (3), and this completes the proof. \square

PROOF OF COROLLARY 1.13. This immediately follows from Theorem 1.12 and Corollary 1.3. \square

EXAMPLE 1.14. Let X be the closure of $\cup_{n \in \mathbb{N}}(\{1/n\} \times [0, 1/n])$ in \mathbb{R}^2 . Then it is easy to see that X is hereditarily unicoherent and almost locally connected. Thus Corollary 1.3 shows that $C(X)$ is algebraically closed. In particular, $C(X)$ is n -th root closed for every $n \in \mathbb{N}$.

EXAMPLE 1.15. Let X be the closure of $\cup_{n \in \mathbb{N}}(\{1/n\} \times [0, 1])$ in \mathbb{R}^2 . Then it is easy to verify that X is not almost locally connected. So, by Corollary 1.13, $C(X)$ is not n -th root closed for any $n \geq 2$.

EXAMPLE 1.16. Let S^1 be the unit circle. Then S^1 is not hereditarily unicoherent. So, by Corollary 1.13, $C(S^1)$ is not n -th root closed for any $n \geq 2$.

REMARK. By Theorem 1.12 and Corollary 1.13, if X is first countable, then n -th root closedness is equivalent to square-root closedness and to algebraic closedness. The same holds for locally connected X (see Corollary 1.2 and Remark in the preceding section).

REMARK. Let us consider the following two conditions:

- (a) The set $\{f^n : f \in C(X)\}$ is uniformly dense in $C(X)$ for *every* $n \in \mathbb{N}$
- (b) The set $\{f^n : f \in C(X)\}$ is uniformly dense in $C(X)$ for *some* $n \in \mathbb{N}$ with $n \geq 2$.

The implications (2) in Theorem 1.12 \Rightarrow (a) \Rightarrow (b) is obviously true for any compact Hausdorff space X . In the preceding section, it was shown that if X is locally connected, then the implication (b) \Rightarrow (2) holds. For first countable X , however, the situation is different. For example, let X be the closure of $\cup_{n \in \mathbb{N}}(\{1/n\} \times [0, 1])$ in \mathbb{R}^2 . Then, from [13], we see that $\dim X = 1$ and $\check{H}^1(X; \mathbb{Z}) = 0$. So, by Lemma 1.8, the set $\{f^n : f \in C(X)\}$ is uniformly dense in $C(X)$ for every $n \geq 2$. But, as in Example 1.15, $C(X)$ is not n -th root closed for any $n \geq 2$.

REMARK. From Theorem 1.12, we see that n -th root closedness for some $n \geq 2$ implies m -th root closedness for all $m \in \mathbb{N}$ whenever X is first countable. The implication is true for locally connected X (see Corollary 1.2 and Remark in the preceding section). So, a natural question arises: is this implication true for all compact Hausdorff spaces? Countryman, Jr. [5, Remarks (2)] noted that there exists a compact Hausdorff space X , which is not first countable nor locally connected, with the following property: there exists $f \in C(X)$ such that f has a continuous 2^n -th root in $C(X)$ for every $n \in \mathbb{N}$ but that no continuous fifth root. Here, continuous

k -th root of f means a continuous function in $C(X)$ whose k -th power equals f . Recently, Kawamura and Miura [24] gave a negative answer to the question above. More explicitly, they showed that for each pair of relatively prime positive integers m and n , there exists a compact Hausdorff space X such that $C(X)$ is n -th root closed but not m -th root closed.

CHAPTER 2

On isomorphisms between algebras of continuous functions

1. Surjections on the algebras of continuous functions which preserve peripheral spectrum

1.1. Introduction. For a locally compact Hausdorff space X , we denote by $C_0(X)$ the Banach algebra of all complex-valued continuous functions on X which vanish at infinity with supremum norm $\|\cdot\|$. Let T be a surjection from $C_0(X)$ onto $C_0(Y)$ and we do not assume that T is linear and multiplicative. Molnár [31] proved that for a first countable compact Hausdorff space X , if T is a surjection from $C_0(X)$ onto itself which satisfies the ‘range multiplicativity’ condition :

$$(Tf\overline{Tg})(X) = (f\overline{g})(X) \quad (f, g \in C_0(X)),$$

where $\overline{\cdot}$ denotes complex conjugate, then there exist a homeomorphism ϕ from X onto itself and a continuous function τ from X into the unit circle S^1 such that $Tf(x) = \tau(x)f(\phi(x))$ ($x \in X, f \in C_0(X)$). In particular, if $T(1) = 1$, then T is an algebra $*$ -isomorphism. Hatori-Miura-Takagi [14] have generalized this result for the case of compact Hausdorff spaces which need not be first countable.

In this section, we will replace the ‘range multiplicativity’ condition for T in above result by a ‘peripheral range multiplicativity’ condition which is inspired by a result of Luttman and Tonev [27]. And we give an extension of above result for the case of arbitrary locally compact Hausdorff spaces.

1.2. Results. Let X be a locally compact Hausdorff space. Let $f \in C_0(X)$. The *peripheral range* of f , denoted by $Ran_\pi(f)$, is defined as follows:

$$Ran_\pi(f) = \{f(x) : |f(x)| = \|f\|, x \in X\}.$$

A net $\{e_\lambda\}_{\lambda \in \Lambda}$ in $C_0(X)$ which satisfies $\lim_{\lambda \in \Lambda} \|e_\lambda f - f\| = 0$ for every $f \in C_0(X)$ is called an approximate identity. $C_0(X)$ always contains an approximate identity.

Our main result is the following.

THEOREM 2.1. *Let X and Y be locally compact Hausdorff spaces, which need not be compact. If T is a surjection from $C_0(X)$ onto $C_0(Y)$ which satisfies the following condition :*

$$(*) \quad Ran_\pi(Tf\overline{Tg}) = Ran_\pi(f\overline{g}) \quad (f, g \in C_0(X)),$$

then there exist a homeomorphism ϕ from Y onto X and a continuous function τ from Y into the unit circle S^1 such that

$$Tf(y) = \tau(y)f(\phi(y)) \quad (y \in Y, f \in C_0(X)).$$

*In particular, if T preserves an approximate identity, then T is an isometric algebra *-isomorphism.*

Let X be a locally compact Hausdorff space. For $r > 0$, we write $D_r = \{z \in \mathbb{C} : |z| < r\}$. For each $x \in X$ we denote by P_x the set of all peak functions in $C_0(X)$ peaking at x , that is, $P_x = \{f \in C_0(X) : f(X) \subset D_1 \cup \{1\}, f(x) = 1\}$ ¹. Let $P_X = \cup_{x \in X} P_x$. In other words, P_X is the set of all peak functions in $C_0(X)$. For each $x \in X$, $PP_x = \{f \in C_0(X) : f(x) = \|f\| = 1\}$. Let $PP_X = \cup_{x \in X} PP_x$. Clearly $P_x \subset PP_x$, so $P_X \subset PP_X$. For $f \in P_X$, the level set of f is defined by $L_f = \{x \in X : f(x) = 1\}$.

¹An element in P_x may have other points where it takes the value one; not merely at the point x .

In the following four lemmata, X is assumed to be a locally compact Hausdorff space.

LEMMA 2.2. *Let $f \in C_0(X)$ and $x_0 \in X$. If $\lambda = f(x_0)$ and $\lambda \neq 0$, then there exists $u \in P_{x_0}$ such that $(1/\lambda)fu \in P_{x_0}$.*

PROOF. Suppose $\lambda \neq 0$. Put $F_0 = \{x \in X : |f(x) - \lambda| \geq |\lambda|/2\}$ and

$$F_n = \left\{ x \in X : \frac{|\lambda|}{2^{n+1}} \leq |f(x) - \lambda| \leq \frac{|\lambda|}{2^n} \right\} \quad (n = 1, 2, \dots).$$

Clearly, $F_0, F_1, \dots, F_n, \dots$ are all closed subsets of X which do not contain x_0 . By Urysohn's lemma, for each $n \in \{0, 1, 2, \dots\}$ there exists $u_n \in P_{x_0}$ such that $u_n = 0$ for all $x \in F_n$. Now we put

$$u = u_0 \sum_{n=1}^{\infty} \frac{u_n}{2^n}.$$

The above series is majorized by the convergent series $\sum \frac{1}{2^n}$, so u is well defined and $u \in C_0(X)$. Moreover it is easy to see that $u \in P_{x_0}$. We put $g = (1/\lambda)fu$. To verify $g \in P_{x_0}$, pick $x \in X$. If $x \in F_0$, since $u(x) = 0$, $g(x) = 0$. If $x \in F_n$ for some $n \in \{1, 2, \dots\}$, then

$$\begin{aligned} |g(x)| &\leq \frac{1}{|\lambda|} |f(x)| \left| \sum_{k \neq n} \frac{u_k}{2^k} \right| \\ &\leq \frac{1}{|\lambda|} (|f(x) - \lambda| + |\lambda|) \left(1 - \frac{1}{2^n} \right) \\ &\leq \frac{1}{|\lambda|} \left(\frac{|\lambda|}{2^n} + |\lambda| \right) \left(1 - \frac{1}{2^n} \right) \\ &< 1. \end{aligned}$$

If $x \in X \setminus \bigcup_{n=0}^{\infty} F_n$, then $f(x) = \lambda$ and so $g(x) = u(x) \in D_1 \cup \{1\}$. Thus we obtain $g(X) \subset D_1 \cup \{1\}$. In particular, $g(x_0) = u(x_0) = 1$. Hence $g \in P_{x_0}$ and the proof is complete. \square

LEMMA 2.3. *For $f, g \in C_0(X)$, $f = g$ if and only if $\text{Ran}_{\pi}(fu) = \text{Ran}_{\pi}(gu)$ for every $u \in P_X$.*

PROOF. The ‘only if’ part is trivial. To show the ‘if’ part, we assume that $f \neq g$ and will find a peak function u such that $\text{Ran}_\pi(fu) \neq \text{Ran}_\pi(gu)$. There is an element $x_0 \in X$ such that $f(x_0) \neq g(x_0)$. Without loss of generality, we can assume that $|f(x_0)| \leq |g(x_0)|$.

If $f(x_0) \neq 0$, then Lemma 2.2 gives a function $u \in P_{x_0}$ such that $(1/\lambda)fu \in P_{x_0}$, where $\lambda = f(x_0)$. Then $fu(X) \subset D_{|\lambda|} \cup \lambda$, so $\text{Ran}_\pi(fu) = \{\lambda\}$. However $|\lambda| \leq |g(x_0)| = |(gu)(x_0)|$ and $(gu)(x_0) = g(x_0) \neq \lambda$, so $\text{Ran}_\pi(fu) \neq \text{Ran}_\pi(gu)$.

On the other hand, if $f(x_0) = 0$, then $g(x_0) \neq 0$. We put $r = |g(x_0)|$ and $F = \{x \in X : |f(x)| \geq r\}$. Since F is a closed subset of X with $x_0 \notin F$, by Urysohn’s lemma, there exists $u \in P_{x_0}$ such that $u(x) = 0$ for $x \in F$. It follows that

$$|(fu)(x)| = |f(x)||u(x)| \begin{cases} = 0 & (x \in F) \\ \leq |f(x)| < r & (x \in X \setminus F). \end{cases}$$

Hence for each $x \in X$, $|(fu)(x)| < r = |g(x_0)| = |(gu)(x_0)|$. This implies that $\text{Ran}_\pi(fu) \neq \text{Ran}_\pi(gu)$. Thus the proof is complete. \square

LEMMA 2.4. *Let $f, g \in P_X$. Then $L_f \subset L_g$ if and only if $1 \in (gu)(X)$ for each $u \in P_X$ with $1 \in (fu)(X)$.*

PROOF. Suppose $L_f \subset L_g$ and assume that $u \in P_X$ satisfies $1 \in (fu)(X)$. $f(x)u(x) = 1$ for some $x \in X$. Since $f, u \in P_X$, $f(x) = u(x) = 1$. Hence $x \in L_f \subset L_g$. This implies $g(x) = 1$ and $g(x)u(x) = 1$. Thus $1 \in (gu)(X)$ and the ‘only if’ part is proved.

Suppose $L_f \not\subset L_g$. Then we find an element $x_0 \in L_f \setminus L_g$. Since L_g is a closed subset with $x_0 \notin L_g$, there is a peak function $u \in P_{x_0}$ such that u vanishes on L_g by Urysohn’s lemma. Thus we have

$$1 \in (fu)(X) \text{ and } 1 \notin (gu)(X),$$

because $f(x_0)u(x_0) = 1$ and $(gu)(X) \subset D_1$. The ‘if’ part is proved. \square

LEMMA 2.5. *Let $x, y \in X$ and $\alpha, \beta \in \mathbb{C}$. If $\alpha PP_x \subset \beta PP_y$, then $x = y$ and $\alpha = \beta$.*

PROOF. If $x = y$ and $\alpha \neq \beta$, then it is clear that $\alpha PP_x \not\subset \beta PP_y$. We assume that $x \neq y$. Then by Urysohn's lemma, for each α and β , there is a function $u \in \alpha PP_x$ such that $u(y) = 0$. Thus $u \in \alpha PP_x \setminus \beta PP_y$. Hence $\alpha PP_x \not\subset \beta PP_y$. \square

1.3. A proof of Theorem 2.1.

CLAIM 1. *T is injective.*

PROOF. Suppose that $Tf = Tg$ for $f, g \in C_0(X)$. Then for each $u \in P$, we apply $(*)$ to see that

$$\text{Ran}_\pi(fu) = \text{Ran}_\pi(Tf\overline{T\bar{u}}) = \text{Ran}_\pi(Tg\overline{T\bar{u}}) = \text{Ran}_\pi(gu).$$

Therefore by Lemma 2.3 we obtain $f = g$. Hence T is injective. \square

Since T is a bijection from $C_0(X)$ onto $C_0(Y)$, we can consider its inverse T^{-1} from $C_0(Y)$ onto $C_0(X)$. Clearly, T^{-1} has the similar property as T :

$$\text{Ran}_\pi\left(T^{-1}\mathfrak{f}\overline{T^{-1}\mathfrak{g}}\right) = \text{Ran}_\pi(\mathfrak{f}\mathfrak{g}) \quad (\mathfrak{f}, \mathfrak{g} \in C_0(Y)).$$

CLAIM 2. *Let $|Tf|, |Tg| \in P_Y$. If $L_{|Tf|} \subset L_{|Tg|}$, then $L_{|f|} \subset L_{|g|}$.*

PROOF. For $|Th| \in P_Y$, we apply $(*)$ to see that $\text{Ran}_\pi(|h|^2) = \text{Ran}_\pi(|Th|^2)$. It follows from this that $|f|, |g| \in P_X$. Hence $L_{|Tf|}, L_{|Tg|}, L_{|f|}, L_{|g|}$ are all well defined. Suppose $L_{|Tf|} \subset L_{|Tg|}$ and assume that $u \in P_X$ with $1 \in (|f|u)(X)$. There is an element $x_0 \in X$ such that $|f(x_0)|u(x_0) = 1$. Since $|f|, u \in P_X$, $|f(x_0)| = u(x_0) = 1$. We put $\lambda = f(x_0)u(x_0)$. Then we see that $|\lambda| = 1$ and λ is contained in $\text{Ran}_\pi(fu)$, so in $\text{Ran}_\pi(Tf\overline{T\bar{u}})$. It follows that $1 \in \text{Ran}_\pi(|Tf||T\bar{u}|)$. Since $|T\bar{u}| \in P_Y$, $1 \in \text{Ran}_\pi(|Tg||T\bar{u}|)$ by Lemma 2.4. There is an element $y_0 \in Y$ such that $|Tg(y_0)||T\bar{u}(y_0)| = 1$. We put $\gamma = Tg(y_0)\overline{T\bar{u}(y_0)}$. Then we see that $|\gamma| = 1$ and

γ is contained in $\text{Ran}_\pi(Tg\overline{T\bar{u}})$, so in $\text{Ran}_\pi(gu)$. Therefore $1 \in \text{Ran}_\pi(|g|u)$, since $u \in P_X$. We apply Lemma 2.4 to conclude that $L_{|f|} \subset L_{|g|}$. \square

CLAIM 3. *For each $y \in Y$, there exist $x \in X$ and $\lambda_x \in S^1$ such that $T^{-1}(\alpha PP_y) \subset \alpha \lambda_x PP_x$ holds for all $\alpha \in S^1$.*

PROOF. Fix $y_0 \in Y$. We put

$$L = \bigcap_{\substack{\alpha \in S^1 \\ f \in T^{-1}(\alpha PP_{y_0})}} L_{|f|}.$$

First we observe that L is non-empty. To see this, it is enough to show that the family $\{L_{|f|} : f \in T^{-1}(\alpha PP_{y_0}), \alpha \in S^1\}$ has the finite intersection property, since each $L_{|f|}$ is compact. Pick $\alpha_1, \dots, \alpha_n \in S^1$ and $f_1^j, \dots, f_{m(j)}^j \in T^{-1}(\alpha_j PP_{y_0})$ for $j = 1, \dots, n$. Since T is surjective, there exists $g \in C_0(X)$ such that

$$Tg = \prod_{\substack{1 \leq j \leq n \\ 1 \leq k \leq m(j)}} T f_k^j.$$

Since $|T f_k^j| \in P_{y_0}$, $|Tg| \in P_{y_0}$. And $L_{|Tg|} \subset L_{|T f_k^j|}$. Thus by Claim 2 we see that $L_{|g|} \subset L_{|f_k^j|}$, namely $L_{|g|} \subset L_{|f_1^1|} \cap \dots \cap L_{|f_{m(n)}^n|}$. Since $1 \in \text{Ran}_\pi(|Tg|^2) = \text{Ran}_\pi(|g|^2)$, $L_{|g|}$ is non-empty and so is $L_{|f_1^1|} \cap \dots \cap L_{|f_{m(n)}^n|}$. Thus $\{L_{|f|} : f \in T^{-1}(\alpha PP_{y_0}), \alpha \in S^1\}$ has the finite intersection property. Hence L is non-empty.

Pick an element x_0 from L arbitrarily. Next we observe that $\bar{\alpha}f(x_0)$ is uniquely determined for $f \in T^{-1}(\alpha PP_{y_0})$ with $\alpha \in S^1$. Let $f \in T^{-1}(\alpha PP_{y_0}), g \in T^{-1}(\beta PP_{y_0})$ with $\alpha, \beta \in S^1$. We show that $\bar{\alpha}f(x_0) = \bar{\beta}g(x_0)$. Let $\varepsilon > 0$ be given. Since Tf and Tg are continuous, there exists an open neighborhood G_ε of y_0 such that

$$Tf(G_\varepsilon) \subset \{z \in \mathbb{C} : |z - \alpha| < \varepsilon\},$$

$$Tg(G_\varepsilon) \subset \{z \in \mathbb{C} : |z - \beta| < \varepsilon\}.$$

$Y \setminus G_\varepsilon$ is a closed subset of Y which does not contain y_0 . By Urysohn's lemma, there exists $H_\varepsilon \in P_{y_0}$ such that $H_\varepsilon^{-1}(1) \subset G_\varepsilon$. Since T is surjective, there exists $h_\varepsilon \in C_0(X)$ such that $Th_\varepsilon = \overline{H_\varepsilon}$. $Th_\varepsilon \in P_{y_0} \subset PP_{y_0}$. Thus $|h_\varepsilon(x_0)| = 1$ and $\|h_\varepsilon\| = 1$.

Now we consider the function $f\overline{h_\varepsilon}$. Since $Tf \in \alpha PP_{y_0}$ with $\alpha \in S^1$, $|f(x_0)| = 1$ and $\|f\| = 1$. Thus $\|f\overline{h_\varepsilon}\| = \left|f(x_0)\overline{h_\varepsilon(x_0)}\right| = 1$. It follows from this that

$$\begin{aligned} f(x_0)\overline{h_\varepsilon(x_0)} &\in \text{Ran}_\pi(f\overline{h_\varepsilon}) = \text{Ran}_\pi(Tf\overline{Th_\varepsilon}) = \text{Ran}_\pi(TfH_\varepsilon) \\ &\subset \{z \in \mathbb{C} : |z - \alpha| < \varepsilon\}, \end{aligned}$$

since $H_\varepsilon^{-1}(1) \subset G_\varepsilon$. Thus $\left|f(x_0)\overline{h_\varepsilon(x_0)} - \alpha\right| < \varepsilon$. Hence

$$(1) \quad \overline{\alpha}f(x_0)\overline{h_\varepsilon(x_0)} \longrightarrow 1 \quad \text{as } \varepsilon \rightarrow 0.$$

We repeat the similar argument for $g\overline{h_\varepsilon}$, we obtain

$$(2) \quad \overline{\beta}g(x_0)\overline{h_\varepsilon(x_0)} \longrightarrow 1 \quad \text{as } \varepsilon \rightarrow 0.$$

It follows from (1) and (2) that $\overline{\alpha}f(x_0) = \overline{\beta}g(x_0)$. Now we put λ_{x_0} this unique number $\overline{\alpha}f(x_0)$, then it is easily seen that $T^{-1}(\alpha PP_{y_0}) \subset \alpha\lambda_{x_0}PP_{x_0}$ for all $\alpha \in S^1$. Thus the proof is complete. \square

CLAIM 4. *For each $y \in Y$, there corresponds a unique pair of $x \in X$ and $\gamma_x \in S^1$ such that $T(\alpha PP_x) = \alpha\gamma_x PP_y$ for all $\alpha \in S^1$.*

PROOF. Fix $y_0 \in Y$. By Claim 3, there exist $x_0 \in X$ and $\lambda_{x_0} \in S^1$ such that

$$(3) \quad T^{-1}(\alpha PP_{y_0}) \subset \alpha\lambda_{x_0}PP_{x_0} \quad (\alpha \in S^1).$$

Since T^{-1} has the similar property as T , we can apply Claim 3 to T^{-1} and find an element $y_1 \in Y$ and a complex number $\lambda_{y_1} \in S^1$ such that

$$(4) \quad T(\alpha PP_{x_0}) \subset \alpha\lambda_{y_1}PP_{y_1} \quad (\alpha \in S^1).$$

It follows from (3) and (4) that

$$(5) \quad \alpha PP_{y_0} = T(T^{-1}(\alpha PP_{y_0})) \subset T(\alpha\lambda_{x_0}PP_{x_0}) \subset \alpha\lambda_{x_0}\lambda_{y_1}PP_{y_1}$$

for all $\alpha \in S^1$. Put $\alpha = 1$, then (5) gives $PP_{y_0} \subset \lambda_{x_0}\lambda_{y_1}PP_{y_1}$. This relation implies that $y_0 = y_1$ and $\lambda_{x_0}\lambda_{y_1} = 1$ by Lemma 2.5. Here we put $\gamma_{x_0} = \overline{\lambda_{x_0}} = \lambda_{y_1}$, then it

follows from (5) that

$$(6) \quad T(\alpha PP_{x_0}) = \alpha \gamma_{x_0} PP_{y_0} \quad (\alpha \in S^1).$$

To show the uniqueness of x_0 and γ_{x_0} , suppose that

$$(7) \quad T(\alpha PP_{x_1}) = \alpha \gamma_{x_1} PP_{y_0} \quad (\alpha \in S^1)$$

for some pair of $x_1 \in X$ and $\gamma_{x_1} \in S^1$. In (6) and (7), we put $\alpha = \overline{\gamma_{x_0}}$ and $\alpha = \overline{\gamma_{x_1}}$ respectively, then we see that

$$\overline{\gamma_{x_0}} PP_{x_0} = T^{-1}(PP_{y_0}) = \overline{\gamma_{x_1}} PP_{x_1}.$$

By Lemma 2.5, we see that $x_0 = x_1$ and $\gamma_{x_0} = \gamma_{x_1}$. □

Claim 4 gives two maps $\phi : Y \rightarrow X$ and $\tau : Y \rightarrow S^1$ which satisfy that

$$(\sharp) \quad T(\alpha PP_{\phi(y)}) = \alpha \tau(y) PP_y \quad (y \in Y, \alpha \in S^1).$$

CLAIM 5. *For every $f \in C_0(X)$, $Tf(y) = \tau(y)f(\phi(y))$ holds for all $y \in Y$.*

PROOF. Let $f \in C_0(X)$ and $y_0 \in Y$. Put $\alpha = f(\phi(y_0))$ and $\beta = Tf(y_0)$. We show that $\tau(y_0)\alpha = \beta$.

First we assume that $\alpha \neq 0$ and $\beta \neq 0$. Since $\alpha \neq 0$, by Lemma 2.2, there exists $u \in P_{\phi(y_0)}$ such that $(1/\alpha)fu \in P_{\phi(y_0)}$. Since $u \in P_{\phi(y_0)} \subset PP_{\phi(y_0)}$, it follows from (\sharp) that $Tu \in \tau(y_0)PP_{y_0}$. Applying $(*)$, we also have that $Ran_\pi(Tf\overline{Tu}) = Ran_\pi(f\overline{u}) = \{\alpha\}$, since $(1/\alpha)fu \in P_{\phi(y_0)}$. Then we see that

$$(8) \quad \beta \overline{\tau(y_0)} = Tf(y_0)\overline{Tu(y_0)} \in (Tf\overline{Tu})(Y) \subset D_{|\alpha|} \cup \{\alpha\}$$

On the other hand, since $\beta \neq 0$, by Lemma 2.2, there exists $v \in P_{y_0}$ such that $(1/\beta)(Tf)v \in P_{y_0}$. By (\sharp) , there exists $w \in PP_{\phi(y_0)}$ such that $Tw = \tau(y_0)v$. It follows from this that

$$Ran_\pi(f\overline{w}) = Ran_\pi(Tf\overline{Tw}) = Ran_\pi(Tf\overline{\tau(y_0)v}) = \{\beta \overline{\tau(y_0)}\}.$$

Combining (8), we see that

$$\beta \overline{\tau(y_0)} \in D_{|\alpha|} \cup \{\alpha\} \quad \text{and} \quad \alpha \in D_{|\beta|} \cup \left\{ \beta \overline{\tau(y_0)} \right\}.$$

Therefore $\alpha = \beta \overline{\tau(y_0)}$, that is, $\tau(y_0)\alpha = \beta$.

Next we consider the case $\alpha = 0$. Let $\varepsilon > 0$ be given. Put $F = \{x \in X : |f(x)| \geq \varepsilon\}$. Since F is a closed subset of X which does not contain $\phi(y_0)$, by Urysohn's lemma, there exists $u \in P_{\phi(y_0)}$ such that u vanishes on F . By (\sharp) , $Tu \in \tau(y_0)PP_{y_0}$. So $Tu(y_0) = \tau(y_0)$. Then we see that

$$|(fu)(x)| = |f(x)||u(x)| \begin{cases} = 0 & (x \in F) \\ \leq |f(x)| < \varepsilon & (x \in X \setminus F). \end{cases}$$

Thus $(f\bar{u})(X) \subset D_\varepsilon$. Combining $(*)$, we see that $(Tf\overline{T\bar{u}})(Y) \subset D_\varepsilon$. So

$$\beta \overline{\tau(y_0)} = Tf(y_0)\overline{T\bar{u}(y_0)} \in (Tf\overline{T\bar{u}})(Y) \subset D_\varepsilon.$$

Thus $|\beta| = |\beta \overline{\tau(y_0)}| < \varepsilon$. Since ε is arbitrary, we conclude that $\beta = 0 = \tau(y_0)\alpha$.

Finally we consider the case $\beta = 0$. Let $\varepsilon > 0$ be given. Put $F = \{y \in Y : |Tf(y)| \geq \varepsilon\}$. F is a closed subset of Y which does not contain y_0 , so by Urysohn's lemma, there exists $u \in P_{y_0}$ such that u vanishes on F . Then, it is easy to see that

$$\left| Tf(y)\overline{\tau(y_0)u(y)} \right| < \varepsilon$$

for every $y \in Y$. By (\sharp) , there exists $v \in PP_{\phi(y_0)}$ such that $Tv = \tau(y_0)u$. It follows from this that

$$(Tf\overline{T\bar{v}})(Y) = (Tf\overline{\tau(y_0)u})(Y) \subset D_\varepsilon.$$

Combining $(*)$, we see that $(f\bar{v})(X) \subset D_\varepsilon$. So

$$\alpha = f(\phi(y_0)) = f(\phi(y_0))\overline{v(\phi(y_0))} \in (f\bar{v})(X) \subset D_\varepsilon.$$

Hence $|\alpha| < \varepsilon$. Since ε is arbitrary, we see that $\alpha = 0$ and conclude that $\tau(y_0)\alpha = 0 = \beta$. □

Since T^{-1} has the similar property as T , we can apply above argument to T^{-1} and obtain two maps $\psi : X \rightarrow Y$ and $\eta : X \rightarrow S^1$ such that

$$(\natural) \quad T^{-1}(\alpha PP_{\psi(x)}) = \alpha \eta(x) PP_x \quad (x \in X, \alpha \in S^1),$$

and T^{-1} is presented as follows :

$$(T^{-1}\mathfrak{f})(x) = \eta(x)\mathfrak{f}(\psi(x)) \quad (\mathfrak{f} \in C_0(Y), x \in X).$$

From (\sharp) and (\natural) , we see that for each $y \in Y$

$$\begin{aligned} PP_{\psi(\phi(y))} &= T(T^{-1}(PP_{\psi(\phi(y))})) \\ &= T(\eta(\phi(y))PP_{\phi(y)}) \\ &= \eta(\phi(y))\tau(y)PP_y. \end{aligned}$$

Applying Lemma 2.5, we see that $\psi(\phi(y)) = y$ for every $y \in Y$. Thus $\psi \circ \phi$ is the identity mapping on Y . In a similar way, we see that $\phi \circ \psi$ is the identity mapping on X . These facts imply that ϕ is a bijection from Y onto X and $\phi^{-1} = \psi$.

Pick $y \in Y$ and let $\{y_\lambda\}_{\lambda \in \Lambda}$ be a convergent net in Y with $\lim_{\lambda \in \Lambda} y_\lambda = y$. Then for every $f \in C_0(X)$, we see that

$$\begin{aligned} \lim_{\lambda \in \Lambda} |f|(\phi(y_\lambda)) &= \lim_{\lambda \in \Lambda} |\tau(y_\lambda)f(\phi(y_\lambda))| = \lim_{\lambda \in \Lambda} |Tf(y_\lambda)| = |Tf(y)| \\ &= |\tau(y)f(\phi(y))| = |f|(\phi(y)). \end{aligned}$$

Thus $\phi(y_\lambda)$ converges to $\phi(y)$ with respect to the weak topology generated by all $|f|$'s. Since this weak topology is equal to the given topology on X by complete regularity of X , ϕ is continuous on Y . We can apply the similar argument to $\psi = \phi^{-1}$ and see that ϕ^{-1} is also continuous. Hence ϕ is a homeomorphism from Y onto X . Pick an element $h \in C_0(X)$ such that $h(\phi(y)) \neq 0$. Without loss of generality, we assume that $h(\phi(y_\lambda)) \neq 0$ for $\lambda \in \Lambda$. Then we see that $\lim_{\lambda \in \Lambda} \tau(y_\lambda) = \lim_{\lambda \in \Lambda} Th(y_\lambda)h(\phi(y_\lambda))^{-1} = Th(y)h(\phi(y))^{-1} = \tau(y)$. Thus τ is continuous.

Suppose that T preserves an approximate identity $\{e_\lambda\}_{\lambda \in \Lambda}$ of $C_0(X)$. For each $x \in X$, we can find a continuous function $f_x \in C_0(X)$ such that $f_x(x) = 1$. Then we see that $|e_\lambda(x) - 1| = |e_\lambda(x)f_x(x) - f_x(x)| \leq \|e_\lambda f_x - f_x\|$. Since $\lim_{\lambda \in \Lambda} \|e_\lambda f_x - f_x\| = 0$, we obtain $\lim_{\lambda \in \Lambda} e_\lambda(x) = 1$. Since $\{T(e_\lambda)\}$ is an approximate identity, in a similar way, we see that $\lim_{\lambda \in \Lambda} (Te_\lambda)(x) = 1$. Then we see that for each $y \in Y$, $\tau(y) =$

$\lim_{\lambda \in \Lambda} \tau(y)e_\lambda(\phi(y)) = \lim_{\lambda \in \Lambda} (Te_\lambda)(y) = 1$. Thus $\tau(y) = 1$ for every $y \in Y$. It follows from this that $Tf = f \circ \phi$ for $f \in C_0(X)$. Hence T is an isometric algebra $*$ -isomorphism.

2. Norm-preserving surjections on algebras of continuous functions

2.1. Introduction. There are many papers dealing with spectrum-preserving mappings between Banach algebras. Molnár [31] initiated the study of multiplicatively spectrum-preserving mappings and essentially showed that a unit preserving surjection $T : C(X) \rightarrow C(X)$ for a first countable compact Hausdorff space X is an algebra isomorphism if

$$\sigma(TfTg) = \sigma(fg)$$

holds for every pair of f and g in $C(X)$. Rao and Roy [35] dealt with uniform algebras on compact Hausdorff spaces which are regarded as the maximal ideal space and generalized the result of Molnár. Hatori, Miura and Takagi [14] extended the result of Molnár by replacing the spectrum with the range and showed that a unit preserving surjection $T : A \rightarrow B$ between two uniform algebras is an algebra isomorphism if

$$\text{Ran}(TfTg) = \text{Ran}(fg)$$

holds for every pair of f and g in A , where $\text{Ran}(h)$ denotes the range of h . Recall that the *peripheral range*, $\text{Ran}_\pi(f)$, of f in a uniform algebra is defined by $\text{Ran}_\pi(f) = \{z \in \text{Ran}(f) : |z| = \|f\|\}$, where $\|f\|$ is the supremum norm of f . Luttmann and Tonev [27] extended the result of Hatori, Miura and Takagi by replacing the ranges with the peripheral ranges and showed that a unit preserving surjection $T : A \rightarrow B$ is an algebra isomorphism if T is *Ran $_\pi$ -multiplicative*, i.e.

$$\text{Ran}_\pi(TfTg) = \text{Ran}_\pi(fg) \quad (f, g \in A).$$

Lambert, Luttman and Tonev [26] considered multiplicatively norm-preserving mappings and showed that if a map $T : A \rightarrow B$ preserves the class of peak functions and is *norm-multiplicative*, i.e.

$$\|TfTg\| = \|fg\| \quad (f, g \in A),$$

then there exists a homeomorphism $\phi : \delta A \rightarrow \delta B$ so that the equality $|(Tf)(\phi(x))| = |f(x)|$ holds for every $f \in A$ and all x in the Choquet boundary δA of A .

Molnár [31] also gave a characterization on algebra $*$ -isomorphisms: A unit preserving surjection $T : C(X) \rightarrow C(X)$ for a first countable compact Hausdorff space X is an algebra $*$ -isomorphism if

$$\sigma(Tf\overline{Tg}) = \sigma(f\overline{g})$$

holds for every pair of f and g in $C(X)$. Hatori, Miura and Takagi [15] generalized this result for certain semisimple commutative Banach $*$ -algebras. In the preceding section, the result of Molnár was extended by considering the peripheral ranges instead of the spectra and in particular, it was proved that a unit preserving surjection $T : C(X) \rightarrow C(Y)$ for compact Hausdorff spaces X and Y (not necessarily first countable) is an algebra $*$ -isomorphism if

$$\text{Ran}_\pi(Tf\overline{Tg}) = \text{Ran}_\pi(f\overline{g})$$

holds for every pair f and g in $C(X)$.

In this section we consider a further extension; multiplicatively norm-preserving mappings. First of all one can modify an example of Lambert, Luttman and Tonev [26, Example 1] and exhibit a mapping which is not linear while the equality

$$\|Tf\overline{Tg}\| = \|f\overline{g}\|$$

holds for every pair of f and g in $C(X)$. In the following we mainly consider the condition

$$\|Tf\overline{Tg} - 1\| = \|f\overline{g} - 1\| \quad (f, g \in C(X))$$

on a mapping T from $C(X)$ onto $C(Y)$. If T satisfies the hypothesis

$$\sigma(Tf\overline{Tg}) = \sigma(f\overline{g}) \quad (f, g \in C(X)),$$

then T satisfies the above condition.

2.2. Results. Our main result is the following.

THEOREM 2.6. *Let X, Y be two compact Hausdorff spaces. If a surjection $T : C(X) \rightarrow C(Y)$ satisfies the conditions*

- (a) $T\lambda = \lambda$ for $\lambda \in \{\pm 1, \pm i\}$ and
- (b) $\|Tf\overline{Tg} - 1\| = \|f\overline{g} - 1\|$ for all $f, g \in C(X)$,

then there exists a homeomorphism ϕ from Y onto X such that $Tf = f \circ \phi$ for every f in $C(X)$; in particular, T is an isometric algebra $$ -isomorphism.*

Let X be a compact Hausdorff space. We denote by $\sigma_\pi(f)$ the *peripheral spectrum* of an element $f \in C(X)$:

$$\sigma_\pi(f) = \{z \in \sigma(f) : |z| = r(f)\},$$

where $r(f)$ denotes the spectral radius of f . Clearly, $\sigma_\pi(f) = \text{Ran}_\pi(f)$. We denote by P_X^{-1} the set of all peak functions in $C(X)^{-1}$, that is, $P_X^{-1} = \{u \in C(X)^{-1} : \sigma_\pi(u) = \{1\}\}$. For each $x \in X$, $P_x^{-1} = \{u \in P_X^{-1} : u(x) = 1\}$ ². Clearly, $P_X^{-1} = \bigcup_{x \in X} P_x^{-1}$. For $x \in X$, set $PP_x^{-1} = C(X)^{-1} \cap \{f \in C(X) : |f(x)| = 1 = \|f\|\}$. Clearly, PP_x^{-1} properly contains P_x^{-1} .

LEMMA 2.7. *Let $x_0 \in X$ and let F be a closed subset of X with $x_0 \notin F$. Then for each $\varepsilon > 0$ there exists $u \in P_{x_0}^{-1}$ such that $|u(x)| < \varepsilon$ for $x \in F$.*

PROOF. Let $\varepsilon > 0$ be given. Since $\{x_0\}$ and F are disjoint closed subsets of X , by Urysohn's lemma there exists a continuous function $v_1 : X \rightarrow [0, 1]$ such that

²An element in P_x^{-1} may have other points where it takes the value one; not merely at the point x .

$v_1(x_0) = 1$ and $v_1 = 0$ on F . Let $v_2 = 2^{-1}(1 + v_1)$. Then $v_2 \in P_{x_0}^{-1}$ and $v_2 = 2^{-1}$ on F . We see that $u = v_2^n$ has the required properties for some sufficiently large n . \square

On the following lemma, the same result for a not necessary invertible, peak function is proved in [26] for arbitrary uniform algebra.

LEMMA 2.8. *Let $f \in C(X)$ and $x_0 \in X$. If $\lambda = f(x_0)$ and $\lambda \neq 0$, then there exists $u \in P_{x_0}^{-1}$ such that $\sigma_\pi((1/\lambda)fu) = \{1\}$.*

PROOF. Suppose $\lambda \neq 0$. Let $F_0 = \{x \in X : |f(x) - \lambda| \geq 2^{-1}|\lambda|\}$ and

$$F_n = \{x \in X : 2^{-n-1}|\lambda| \leq |f(x) - \lambda| \leq 2^{-n}|\lambda|\}$$

for $n = 1, 2, \dots$. Clearly, F_0, F_1, \dots are closed subsets of X that do not contain x_0 ; so by Urysohn's lemma, there exist continuous functions v_0, v_1, \dots such that $0 \leq v_j \leq 1$, $v_j(x_0) = 1$ and $v_j = 0$ on F_j for $j = 0, 1, \dots$. For each j we take a positive integer n_j so that $u_j = 2^{-n_j}(1 + v_j)^{n_j}$ may have the property: If $j = 0$,

$$|u_0(x)| < \frac{|\lambda|}{\|f\|} \quad (x \in F_0)$$

and if $j > 0$,

$$|u_j(x)| < \frac{1}{2^j + 1} \quad (x \in F_j).$$

Now put $u = u_0 \sum_{n=1}^{\infty} 2^{-n} u_n$. This series is majorized by the convergent series $\sum 2^{-n}$, so u is well defined and $u \in C(X)$. Moreover, u is easily seen to be a function in $P_{x_0}^{-1}$.

Put $g = (1/\lambda)fu$. To verify $\sigma_\pi(g) = \{1\}$, pick $x \in X$. If $x \in F_0$, then we see

$$|g(x)| = \frac{1}{|\lambda|} |f(x)| |u_0(x)| \sum_{n=1}^{\infty} \frac{|u_n(x)|}{2^n} < \frac{1}{|\lambda|} \|f\| \frac{|\lambda|}{\|f\|} \sum_{n=1}^{\infty} \frac{1}{2^n} = 1.$$

If $x \in F_n$ for some positive integer n , then

$$\begin{aligned} |g(x)| &= \frac{1}{|\lambda|} |f(x)| |u_0(x)| \left(\frac{|u_n(x)|}{2^n} + \sum_{j \neq n} \frac{|u_j(x)|}{2^j} \right) \\ &\leq \frac{1}{|\lambda|} (|f(x) - \lambda| + |\lambda|) \left(\frac{|u_n(x)|}{2^n} + \sum_{j \neq n} \frac{1}{2^j} \right) \\ &< \frac{1}{|\lambda|} \left(\frac{|\lambda|}{2^n} + |\lambda| \right) \left(\frac{1}{2^n} \frac{1}{2^n + 1} + 1 - \frac{1}{2^n} \right) = 1. \end{aligned}$$

If $x \in X \setminus \cup_{j=0}^{\infty} F_j$, then $f(x) = \lambda$ and $g(x) = u(x) \in D \cup \{1\}$, where $D = \{z \in \mathbb{C} : |z| < 1\}$. Thus $g(X) \subset D \cup \{1\}$. In particular $g(x_0) = u(x_0) = 1$. Hence $\sigma_{\pi}(g) = \{1\}$. This completes the proof. \square

LEMMA 2.9. For $x_1, x_2 \in X$, $x_1 = x_2$ if and only if $PP_{x_1}^{-1} \subset PP_{x_2}^{-1}$.

PROOF. The ‘only if’ part is trivial. We will show the ‘if’ part. Assume that $x_1 \neq x_2$. Then, by Lemma 2.7, there exists $u \in P_{x_1}^{-1}$ such that $|u(x_2)| < 1$. Thus we see that $u \in PP_{x_1}^{-1} \setminus PP_{x_2}^{-1}$, that is, $PP_{x_1}^{-1} \not\subset PP_{x_2}^{-1}$. \square

2.3. A proof of Theorem 2.6. Throughout this section, T denotes a surjection which satisfies the hypotheses of Theorem 2.6:

- (a) $T\lambda = \lambda$ for $\lambda \in \{\pm 1, \pm i\}$ and
- (b) $\|Tf\overline{Tg} - 1\| = \|f\overline{g} - 1\|$ for all $f, g \in C(X)$,

CLAIM 1. $T(C(X)^{-1}) = C(Y)^{-1}$.

PROOF. Let $f \in C(X)^{-1}$. Then we see that $\|Tf\overline{T(f^{-1})} - 1\| = \|ff^{-1} - 1\| = 0$, thus $Tf\overline{T(f^{-1})} = 1$. Hence $Tf \in C(Y)^{-1}$. Let $f \in C(Y)^{-1}$. Since T is surjective, there is f and g in $C(X)$ such that $Tf = f$ and $Tg = \overline{f^{-1}}$. Then we see that $\|f\overline{g} - 1\| = \|Tf\overline{Tg} - 1\| = \|ff^{-1} - 1\| = 0$. Thus we have that $f\overline{g} = 1$, and $f \in C(X)^{-1}$. \square

CLAIM 2. T is injective.

PROOF. Suppose $Tf = Tg$ for $f, g \in C(X)$. We will show that $f = g$. Let $x \in X$. First we consider the case where $f(x) \neq 0$ and $g(x) \neq 0$. By Lemma 2.8, there exist $u_f, u_g \in P_x^{-1}$ such that $\sigma_\pi(fu_f) = \{f(x)\}$ and $\sigma_\pi(gu_g) = \{g(x)\}$. Let $u = u_f u_g$. Then u is an element in P_x^{-1} such that $\sigma_\pi(fu) = \{f(x)\}$ and $\sigma_\pi(gu) = \{g(x)\}$. Then we see that

$$\begin{aligned} 2 &= \left\| f \frac{-1}{f(x)} u - 1 \right\| = \left\| T f T \left(\overline{\frac{-u}{f(x)}} \right) - 1 \right\| \\ &= \left\| T g T \left(\overline{\frac{-u}{f(x)}} \right) - 1 \right\| = \left\| g \frac{-1}{f(x)} u - 1 \right\| \\ &\leq \frac{1}{|f(x)|} \|gu\| + 1 = \frac{|g(x)|}{|f(x)|} + 1. \end{aligned}$$

Therefore $|f(x)| \leq |g(x)|$ holds. In a similar way, we see that $|g(x)| \leq |f(x)|$. So we have $|f(x)| = |g(x)|$. This fact implies that $\left\| g \frac{-1}{f(x)} u \right\| = 1$. It follows that $-1 \in \sigma_\pi \left(g \frac{-1}{f(x)} u \right)$, since $\left\| g \frac{-1}{f(x)} u - 1 \right\| = 2$. Thus $f(x) \in \sigma_\pi(gu) = \{g(x)\}$, and $f(x) = g(x)$.

Next we consider the case where $f(x) = 0$ or $g(x) = 0$. Without loss of generality we may assume $f(x) = 0$. We will show that $g(x) = 0$. Suppose not. Let ε be a positive number with $\varepsilon < |g(x)|$. Let $F = \{x' \in X : |f(x')| \geq \varepsilon\}$. Since F is a closed subset of X with $x \notin F$, by Lemma 2.7 there exists $u_f \in P_x^{-1}$ such that $|u_f(x')| < \varepsilon(\|f\| + 1)^{-1}$ for all $x' \in F$. We have that $|fu_f| < \varepsilon$ on X . Since $g(x) \neq 0$, by Lemma 2.8 there exists $u_g \in P_x^{-1}$ such that $\sigma_\pi(gu_g) = \{g(x)\}$. Let $u = u_f u_g$. Then u is an element of P_x^{-1} such that $|fu| < \varepsilon$ on X and $\sigma_\pi(gu) = \{g(x)\}$. Choose $\alpha \in \mathbb{C}$ such that $|\alpha| = 1$ and $(\alpha gu)(x) = -|g(x)|$. Then we see that

$$\begin{aligned} |g(x)| + 1 &= \|\alpha gu - 1\| = \left\| T g T \left(\overline{\alpha u} \right) - 1 \right\| \\ &= \left\| T f T \left(\overline{\alpha u} \right) - 1 \right\| = \|\alpha fu - 1\| \\ &\leq 1 + \varepsilon. \end{aligned}$$

Thus we have $|g(x)| \leq \varepsilon$. This contradicts $\varepsilon < |g(x)|$. Hence $g(x) = 0$. This completes the proof. \square

By Claim 2, we can consider the inverse T^{-1} from $C(Y)$ onto $C(X)$. Clearly T^{-1} has the same properties as T :

$$(a') \quad T^{-1}\lambda = \lambda \quad (\lambda \in \{\pm 1, \pm i\}),$$

$$(b') \quad \left\| T^{-1}\mathfrak{f}\overline{T^{-1}\mathfrak{g}} - 1 \right\| = \|\mathfrak{f}\overline{\mathfrak{g}} - 1\| \quad (\mathfrak{f}, \mathfrak{g} \in C(Y)).$$

CLAIM 3. *If $f, g \in C(X)^{-1}$, then the equation $\|Tf\overline{Tg}\| = \|f\overline{g}\|$ holds; in particular, $\|Tf\| = \|f\|$ holds for every $f \in C(X)^{-1}$. Since $|z\overline{w}| = |zw|$ for any $z, w \in \mathbb{C}$, as a consequence of it T is norm-multiplicative on $C(X)^{-1}$, that is, $\|TfTg\| = \|fg\|$ for every pair f and g in $C(X)^{-1}$.*

PROOF. Let $f, g \in C(X)^{-1}$. We will show that $\|f\overline{g}\| \leq \|Tf\overline{Tg}\|$. From Claim 1, $Tf, Tg \in C(Y)^{-1}$. Let $K_n = T(nf)(Tf)^{-1}$ for $n = 1, 2, \dots$. In the proof of Claim 1, we have shown that $(Tf)^{-1} = \overline{T(f^{-1})}$. It follows that $K_n = T(nf)\overline{T(f^{-1})}$. Then we have that $\|K_n\| \leq n$ for each n , since $\|K_n\| - 1 \leq \|K_n - 1\| = \left\| T(nf)\overline{T(f^{-1})} - 1 \right\| = \|(nf)f^{-1} - 1\| = n - 1$. For each n we have

$$\begin{aligned} n\|f\overline{g}\| - 1 &\leq \|(nf)\overline{g} - 1\| = \|T(nf)\overline{Tg} - 1\| \\ &\leq \|T(nf)\overline{Tg}\| + 1 \leq \|K_n\| \|Tf\overline{Tg}\| + 1. \end{aligned}$$

Since $\|K_n\| \leq n$ for every n , it follows that

$$\|f\overline{g}\| - \frac{1}{n} \leq \|Tf\overline{Tg}\| + \frac{1}{n}.$$

Letting n tend to ∞ , gives $\|f\overline{g}\| \leq \|Tf\overline{Tg}\|$. Applying a similar argument to T^{-1} , yields $\|Tf\overline{Tg}\| \leq \|f\overline{g}\|$. Hence $\|Tf\overline{Tg}\| = \|f\overline{g}\|$ holds for every pair f and g in $C(X)^{-1}$. \square

CLAIM 4. *If f and g are elements in $C(X)^{-1}$ which satisfies that $\|Tf\| = 1 = \|Tg\|$ and $|Tf|^{-1}(1) \subset |Tg|^{-1}(1)$, then $|f|^{-1}(1) \subset |g|^{-1}(1)$.*

PROOF. We assume that $|Tf|^{-1}(1) \subset |Tg|^{-1}(1)$. We will show that $|f|^{-1}(1) \subset |g|^{-1}(1)$. Suppose not. Then there exists $x \in X$ such that $x \in |f|^{-1}(1) \setminus |g|^{-1}(1)$. By Claim 3 we have that $\|f\| = \|Tf\| = 1$ and $\|g\| = \|Tg\| = 1$. $|g|^{-1}(1)$ is a closed subset of X which does not contain x . By Lemma 2.7 there exists $u \in P_x^{-1}$ such that $|u(x')| < 1$ for $x' \in |g|^{-1}(1)$. Then we have that $|ug| < 1$ on X , that is, $\|ug\| < 1$. Then we have

$$(1) \quad \|Tu\overline{Tg}\| = \|u\overline{g}\| = \|ug\| < 1.$$

On the other hand, since $|(u\overline{f})(x)| = 1$ and $\|f\| = 1 = \|u\|$, we have $\|Tu\overline{Tf}\| = \|u\overline{f}\| = 1$. Hence there exists $y \in Y$ such that $|Tu(y)||Tf(y)| = 1$. Since by Claim 3 $\|Tu\| = \|u\| = 1 = \|Tf\|$, we have $|Tu(y)| = 1 = |Tf(y)|$. This implies $|(Tu\overline{Tg})(y)| = |Tu(y)||Tg(y)| = 1$ because $|Tf|^{-1}(1) \subset |Tg|^{-1}(1)$. Hence we have that $\|Tu\overline{Tg}\| = 1$. This contradicts the inequality (1). \square

We will construct a homeomorphism from Y onto X which satisfies the resulting conditions of Theorem 2.6. Lambert, Luttmann and Tonev [26] proved that if a mapping $T : A \rightarrow B$ between two uniform algebras preserves the class of peak functions and is norm-multiplicative, then there exists a homeomorphism $\phi : \delta A \rightarrow \delta B$ so that the equality $|(Tf)(\phi(x))| = |f(x)|$ holds for every $f \in A$ and all x in the Choquet boundary δA of A . A similar argument was used in [27, 35]. The idea of our construction of a homeomorphism has the same vein. First, we will show that there exists a homeomorphism ϕ from Y onto X such that $|Tf(y)| = |f(\phi(y))|$ holds for every $f \in C(X)^{-1}$ and $y \in Y$. Next, we will show that the indicated mapping has the desired properties.

CLAIM 5. *For each $y \in Y$, there corresponds a unique element $x \in X$ such that $T(PP_x^{-1}) = PP_y^{-1}$.*

PROOF. Let $y \in Y$. First, we will show that there exists $x \in X$ such that $T^{-1}(PP_y^{-1}) \subset PP_x^{-1}$. Set

$$L_y = \bigcap_{f \in T^{-1}(PP_y^{-1})} |f|^{-1}(1).$$

Let $f_1, \dots, f_n \in T^{-1}(PP_y^{-1})$. Let $\mathfrak{f} = Tf_1 \cdots Tf_n$ and $f = T^{-1}\mathfrak{f}$. Then we see that $\mathfrak{f} \in C(Y)^{-1}$ and $|\mathfrak{f}(y)| = 1$. Since $\|Tf_j\| = 1$ for every j , we have that $\|\mathfrak{f}\| = 1$. Thus by Claim 3, we have that $\|f\| = \|Tf\| = \|\mathfrak{f}\| = 1$. Thus $|f|^{-1}(1)$ is nonempty. Since $\|Tf_j\| = 1$, we have that $|Tf|^{-1}(1) \subset |Tf_j|^{-1}(1)$ for every j . By Claim 4 we see that $|f|^{-1}(1) \subset |f_j|^{-1}(1)$ for every j , that is, the intersection of all $|f_j|^{-1}(1)$ contains the non-empty set $|f|^{-1}(1)$. Thus the class $\{|f|^{-1}(1) : f \in T^{-1}(PP_y^{-1})\}$ has the finite intersection property. Since $|g|^{-1}(1)$ is compact for every $g \in T^{-1}(PP_y^{-1})$, L_y is non-empty. Take an element, say x , from L_y , then we see that $T^{-1}(PP_y^{-1}) \subset PP_x^{-1}$.

Secondly, we show that $T(PP_x^{-1}) = PP_y^{-1}$. Applying the above argument to T^{-1} , there exists $y' \in Y$ such that $T(PP_x^{-1}) \subset PP_{y'}^{-1}$ because T^{-1} has the same properties as T . Since T is surjective, we have that $PP_{y'}^{-1} = T(T^{-1}(PP_{y'}^{-1})) \subset T(PP_x^{-1}) \subset PP_{y'}^{-1}$. By Lemma 2.9, we see that $y = y'$ and hence $T(PP_x^{-1}) = PP_y^{-1}$.

Finally, we show the uniqueness of x . Suppose that there is $x' \in X$ such that $T(PP_{x'}^{-1}) = PP_y^{-1}$. Then the injectivity of T implies that $PP_x = T^{-1}(PP_y^{-1}) = PP_{x'}^{-1}$. Lemma 2.9 yields that $x = x'$. This completes the proof. \square

By Claim 5, we can consider a mapping ϕ of Y into X such that $T(PP_{\phi(y)}^{-1}) = PP_y^{-1}$ for every $y \in Y$.

CLAIM 6. ϕ is a homeomorphism from Y onto X such that $|Tf(y)| = |f(\phi(y))|$ holds for every $f \in C(X)^{-1}$ and $y \in Y$.

PROOF. First, we will show that $|Tf(y)| = |f(\phi(y))|$ holds for every $f \in C(X)^{-1}$ and $y \in Y$. Let $f \in C(X)^{-1}$ and $y \in Y$. Then $f(\phi(y)) \neq 0$. By Lemma 2.8, there

exists an $u \in P_{\phi(y)}^{-1}$ such that $\sigma_\pi(fu) = \{f(\phi(y))\}$. Thus

$$(2) \quad \|fu\| = |f(\phi(y))|.$$

Since $P_{\phi(y)}^{-1} \subset PP_{\phi(y)}^{-1}$, by the definition of ϕ , we see that $|Tu(y)| = 1 = \|Tu\|$. Since $f, u \in C(X)^{-1}$ and $|Tu(y)| = 1$, by Claim 3 we have that

$$\|fu\| = \|TfTu\| \geq |Tf(y)||Tu(y)| = |Tf(y)|.$$

This fact and the equality (2) imply that

$$(3) \quad |Tf(y)| \leq |f(\phi(y))|.$$

Since $Tf \in C(X)^{-1}$, $Tf(y) \neq 0$. By Lemma 2.8 there exists $\mathbf{v} \in P_y^{-1}$ such that $\sigma_\pi((Tf)\mathbf{v}) = \{Tf(y)\}$. Let $v = T^{-1}\mathbf{v}$. Then \mathbf{v} is also in PP_y^{-1} , thus $|v(\phi(y))| = 1$ from the definition of ϕ . Since $\mathbf{v} \in C(Y)^{-1}$, we have that $v \in C(X)^{-1}$ by Claim 1. Thus by Claim 3 we have that $|Tf(y)| = \|(Tf)\mathbf{v}\| = \|fv\| \geq |f(\phi(y))|$. This fact and the inequality (3) imply that $|Tf(y)| = |f(\phi(y))|$.

Secondly, we will show that ϕ is continuous. Let \mathbf{T}_1 be the given topology on X and let $\{y_\alpha\}$ be a convergent net in Y with $\lim y_\alpha = y$. Then the first part shows that $\lim |f(\phi(y_\alpha))| = \lim |Tf(y_\alpha)| = |Tf(y)| = |f(\phi(y))|$ for every $f \in C(X)^{-1}$. Thus $\phi(y_\alpha)$ converges to $\phi(y)$ with respect to the weak topology \mathbf{T}_2 on X generated by $|C(X)^{-1}| = \{|f| : f \in C(X)^{-1}\}$. The identity mapping of (X, \mathbf{T}_1) onto (X, \mathbf{T}_2) is continuous, and (X, \mathbf{T}_2) is Hausdorff because $|C(X)^{-1}|$ separates the points of X ; since (X, \mathbf{T}_1) is compact, the mapping is a homeomorphism. Hence ϕ is continuous.

Finally, we will show that ϕ is a homeomorphism from Y onto X . Since T^{-1} has the same properties as T , there exists a continuous mapping ψ from X into Y such that $T^{-1}(PP_{\psi(x)}^{-1}) = PP_x^{-1}$ and $|T^{-1}\mathbf{f}(x)| = |\mathbf{f}(\psi(x))|$ for every $\mathbf{f} \in C(Y)^{-1}$. Let $y \in Y$ and $f \in C(X)^{-1}$ with $\mathbf{f} = Tf$. Then we have that

$$\begin{aligned} |Tf(y)| &= |f(\phi(y))| = |T^{-1}\mathbf{f}(\phi(y))| = |\mathbf{f}(\psi(\phi(y)))| \\ &= |Tf(\psi(\phi(y)))|. \end{aligned}$$

Thus $y = \psi(\phi(y))$, since $|C(Y)^{-1}| = |T(C(X)^{-1})|$ separates the points of Y . In a similar way, we have $x = \phi(\psi(x))$ for every $x \in X$. Hence ϕ is bijection from Y onto X with $\phi^{-1} = \psi$. Since ψ is continuous, ϕ is a homeomorphism from Y onto X . \square

CLAIM 7. $T\lambda = \lambda$ holds for every $\lambda \in S^1$, where $S^1 = \{z \in \mathbb{C} : |z| = 1\}$.

PROOF. Let $\lambda \in S^1$. We may assume that $\lambda \notin \{\pm 1, \pm i\}$. From the condition (b), we have that $\| |T\lambda|^2 - 1 \| = \| T\lambda \overline{T\lambda} - 1 \| = \| \lambda \overline{\lambda} - 1 \| = 0$, thus $|T\lambda| = 1$, or equivalently, $(T\lambda)(Y) \subset S^1$. Since $T1 = 1$, we have

$$(4) \quad \|T\lambda - 1\| = \|T\lambda \overline{T1} - 1\| = \|\lambda \overline{1} - 1\| = |\lambda - 1|.$$

Since $T(-1) = -1$, we have

$$(5) \quad \|T\lambda + 1\| = \|T\lambda \overline{T(-1)} - 1\| = \|\lambda \overline{(-1)} - 1\| = |\lambda + 1|.$$

Since $(T\lambda)(Y) \subset S^1$, (4) and (5) imply that $(T\lambda)(Y) \subset \{\lambda, \overline{\lambda}\}$. If $\text{Im } \lambda > 0$, the condition $Ti = i$ gives

$$\|T\lambda - i\| = \|T\lambda \overline{Ti} - 1\| = \|\lambda \overline{i} - 1\| = |\lambda - i|.$$

This implies that $(T\lambda)(Y) = \{\lambda\}$, that is, $T\lambda = \lambda$. If $\text{Im } \lambda < 0$, in a similar way, we have that $\|T\lambda + i\| = |\lambda + i|$ because $T(-i) = -i$. It follows that $T\lambda = \lambda$. Thus the proof is complete. \square

CLAIM 8. $T\left(\alpha P_{\phi(y)}^{-1}\right) = \alpha P_y^{-1}$ holds for every $\alpha \in S^1$ and $y \in Y$.

PROOF. Let $\alpha \in S^1$. First, we will show that $T(\alpha P_X^{-1}) = \alpha P_Y^{-1}$. Let $f \in P_X^{-1}$. From Claim 7 we have

$$\begin{aligned} 2 &= \| -\overline{\alpha} \alpha f - 1 \| = \| T(\alpha f) \overline{T(-\alpha)} - 1 \| \\ &= \| -\overline{\alpha} T(\alpha f) - 1 \|. \end{aligned}$$

Thus $\alpha \in \sigma_\pi(T(\alpha f))$ since $\|T(\alpha f)\| = \|\alpha f\| = 1$ by Claim 3. Let $\beta \in \sigma_\pi(T(\alpha f))$. Claim 7 gives

$$\begin{aligned} 2 &= \left\| -\bar{\beta} T(\alpha f) - 1 \right\| = \left\| \overline{T(-\beta)} T(\alpha f) - 1 \right\| \\ &= \left\| -\bar{\beta} \alpha f - 1 \right\|. \end{aligned}$$

Thus $\beta \in \sigma_\pi(\alpha f)$ since $\|\alpha f\| = 1$. Since $\sigma_\pi(\alpha f) = \{\alpha\}$, we have that $\beta = \alpha$ and $\sigma_\pi(T(\alpha f)) = \{\alpha\}$. Thus $T(\alpha P_X^{-1}) \subset \alpha P_Y^{-1}$. In a similar way, we have that $T^{-1}(\alpha P_Y^{-1}) \subset \alpha P_X^{-1}$. Hence $T(\alpha P_X^{-1}) = \alpha P_Y^{-1}$.

Let $y \in Y$ and $f \in P_{\phi(y)}^{-1}$. From the above argument, we have that $T(\alpha f) \in \alpha P_Y^{-1}$. We show that $T(\alpha f)(y) = \alpha$. Since $f \in C(X)^{-1}$, by Claim 3 and 6 we see that

$$|T(\alpha f)(y)| = |\alpha f(\phi(y))| = 1 = \|\alpha f\| = \|T(\alpha f)\|.$$

Thus we have $T(\alpha f)(y) = \alpha$. Since $f \in P_{\phi(y)}^{-1}$ is arbitrary, we have that $T(\alpha P_{\phi(y)}^{-1}) \subset \alpha P_y^{-1}$. In a similar way, it holds for T^{-1} that $T^{-1}(\alpha P_{\phi^{-1}(x)}^{-1}) \subset \alpha P_x^{-1}$ for every $x \in X$. Let $x = \phi(y)$. Then $T^{-1}(\alpha P_y^{-1}) \subset \alpha P_{\phi(y)}^{-1}$, so we see that

$$\alpha P_y^{-1} = T(T^{-1}(\alpha P_y^{-1})) \subset T(\alpha P_{\phi(y)}^{-1}) \subset \alpha P_y^{-1}.$$

Thus we have that $T(\alpha P_{\phi(y)}^{-1}) = \alpha P_y^{-1}$. □

CLAIM 9. *If $f \in C(X)^{-1}$, then $Tf(y) = f(\phi(y))$ holds for every $y \in Y$.*

PROOF. Let $f \in C(X)^{-1}$ and $y \in Y$. From Claim 6, we have $|Tf(y)| = |f(\phi(y))|$. Suppose $Tf(y) \neq f(\phi(y))$. Since $Tf(y) \neq 0$, there exists $\mathbf{u} \in P_y^{-1}$ such that $\sigma_\pi((Tf)\mathbf{u}) = \{Tf(y)\}$ by Lemma 2.8. Since T is surjective, there exists $u \in C(X)$ such that $Tu = \bar{\mathbf{u}}$. Since $\bar{\mathbf{u}}$ is also in P_y^{-1} , Claim 8 implies that $u \in P_{\phi(y)}^{-1}$ and $\sigma_\pi(Tf\bar{T}u) = \sigma_\pi((Tf)\mathbf{u}) = \{Tf(y)\}$. Let $\alpha = -|f(\phi(y))|^{-1}\overline{f(\phi(y))}$. Then we have $\sigma_\pi(\alpha Tf\bar{T}u) = \{\alpha Tf(y)\}$. Our assumption $Tf(y) \neq f(\phi(y))$ implies that $\alpha Tf(y) \neq -|Tf(y)|$. Thus we see that $-|Tf(y)| \notin (\alpha Tf\bar{T}u)(Y)$. It follows that

$$(6) \quad \left\| \alpha Tf\bar{T}u - 1 \right\| < |Tf(y)| + 1.$$

Since $\overline{\alpha}Tu \in \overline{\alpha}P_y^{-1}$, from Claim 8, there exists $u' \in P_{\phi(y)}^{-1}$ such that $T(\overline{\alpha}u') = \overline{\alpha}Tu$.

Then we see that

$$\begin{aligned} \|\alpha Tf \overline{Tu} - 1\| &= \|Tf \overline{T(\overline{\alpha}u')} - 1\| = \|\alpha f \overline{u'} - 1\| \\ &\geq \left| \alpha f(\phi(y)) \overline{u'(\phi(y))} - 1 \right| = |-|f(\phi(y))| - 1| \\ &= |-|Tf(y)| - 1| = |Tf(y)| + 1. \end{aligned}$$

Thus $\|\alpha Tf \overline{Tu} - 1\| \geq |Tf(y)| + 1$. This contradicts the inequality (6). Thus we have that $Tf(y) = f(\phi(y))$. \square

CLAIM 10. *If $f \in C(X)$, then $Tf(y) = f(\phi(y))$ holds for every $y \in Y$. In particular, T is an isometric algebra $*$ -isomorphism from $C(X)$ onto $C(Y)$.*

PROOF. First, we consider the case that $f(\phi(y)) \neq 0$ and $Tf(y) \neq 0$. Using Lemma 2.8, gives $u_1 \in P_{\phi(y)}^{-1}$ such that $\sigma_\pi(fu_1) = \{f(\phi(y))\}$. Let $\alpha = -\overline{f(\phi(y))}|f(\phi(y))|^{-1}$. Then we have $\sigma_\pi(\alpha fu_1) = \{-|f(\phi(y))|\}$. Also, there exists $u_2 \in P_y^{-1}$ such that $\sigma_\pi((Tf)u_2) = \{Tf(y)\}$. Let $u_2 = T^{-1}u_2$. Since $u_2 \in P_y^{-1}$, by Claim 8, $u_2 \in P_{\phi(y)}^{-1}$. Thus we have that $u_2(\phi(y)) = 1 = \|u_2\|$. It follows that $\sigma_\pi(\alpha fu_1 u_2) = \{-|f(\phi(y))|\}$. Thus we have

$$(7) \quad \left\| Tf \overline{T(\overline{\alpha u_1 u_2})} - 1 \right\| = \|\alpha f u_1 u_2 - 1\| = |f(\phi(y))| + 1.$$

Since $u_1, u_2 \in C(X)^{-1}$, Claim 9 shows that $T(\overline{\alpha u_1 u_2}) = (\overline{\alpha u_1 u_2}) \circ \phi$ and $u_2 \circ \phi = Tu_2 = u_2$. So we see that

$$\begin{aligned} \left\| Tf \overline{T(\overline{\alpha u_1 u_2})} - 1 \right\| &= \|(Tf)((\alpha u_1 u_2) \circ \phi) - 1\| \\ &\leq |\alpha| \|(Tf)u_2\| \|u_1 \circ \phi\| + 1 \\ &= \|(Tf)u_2\| + 1 \\ &= |Tf(y)| + 1. \end{aligned}$$

Combining this inequality and the equality (7), gives that $|f(\phi(y))| \leq |Tf(y)|$. In a similar way, we see $|\mathfrak{f}(\psi(x))| \leq |T^{-1}\mathfrak{f}(x)|$, where $\mathfrak{f} = Tf$ and $x = \phi(y)$. Thus

$|Tf(y)| = |f(\phi(y))|$. Since $\sigma_\pi(\alpha(Tf)\mathbf{u}_2) = \{\alpha Tf(y)\}$ and $u_1(\phi(y)) = 1 = \|u_1\|$, we have that $\sigma_\pi\left(Tf\overline{T(\overline{\alpha u_1 u_2})}\right) = \sigma_\pi((Tf)((\alpha u_1 u_2) \circ \phi)) = \sigma_\pi(\alpha(Tf)\mathbf{u}_2(u_1 \circ \phi)) = \{\alpha Tf(y)\}$. Thus $\left\|Tf\overline{T(\overline{\alpha u_1 u_2})}\right\| = |\alpha Tf(y)| = |f(\phi(y))|$. The equality (7) gives that $-|f(\phi(y))| \in \sigma_\pi\left(Tf\overline{T(\overline{\alpha u_1 u_2})}\right)$. Hence $-|f(\phi(y))| = \alpha Tf(y)$ because $\sigma_\pi\left(Tf\overline{T(\overline{\alpha u_1 u_2})}\right) = \{\alpha Tf(y)\}$. Since $-|f(\phi(y))| = \alpha f(\phi(y))$ holds from the definition of α , the equality $Tf(y) = f(\phi(y))$ holds.

Next, we consider the case where $Tf(y) = 0$ and suppose $f(\phi(y)) \neq 0$. Let ε be a positive number with $\varepsilon < |f(\phi(y))|$. Then there exists $\mathbf{u}_1 \in P_y^{-1}$ such that $|(Tf)\mathbf{u}_1| < \varepsilon$ on Y . Also, there exists $u_2 \in P_{\phi(y)}^{-1}$ such that $\sigma_\pi(fu_2) = \{f(\phi(y))\}$. Choose a complex number α with $|\alpha| = 1$ such that $\alpha f(\phi(y)) u_2(\phi(y)) = -|f(\phi(y))|$. Let $u_1 = T^{-1}\mathbf{u}_1$, then since $u_1 \in P_{\phi(y)}^{-1}$, we see that $\sigma_\pi(\alpha f u_1 u_2) = \{-|f(\phi(y))|\}$. It follows that

$$(8) \quad \left\|Tf\overline{T(\overline{\alpha u_1 u_2})} - 1\right\| = \|\alpha f u_1 u_2 - 1\| = |f(\phi(y))| + 1.$$

Applying Claim 9 to $\overline{\alpha u_1 u_2}$ and u_1 ,

$$\begin{aligned} \left\|Tf\overline{T(\overline{\alpha u_1 u_2})} - 1\right\| &= \|\alpha(Tf)(u_1 \circ \phi)(u_2 \circ \phi) - 1\| \\ &= \|\alpha(Tf)\mathbf{u}_1(u_2 \circ \phi) - 1\| \\ &\leq \|(Tf)\mathbf{u}_1\| \|u_2 \circ \phi\| + 1 < \varepsilon + 1. \end{aligned}$$

This inequality and (8) show that $|f(\phi(y))| < \varepsilon$, and this is a contradiction, so $f(\phi(y)) = 0 = Tf(y)$.

Finally, the case $f(\phi(y)) = 0$. Let $f = T^{-1}\mathfrak{f}$, $\mathfrak{f} \in C(Y)$ and $y = \psi(x)$, $x \in X$. Then the hypothesis implies that $T^{-1}\mathfrak{f}(x) = 0$. Noticing that $\phi = \psi^{-1}$, we can conclude from the argument in the previous paragraph that $\mathfrak{f}(\psi(x)) = 0 = T^{-1}\mathfrak{f}(x)$. This completes the proof. \square

Bibliography

- [1] R. B. Burckel, *Characterizations of $C(X)$ among its subalgebras*, Marcel Dekker, N. Y. 1972.
- [2] N. Brodskiy, J. Dydak, A. Karasev and K. Kawamura, *Root closed function algebras on compacta of large dimension*, Proc. Amer. Math. Soc., **135** (2007), 587–596.
- [3] A. Chigogidze, A. Karasev, K. Kawamura and V. Valov, *On C^* -algebras with the approximate n -th root property*, Bull. Austral. Math. Soc., **72** (2005), 197–212.
- [4] E. M. Čirka, *Approximation of continuous functions by functions holomorphic on Jordan arcs in \mathbb{C}^n* , Soviet Math., **7** (1966), 336–338.
- [5] R. S. Countryman Jr., *On the characterization of compact Hausdorff X for which $C(X)$ is algebraically closed*, Pacific J. Math., **20** (1967), 433–448.
- [6] T. W. Dawson, *A survey of algebraic extensions of commutative, unital normed algebras*, Contemp. Math., **328** (2003), 157–170.
- [7] D. Deckard and C. Pearcy, *On matrices over the ring of continuous complex valued functions on a Stonian space*, Proc. Amer. Math. Soc., **14** (1963), 322–328.
- [8] D. Deckard and C. Pearcy, *On algebraic closure in function algebras*, Proc. Amer. Math. Soc., **15** (1964), 259–263.
- [9] J. F. Feinstein and T. J. Oliver, *Extensions of endomorphisms of $C(X)$* , Proc. Amer. Math. Soc., **135** (2007), 109–117.
- [10] M. K. Fort, Jr., *Images of plane continua*, Amer. J. Math., **81** (1959), 541–546.
- [11] T. W. Gamelin, *Uniform algebras*, Prentice-Hall, N.J. 1969.
- [12] E. A. Gorin and M. I. Karahanjan, *Some certain characteristic properties of the algebra of all continuous functions on a locally connected compactum*, Izv. Akad. Nauk Armjan. SSR Ser. Mat., **11** (1976), 237–255.
- [13] O. Hatori and T. Miura, *On a characterization of the maximal ideal spaces of commutative C^* -algebras in which every element is the square of another*, Proc. Amer. Math. Soc., **128** (2000), 1185–1189.
- [14] O. Hatori, T. Miura and H. Takagi, *Characterizations of isometric isomorphisms between uniform algebras via nonlinear range-preserving properties*, Proc. Amer. Math. Soc., **134** (2006) 2923–2930.
- [15] O. Hatori, T. Miura and H. Takagi, *Unital and multiplicatively spectrum-preserving surjections between semi-simple commutative Banach algebras are linear and multiplicative*, J. Math. Anal. Appl., **326** (2007), 281–296.
- [16] O. Hatori, T. Miura and H. Takagi, *Multiplicatively spectrum-preserving and norm-preserving maps between invertible groups of commutative Banach algebras*, preprint.

- [17] S. H. Hochwald, *Multiplicative maps on matrices that preserve the spectrum*, Linear Algebra Appl., **212/213** (1994), 339–351
- [18] D. Honma, *Surjections on the algebras of continuous functions which preserve peripheral spectrum*, Contemp. Math., **435** (2007), 199–205
- [19] D. Honma and T. Miura, *On a characterization of compact Hausdorff space X for which certain algebraic equations are solvable in $C(X)$* , Tokyo J. Math., **30** (2007), 403–416.
- [20] D. Honma and T. Miura, *Remarks on n -th root closedness for commutative C^* -algebras*, Nihonkai J. Math., **18** (2007), 51–55.
- [21] D. Honma, *Norm-preserving surjections on algebras of continuous functions*, to appear in Rocky Mountain J. Math.
- [22] M. I. Karahanjan, *Some algebraic characterizations of the algebra of all continuous functions on a locally connected compactum*, Math. USSR-Sb., **35** (1979), 681–696.
- [23] K. Kawamura and T. Miura, *On the existence of continuous (approximate) roots of algebraic equations*, Topology Appl., **154** (2007), 434–442.
- [24] K. Kawamura and T. Miura, *On the root closedness of continuous function algebras*, preprint.
- [25] K. Kuratowski, *Topology vol.II*, New edition 1968 Academic Press.
- [26] S. Lambert, A. Luttman and T. Tonev, *Weakly peripherally-multiplicative mappings between uniform algebras*, Contemp. Math., **435** (2007), 265–281
- [27] A. Luttman and T. Tonev, *Uniform algebra isomorphisms and peripheral-multiplicativity*, Proc. Amer. Math. Soc., **135** (2007), 3589–3598.
- [28] T. Miura, *On commutative C^* -algebras in which every element is almost the square of another*, Contemp. Math., **232** (1999), 239–242.
- [29] T. Miura and K. Nijima, *On a characterization of the maximal ideal spaces of algebraically closed commutative C^* -algebras*, Proc. Amer. Math. Soc., **131** (2003), 2869–2876.
- [30] T. Miura and D. Honma, *A survey of certain algebraic equations in commutative C^* -algebras*, Contemp. Math., **435** (2007), 327–333.
- [31] L. Molnár, *Some characterizations of the automorphisms of $B(H)$ and $C(X)$* , Proc. Amer. Math. Soc., **130** (2002), 111–120.
- [32] K. Morita, *Dimension of general topological spaces*, Surveys in general topology (G. M. Reed ed.) Academic Press N.Y.1980.
- [33] T. Petek and P. Šemrl, *Characterization of Jordan homomorphisms on M_n using preserving properties*, Linear Algebra Appl., **269** (1998), 33–46.
- [34] T. Ransford, *A Cartan theorem for Banach algebras*, Proc. Amer. Math. Soc., **124** (1996), 243–247
- [35] N.V. Rao and A.K. Roy, *Multiplicatively spectrum-preserving maps of function algebras*, Proc. Amer. Math. Soc., **133** (2005), 1135–1142
- [36] N.V. Rao and A.K. Roy, *Multiplicatively spectrum-preserving maps of function algebras.II*, Proc. Edin. Math. Soc., **48** (2005), 219–229