Invariant subspaces of $L^2(\mathbb{T}^2)$ for irrational rotation unitary systems

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Chapter 1

Introduction

In [2], Beurling characterized the invariant subspaces ($z\mathfrak{M} \subset \mathfrak{M}$) of the Hardy space $H^2(\mathbb{T})$ ($\mathbb{T}$ the unit circle), that is, if $\mathfrak{M}$ is an invariant subspace of $H^2(\mathbb{T})$, then $\mathfrak{M}$ is of the form $\phi H^2(\mathbb{T})$, where ϕ is an inner function. Wiener proved that the doubly invariant subspaces ($z\mathfrak{M} = \mathfrak{M}$) of the Lebesgue space $L^2(\mathbb{T})$ are of the form $\chi_E L^2(\mathbb{T})$, where χ_E is the characteristic function of a Borel subset E in $\mathbb{T}$, while the simply invariant subspaces ($z\mathfrak{M} \subsetneq \mathfrak{M}$) of $L^2(\mathbb{T})$ are of the form $\phi H^2(\mathbb{T})$, where ϕ is a unimodular function. The most general technique indicated was the Wold-type decomposition.

We let $\mathbb{T}^2$ be the torus that is the cartesian product of 2 unit circles $\mathbb{T}$. The usual Lebesgue spaces on $\mathbb{T}^2$ are denoted by $L^2(\mathbb{T}^2)$. A closed subspace $\mathfrak{M}$ of $L^2(\mathbb{T}^2)$ is said to be invariant if $M_z\mathfrak{M} \subset \mathfrak{M}$ and $M_w\mathfrak{M} \subset \mathfrak{M}$, where M_z and M_w are the multiplication operators with the coordinate functions on $L^2(\mathbb{T}^2)$, that is, $M_z f = zf$ and $M_w f = wf$ ($f \in L^2(\mathbb{T}^2)$). As is well known, the form of invariant subspaces of $L^2(\mathbb{T}^2)$ or even $H^2(\mathbb{T}^2)$ is much more complicated. In general, the invariant subspaces of $L^2(\mathbb{T}^2)$ are not necessarily of the form $\phi H^2(\mathbb{T}^2)$ with some unimodular function ϕ. But the structure of Beurling-type invariant subspaces has been studied in recent years and, in particular, some necessary and sufficient condition for invariant subspaces to be Beurling-type have been given by many researchers (cf. [9, 10, 19, 20, 21, 22, 28, 32, 33], etc.).

In [28], the invariant subspaces $\mathfrak{M}$ of $H^2(\mathbb{T}^2)$ of the Beurling-type were characterized as the subspaces on which M_z and M_w are doubly commuting (that is, $M_z|_{\mathfrak{M}}$ commutes with $M_w^*|_{\mathfrak{M}}$).

In [42], an extension of the Wold-type decomposition was given for two doubly commuting isometries (this extension was further extended for an arbitrary commuting isometries by Popovici in [41]). In [10], this decomposition was used to characterize the form of invariant subspaces of $L^2(\mathbb{T}^2)$ on which M_z and M_w are doubly commuting.

An irrational rotation C^*-algebra $\mathcal{A}_\theta$ is a C^*-algebra generated by a pair of unitary elements U and V which satisfy the relation $UV = e^{2\pi i\theta}VU$, where θ is an irrational number in $(0,1)$. It has received a lot of special attention in recent years(cf. [6, 7, 8, 25, 40], etc.). A unitary system $\mathcal{U}$ is a subset of the unitary operators acting on a separable Hilbert space $\mathcal{H}$ which contains the identity operator I. A norm one element $\psi \in \mathcal{H}$ is called a complete wandering vector for $\mathcal{U}$ if $\mathcal{U}\psi = \{U\psi : U \in \mathcal{U}\}$ is an orthonormal basis for $\mathcal{U}$. The set of all complete wandering vector for $\mathcal{U}$ is denoted by $\mathcal{W}(\mathcal{U})$. In [13], Han introduced a notion of an irrational rotation unitary system $\mathcal{U}_{U,V} = \{U^mV^n : m, n \in \mathbb{Z}\}$, where U and V are defined above rule and claimed that, up to unitary equivalence, there exists a unique faithful representation π of $\mathcal{A}_\theta$ on $L^2(\mathbb{T}^2)$. Operators L_z and L_w constructed in the proof are very interesting, because $\mathcal{U} = \{L_z^m L_w^n : m, n \in \mathbb{Z}\}$ is an irrational rotation unitary system and has some properties, that is, $\mathcal{W}(\mathcal{U})$ is a non-empty closed connected subset of $L^2(\mathbb{T}^2)$ and the closure of the linear span of $\mathcal{W}(\mathcal{U})$ is $L^2(\mathbb{T}^2)$.

Motivated by these facts, in this paper, we consider a von Neumann algebra $\mathfrak{L}$ generated by L_z and L_w and study invariant subspace structure of $L^2(\mathbb{T}^2)$ with respect to $\mathfrak{L}$, whose meaning is different from "usual" invariant subspace, that is, our setting is the following. Let θ be an irrational number in $(0,1)$. As in the proof of [13, Theorem 1], we define the unitary operators on $L^2(\mathbb{T}^2)$ satisfying:

$$L_z(z^m w^n) = z^{m+1} w^n \quad and \quad L_w(z^m w^n) = e^{-2\pi i m\theta} z^m w^{n+1},$$

where $(z, w) \in \mathbb{T}^2$. Moreover we also define the unitary operators R_z and R_w as follows:

$$R_z(z^m w^n) = e^{-2\pi i n\theta} z^{m+1} w^n \quad and \quad R_w(z^m w^n) = z^m w^{n+1},$$

where $(z, w) \in \mathbb{T}^2$. Let $\mathfrak{L}$ (resp. $\mathfrak{R}$) denote the von Neumann algebra generated by L_z and L_w (resp. R_z and R_w). Then $\mathfrak{L}$ and $\mathfrak{R}$ are II_1-factors, which are important classes of von Neumann algebras. Let $\mathfrak{M}$ be a closed subspace of $L^2(\mathbb{T}^2)$. Then we say that $\mathfrak{M}$ is left-invariant (resp. right-invariant) if $L_z\mathfrak{M} \subset \mathfrak{M}$ and $L_w\mathfrak{M} \subset \mathfrak{M}$ (resp. $R_z\mathfrak{M} \subset \mathfrak{M}$ and $R_w\mathfrak{M} \subset \mathfrak{M}$). If $\mathfrak{M}$ is both left-invariant and right-invariant, then $\mathfrak{M}$ is called two-sided invariant.

In chapter 2, we recall some notions about the theory of von Neumann algebra which will be used later. We also recall the invariant subspaces of L^2-space on both $\mathbb{T}$ and $\mathbb{T}^2$. About the invariant subspaces of $L^2(\mathbb{T})$, we only recall the Beurling theorem. On the other hand, about the invariant subspaces $\mathfrak{M}$ of $L^2(\mathbb{T}^2)$, at first we recall the result of the doubly invariant case ($z\mathfrak{M} = w\mathfrak{M} = \mathfrak{M}$), and then recall some results of the simply invariant case ($z\mathfrak{M} \subsetneq \mathfrak{M}$, $w\mathfrak{M} \subsetneq \mathfrak{M}$), which have not been complete characterization of the invariant subspaces of $L^2(\mathbb{T}^2)$ yet. These results of the invariant subspaces of $L^2(\mathbb{T}^2)$ should be compared with the results in §4.2. We also describe the Popovici's Wold-type decomposition for commuting isometries which will be used in §4.4. Needed in §3.2, we prepare for the notion of unitary systems. Although unitary systems were first introduced as a generalization of wavelet systems, conversely we introduce wavelet systems as a example of unitary systems.

In chapter 3, we study irrational rotation C*-algebras $\mathcal{A}_\theta$. In particular, we prove the existence of a unique trace on $\mathcal{A}_\theta$ and prove that $\mathcal{A}_\theta$ is simple. We also prove the Han's theorem that an irrational rotation C*-algebra can be representated on $L^2(\mathbb{T}^2)$ and that there is an irrational rotation unitary system $\mathcal{U}$ which has complete wandering vectors, $\mathcal{W}(\mathcal{U})$ is closed connected subset of $L^2(\mathbb{T}^2)$ and the closure of the linear span of $\mathcal{U}$ is $L^2(\mathbb{T}^2)$.

In chapter 4, we have the invariant subspace structure of $L^2(\mathbb{T}^2)$. At first, we give a characterization of Beurling-type left-invariant subspaces of $L^2(\mathbb{T}^2)$, which is a generalization of the Mandrekar's result ([28]) in a sense. We also give a structure theorem of a non-trivial two-sided invariant subspace of $L^2(\mathbb{T}^2)$. Finally, let $\mathfrak{M}$ be a non-trivial two-sided invariant subspace of $L^2(\mathbb{T}^2)$. We consider the Popovici's Wold-type decomposition for certain commuting isometries $U = (L_z L_w)|_{\mathfrak{M}}$ and $V = (R_z R_w)|_{\mathfrak{M}}$, and prove that the couple $W = (U, V)$ is a weak bi-shift on $\mathfrak{M}$.

Chapter 2

Notations and preliminaries

2.1 von Neumann algebras

In this section we recall some notions about the theory of von Neumann algebra which will be often used later. For a von Neumann algebra $\mathcal{A}$, $\mathcal{P}(\mathcal{A})$ denotes a set of all projections of $\mathcal{A}$. Let $\mathcal{B}(\mathcal{H})$ be the set of all bounded linear operators on a Hilbert space $\mathcal{H}$ and I denotes an identity operator on $\mathcal{H}$. For an element A in $\mathcal{A}$, we call it *positive*, we write $A \geq 0$, if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathcal{H}$. For a subset $\mathcal{S}$ of $\mathcal{B}(\mathcal{H})$, we use $\mathrm{W}^*(\mathcal{S})$ to denote the von Neumann algebra generated by $\mathcal{S}$, use $\mathbb{U}(\mathcal{S})$ to denote the set of all unitary operators in $\mathcal{S}$, and use $\mathcal{S}_+$ to denote the set of all positive elements in $\mathcal{S}$. The commutant of $\mathcal{S}$ is $\mathcal{S}' = \{T \in \mathcal{B}(\mathcal{H}) : TS - ST = 0, {}^\forall S \in \mathcal{S}\}$. $\mathcal{Z} = \mathcal{A} \cap \mathcal{A}'$ is called the *center* of $\mathcal{A}$. If $\mathcal{Z} = \mathbb{C}I$, then $\mathcal{A}$ is called a *factor*. For a subset $\mathcal{M}$ of $\mathcal{H}$, we use $[\mathcal{M}]$ to denote the closure of the linear span of $\mathcal{M}$.

Definition 2.1.1 Let $\mathcal{A}$ be a von Neumann algebra. Let P and Q be two projections in $\mathcal{P}(\mathcal{A})$. P and Q are said to be ; *equivalent* (relative to $\mathcal{A}$), and we write $P \sim Q$, if there exists an element $V \in \mathcal{A}$ such that $V^*V = P$ and $VV^* = Q$; *partially equivalent* (relative to $\mathcal{A}$), and we write $P \preceq Q$, if there exists an element $R \in \mathcal{P}(\mathcal{A})$ such that $P \sim R \leq Q$.

Definition 2.1.2 Let $\mathcal{A}$ be a von Neumann algebra. A projection $P \in \mathcal{P}(\mathcal{A})$ is said to be ;*abelian* if $P\mathcal{A}P$ is abelian ; *finite* if any projection $Q \in \mathcal{P}(\mathcal{A})$ with $P \preceq Q$ and $P \sim Q$ implies $Q = P$. Moreover $\mathcal{A}$ is said to be ; *finite* if I is finite ; *semi-finite* if for all non-zero element $P \in \mathcal{Z}$ there exists a non-zero finite projection $Q \in \mathcal{P}(\mathcal{A})$ such that $Q \leq P$; *typeII* if $\mathcal{A}$ contains no non-zero abelian projection and is semi-finite ; *typeII$_1$* if $\mathcal{A}$ is type II and finite.

Proposition 2.1.3 ([27, Proposition 6.3.1]) *Let $\mathcal{A}$ be a von Neumann algebra. Then $\mathcal{A}$ is finite if and only if any isometry in $\mathcal{A}$ is unitary.*

Proof. Suppose that $\mathcal{A}$ is finite. Putting $VV^* = P$, since $P \in \mathcal{P}(\mathcal{A})$ and $V^*V = I$, we have $P \sim I$. It is clear that $P \leq I$. Since I is finite, we see $P = I$, that is, $VV^* = I$. Conversely suppose that $P \leq I, P \sim I$ and $P \in \mathcal{P}(\mathcal{A})$. Then there exists a isometry V such that $VV^* = P$, $V^*V = I$. But by assumption we have $VV^* = I$. Thus we see $P = I$, that is, $\mathcal{A}$ is finite. ∎

Definition 2.1.4 Let $\mathcal{A}$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$. A vector $x \in \mathcal{H}$ is said to be ; a *cyclic vector* for $\mathcal{A}$ if $[\mathcal{A}x] = \mathcal{H}$; a *trace vector* for $\mathcal{A}$ if $\langle ABx, x\rangle = \langle BAx, x\rangle$ for all A and B in $\mathcal{A}$.

The next two results are used to prove Theorem 3.2.1. They are proved in detail in [24], and so we omit the proof.

Lemma 2.1.5 ([24, Lemma 7.2.14]) *Let $\mathcal{A}$ be a von Neumann algebra. If x is a cyclic trace vetctor for $\mathcal{A}$, Then x is a cyclic trace vetctor for $\mathcal{A}'$.*

Theorem 2.1.6 ([24, Theorem 7.2.15]) *Let $\mathcal{A}$ be a von Neumann algebra. If $\mathcal{A}$ has a cyclic trace vector x, then $\mathcal{A}$ is finite.*

Definition 2.1.7 ([27, Definition 6.5.1]) Let $\mathcal{A}$ be a von Neumann algebra. A *trace* on $\mathcal{A}_+$ is a function τ on $\mathcal{A}_+$, taking non-negative, possibly infinite, real values, possessing the following properties:

(1) $\tau(A+B) = \tau(A) + \tau(B)$, $A, B \in \mathcal{A}_+$;

(2) $\tau(\alpha A) = \alpha\tau(A)$, $A \in \mathcal{A}_+$, $\alpha \geq 0$ (with the convention that $0 \cdot +\infty = 0$);

(3) $\tau(A^*A) = \tau(AA^*)$, $A \in \mathcal{A}_+$.

A trace τ on $\mathcal{A}_+$ is said to be ; *faithful* if $A \in \mathcal{A}_+$ is such that $\tau(A) = 0$, then $A = 0$; *finite* if $\tau(I) < \infty$; *semi-finite* if for all non-zero element $A \in \mathcal{A}_+$, there exists a non-zero element $B \in \mathcal{A}_+$ and $B \leq A$ such that $\tau(B) < +\infty$. ; *normal* if for all bounded increasing net A_λ of $\mathcal{A}_+$, we have

$$\tau(sup_\lambda A_\lambda) = sup_\lambda \tau(A_\lambda).$$

The next theorems is needed in §4.1.

Theorem 2.1.8 ([44, Theorem 4.64]) *A von Neumann algebra $\mathcal{A}$ is finite if and only if there is a family of finite normal trace.*

Theorem 2.1.9 ([44, Theorem 4. 67],[27, Theorem 6.5.8]) *A von Neumann algebra $\mathcal{A}$ is semi-finite if and only if there is a faithful semi-finite normal trace.*

More about von Neumann algebras, we refer to [4, 24, 27, 43, 44], etc..

2.2 Invariant subspaces of $L^2(\mathbb{T})$ and $L^2(\mathbb{T}^2)$

Let $\mathbb{T}$ be the unit circle in $\mathbb{C}$. The usual Lebesgue spaces, with respect to the Haar Measure m on $\mathbb{T}$, are denoted by $L^2(\mathbb{T})$, and $H^2(\mathbb{T})$ is the space of all f in $L^2(\mathbb{T})$ whose Fourier coefficients

$$\hat{f}(m) = \langle f, z^m \rangle = \int_{\mathbb{T}} f(z)\overline{z}^m dm$$

satisfy $\hat{f}(m) = 0$ for $m < 0$.

A closed subspace $\mathfrak{M}$ of $L^2(\mathbb{T})$ is said to be invariant if

$$z\mathfrak{M} \subset \mathfrak{M}.$$

In [2], Beurling characterized the invariant subspaces of the Hardy space. In [17], Beurling theorem was extended to obtain the invariant subspaces of $L^2(\mathbb{T})$.

Theorem 2.2.1 (Beurling [17]) *Let $\mathfrak{M}$ be an invariant subspace of $L^2(\mathbb{T})$.*

(1) *If $z\mathfrak{M} = \mathfrak{M}$, then*

$$\mathfrak{M} = \chi_E L^2(\mathbb{T})$$

where χ_E is a characteristic function of Borel set $E \subset \mathbb{T}$.

(2) *If $z\mathfrak{M} \neq \mathfrak{M}$, then*

$$\mathfrak{M} = \phi H^2(\mathbb{T})$$

where ϕ is a unimodular function, that is, $|\phi| = 1$ a.e. on $\mathbb{T}$.

In particular, if $\mathfrak{M}$ be an invariant subspace of $H^2(\mathbb{T})$, then

$$\mathfrak{M} = \phi H^2(\mathbb{T})$$

where ϕ is an inner function, that is, a unimodular function in $H^\infty(\mathbb{T})$.

We let $\mathbb{T}^2$ be the torus that is the cartesian product of 2 unit circles in $\mathbb{C}$. The usual Lebesgue spaces, with respect to the Haar Measure m on $\mathbb{T}^2$, are denoted by $L^2(\mathbb{T}^2)$, and $H^2(\mathbb{T}^2)$ is the space of all f in $L^2(\mathbb{T}^2)$ whose Fourier coefficients

$$\hat{f}(m,n) = \langle f, z^m w^n \rangle = \int_{\mathbb{T}^2} f(z,w)\overline{z}^m \overline{w}^n \, dm,$$

satisfy $\hat{f}(m,n) = 0$ for $m < 0$ or $n < 0$.

A closed subspace $\mathfrak{M}$ of $L^2(\mathbb{T}^2)$ is said to be invariant if

$$M_z\mathfrak{M} \subset \mathfrak{M} \quad and \quad M_w\mathfrak{M} \subset \mathfrak{M}$$

where M_z and M_w are the multiplication operators with the coordinate functions on $L^2(\mathbb{T}^2)$, that is, for $f \in L^2(\mathbb{T}^2)$

$$M_z f = zf \quad and \quad M_w f = wf.$$

More simply, we also say $\mathfrak{M}$ is invariant if

$$z\mathfrak{M} \subset \mathfrak{M} \quad and \quad w\mathfrak{M} \subset \mathfrak{M}.$$

As is well known, the problem of invariant subspaces of $L^2(\mathbb{T}^2)$ on the 2-dimensional torus or even in the corresponding $H^2(\mathbb{T}^2)$ is more complicated than that of $L^2(\mathbb{T})$.

If $M_z\mathfrak{M} = \mathfrak{M}$ and $M_w\mathfrak{M} = \mathfrak{M}$, $\mathfrak{M}$ is called *doubly invariant*. It is well known that:

Lemma 2.2.2 ([10, Lemma 3]) *Every doubly invariant subspace of $L^2(\mathbb{T}^2)$ is of the form $\chi_E L^2(\mathbb{T}^2)$, where χ_E is a characteristic function of Borel set $E \subset \mathbb{T}^2$.*

We shall say $\mathfrak{M}$ is *simply invariant* if $M_z\mathfrak{M} \subsetneq \mathfrak{M}$ and $M_w\mathfrak{M} \subsetneq \mathfrak{M}$. The simply invariant subspaces of $L^2(\mathbb{T}^2)$ are not fully known, their structure being much more complicated. But they have been studied in various ways (cf. [9, 10, 19, 20, 21, 22, 28, 32, 33], etc.). In [28], the invariant subspaces $\mathfrak{M}$ of $H^2(\mathbb{T}^2)$ of the Beurling form were characterized as the subspaces on which multiplication operators M_z and M_w are doubly commuting (that is, $M_z|_{\mathfrak{M}}$ commutes with $M_w^*|_{\mathfrak{M}}$). We note that if $\mathfrak{M}$ is an invariant subspace, then $M_z|_{\mathfrak{M}}$ commutes with $M_w|_{\mathfrak{M}}$.

Theorem 2.2.3 ([28, Theorem 2]) *An invariant subspace $\mathfrak{M} \neq \{0\}$ of $H^2(\mathbb{T}^2)$ is of form $qH^2(\mathbb{T}^2)$ with q inner function if and only if M_z and M_w are doubly commuting on $\mathfrak{M}$.*

By using Słociński's Wold-type decomposition (cf. Theorem 2.3.2), the generalization of Theorem 2.2.3 follows. H_1 (resp. H_2) is the space of all $f \in L^2(\mathbb{T}^2)$ so that $\hat{f}(m,n) = 0$ for $m < 0$ (resp. $n < 0$) and L_1 (resp. L_2) is the space of all $f \in L^2(\mathbb{T}^2)$ so that $\hat{f}(m,n) = 0$ for $n \neq 0$ (resp. $m \neq 0$).

Theorem 2.2.4 ([10, Theorem 2]) Let $\mathfrak{M}$ be an invariant subspace of $L^2(\mathbb{T}^2)$. Then M_z and M_w are doubly commuting on $\mathfrak{M}$ if and only if

$$\mathfrak{M} = qH^2(\mathbb{T}^2) + \chi_{E_1} q_1 H_1 + \chi_{E_2} q_2 H_2 + \chi_E L^2(\mathbb{T}^2)$$

where q, q_1, q_2 are unimodular functions, $\chi_E \in L^2(\mathbb{T}^2)$, $\chi_{E_1} \in L_2$ and $\chi_{E_2} \in L_1$.

In [32], the invariant subspace $\mathfrak{M}$ of $L^2(\mathbb{T}^2)$ with $\mathfrak{M} \ominus M_w\mathfrak{M} = \{0\}$ was paid attention to.

Theorem 2.2.5 ([32, Theorem 5]) *Let $\mathfrak{M}$ be an invariant subspace of $L^2(\mathbb{T}^2)$ and $\mathfrak{M} \ominus M_w\mathfrak{M} = \mathfrak{S}_w \neq \{0\}$.*

(1) *$M_z\mathfrak{S}_w = \mathfrak{S}_w$ if and only if*

$$\mathfrak{M} = \chi_{E_1} qH_2 \oplus \chi_{E_2} L^2(\mathbb{T}^2)$$

where q is a unimodular function, $\chi_{E_1} \in L_1$, $\chi_{E_1} + \chi_{E_2} \leq 1$ a.e..

(2) *$M_z\mathfrak{S}_w \subsetneq \mathfrak{S}_w$ if and only if*

$$\mathfrak{M} = qH^2(\mathbb{T}^2)$$

where q is a unimodular function.

Remark 2.2.6 As is written in [10], Theorem 2.2.5 and Theorem 2.2.3 are corollaries of Theorem 2.2.4.

2.3 Decompositions

Wold introduced, in a probabilistic language, a remarkable decomposition for stationary stochastic processes ([45]). The study of isometric operators on Hilbert spaces is reduced to the study of unitary operators and unilateral shifts.

A *(unilateral) shift* is an operator S on a Hilbert space $\mathcal{H}$ unitarily equivalent to multiplication by the independent variable z on a certain Hardy space on the torus $\mathbb{T}$. More precisely, there exists a Hilbert space $\mathcal{W}$ and a unitary operator $U : \mathcal{H} \to H^2(\mathbb{T}) \otimes \mathcal{W}$ such that $S = U^*(T_z \otimes I)U$ (the symbol "$\otimes$" denotes the Hilbertian tensor product). The following characterization illustrates the shift's geometrical structure: an isometry S on $\mathcal{H}$ is a shift if and only if there exists a subspace $\mathcal{W}$ such that $\mathcal{H} = \sum \oplus_{n \geq 0} S^n \mathcal{W}$. $\mathcal{W}$ is unique ($\mathcal{W} = \ker S^*$) and is said to be the *defect* of S.

An operator V on $\mathcal{H}$ is said to be *reduced* by a (closed) subspace $\mathcal{H}_0 \subset \mathcal{H}$ if $\mathcal{H}_0$ is invariant under both V and V^*, that is, $V\mathcal{H}_0 \subset \mathcal{H}_0$ and $V^*\mathcal{H}_0 \subset \mathcal{H}_0$.

Theorem 2.3.1 (Wold [37], Chapter 1) *For any isometry V on $\mathcal{H}$ there corresponds a unique orthogonal decomposition of the form*

$$\mathcal{H} = \mathcal{H}_u \oplus \mathcal{H}_s$$

such that V is reduced by $\mathcal{H}_u$ to a unitary operator and by $\mathcal{H}_s$ to a shift. More exactly,

$$\mathcal{H}_u = \bigcap_{n \geq 0} V^n \mathcal{H}$$

and

$$\mathcal{H}_s = \sum_{n \geq 0} \oplus V^n \mathcal{W}$$

where $\mathcal{W} = \ker V^*$.

We want to underline the importance of this decomposition in invariant subspace theory. In [42], an extension of the Wold-type decomposition is given for two doubly commuting isometries. Let $V = (V_1, V_2)$ be a commuting isometries on a Hilbert space $\mathcal{H}$ and $p = (p_1, p_2)$ be a pair of integers. We use the notation V^p for ; $V_1^{p_1}V_2^{p_2}$ if $p_1, p_2 \geq 0$; $V_1^{*|p_1|}V_2^{p_2}$ if $p_1 < 0, p_2 \geq 0$; $V_2^{*|p_2|}V_1^{p_1}$ if $p_1 \geq 0, p_2 < 0$; $V_1^{*|p_1|}V_2^{*|p_2|}$ if $p_1, p_2 < 0$.

Theorem 2.3.2 ([42, Theorem 3]) *To any doubly commuting isometric pair $V = (V_1, V_2)$ there corresponds a unique orthogonal decomposition of the form*

$$\mathcal{H} = \mathcal{H}_{uu} \oplus \mathcal{H}_{us} \oplus \mathcal{H}_{su} \oplus \mathcal{H}_{ss}$$

such that $\mathcal{H}_{\alpha_1\alpha_2}$ reduces V_i to a unitary operator if $\alpha_i = u$ and to a unilateral shift if $\alpha_i = s$, $i = 1, 2$. More exactly,

$$\mathcal{H}_{uu} = \bigcap_{m\geq 0} V_1^m\mathcal{H} \cap \bigcap_{n\geq 0} V_2^n\mathcal{H}$$

$$\mathcal{H}_{us} = \sum_{n\geq 0} \oplus V_2^n\Big(\bigcap_{m\geq 0} V_1^m\mathcal{W}_2\Big)$$

$$\mathcal{H}_{su} = \sum_{m\geq 0} \oplus V_1^m\Big(\bigcap_{n\geq 0} V_2^n\mathcal{W}_1\Big)$$

and

$$\mathcal{H}_{ss} = \sum_{p\in\mathbb{Z}_+^2} \oplus V^p\mathcal{W}$$

where $\mathcal{W}_1 = \ker V_1^*$, $\mathcal{W}_2 = \ker V_2^*$ and $\mathcal{W} = \mathcal{W}_1 \cap \mathcal{W}_2$.

Definition 2.3.3 ([41]) Let $\mathcal{H}$ be a Hilbert space and $V = (V_1, V_2)$ be a commuting pair of isometries (in short, a *bi-isometry*). We shall say that V is ; a *bi-unitary* on $\mathcal{H}$ if the both V_1 and V_2 are unitary operators on $\mathcal{H}$; a *unitary-shift* on $\mathcal{H}$ if V_1 is a unitary and V_2 is a shift on $\mathcal{H}$; a *shift-unitary* on $\mathcal{H}$ if V_1 is a shift and V_2 is a unitary on $\mathcal{H}$; a *weak bi-shift* if

$$V_1|_{\bigcap_{i\geq 0}\ker V_2^*V_1^i},\ V_2|_{\bigcap_{j\geq 0}\ker V_1^*V_2^j} \text{ and } V_1V_2 \text{ are shifts on } \mathcal{H}.$$

Popovici obtained such a Wold-type decomposition for an arbitary bi-isometry, the bi-shift part $\mathcal{H}_{ss}$ being replaced by the weak bi-shift part $\mathcal{H}_{ws}$.

Theorem 2.3.4 ([41, Theorem 2.8]) *Let $V = (V_1, V_2)$ be a bi-isometry on $\mathcal{H}$. Then there is a unique orthogonal decomposition of the form*

$$\mathcal{H} = \mathcal{H}_{uu} \oplus \mathcal{H}_{us} \oplus \mathcal{H}_{su} \oplus \mathcal{H}_{ws}$$

into reducing subspaces for V such that $V|_{\mathcal{H}_{uu}}$ is a bi-unitary, $V|_{\mathcal{H}_{us}}$ is a unitary-shift, $V|_{\mathcal{H}_{su}}$ is a shift-unitary and $V|_{\mathcal{H}_{ws}}$ is a weak bi-shift. More exactly,

$$\mathcal{H}_{uu} = \bigcap_{n\geq 0}(V_1V_2)^n\mathcal{H},$$

$$\mathcal{H}_{us} = \sum_{n\geq 0} \oplus V_2^n(\bigcap_{m\geq 0} V_1^m(\bigcap_{i\geq 0} \ker V_2^* V_1^i))$$

and

$$\mathcal{H}_{su} = \sum_{m\geq 0} \oplus V_1^m(\bigcap_{n\geq 0} V_2^n(\bigcap_{j\geq 0} \ker V_1^* V_2^j)).$$

Remark 2.3.5 In the above theorem, putting $\mathcal{H}_{uu}^{\perp} = \mathcal{H}_{us} \oplus \mathcal{H}_{su} \oplus \mathcal{H}_{ws}$, the orthogonal decomposition

$$\mathcal{H} = \mathcal{H}_{uu} \oplus \mathcal{H}_{uu}^{\perp}$$

coinscides with the Wold decompositon attached to the isometry V_1V_2.

2.4 Unitary systems

Following Dai and Larson [5], a *unitary system* $\mathcal{U}$ is a subset of the unitary operators acting on a separable Hilbert space $\mathcal{H}$ which contains the identity operator I. A norm one element $\psi \in \mathcal{H}$ is called a *wandering vector* for $\mathcal{U}$ if $\mathcal{U}\psi = \{U\psi : U \in \mathcal{U}\}$ is an orthonormal set; that is, $\langle U\psi, V\psi\rangle = 0$ if $U, V \in \mathcal{U}$ and $U \neq V$. If $\mathcal{U}\psi$ is an orhonormal basis for $\mathcal{H}$, then ψ is called a *complete wandering vector* for $\mathcal{U}$. The set of all complete wandering vector for $\mathcal{U}$ is denoted by $\mathcal{W}(\mathcal{U})$.

If U and V are unitary operators in $\mathcal{B}(\mathcal{H})$, we write $\mathcal{U}_{U,V} = \{U^m V^n : m, n \in \mathbb{Z}\}$. Unitary systems of this form have an importance in wavelet theory (about wavelet theory, we refer to [1, 11, 26, 36, 46], etc.). If U and V satisfy the relation $UV = e^{2\pi i\theta}VU$ with $\theta \in (0,1)$ an irrational number, then we call $\mathcal{U}_{U,V}$ an irrational rotation unitary system.

If $\mathcal{U}$ is a unitary system and $\psi \in \mathcal{W}(\mathcal{U})$, the *local commutant* $C_\psi(\mathcal{U})$ at ψ is defined by $\{V \in \mathcal{B}(\mathcal{H}) : (VU - UV)\psi = 0, {}^\forall U \in \mathcal{U}\}$. A useful result is the one-to-one correspondence between the complete wandering vectors and the unitary operators in $C_\psi(\mathcal{U})$:

Proposition 2.4.1 ([5, Proposition 1.3]) *Let $\mathcal{U}$ be a unitary system. Suppose $\psi \in \mathcal{W}(\mathcal{U})$. Then*

$$\mathcal{W}(\mathcal{U}) = \mathbb{U}(C_\psi(\mathcal{U}))\psi = \{V\psi : V \in \mathbb{U}(C_\psi(\mathcal{U}))\}.$$

Moreover, the correspondence

$$U(C_\psi(\mathcal{U})) \ni V \longmapsto V\psi \in \mathcal{W}(\mathcal{U})$$

is one-to-one.

The following result is also interesting.

Lemma 2.4.2 ([5, Lemma 1.1]) *If $\mathcal{U}$ be a unitary system and if x is a cyclic vector for $\mathcal{U}$, then*

(1) *x is separating for $C_x(\mathcal{U})$.*

(2) *If $\mathcal{U}$ is a semigroup, then $C_x(\mathcal{U}) = \mathcal{U}'$.*

Proof. (1)If $A \in C_x(\mathcal{U})$ and if $Ax = 0$, then for any $U \in \mathcal{U}$ we have $AUx = UAx = 0$. So $AUx = 0$, hence $A = 0$.

(2)The inclusion "$\supset$" is trivial. For "$\subset$", suppose $A \in C_x(\mathcal{U})$. Then for each $U, V \in \mathcal{U}$ we have $UV \in \mathcal{U}$, and so

$$AU(Vx) = (UV)Ax = U(AVx) = UA(Vx).$$

So since $V \in \mathcal{U}$ was arbitrary and $[\mathcal{U}x] = \mathcal{H}$, it follows that $AU = UA$. ∎

This section will be concluded with a few examples of a unitary system.

Example 2.4.3 ([5]) Let T and D be the operators on $L^2(\mathbb{R})$ defined by

$$(Tf)(t) = f(t-1) \ \ \textit{and} \ \ (Df)(t) = \sqrt{2}f(2t), \ \ f \in L^2(\mathbb{R}), \ \ f \in \mathbb{R}.$$

Then they fail to commute, but for $f \in L^2(\mathbb{R})$ we have

$$(TDf)(t) = T(\sqrt{2}f(2t)) = \sqrt{2}f(2(t-1)) = \sqrt{2}f(2t-2) = (DT^2f)(t),$$

so $TD = DT^2$. Let

$$\mathcal{U}_{D,T} = \{D^mT^n : m, n \in \mathbb{Z}\}.$$

Then $\mathcal{U}_{D,T}$ is a unitary system. Particularly, $\mathcal{U}_{D,T}$ is called a *wavelet system*. If $\psi \in \mathcal{W}(\mathcal{U}_{D,T})$, then ψ is called a *wavelet*.

Example 2.4.4 ([5, Example 1.9]) Let $\{e_n\}_{n=-\infty}^{\infty}$ be an orthonormal basis for $l^2(\mathbb{Z})$, and let $Se_n = e_{n+1}$ be the bilateral shift of multiplicity one. Let $\mathcal{U} = \{S^n : n \in \mathbb{Z}\}$ be the group generated by S. Each e_n is in $\mathcal{W}(\mathcal{U})$. By Proposition 2.4.1 and Lemma 2.4.2 part (2),

$$\mathcal{W}(\mathcal{U}) = \{Ve_0 : V \in \mathbb{U}(\{S\}')\}.$$

Here $\{S\}'$ coincides with the set of Laurent operators. Let $\mathbb{T}$ be the unit circle. If we represent S on $L^2(\mathbb{T})$ in the usual way by identifying it with the multiplication operator M_z, then $\mathbb{U}(\{S\}')$ is identified with (multiplication by) the set of unimodular functions on $\mathbb{T}$, and e_0 is identified with the constant function 1. Then Proposition 2.4.1 just recovers the well-known fact that the set of complete wandering vectors for the shift coincides (under this representation) with the set of unimodular functions on $\mathbb{T}$. In this case $\mathcal{W}(\mathcal{U})$ is clearly a closed, connected subset of the unit ball of $\mathcal{H}$ in the norm topology with dense linear span.

Chapter 3

Irrational rotation unitary systems

3.1 Irrational rotation C*-algebras

A *C^*-algebra* is a Banach $*$-algebra $\mathcal{A}$ with the additional norm condition

$$||A^*A|| = ||A||^2 \quad \textit{for all} \quad A \in \mathcal{A}.$$

We say that an element A of a C^*-algebra $\mathcal{A}$ is *self-adjoint* if $A = A^*$; *normal* if $A^*A = AA^*$; *unitary* if $A^*A = AA^* = I$; *positive*, we write $A \geq 0$, if $A = A^*$ and the spectrum $\sigma(A) = \{\lambda \in \mathbb{C} : \lambda I - A \text{ is not invertible}\}$ is contained in the non-negative real line $[0, \infty)$. A *positive linear functional* f on a C^*-algebra is a linear functional such that $f(A) \geq 0$ whenever $A \geq 0$. A *state* is a positive linear functional of norm 1. A *representation* of a C^*-algebra $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ is a $*$-homomorphism from $\mathcal{A}$ into $\mathcal{B}(\mathcal{H})$. More about C^*-algebra, we refer to [3, 4, 6, 23, 27, 31, 39, 43, 44], etc..

An irrational rotation C^*-algebra is a C^*-algebra generated by a pair of unitary elements U and V which satisfy the relation

$$UV = e^{2\pi i\theta} VU \qquad (\dagger)$$

where $\theta \in (0, 1)$ is an irrational number.

For any constants λ, μ on the unit circle ($|\lambda| = |\mu| = 1$), the unitary pair $(\lambda U, \mu V)$ satisfies (†). Thus there is an endomorphism $\rho_{\lambda,\mu}$ of $\mathcal{A}_\theta$ such that $\rho_{\lambda,\mu}(U) = \lambda U$ and $\rho_{\lambda,\mu}(V) = \mu V$. Let $\sigma = \rho_{\overline{\lambda},\overline{\mu}}\rho_{\lambda,\mu}$. Since $\sigma(U) = U$ and $\sigma(V) = V$, we have $\sigma = I$. Thus $\rho_{\lambda,\mu}$ is an automorphism.

For each fixed A in $\mathcal{A}_\theta$, the map from $\mathbb{T}^2$ to $\mathcal{A}_\theta$ given by $f_A(\lambda,\mu) = \rho_{\lambda,\mu}(A)$ is norm continuous. To verify this, notice that it is true for all non-commuting polynomials in U, V, U^* and V^*. These are dense and automorphisms are contractive; so the rest follows from a simple approximation argument.

Define two maps of $\mathcal{A}_\theta$ into itself by the formula

$$\Phi_1(A) = \int_0^1 \rho_{1,e^{2\pi i t}}(A)dt$$

and

$$\Phi_2(A) = \int_0^1 \rho_{e^{2\pi i t},1}(A)dt.$$

These integrals make sense as Riemann sums because the integrand is a norm continuous function. Some of the nice properties of these maps are captured in the following theorem.

An *expectation* of C^*-algebra onto a subalgebra is a positive, unital idempotent map. Expectations occur frequently in the study of operator algebras, and have many nice properties. The point of this next theorem is to show that Φ_1 and Φ_2 are expectations. Recall that a map Φ is *contractive* if $||\Phi|| \leq 1$, *idempotent* if $\Phi^2 = \Phi$, and a positive map is *faithful* if $A \geq 0$ and $\Phi(A) = 0$ implies that $A = 0$.

Theorem 3.1.1 ([6, Theorem VI.1.1]) Φ_1 *is positive contractive idempotent and faithful, and maps $\mathcal{A}_\theta$ onto $C^*(U)$. Moreover,*

$$\Phi_1(f(U)Ag(U)) = f(U)\Phi_1(A)g(U)$$

for all f, g in $C(\mathbb{T})$. For any finite linear combination of $\{U^kV^l : k,l \in \mathbb{Z}\}$,

$$\Phi_1(\sum_{k,l} a_{kl}U^kV^l) = \sum_k a_{k0}U^k.$$

In addeition, for every A in $\mathcal{A}_\theta$,

$$\Phi_1(A) = \lim_{n\to\infty} \frac{1}{2n+1} \sum_{j=-n}^{n} U^j A U^{-j}.$$

The corresponding results for Φ_2 also hold. Combining them, we obtain:

Corollary 3.1.2 ([6, Corollary VI.1.2]) *The map $\tau = \Phi_1\Phi_2 = \Phi_2\Phi_1$ is a faithful unital scalar valued trace on $\mathcal{A}_\theta$.*

We have enough structure to show that τ is in fact the only trace on $\mathcal{A}_\theta$.

Proposition 3.1.3 ([6, Proposition VI.1.3]) *τ is the unique trace on $\mathcal{A}_\theta$.*

Proof. Suppose that κ is another trace on $\mathcal{A}_\theta$. Then for any A in $\mathcal{A}_\theta$, we have $\kappa(A) = \kappa(AU^{-j}U^j) = \kappa(U^jAU^{-j})$. So by Theorem 3.1.1,

$$\begin{aligned}
\kappa(A) &= \lim_{n\to\infty} \frac{1}{2n+1} \sum_{j=-n}^{n} \kappa(A) \\
&= \lim_{n\to\infty} \frac{1}{2n+1} \sum_{j=-n}^{n} \kappa(U^jAU^{-j}) \\
&= \lim_{n\to\infty} \kappa(\frac{1}{2n+1} \sum_{j=-n}^{n} U^jAU^{-j}) \\
&= \kappa(\lim_{n\to\infty} \frac{1}{2n+1} \sum_{j=-n}^{n} U^jAU^{-j}) \\
&= \kappa(\Phi_1(A)).
\end{aligned}$$

Similarly, $\kappa(A) = \kappa(\Phi_2(A))$. Thus we have

$$\kappa(A) = \kappa(\Phi_2(A)) = \kappa(\Phi_1\Phi_2(A)) = \kappa(\tau(A)) = \tau(A)$$

because $\kappa(I) = I$ and $\tau(A)$ is always a scalar. ∎

Now we are prepared to prove the main result of this section, which is the uniqueness of the C^*-algebra generated by unitaries satisfying (†).

Theorem 3.1.4 ([6, Theorem VI.1.4]) *$\mathcal{A}_\theta$ is simple. Thus if U' and V' are any unitary elements satisfying* (†)*, then $C^*(U', V')$ is canonically isomorphic to $\mathcal{A}_\theta$.*

Proof. Suppose that $\mathfrak{J}$ is a non-zero ideal of $\mathcal{A}_\theta$. Then there is a positive, non-zero element X in $\mathfrak{J}$. Since $U^j X U^{-j}$ belongs to $\mathfrak{J}$, the limit formula for Φ_1 shows that

$$\Phi_1(X) = \lim_{n \to \infty} \frac{1}{2n+1} \sum_{j=-n}^{n} U^j A U^{-j} \in \mathfrak{J}.$$

Similarly we have $\Phi_2(X) \in \mathfrak{J}$. Hence $\tau(X)$ belongs to $\mathfrak{J}$. But since τ is a faithful trace, $\tau(X)$ is non-zero multiple of the identity. Therefore $\mathfrak{J} = \mathcal{A}_\theta$.

If U' and V' are any unitary elements satisfying (†), then there is a canonical homomorphism ϕ of $\mathcal{A}_\theta$ onto $C^*(U', V')$ such that $\phi(U) = U'$ and $\phi(V) = V'$. Since $\mathcal{A}_\theta$ is simple, this homomorphism must be an isomorphism. ∎

3.2 Representation of irrational rotation unitary systems

Let $\mathcal{A}_\theta$ be an irrational rotation C^*-algebra generated by a pair of unitary elements U and V satisfying (†). We will call the set $\mathcal{U}_{U,V} = \{U^m V^n : m, n \in \mathbb{Z}\}$ an *irrational rotation unitary system.* If $\mathcal{B}$ is a C^*-algebra and A and B are two elements in $\mathcal{B}$ satisfying (†), then it is known that there is a faithful $*$-isomorphism π from $\mathcal{A}_\theta$ into $\mathcal{B}$ satisfying $\pi(U) = A$ and $\pi(V) = B$ (see [12] or [40]).

Two representations π_1 and π_2 of C^*-algebra $\mathcal{A}$ are called *unitarily equivalent* if there exists a corresponding unitary operator W such that $W\pi_1(A)W^* = \pi_2(A)$ for every $A \in \mathcal{A}$. A representation is called *faithful* if the mapping is injective. The next theorem shows that $\mathcal{A}_\theta$ can be represented on $L^2(\mathbb{T}^2)$.

Theorem 3.2.1 ([13, Theorem 1]) *Let $\mathcal{A}_\theta$ be an irrational rotation C^*-algebra with unitary generators u, v for which $uv = e^{2\pi i\theta}vu$ for some irrational number $\theta \in (0,1)$. Then, up to unitary equivalence, there exists a unique faithful representation π of $\mathcal{A}_\theta$ on $L^2(\mathbb{T}^2)$ such that irrational rotation unitary system $\mathcal{U} = \{U^m V^n : m, n \in \mathbb{Z}\}$, where $U = \pi(u)$ and $V = \pi(v)$, has a complete wandering vector. Moreover, $\mathcal{W}(\mathcal{U})$ is a closed and connected subset of $\mathcal{H}$ and $[\mathcal{W}(\mathcal{U})] = L^2(\mathbb{T}^2)$.*

Proof. For existence of such a representation π, let us consider the following unitary system. Let $\{z^m w^n : m, n \in \mathbb{Z}\}$ be the basis for $L^2(\mathbb{T}^2)$. Define unitary operators U, V on $L^2(\mathbb{T}^2)$ by

$$U(z^m w^n) = z^{m+1} w^n \quad \textit{and} \quad V(z^m w^n) = \lambda^m z^m w^{n+1},$$

where $\lambda = e^{2\pi i\theta}$. Then $UV = \lambda VU$ follows from

$$UVz^m w^n = U(\lambda^{-m} z^m w^{n+1}) = \lambda^{-m} z^{m+1} w^{n+1}$$

$$= \lambda\lambda^{-(m+1)}z^{m+1}w^{n+1} = \lambda VUz^m w^n.$$

Thus $\mathcal{U}_{U,V}$ is an irrational rotation unitary system. Let π be the faithful $*$-isomorphism from $\mathcal{A}_\theta$ into $\mathcal{B}(L^2(\mathbb{T}^2))$ such that $\pi(u) = U$ and $\pi(v) = V$. We will show that $\mathcal{W}(\mathcal{U})$ is a closed and connected subset of $L^2(\mathbb{T}^2)$ and $[\mathcal{W}(\mathcal{U})] = L^2(\mathbb{T}^2)$, where $\mathcal{U} = \{U^mV^n : m,n \in \mathbb{Z}\}$.

We have that $\mathcal{U}(1) = \{z^kw^l : k,l \in \mathbb{Z}\}$. So 1 is a complete wandering vector for $\mathcal{U}$. Moreover, for any $m,n \in \mathbb{Z}$, we have $\mathcal{U}(z^mw^n) = \{\lambda^{-ml}z^{m+k}w^{b+l} : k,l \in \mathbb{Z}\}$, which is an orthonormal basis for $L^2(\mathbb{T}^2)$. Thus in fact $z^mw^n \in \mathcal{W}(\mathcal{U})$ for all $m,n \in \mathbb{Z}$. So $[\mathcal{W}(\mathcal{U})] = L^2(\mathbb{T}^2)$, since $\{z^mw^n : m,n \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{T}^2)$.

Let $A \in \mathcal{C}_\psi(\mathcal{U})$ for some $\psi \in \mathcal{W}(\mathcal{U})$. The relation $UV = \lambda VU$ implies that $span(\mathcal{U})$ is an algebra. So for each $S,T \in \mathcal{U}$, we have $ST \in span(\mathcal{U})$. So $AS(T\psi) = (ST)A\psi = S(AT)\psi = SA(T\psi)$. Since $T \in \mathcal{U}$ is arbitary and $[\mathcal{U}\psi] = L^2(\mathbb{T}^2)$, it follows that $AS = SA$. Thus $\mathcal{C}_\psi(\mathcal{U}) \subset \mathcal{U}'$. The inclusion "$\supset$" is trivial. Thus $\mathcal{C}_\psi(\mathcal{U}) = \mathcal{U}'$. So $\mathcal{C}_\psi(\mathcal{U})$ is a von Neumann algebra. Since the unitary group of a von Neumann algebra is norm connected, $\mathcal{W}(\mathcal{U}) = \mathbb{U}(\mathcal{U}')\psi$ is norm-pathwise connected.

We claim that the von Neumann algebra $\mathrm{W}^*(\mathcal{U})$ generated by U and V is finite and so is its commutant $\mathcal{U}'$. Let $\psi \in \mathcal{W}(\mathcal{U})$ be arbitary. First we show that $\langle AB\psi,\psi\rangle = \langle BA\psi,\psi\rangle$ for all $A,B \in \mathrm{W}^*(\mathcal{U})$. It is enough to verify that this holds for $A = U^mV^n$, $B = U^kV^l$ with $m,n,k,l \in \mathbb{Z}$, since the linear span of $\mathcal{U}$ is an algebra. In fact, this follows from

$$\begin{aligned}\langle U^mV^nU^kV^l\psi,\psi\rangle &= e^{-2nk\pi i\theta}\langle U^{m+k}V^{n+l}\psi,\psi\rangle \\ &= \begin{cases} 0, & (m+k,n+l) \neq (0,0), \\ e^{-2nk\pi i\theta}, & (m+k,n+l) = (0,0), \end{cases}\end{aligned}$$

and

$$\langle U^kV^lU^mV^n\psi,\psi\rangle = \begin{cases} 0, & (m+k,n+l) \neq (0,0), \\ e^{-2lm\pi i\theta}, & (m+k,n+l) = (0,0). \end{cases}$$

Thus ψ is a trace vector of $W^*(\mathcal{U})$. Note that ψ is also a cyclic vector for $W^*(\mathcal{U})$, since $\mathcal{U}\psi$ is an orthonormal basis for $L^2(\mathbb{T}^2)$. Thus, by Lemma 2.1.5, ψ is a joint cyclic trace vector for $W^*(\mathcal{U})$ and $\mathcal{U}'$. By Theorem 2.1.6, this implies that both $w^*(\mathcal{U})$ and $\mathcal{U}'$ are finite von Neumann algebras.

For the closedness of $\mathcal{W}(\mathcal{U})$, suppose that $\{\psi_n\}$ is a sequence in $\mathcal{W}(\mathcal{U})$ converging in norm to a vector η. Fix $\psi \in \mathcal{W}(\mathcal{U})$. Then by Proposition 2.4.1, since $\mathcal{C}_\psi(\mathcal{U}) = \mathcal{U}'$, there are unitary operators $V_n \in \mathcal{U}'$ with $\psi_n = V_n\psi$. In order to show that $\eta \in \mathcal{W}(\mathcal{U})$, again by Proposition 2.4.1, it is enough to show that $\eta = W\psi$ for some unitary operator W in $\mathcal{U}'$.

Let $\{U_{n_k}\}$ be a subsequence of $\{V_n\}$ such that $U_{n_k} \longrightarrow U_0$ in the weak operator topology for some operator $U_0 \in \mathcal{U}'$. Then $U_{n_k}\psi \longrightarrow \eta$ in norm and $U_{n_k}\psi \longrightarrow U_0\psi$ in the weak topology on $L^2(\mathbb{T}^2)$. So $\eta = U_0\psi$. Now for any $f \in L^2(\mathbb{T}^2)$, we have $|\langle U_{n_k}^*(U_{n_k}\psi - U_0\psi), f\rangle| \leq ||U_{n_k}\psi - U_0\psi||\ ||f|| \longrightarrow 0$ and $\langle U_0\psi, U_{n_k}f\rangle \longrightarrow \langle U_0\psi, U_0 f\rangle = \langle U_0^*U_0\psi, f\rangle$. Thus

$$\begin{aligned}
\langle \psi, f\rangle &= \langle U_{n_k}^*U_{n_k}\psi, f\rangle \\
&= \langle U_{n_k}^*(U_{n_k}\psi - U_0\psi), f\rangle + \langle U_0\psi, U_{n_k}f\rangle \\
&\longrightarrow \langle U_0^*U_0\psi, f\rangle.
\end{aligned}$$

which implies that $U_0^*U_0\psi = \psi$.

Since ψ is cyclic for $span(\mathcal{U})$, it follows that ψ separates $\mathcal{U}'$. So since $U_0^*U_0 \in \mathcal{U}'$ and $(U_0^*U_0 - I)\psi = 0$, we get $U_0^*U_0 = I$. But $\mathcal{U}'$ is finite, so U_0 is a unitary in $\mathcal{U}'$ as required.

Let π_1 and π_2 be faithful representations on Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, suth that $\mathcal{U}_{U_i,V_i}$ has a complete wandering vector ψ_i, where $U_i = \pi_i(u)$, $V_i = \pi_i(v)$, $i = 1, 2$. Since u, v are generators for $\mathcal{A}_\theta$, we only need to prove that there is a unitary operator W satisfying $WU_1W^* = U_2$ and $WV_1W^* = V_2$. For this purpose, write $\psi_{m,n}^{(i)} = U_i^m V_i^n \psi_i$ for $i = 1, 2$ and $m, n \in \mathbb{Z}$. Then $\{\psi_{m,n}^{(i)} : m, n \in \mathbb{Z}\}$ is an orthonormal basis for $\mathcal{H}_i$. Define $W : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ by $W\psi_{m,n}^{(1)} = \psi_{m,n}^{(2)}$ for all $m, n \in \mathbb{Z}$. Then W is a unitary operator, and we have

$$WU_1\psi^{(1)}_{m,n} = WU_1U_1^mV_1^n\psi_1 = U_2U_2^mV_2^n\psi_2 = U_2W\psi^{(1)}_{m,n}$$

and

$$\begin{aligned} WV_1\psi^{(1)}_{m,n} &= WV_1U_1^mV_1^n\psi_1 = e^{-2\pi im\theta}WU_1^mV_1^{n+1}\psi_1 \\ &= e^{-2\pi im\theta}U_2^mV_2^{n+1}\psi_2 = V_2U_2^mV_2^n\psi_2 = V_2W\psi^{(1)}_{m,n}. \end{aligned}$$

Thus $WU_1W^* = U_2$ and $WV_1W^* = V_2$, since these relations hold on an orthonormal basis for $\mathcal{H}_1$. Hence π_1 and π_2 are unitarily equivalent. ∎

Chapter 4

Invariant subspaces of $L^2(\mathbb{T}^2)$ for certain von Neumann algebras

4.1 Introduction

In this section, we introduce the four unitary operators and construct the certain von Neumann algebras which are II_1-factors. Let θ be an irrational number in $(0,1)$. We consider the unitary operators on $L^2(\mathbb{T}^2)$ satisfying:

$$L_z(z^m w^n) = z^{m+1} w^n,$$

$$L_w(z^m w^n) = e^{-2\pi i m\theta} z^m w^{n+1},$$

$$R_z(z^m w^n) = e^{-2\pi i n\theta} z^{m+1} w^n$$

and

$$R_w(z^m w^n) = z^m w^{n+1},$$

where $(z,w) \in \mathbb{T}^2$. As in the proof of Theorem 3.2.1, we have

$$L_z L_w = e^{2\pi i\theta} L_w L_z \quad \text{and} \quad R_w R_z = e^{2\pi i\theta} R_z R_w.$$

If we define $JA1 = A^*1$ for all $A \in \mathfrak{L}$, then J is a conjugate linear isometry from $L^2(\mathbb{T}^2)$ onto $L^2(\mathbb{T}^2)$. Since $L_w^n L_z^m(1) = e^{-2\pi imn\theta} z^m w^n$, we have

$$J(z^m w^n) = J(e^{2\pi imn\theta} L_w^n L_z^m(1)) = (e^{2\pi imn\theta} L_w^n L_z^m)^*(1)$$
$$= e^{-2\pi imn\theta} L_z^{-m} L_w^{-n}(1) = e^{-2\pi imn\theta} L_z^{-m} w^{-n} = e^{-2\pi imn\theta} z^{-m} w^{-n}.$$

Thus we see $JL_zJ = R_z{}^*$ and $JL_wJ = R_w{}^*$. Let $\mathfrak{L}$ (resp. $\mathfrak{R}$) denote the von Neumann algebra generated by L_z and L_w (resp. R_z and R_w), then $J\mathfrak{L}J = \mathfrak{R}$ and $J\mathfrak{R}J = \mathfrak{L}$. If we define $\tau(A) =< A1, 1 >$ for all $A \in \mathfrak{L}$, then τ is a unique faithful tracial state on $\mathfrak{L}$. So we have

Proposition 4.1.1 *$\mathfrak{L}$ and $\mathfrak{R}$ are II_1-factors. Moreover, $\mathfrak{L} = \mathfrak{R}'$ and $\mathfrak{R} = \mathfrak{L}'$.*

Thus we shall call $\mathfrak{L}$ and $\mathfrak{R}$ the left von Neumann algebra and the right von Neuman algebra, respectively. II_1-factor is an important class in the theory of von Neumann algebras. If θ should be an integer, then $\mathfrak{L}$ and $\mathfrak{R}$ are equal to $\mathrm{W}^*(M_z, M_w)$, which is isometric to $L^\infty(\mathbb{T}^2)$. Thus it is a maximal abelian von Neumann algebra (*masa*). Masa is also an important class in the theory of von Neumann algebras, but it is the opposite side of II_1-factor.

4.2 Beurling-Type invariant subspace of $L^2(\mathbb{T}^2)$

In this section, we introduce the notions of left-invariant and right invariant. Our goal of this section is to characterize the Beurling-type left-invariant subspaces of $L^2(\mathbb{T}^2)$.

Let $\mathfrak{L}_+$ (resp. $\mathfrak{R}_+$) denote the σ-weakly closed subalgebra of $\mathfrak{L}$ (resp. $\mathfrak{R}$) generated by the positive powers of L_z and L_w (resp. R_z and R_w).

Definition 4.2.1 *Let $\mathfrak{M}$ be a closed subspace of $L^2(\mathbb{T}^2)$. We shall say that $\mathfrak{M}$ is; left-invariant, if $\mathfrak{L}_+\mathfrak{M} \subset \mathfrak{M}$; left-reducing, if $\mathfrak{L}\mathfrak{M} \subset \mathfrak{M}$; left-pure, if $\mathfrak{M}$ contains no left-reducing subspace; left-full, if the smallest left-reducing subspace containing $\mathfrak{M}$ is all of $L^2(\mathbb{T}^2)$. The right-hand versions of these concepts are defined similarly.*

Remark 4.2.2 *Let $\mathfrak{M}$ be a closed subspace of $L^2(\mathbb{T}^2)$. Then $\mathfrak{M}$ is left-invariant if and only if $L_z\mathfrak{M} \subset \mathfrak{M}$ and $L_w\mathfrak{M} \subset \mathfrak{M}$, left-reducing if and only if there exists a projection $P \in \mathfrak{R}$ such that $\mathfrak{M} = PL^2(\mathbb{T}^2)$, left-pure if and only if $\cap_{m,n\geq 0} L_z^m L_w^n \mathfrak{M} = \{0\}$, and left-full if and only if $\overline{\cup_{m,n<0} L_z^m L_w^n \mathfrak{M}} = L^2(\mathbb{T}^2)$. The right-hand versions of this property hold similarly.*

Lemma 4.2.3 *Let $\mathfrak{M}_0 = \sum \oplus_{m,n\geq 0} {L_z}^m {L_w}^n [q]$ for some norm one element q of $L^2(\mathbb{T}^2)$. Then there exists a unitary operator $V \in \mathfrak{R}$ such that $\mathfrak{M}_0 = VH^2(\mathbb{T}^2)$.*

Proof. Suppose that $\mathfrak{M}_0 = \sum \oplus_{m,n\geq 0} {L_z}^m {L_w}^n [q]$ for some norm one element q of $L^2(\mathbb{T}^2)$. Then we note that $< {L_z}^m {L_w}^n q, {L_z}^k {L_w}^l q >= 0$ for all $m, n, k, l \in \mathbb{Z}$ such that $(m, n) \neq (k, l)$. Now we define an operator V by

$$V(\sum_{m,n\geq 0} \oplus \alpha_{m,n} {L_z}^m {L_w}^n 1) = \sum_{m,n\geq 0} \oplus \alpha_{m,n} {L_z}^m {L_w}^n q.$$

Then V is an isometry and $VL_z = L_zV, VL_w = L_wV$. Hence V is in the commutant of $\mathfrak{L}$. That is, V is in $\mathfrak{R}$. Since $\mathfrak{R}$ is a finite von Neumann algebra, V is unitary. Since $q = V1$, $\mathfrak{M}_0 = \sum \oplus_{m,n\geq 0} {L_z}^m {L_w}^n [V1] = VH^2(\mathbb{T}^2)$. This completes the proof. ∎

We note that subspaces of the form $VH^2(\mathbb{T}^2)$ can be represented:

$$VH^2(\mathbb{T}^2) = \sum_{m,n\geq 0} \oplus {L_z}^m {L_w}^n [V1] \tag{4.1}$$

where V is a partial isometry in the commutant $\mathfrak{R}$ of $\mathfrak{L}$. From above lemmas we now get the following Beurling-type theorem.

Let $\mathfrak{F}$ be a closed subspace of $L^2(\mathbb{T}^2)$. We shall say that $\mathfrak{F}$ is *a wandering subspace*, if $L_z^m L_w^n \mathfrak{F}$ and $L_z^{m'} L_w^{n'} \mathfrak{F}$ are orthogonal for any different (m,n) and (m',n') in $\mathbb{Z}^2$.

Theorem 4.2.4 *Let $\mathfrak{M}$ be a left-invariant subspace of $L^2(\mathbb{T}^2)$ and put $V_z = L_z|_{\mathfrak{M}}$, $V_w = L_w|_{\mathfrak{M}}$, $\mathfrak{F}_z = \mathfrak{M} \ominus V_z\mathfrak{M}$ and $\mathfrak{F}_w = \mathfrak{M} \ominus V_w \mathfrak{M}$. Then the following statements are equivalent:*

(1) There exists a wandering subspace $\mathfrak{F}$ such that $\mathfrak{M} = \sum \oplus_{m,n\geq 0} {V_z}^m {V_w}^n \mathfrak{F}$,

(2) V_z, V_w are shift operators on $\mathfrak{M}$ and $V_w {V_z}^* = e^{2\pi i\theta} {V_z}^* V_w$,

(3) V_w is a shift operator on $\mathfrak{M}$ and $\mathfrak{F}_w = \sum \oplus_{n\geq 0} {V_z}^n (\mathfrak{F}_z \cap \mathfrak{F}_w)$, or V_z is a shift operator on $\mathfrak{M}$ and $\mathfrak{F}_z = \sum \oplus_{m\geq 0} {V_w}^m (\mathfrak{F}_z \cap \mathfrak{F}_w)$,

(4) $\mathfrak{F}_z \cap \mathfrak{F}_w$ is a wandering subspace and $\mathfrak{M} = \sum \oplus_{m,n\geq 0} {V_z}^m {V_w}^n (\mathfrak{F}_z \cap \mathfrak{F}_w)$,

(5) $\mathfrak{M}$ is of the form $VH^2(\mathbb{T}^2)$, where V is a unitary operator in $\mathfrak{R}$.

In this case, $dim(\mathfrak{F}_z \cap \mathfrak{F}_w) = 1$

Proof. (1) $\Rightarrow$ (2). Let $\mathfrak{F}$ be a wandering subspace such that $\mathfrak{M} = \sum \oplus_{m,n\geq 0} {V_z}^m {V_w}^n \mathfrak{F}$. We define

$$\mathfrak{F}_z' = \sum_{m\geq 0} \oplus {V_w}^m \mathfrak{F}$$

and

$$\mathfrak{F}_w' = \sum_{n\geq 0} \oplus {V_z}^n \mathfrak{F}.$$

Since

$$\mathfrak{M} = \sum_{n\geq 0} \oplus V_z{}^n \mathfrak{F}_z{}' = \sum_{m\geq 0} \oplus V_w{}^m \mathfrak{F}_w{}',$$

V_z and V_w are shift operators. It follows that $\mathfrak{F}_z = \mathfrak{F}_z{}'$ and $\mathfrak{F}_w = \mathfrak{F}_w{}'$. Now we shall show $V_w V_z{}^* = e^{2\pi i\theta} V_z{}^* V_w$. If $x \in \mathfrak{M}$, then $x = \sum_{m\geq 0} V_z{}^m x_m$, where $x_m \in \mathfrak{F}_z$. Then we have

$$\begin{aligned} V_z{}^* V_w x = \sum_{m\geq 0} V_z{}^* V_w V_z{}^m x_m &= \sum_{m\geq 0} e^{-2\pi i m\theta} V_z{}^* V_z{}^m V_w x_m \\ &= \sum_{m\geq 1} e^{-2\pi i m\theta} V_z{}^{m-1} V_w x_m + V_z{}^* V_w x_0. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} V_w V_z{}^* x = \sum_{m\geq 0} V_w V_z{}^* V_z{}^m x_m &= \sum_{m\geq 1} V_w V_z{}^{m-1} x_m + V_w V_z{}^* x_0 \\ &= e^{2\pi i\theta} \sum_{m\geq 1} e^{-2\pi i m\theta} V_z{}^{m-1} V_w x_m + V_w V_z{}^* x_0. \end{aligned}$$

Since $V_z{}^* V_w x_0 = 0$ and $V_w V_z{}^* x_0 = 0$, we have $V_w V_z{}^* = e^{2\pi i\theta} V_z{}^* V_w$.

(2) $\Rightarrow$ (3). We shall prove that $\mathfrak{F}_w = \sum \oplus_{n\geq 0} V_z{}^n(\mathfrak{F}_z \cap \mathfrak{F}_w)$. The second assertion can be obtained in the same way. First we notice that $\mathfrak{F}_w$ reduces V_z. Hence for all $n \geq 0$,

$$V_z{}^n(\mathfrak{F}_z \cap \mathfrak{F}_w) \subset \mathfrak{F}_w.$$

Evidently $\mathfrak{F}_z \cap \mathfrak{F}_w$ is a wandering subspace for V_z. Then we have

$$\sum_{n\geq 0} \oplus V_z{}^n(\mathfrak{F}_z \cap \mathfrak{F}_w) \subset \mathfrak{F}_w.$$

Let $\mathfrak{F}_0 = \mathfrak{F}_w \ominus V_z \mathfrak{F}_w$. If we prove that $\mathfrak{F}_0 \subset \mathfrak{F}_z \cap \mathfrak{F}_w$, then we get

$$\mathfrak{F}_w = \sum_{n\geq 0} \oplus V_z{}^n \mathfrak{F}_0 \subset \sum_{n\geq 0} \oplus V_z{}^n(\mathfrak{F}_z \cap \mathfrak{F}_w) \subset \mathfrak{F}_w,$$

which finishes this part of the proof. Suppose that $x \in \mathfrak{F}_0$. Then $x \perp V_z \mathfrak{F}_w$ and consequently $V_z{}^* x \perp \mathfrak{F}_w$. On the other hand $x \in \mathfrak{F}_w$. Since $\mathfrak{F}_w$ reduces V_z, we have $V_z{}^* x \in \mathfrak{F}_w$. This implies that $V_z{}^* x = 0$ and so $x \in \mathfrak{F}_z$. Since $x \in \mathfrak{F}_w$, our proof is complete.

(3) $\Rightarrow$ (4). Suppose that the first condition of (3) is fullfilled. Since V_w is a shift, we have $\mathfrak{M} = \sum_{m\geq 0} \oplus V_w{}^m \mathfrak{F}_w$. Then

$$\mathfrak{M} = \sum_{m\geq 0} \oplus V_w{}^m (\sum_{n\geq 0} \oplus V_z{}^n (\mathfrak{F}_z \cap \mathfrak{F}_w)) = \sum_{m,n\geq 0} \oplus V_z{}^m V_w{}^n (\mathfrak{F}_z \cap \mathfrak{F}_w).$$

In the second case the proof is the same.

(4) $\Rightarrow$ (1). (1) follows (4) immediately.

(5) $\Rightarrow$ (2). It is clear from (2.1).

(4) $\Rightarrow$ (5). Suppose

$$\mathfrak{M} = \sum_{m,n\geq 0} \oplus V_z{}^m V_w{}^n (\mathfrak{F}_z \cap \mathfrak{F}_w).$$

We shall now prove that $\mathfrak{F}_z \cap \mathfrak{F}_w$ is one-dimensional. Suppose $\dim(\mathfrak{F}_z \cap \mathfrak{F}_w) > 1$, and fix norm one orthogonal elements q_1, q_2 in $\mathfrak{F}_z \cap \mathfrak{F}_w$. Let

$$\mathfrak{M}_1 = \sum_{m,n\geq 0} \oplus L_z{}^m L_w{}^n [q_1]$$

and

$$\mathfrak{M}_2 = \sum_{m,n\geq 0} \oplus L_z{}^m L_w{}^n [q_2].$$

By Lemma 4.2.3 there exists unitary operators U_1 and U_2 in $\mathfrak{R}$ such that

$$\mathfrak{M}_1 = U_1 H^2(\mathbb{T}^2) \text{ and } \mathfrak{M}_2 = U_2 H^2(\mathbb{T}^2).$$

Since $q_1 \perp q_2$, we have

$$U_1 H^2(\mathbb{T}^2) \perp U_2 H^2(\mathbb{T}^2).$$

Putting $U_0 = U_1{}^* U_2$, then U_0 is a unitary operator in $\mathfrak{R}$. Moreover we have

$$H^2(\mathbb{T}^2) \perp U_0 H^2(\mathbb{T}^2).$$

So we see that $L_z{}^m L_w{}^n 1 \perp U_0 H^2(\mathbb{T}^2)$ for all $m, n \in \mathbb{Z}$. Therefore we see

$$L^2(\mathbb{T}^2) \perp U_0 H^2(\mathbb{T}^2).$$

That is $U_0 H^2(\mathbb{T}^2) = \{0\}$, a contradiction. So we have norm one element q in $\mathfrak{F}_z \cap \mathfrak{F}_w$. Again from Lemma 4.2.3 we have $\mathfrak{M} = VH^2(\mathbb{T}^2)$ for some unitary operator $V \in \mathfrak{R}$. This completes the proof. ∎

Thus we can describe the result of Beurling-type left-invariant subspaces.

Corollary 4.2.5 *A nonzero left-pure and left-invariant subspace $\mathfrak{M}$ of $L^2(\mathbb{T}^2)$ is of the form $VH^2(\mathbb{T}^2)$ with $V \in \mathfrak{R}$ if and only if $V_w V_z{}^* = e^{2\pi i\theta} V_z{}^* V_w$, where $V_z = L_z|_{\mathfrak{M}}$, $V_w = L_w|_{\mathfrak{M}}$.*

Corollary 4.2.6 *A nonzero left-invariant subspace $\mathfrak{M}$ of $H^2(\mathbb{T}^2)$ is of the form $VH^2(\mathbb{T}^2)$ with $V \in \mathfrak{R}$ if and only if $V_w V_z{}^* = e^{2\pi i\theta} V_z{}^* V_w$, where $V_z = L_z|_{\mathfrak{M}}$, $V_w = L_w|_{\mathfrak{M}}$.*

Similarly, we have the following result about right-invariant subspace.

Corollary 4.2.7 *A nonzero right-pure and right-invariant subspace $\mathfrak{M}$ of $L^2(\mathbb{T}^2)$ is of the form $UH^2(\mathbb{T}^2)$ with $U \in \mathfrak{L}$ if and only if $U_w U_z{}^* = e^{2\pi i\theta} U_z{}^* U_w$, where $U_z = R_z|_{\mathfrak{M}}$, $U_w = R_w|_{\mathfrak{M}}$.*

Corollary 4.2.8 *A nonzero right-invariant subspace $\mathfrak{M}$ of $H^2(\mathbb{T}^2)$ is of the form $UH^2(\mathbb{T}^2)$ with $U \in \mathfrak{L}$ if and only if $U_w U_z{}^* = e^{2\pi i\theta} U_z{}^* U_w$, where $U_z = R_z|_{\mathfrak{M}}$, $U_w = R_w|_{\mathfrak{M}}$.*

Remark 4.2.9 Since unitary operators in $L^\infty(\mathbb{T}^2)$ are inner functions, Corollary 4.2.6 and Corollary 4.2.8 are generalization of Theorem 2.2.3 in a sense.

4.3 Two-sided invariant subspaces of $L^2(\mathbb{T}^2)$

In this section we shall study about two-sided invariant subspaces of $L^2(\mathbb{T}^2)$. We also prove that a non-trivial two-sided invariant subspace of $L^2(\mathbb{T}^2)$ is two-sided pure and two-sided full.

Definition 4.3.1 *Let $\mathfrak{M}$ be a closed subspace of $L^2(\mathbb{T}^2)$. We shall say that $\mathfrak{M}$ is; two-sided invariant, if $\mathfrak{M}$ is both left-invariant and right-invariant; two-sided reducing, if $\mathfrak{M}$ is both left-reducing and right-reducing, two-sided pure, if $\mathfrak{M}$ is both left-pure and right-pure; two-sided full, if $\mathfrak{M}$ is both left-full and right-full.*

To prove the theorem about two-sided invariant subspaces of $L^2(\mathbb{T}^2)$, we need the following lemma.

Lemma 4.3.2 *If $\mathfrak{M}$ is a right-invariant subspace of $L^2(\mathbb{T}^2)$ and a left-reducing subspace of $L^2(\mathbb{T}^2)$, then $\mathfrak{M}$ is either $\{0\}$ or $L^2(\mathbb{T}^2)$.*

Proof. Let P be the projection with range $\mathfrak{M}$. Then since $\mathfrak{M}$ is left reducing, P belongs to $\mathfrak{R}$. Since $\mathfrak{M}$ is right-invariant, we have $R_zPR_z^* \leq P$. It is easy to see $R_zPR_z^* \sim P$. Since $\mathcal{R}$ is a finite von Neumann algebra, we have $R_zPR_z^* = P$, that is, $R_zP = PR_z$. Similarly, we have $R_wP = PR_w$. Hence P lies in $\mathfrak{R}'$. Therefore P belongs to the center of $\mathfrak{R}$. Since $\mathfrak{R}$ is a factor, P is either 0 or 1 . This completes the proof. ∎

Reverse version of the previous lemma is valid.

Lemma 4.3.3 *If $\mathfrak{M}$ is a left-invariant subspace of $L^2(\mathbb{T}^2)$ and a right-reducing subspace of $L^2(\mathbb{T}^2)$, then $\mathfrak{M}$ is either $\{0\}$ or $L^2(\mathbb{T}^2)$.*

Remark 4.3.4 If θ is an integer, then the assumption of the above lemmas is that "$\mathfrak{M}$ is doubly invariant ($M_z\mathfrak{M} = \mathfrak{M}, M_w\mathfrak{M} = \mathfrak{M}$)". In this case $\mathfrak{M}$ is of the form $\chi_E L^2(\mathbb{T}^2)$ where χ_E is a characteristic function of Borel set $E \subset \mathbb{T}^2$ (see Lemma 2.2.2).

Theorem 4.3.5 *A non-trivial two-sided invariant subspace of $L^2(\mathbb{T}^2)$ is two-sided pure and two-sided full.*

Proof. Let $\mathfrak{M}$ be a non-trivial two-sided invariant subspace of $L^2(\mathbb{T}^2)$. Put $\mathfrak{M}_\infty = \bigcap_{m,n\geq 0} L_z^m L_w^n \mathfrak{M}$ and let P_∞ be the projection from $L^2(\mathbb{T}^2)$ onto $\mathfrak{M}_\infty$. Then we have that $P_\infty \neq I$, $P \in \mathfrak{R}$ and $\mathfrak{M}_\infty$ is right-invariant and left-reducing. Indeed,

$$\begin{aligned} L_z \mathfrak{M}_\infty = L_z \bigcap_{m,n\geq 0} L_z^m L_w^n \mathfrak{M} &= \bigcap_{m,n\geq 0} L_z^{m+1} L_w^n \mathfrak{M} \\ = \bigcap_{m,n\geq 0} L_z^m L_w^n \lambda^n L_z \mathfrak{M} &\subset \bigcap_{m,n\geq 0} L_z^m L_w^n \mathfrak{M} = \mathfrak{M}_\infty \end{aligned}$$

Similarly we have that $L_w \mathfrak{M}_\infty \subset \mathfrak{M}_\infty$, $L_z^* \mathfrak{M}_\infty \subset \mathfrak{M}_\infty$, $L_w^* \mathfrak{M}_\infty \subset \mathfrak{M}_\infty$, $R_z \mathfrak{M}_\infty \subset \mathfrak{M}_\infty$ and $R_w \mathfrak{M}_\infty \subset \mathfrak{M}_\infty$. From Lemma 4.3.2 we have $\mathfrak{M}_\infty = \{0\}$. Thus $\mathfrak{M}$ is left-pure.

The right-pureness is similarly proved by considering a projection from $L^2(\mathbb{T}^2)$ onto $\bigcap_{m,n\geq 0} R_z^m R_w^n \mathfrak{M}$. The left-fullness and the right-fullness is similarly proved by cosidering projections onto $\overline{\bigcup_{m,n<0} L_z^m L_w^n \mathfrak{M}}$ and onto $\overline{\bigcup_{m,n<0} R_z^m R_w^n \mathfrak{M}}$ respectively. This completes the proof. ∎

4.4 Popovici Decomposition

In this section we shall characterize two-sided invariant subspaces of $L^2(\mathbb{T}^2)$ by using Popovici's decomposition with respect to a bi-isometry.

Definition 4.4.1 *Let S be an isometry on $L^2(\mathbb{T}^2)$ and $\mathfrak{M}$ be a closed subspace of $L^2(\mathbb{T}^2)$. We shall say that $\mathfrak{M}$ is; S-invariant, if $S\mathfrak{M} \subset \mathfrak{M}$.*

Let $\mathfrak{M}$ be a non-trivial two-sided invariant subspace of $L^2(\mathbb{T}^2)$. Then $\mathfrak{M}$ is both $(L_z L_w)$-invariant and $(R_z R_w)$-invariant. So putting $U = (L_z L_w)|_{\mathfrak{M}}$ and $V = (R_z R_w)|_{\mathfrak{M}}$, then the couple $W = (U, V)$ is a bi-isometry on $\mathfrak{M}$, but U^* is not commuting with V. We note that $\mathfrak{M}$ is both U-invariant and V-invariant.

By Popovici's decomposition of $\mathfrak{M}$ with respect to W, we have

$$\mathfrak{M} = \mathfrak{M}_{uu} \oplus \mathfrak{M}_{us} \oplus \mathfrak{M}_{su} \oplus \mathfrak{M}_{ws}.$$

such that $W|_{\mathfrak{M}_{uu}}$ is a bi-unitary (that is, both $U|_{\mathfrak{M}_{uu}}$ and $V|_{\mathfrak{M}_{uu}}$ are unitary operators), $W|_{\mathfrak{M}_{us}}$ is a unitary-shift (that is, $U|_{\mathfrak{M}_{us}}$ is a unitary and $V|_{\mathfrak{M}_{us}}$ is a shift), $W|_{\mathfrak{M}_{su}}$ is a shift-unitary (that is, $U|_{\mathfrak{M}_{su}}$ is a shift and $V|_{\mathfrak{M}_{su}}$ is a unitary) and $W|_{\mathfrak{M}_{ws}}$ is a weak bi-shift (that is, $U|_{\bigcap_{i\geq 0} \ker V^* U^i}$, $V|_{\bigcap_{j\geq 0} \ker U^* V^j}$ and $(U|_{\mathfrak{M}_{ws}})(V|_{\mathfrak{M}_{ws}})$ are shift operators).

We have the following:

Theorem 4.4.2 *Let $\mathfrak{M}$ be a non-trivial two-sided invariant subspace of $L^2(\mathbb{T}^2)$. Then the couple $W = (U, V)$ is a weak bi-shift on $\mathfrak{M}$, that is, $\mathfrak{M} = \mathfrak{M}_{ws}$.*

Proof. Both U and V are unitary on $\mathfrak{M}_{uu}$, thus $\mathfrak{M}_{uu}$ is two-sided reducing by [22, Proposition 1]. By Lemma 4.3.2, we have that $\mathfrak{M}_{uu} = \{0\}$. Since $\mathfrak{M}$ is U-invariant, we have the Wold-type decomposition of $\mathfrak{M}$ with respect to U as follows;

$$\mathfrak{M} = \bigcap_{n\geq 0} U^n \mathfrak{M} \oplus \sum_{n\geq 0} \oplus U^n \mathfrak{F}^U,$$

where $\mathfrak{F}^U = \mathfrak{M} \ominus U\mathfrak{M}$. Define $\mathfrak{M}_u^U = \bigcap_{n\geq 0} U^n \mathfrak{M}$ and $\mathfrak{M}_s^U = \sum \oplus_{n\geq 0} U^n \mathfrak{F}^U$. Then it is clear that $\mathfrak{M}_u^U$ is right-invariant.

For each $n \in \mathbb{Z}$, we have

$$\begin{aligned} L_z(L_zL_w)^n &= L_z(e^{2\pi i\theta} L_wL_z)^n \\ &= e^{2\pi in\theta}(L_zL_w)^n L_z. \end{aligned}$$

Since $\mathfrak{M}$ is two-sided invariant, we have

$$\begin{aligned} L_z\mathfrak{M}_u^U &= \bigcap_{n\geq 0} L_z(L_zL_w)^n \mathfrak{M} \\ &= \bigcap_{n\geq 0} (L_zL_w)^n L_z \mathfrak{M} \\ &\subset \bigcap_{n\geq 0} U^n \mathfrak{M} \\ &= \mathfrak{M}_u^U \end{aligned}$$

Similarly we see $L_w\mathfrak{M}_u^U \subset \mathfrak{M}_u^U$. On the other hand, for each $n \in \mathbb{Z}$, we have

$$\begin{aligned} L_z^*(L_zL_w)^n &= L_z^*(L_zL_w)(L_zL_w)^{n-1} \\ &= L_w(L_zL_w)^{n-1}. \end{aligned}$$

Thus we have

$$\begin{aligned} L_z^*\mathfrak{M}_u^U &= \bigcap_{n\geq 0} L_z^*(L_zL_w)^n \\ &= L_w \bigcap_{n\geq 0} (L_zL_w)^{n-1} \mathfrak{M} \\ &= L_w \mathfrak{M}_u^U \\ &\subset \mathfrak{M}_u^U \end{aligned}$$

Moreover we have

$$
\begin{aligned}
L_w^*(L_zL_w)^n &= L_w^*(L_zL_w)(L_zL_w)^{n-1} \\
&= L_w^*(e^{2\pi i\theta}L_wL_z)(L_zL_w)^{n-1} \\
&= e^{2\pi i\theta}L_z(L_zL_w)^{n-1}.
\end{aligned}
$$

It follows $L_w^*\mathfrak{M}_u^U \subset \mathfrak{M}_u^U$. Thus $\mathfrak{M}_u^U$ is right-invariant and left-reducing. By Lemma 4.3.2 and the assumption, $\mathfrak{M}_u^U = \{0\}$. Similarly, if we consider the Wold-type decomposition $\mathfrak{M} = \mathfrak{M}_u^V \oplus \mathfrak{M}_s^V$ of $\mathfrak{M}$ with respect to V, then we have $\mathfrak{M}_u^V = \{0\}$. As in the proof of [41, Theorem 2.8], we have

$$
\mathfrak{M}_{us} \subset \mathfrak{M}_u^U \cap \mathfrak{M}_s^V \quad \text{and} \quad \mathfrak{M}_{su} \subset \mathfrak{M}_s^U \cap \mathfrak{M}_u^V.
$$

It follows $\mathfrak{M}_{uu} \oplus \mathfrak{M}_{us} \oplus \mathfrak{M}_{su} = \{0\}$ and so $\mathfrak{M} = \mathfrak{M}_{ws}$. This completes the proof. ∎

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