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Curvature and Almost Complex Structure

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Introduction

Let M be a C^∞-manifold. A Riemannian metric on M is an assignment to each point $p \in M$ of a symmetric positive-definite bi-linear form g_p on the tangent space T_pM of M at p such that g_p depends differentiably on p. Consequently, for a Riemannian manifold $M = (M, g)$, the length of a curve on M is defined and a distance d of M is defied as the infimum of the length of all piecewise smooth curves joining given two points of M. It is well-known that the topology of M as a topological space coincides with that of M as a metric space with the distance d. On one hand, for a Riemannian manifold $M = (M, g)$, there exists a unique affine connection ∇ on M, called the Riemannian connection or the Levi-Civita connection, compatible with the given Riemannian metric g. The Riemannian connection ∇ defines the notion of differentiation on the tensor algebra over M.

The Riemannian curvature tensor, which locally determines the Riemannian metric, plays an important role in Riemannian geometry. The Riemannian curvature tensor R is a tensor field of type (1,3) on M define by

$$R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]},$$

where X and Y are smooth vector fields on M. The Riemannian curvature tensor R often appears through the sectional curvature, the Ricci tensor and the scalar curvature, and many important classes of Riemannian manifolds, such as space of constant curvature, Einstein space, locally symmetric space, are defined by making use of them. The sectional curvature, which is regarded as a function on the Grassmannian manifold of 2-planes, inform us the "shape" of M, intuitively. In fact, if the sectional curvature is particularly constant, then M is called a space of constant curvature and locally isometric to the sphere, the Euclidean space or the hyperbolic space according to the sign of the constant. The notion of the Einstein space is deeply concerned with mathematical physics. Geometrically, the class of Einstein spaces contains the class of spaces of constant curvature. A locally symmetric space, which is defined by the condition $\nabla R = 0$, is characterized by the property that every local geodesic symmetry is a local isometry. A locally symmetric space is called a symmetric space if all local geodesic symmetries can be extended to global ones. It is well-known that a connected, simply connected, complete locally symmetric space is a symmetric space. It is also well-known that an irreducible symmetric space is an Einstein space.

In Riemannian geometry, it is very important to study the relationship between curvatures and geometric structures. In the present thesis, in particular, we study curvatures and almost complex structures.

H. Whitney introduced the notion of fiber bundle, which is a generalization of the notion of product manifold. Many important manifolds have fiber bundle structures, and hence each of them is regarded as a fiber bundle associated with a suitable principal fiber bundle. For example, some exotic 7-sphere Σ^7 is realized as a fiber bundle associated with a $Sp(1) \times Sp(1)$-principal bundle over $S^4 = Sp(2)/Sp(1) \times Sp(1)$ with standard fiber $S^3 = Sp(1) \times Sp(1)/\Delta$, where $Sp(n)$ denotes the symplectic group of order n and Δ is the diagonal set of $Sp(1) \times Sp(1)$ ([14], [25]). D. Gromoll and W. Meyer ([14]) made a interesting observation on the problem whether Σ^7 admits a Riemannian metric of positive sectional curvature or not. In the first half of the present thesis, we shall construct a Riemannian metric on some fiber bundle. Typical example is the one on the metric twistor bundle. The notion of the twistor space was introduced by P. Penrose. Inspired by the Penrose program, J. Rawnsley ([32]) and others consider the twistor bundle $\tilde{J}(M)$ over M whose fiber $\tilde{J}_p(M)$ over $p \in M$ consists of all complex structures $j \colon T_pM \to T_pM$, $j^2 = -id$, where T_pM denotes the tangent space of M at $p \in M$. When M is a $2n$-dimensional oriented Riemannian manifold, we can consider the subbundle $J(M)$ of $\tilde{J}(M)$, called the metric twistor bundle, whose fiber $J_p(M)$ over $p \in M$ consists of all orthogonal complex structures on T_pM. The bundle $J(M)$ is regarded as a fiber bundle over M associated with the oriented orthonormal frame bundle $F(M)$ with standard fiber $SO(2n)/U(n)$, where $SO(2n)$ and $U(n)$ denote the special orthogonal group of order $2n$ and the unitary group of order n, respectively ([4], [38]).

In the second half of the present thesis, we shall study the integrability of almost complex structure. We shall also state the relation to the first half by making use of the metric twistor bundle. An almost complex structure J on a differentiable manifold M is a tensor field of type (1,1) on M satisfying $J^2 = -id$. The pair (M, J) is called an almost complex manifold. The almost complex structure J of an almost complex manifold $M = (M, J)$ can be regarded as a smooth cross section of the twistor bundle $\tilde{J}(M)$ over M. Any complex manifold is automatically an almost complex manifold with natural almost complex structure induced by the given complex structure. An almost complex manifold is of even-dimensional and orientable. However, the converse is not always true. In fact, it is well-known that $2n$-dimensional standard sphere S^{2n} admits an almost complex structure only when $n = 1$ or 3. An almost complex structure J is said to be integrable if M admits a complex structure and the derived almost complex structure coincides with J. We also say that almost complex manifold $M = (M, J)$ is integrable if J is integrable. We define a tensor field N of type (1,2) on M, called the Nijenhuis tensor field or torsion tensor field of J, by

$$N(X,Y) = [JX, JY] - [X,Y] - J[JX,Y] - J[X,JY],$$

where X and Y are smooth vector fields on M. The well-known result due to Newlander-Nirenberg says that an almost complex structure J is integrable if the corresponding Nijenhuis tensor field vanishes identically on M. An almost complex manifold (M, J) equipped with a Riemannian metric g satisfying $g(JX, JY) = g(X,Y)$ for any smooth vector fields X, Y on M is called an almost Hermitian manifold. The almost complex structure J of an almost Hermitian manifold $M = (M, J, g)$ can be regarded as a smooth cross section of the metric twistor bundle $J(M)$ over the Riemannian manifold $M = (M, g)$. A. Gray and L. M. Hervella ([13]) defined a sixteen classes of almost Hermitian manifolds. The class of almost Kähler manifolds and the class of Kähler manifolds are two of them and these classes have been studied by many authors. It is well-known that an almost Kähler manifold with integrable almost complex structure is a Kähler manifold. A non-Kähler almost Kähler manifold is called a strictly almost Kähler manifold. Many interesting examples of strictly almost Kähler manifold are constructed by many authors ([1], [5], [6], [15], [28], [43], [44], etc.). Concerning the integrability of almost Kähler manifolds, the following conjecture by S. I. Goldberg is known ([11]).

Conjecture. *Any compact Einstein almost Kähler manifold is a Kähler manifold.*

An important progress was made by K. Sekigawa who proved the following theorem ([37]).

Theorem. *The conjecture is true if we additionally assume the scalar curvature is non-negative.*

However, the conjecture is still open the case of negative scalar curvature.

The present thesis consists of three chapters. In Chapter 1, first of all, we shall give a brief survey on the fundamental notions and facts in Riemannian geometry. Next, we introduce the notion of almost complex manifold and define some special classes of Riemannian manifold with almost complex structure, such as almost Hermitian manifold, Kähler manifold, almost Kähler manifold and so on. Chapter 2 is devoted to calculate the curvatures with respect to some Riemannian metric g_E on a fiber bundle E associated with a principal G-bundle P whose standard fiber is a homogeneous space G/K under the assumption that the Lie group G admits an $Ad(K)$-invariant Riemannian metric. The metric construction in [18] includes our construction of g_E, but our aim is to write down explicit formulas for curvatures. As an application, we give a sufficient condition for an associated fiber bundle with a symmetric fiber to be an Einstein space. Moreover, in the last section of Chapter 2, we apply the arguments in preceding sections to the metric twistor bundle over an even-dimensional oriented Riemannian manifold and also to construct a new metric on the tangent bundle over a Riemannian manifold. In Chapter 3, concerning the Goldberg conjecture, we study the integrability of almost Kähler manifolds. As we have mentioned above, a space of constant curvature is necessarily an Einstein space. So, we study the integrability of almost Kähler manifolds of constant sectional curvature in Section 3. We shall prove that the hyperbolic space $\mathbb{H}^{2n}$ of dimension $2n(\geq 4)$ cannot admit an compatible almost Kähler structure (Theorem 3.3.2). Consequently, we can prove that a complete almost Kähler manifold of constant sectional curvature is a flat Kähler manifold (Corollary 3.3.3).

Moreover, we prove that the hypothesis of completeness in Corollary 3.3.3 is needless in dimension ≥ 8 (Theorem 3.3.4). This result is nothing but the Z. Olszak's one ([30]), but we shall give another short proof. Taking account of the fact that an irreducible symmetric space is an Einstein space, it is interesting to study the integrability of almost Kähler structure on a locally symmetric space. In Section 4, we shall construct a strictly almost Kähler structure on the symmetric space $\mathbb{H}^3 \times \mathbb{R}$ and determine the corresponding automorphism group. Moreover, we prove that $\mathbb{H}^3 \times \mathbb{R}$ cannot be a universal almost Hermitian covering of any four-dimensional compact almost Kähler manifold (Theorem 3.4.3). In Section 5, more generally, we shall prove that a four-dimensional compact almost Kähler locally symmetric space is a Kähler manifold (Theorem 3.5.1).

Through the present thesis, all manifolds are assumed to be connected and of class C^∞ unless otherwise stated.

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Chapter 1

Preliminaries

In the present chapter, we shall give a brief survey on the fundamental notions and facts in Riemannian geometry.

Let $M = (M, g)$ be an n-dimensional Riemannian manifold, namely, M is an n-dimensional C^∞-manifold provided with a Riemannian metric g. It is well-known that there exists a unique affine connection ∇ on M, called the Riemannian connection or Levi-Civita connection, which is metrical and torsion-free, namely,

$$Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z),$$
$$\nabla_X Y - \nabla_Y X - [X, Y] = 0,$$

for X, Y, $Z \in \mathfrak{X}(M)$, where $\mathfrak{X}(M)$ denotes the Lie algebra of all smooth vector fields on M. In fact, for any X, $Y \in \mathfrak{X}(M)$, $\nabla_X Y$ is determined by the following equality.

$$\begin{aligned} g(\nabla_X Y, Z) = \frac{1}{2}\{&Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ &+ g([X, Y], Z) - g([Y, Z], X) + g([Z, X], Y)\} \end{aligned}$$

for all $Z \in \mathfrak{X}(M)$. We call $\nabla_X Y$ the covariant derivative of Y in the direction X. The notion of covariant differentiation ∇ can be uniquely extended to that of tensor algebra $\mathfrak{T}(M)$ over M so that ∇ satisfies the following properties (a) $\sim$ (f). For any X, $Y \in \mathfrak{X}(M)$,

(a) $\nabla_X : \mathfrak{T}(M) \to \mathfrak{T}(M)$ is type-preserving, namely, if K is a tensor field of type (r, s), so is $\nabla_X K$;

(b) ∇_X is a derivation on $\mathfrak{T}(M)$, namely, ∇_X is a linear endomorphism of $\mathfrak{T}(M)$ satisfying $\nabla_X(K \otimes L) = (\nabla_X K) \otimes L + K \otimes (\nabla_X L)$ for K, $L \in \mathfrak{T}(M)$;

(c) ∇_X commute with every contraction on $\mathfrak{T}(M)$;

(d) $\nabla_X f = Xf$ for every differentiable function f on M;

(e) $\nabla_{X+Y} = \nabla_X + \nabla_Y$;

(f) $\nabla_{fX} K = f(\nabla_X K)$ for every differentiable function f on M and $K \in \mathfrak{T}(M)$.

For example, if K is a tensor field of type $(0, s)$ or $(1, s)$ on M, then $\nabla_X K$ is uniquely determined by

$$(\nabla_X K)(X_1, \dots, X_s) = \nabla_X(K(X_1, \dots, X_s)) - \sum_{i=1}^{s} K(X_1, \dots, \nabla_X X_i, \dots, X_s)$$

for $X_1, \dots, X_s \in \mathfrak{X}(M)$. For a tensor field K of type (r, s), the covariant differential ∇K of K is a tensor field of type $(r, s+1)$ defined by

$$(\nabla K)(Y_1, \dots, Y_s; X) = (\nabla_X K)(Y_1, \dots, Y_s)$$

for X, Y_1, $\dots$, $Y_s \in \mathfrak{X}(M)$. Inductively, the k-th covariant differential $\nabla^k K$ is defined to be $\nabla(\nabla^{k-1} K)$. We use the notation

$$(\nabla^m K)(; X_1; \dots ; X_k) = \nabla^m_{X_1 \dots X_k} K$$

for X_1, $\dots$, $X_k \in \mathfrak{X}(M)$.

Now, we define the Riemannian curvature tensor. The Riemannian curvature tensor (or briefly the curvature tensor) is a tensor field R of type (1,3) on M defined by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

for X, Y, $Z \in \mathfrak{X}(M)$. The curvature tensor R has the following properties:

(i) $$R(X, Y)Z = -R(Y, X)Z,$$

(ii) $$\underset{X,Y,Z}{\mathfrak{S}} R(X, Y)Z = 0, \qquad \text{(1st Bianchi identity)}$$

(iii) $$g(R(X, Y)Z, W) = -g(R(X, Y)W, Z),$$

(iv) $$g(R(X, Y)Z, W) = g(R(Z, W)X, Y),$$

(v) $$\underset{X,Y,Z}{\mathfrak{S}} (\nabla_X R)(Y, Z)W = 0, \qquad \text{(2nd Bianchi identity)}$$

for any X, Y, Z, $W \in \mathfrak{X}(M)$, where $\mathfrak{S}$ denotes the cyclic sum.

Let σ be a 2-dimensional subspace of the tangent space T_pM of M at $p \in M$ and $\{u, v\}$ a basis of σ. We define the sectional curvature $K_\sigma = K(u, v)$ of the plane σ by

$$K_\sigma = K(u, v) = \frac{g(R(u, v)v, u)}{\|u \wedge v\|^2},$$

where $\|u \wedge v\|^2 = \|u\|^2\|v\|^2 - g(u, v)^2$ and $\|u\|^2 = g(u, u)$. It is easily verified that K_σ is independent of the choice of the basis $\{u, v\}$ of σ.

Definition. A Riemannian manifold M is called a space of constant curvature if K_σ is a constant for all planes $\sigma \subset T_pM$ and all points $p \in M$.

If M is a space of constant curvature c, then the curvature tensor R satisfies

$$R(X, Y)Z = c\{g(Y, Z)X - g(X, Z)Y\}$$

for X, Y, $Z \in \mathfrak{X}(M)$. Typical examples of such spaces are n-dimensional Euclidean space $\mathbb{R}^n$ $(c = 0)$, n-dimensional standard sphere S^n $(c > 0)$, and n-dimensional hyperbolic space $\mathbb{H}^n$ $(c < 0)$. These three examples are all connected, simply connected and complete ones. Any n-dimensional space of constant curvature is locally isometric to $\mathbb{R}^n$, S^n or $\mathbb{H}^n$ according as the sectional curvature is zero, positive or negative.

The Ricci tensor ρ, the Ricci transformation Q and the scalar curvature τ of M are defined respectively by

$$\rho(x, y) = \mathrm{trace}\big(z \mapsto R(z, x)y\big),$$
$$g(Qx, y) = \rho(x, y),$$
$$\tau = \mathrm{trace}\, Q,$$

for x, y, $z \in T_pM$, $p \in M$. It is obvious that ρ is a symmetric tensor field of type (0,2) on M.

Definition. A Riemannian manifold $M = (M, g)$ satisfying $g = c\rho$ for some constant c is called an Einstein space.

It is easy to verify that Einstein constant c is equal to $\dfrac{\tau}{n}$. A space of constant curvature is necessarily an Einstein space. In fact, for an n-dimensional space of constant curvature c, we have $\rho = c(n-1)g$.

A locally symmetric space is one of the important object in Riemannian geometry.

Definition. A Riemannian manifold is called a locally symmetric space if $\nabla R = 0$.

Let $\exp_p\colon T_pM \to M$ be the exponential map at a point p of a Riemannian manifold M. There exists a neighborhood $U_p \subset M$ of p such that the map $\sigma_p\colon U_p \to U_p$ defined by

$$\sigma_p = \exp_p \circ (-id_{T_pM}) \circ \exp_p^{-1}$$

is a diffeomorphism on U_p. We call the map σ_p the geodesic symmetry at p. A locally symmetric space is characterized by the property that, at each point $p \in M$, the geodesic symmetry σ_p is a local isometry. By definition, it is clear that $\sigma_p^2 = id$ and the point p is an isolated fixed point of σ_p.

Definition. A Riemannian manifold is called a symmetric space if, for every point $p \in M$, there exists an isometry σ_p of M such that $\sigma_p^2 = id$ and p is an isolated fixed point of σ_p.

Since each isometry σ_p on a symmetric space induces an isometric geodesic symmetry, a symmetric space is a locally symmetric space. It is well-known that a connected, simply connected, complete locally symmetric space is a symmetric space. It is also well-known that an irreducible symmetric space is an Einstein space.

Next, we introduce the notion of almost complex manifold which is a generalization of that of complex manifold, and define some special classes of Riemannian manifold with almost complex structure.

Let M be an n-dimensional complex manifold and $(U; z^1, \dots, z^n)$ a holomorphic local coordinate neighborhood of M. We put $z^i = x^i + \sqrt{-1}\,y^i$ for $i = 1, \dots, n$. Then, $(U; x^1, \dots, x^n, y^1, \dots, y^n)$ can be regarded as a real local coordinate neighborhood of M, namely, through the above correspondence, M can be regarded as a $2n$-dimensional real differentiable manifold. At each point $p \in U$, we define a linear transformation J_p on T_pM by

$$J_p\left(\frac{\partial}{\partial x^i}\right)_p = \left(\frac{\partial}{\partial y^i}\right)_p, \qquad J_p\left(\frac{\partial}{\partial y^i}\right)_p = -\left(\frac{\partial}{\partial x^i}\right)_p$$

for $i = 1, \ldots, n$. It is easily verified that the definition of J_p is independent of the choice of the complex coordinate system $(z^1, \ldots, z^n)$. The linear transformation J_p thus induces a tensor field J of type (1,1) on M satisfying

$$J^2 = -id.$$

The tensor field J is called the almost complex structure corresponding to the complex structure on M.

Definition. A tensor field J of type (1,1) on a differentiable manifold M satisfying $J^2 = -id$ is call an almost complex structure on M. Then the pair $M = (M, J)$ is called an almost complex manifold.

Definition. The almost complex structure J of an almost complex manifold $M = (M, J)$ is said to be integrable if M admits a complex structure and the derived almost complex structure coincides with J.

We also say that an almost complex manifold $M = (M, J)$ is integrable if J is integrable. Let N be a tensor field of type (1,2) on an almost complex manifold $M = (M, J)$ defined by

$$N(X, Y) = [JX, JY] - [X, Y] - J[JX, Y] - J[X, JY]$$

for X, $Y \in \mathfrak{X}(M)$. The tensor field N is called the Nijenhuis tensor field or the torsion tensor field of the almost complex structure J. It is easy to verify that N satisfies the following identities.

$$N(X, Y) = -N(Y, X), \qquad N(JX, Y) = N(X, JY) = -JN(X, Y)$$

for any $X, Y \in \mathfrak{X}(M)$. By the well-known result due to Newlander-Nirenberg ([24]), the almost complex structure J of an almost complex manifold $M = (M, J)$ is integrable if and only if the corresponding Nijenhuis tensor field N vanishes identically on M. An immediate consequence of this fact is that every two-dimensional almost complex manifold is integrable.

Now, we shall define an almost Hermitian manifold.

Definition. An almost Hermitian manifold $M = (M, J, g)$ is an almost complex manifold $M = (M, J)$ equipped with a Riemannian metric g satisfying

$$g(JX, JY) = g(X, Y)$$

for all X, $Y \in \mathfrak{X}(M)$.

The geometry of almost Hermitian manifold is deeply concerned with other fields in mathematics and mathematical physics.

Let $M = (M, J, g)$ be an almost Hermitian manifold. We assume that M is oriented by the volume form $dM = \dfrac{(-1)^n}{n!}\Omega^n$, where Ω is the Kähler form of M defined by

$$\Omega(X, Y) = g(X, JY)$$

for X, $Y \in \mathfrak{X}(M)$.

Here, we recall the definition of some classes of almost Hermitian manifolds.

Definition. Let $M = (M, J, g)$ be an almost Hermitian manifold. We call M a Hermitian manifold if J is integrable, or equivalently

$$(\nabla_X J)Y - (\nabla_{JX} J)JY = 0$$

for all X, $Y \in \mathfrak{X}(M)$. We call M a Kähler manifold if

$$\nabla J = 0.$$

We call M an almost Kähler manifold if

$$d\Omega = 0,$$

or equivalently

$$\underset{X,Y,Z}{\mathfrak{S}}\, g((\nabla_X J)Y, Z) = 0$$

for all X, Y, $Z \in \mathfrak{X}(M)$. We call M a nearly-Kähler manifold if

$$(\nabla_X J)X = 0$$

for all $X \in \mathfrak{X}(M)$. We call M a quasi-Kähler manifold if

$$(\nabla_X J)Y + (\nabla_{JX} J)JY = 0$$

for all X, $Y \in \mathfrak{X}(M)$. We call M a semi-Kähler manifold if

$$\delta\Omega = 0,$$

where δ denotes the coderivative.

Let $\mathcal{H}$, $\mathcal{K}$, $\mathcal{AK}$, $\mathcal{NK}$, $\mathcal{QK}$ and $\mathcal{SK}$ the class of Hermitian, Kähler, almost Kähler, nearly Kähler, quasi-Kähler and semi-Kähler manifolds, respectively. Then, following inclusion relations between these classes are known.

$$\mathcal{K} \begin{smallmatrix} \subset \mathcal{NK} \subset \\ \subset \mathcal{AK} \subset \end{smallmatrix} \mathcal{QK} \subset \mathcal{SK},$$

$$\mathcal{K} = \mathcal{QK} \cap \mathcal{H}, \qquad \mathcal{K} = \mathcal{NK} \cap \mathcal{AK}.$$

In particular, we have $\mathcal{K} = \mathcal{NK}$ and $\mathcal{AK} = \mathcal{QK} = \mathcal{SK}$ in dimension four. A non-Kähler almost Kähler manifold is called a strictly almost Kähler manifold. For an almost Kähler manifold $M = (M, J, g)$, the equality

$$2g((\nabla_X J)Y, Z) = g(JX, N(Y, Z))$$

is valid for all X, Y, $Z \in \mathfrak{X}(M)$. Therefore, by the result due to Newlander-Nirenberg, an almost Kähler manifold with integrable almost complex structure is a Kähler manifold.

In [42], S. Tachibana introduce the Ricci $*$-tensor of an almost Hermitian manifold. The Ricci $*$-tensor ρ^*, Ricci $*$-transformation Q^* and the $*$-scalar curvature τ^* are defined respectively by

$$\rho^*(x, y) = \operatorname{trace}\big(z \mapsto R(x, Jz)Jy\big),$$

$$g(Q^*x, y) = \rho^*(x, y),$$

$$\tau^* = \operatorname{trace} Q^*,$$

for x, y, $z \in T_pM$, $p \in M$. Using the first Bianchi identity, we have

$$\rho^*(x, y) = -\frac{1}{2}\sum_{i=1}^{2n} g(R(x, Jy)e_i, Je_i)$$

for an arbitrary orthonormal basis $\{e_i\}_{i=1,\dots,2n}$ of T_pM. From this equality, we see that

$$\rho^*(x, y) = \rho^*(Jy, Jx).$$

Definition. An almost Hermitian manifold (M, J, g) is said to be $*$-Einstein if the Ricci $*$-tensor ρ^* is a constant multiple of the Riemannian metric g.

Concerning the Goldberg conjecture, K. Sekigawa and L. Vanhecke proved the following ([40]).

Theorem 1.1. *A four-dimensional compact almost Kähler manifold which is Einstein and $*$-Einstein is a Kähler manifold.*

Consequently, we have the following ([40]).

Corollary 1.2. *A compact four-dimensional almost Kähler manifold of constant sectional curvature is a locally flat Kähler manifold.*

This extends a result of Z. Olszak who proved a similar result for arbitrary almost Kähler manifolds M whit $\dim M \geq 8$ ([30]).

Chapter 2

Reimannian metrics on some fiber bundles

1. Definition of the metric

Let $M = (M, g)$ be an n-dimensional Riemannian manifold and $\pi\colon P \to M$ be a principal G-bundle. We denote by $\mathfrak{g}$ the Lie algebra of G. Since $(R_a)_* A^* = (Ad(a^{-1})A)^*$ for any fundamental vector field A^* corresponding to $A \in \mathfrak{g}$ and $a \in G$, we have

$$[A^*, B^*] = [A, B]^* \tag{2.1.1}$$

for any A, $B \in \mathfrak{g}$.

Let Γ be a connection on P. We denote by ω and Ω the connection form and the curvature form corresponding to the connection Γ, respectively. Then, we have the following formulas ([20]).

$$[A^*, X^*] = 0, \tag{2.1.2}$$

$$\omega([X^*, Y^*]) = -\Omega(X^*, Y^*), \tag{2.1.3}$$

$$\pi_*([X^*, Y^*]) = [X, Y], \tag{2.1.4}$$

$$R_a^*\Omega = Ad(a^{-1})\Omega \tag{2.1.5}$$

for any $A \in \mathfrak{g}$, $a \in G$ and holizontal lifts X^*, $Y^* \in \mathfrak{X}(P)$ of X, $Y \in \mathfrak{X}(M)$ with respect to Γ.

Now, let K be a closed subgroup of G and we consider the fiber bundle $E = P \times_G (G/K) = P/K$ over M with standard fibre G/K which is associated with P and denote by $\pi_1\colon E \to M$ the projection defined by $\pi_1(uK) = \pi(u)$ $(u \in P)$. If we define a map $j\colon P \to E$ by $j(u) = uK$, then we see that $j\colon P \to E$ is a principal K-bundle and that the following diagram is commutative.

$$\begin{array}{ccc} P & \xrightarrow{\ j\ } & E = P/K \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\pi_1} \\ M & = & M \end{array}$$

Throughout the present chapter, we fix an $Ad(K)$-invariant Riemannian metric $\langle\ ,\ \rangle$ on G.

We denote by $\mathfrak{k}$ the Lie algebra of K. Then $\mathfrak{k}$ is a Lie subalgebra of $\mathfrak{g}$ and we obtain an orthogonal decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ with respect to the metric $\langle\ ,\ \rangle$. It is easily verified that $ad(\mathfrak{k})$ is skew-symmetric with respect to $\langle\ ,\ \rangle$.

First, we define a Riemannian metric g_P on P. Taking account of

$$T_uP = \operatorname{span}_{\mathbb{R}}\{\, X_u^*, A_u^* \mid X \in \mathfrak{X}(M),\ A \in \mathfrak{g}\,\}$$

at each point $u \in P$, we define g_P by

$$\begin{aligned} g_P(A_u^*, B_u^*) &= c^2\langle A, B\rangle, \\ g_P(A_u^*, X_u^*) &= 0, \\ g_P(X_u^*, Y_u^*) &= g(X_{\pi(u)}, Y_{\pi(u)}), \end{aligned} \tag{2.1.6}$$

for A, $B \in \mathfrak{g}$ and X, $Y \in \mathfrak{X}(M)$, where c is a fixed positive real number.

Next, we define a Riemannian metric g_E on E. Taking account of

$$T_{j(u)}E = \operatorname{span}_{\mathbb{R}}\{\, j_*(X_u^*), j_*(A_u^*) \mid X \in \mathfrak{X}(M),\ A \in \mathfrak{m}\,\}$$

at each point $j(u) \in E$ $(u \in P)$, we define g_E by

$$\begin{aligned} g_E(j_*(A_u^*), j_*(B_u^*)) &= c^2\langle A, B\rangle, \\ g_E(j_*(A_u^*), j_*(X_u^*)) &= 0, \\ g_E(j_*(X_u^*), j_*(Y_u^*)) &= g(X_{\pi(u)}, Y_{\pi(u)}), \end{aligned} \tag{2.1.7}$$

for A, $B \in \mathfrak{m}$ and $X, Y \in \mathfrak{X}(M)$. From (2.1.6) and (2.1.7), we see that $j\colon (P, g_P) \to (E, g_E)$ is a Riemannian submersion.

2. Curvatures

We denote by D and ∇ the Riemannian connection on (M, g) and (P, g_P), respectively. Taking account of (2.1.1) $\sim$ (2.1.4), by direct calculation, we have the following.

Proposition 2.2.1. *For $A, B, C \in \mathfrak{g}$ and $X, Y, Z \in \mathfrak{X}(M)$, we have*

(i) $$g_P(\nabla_{A^*}B^*, C^*) = \frac{c^2}{2}\{\langle [A,B], C\rangle - \langle [B,C], A\rangle + \langle [C,A], B\rangle\},$$

(ii) $$g_P(\nabla_{A^*}B^*, X^*) = g_P(\nabla_{A^*}X^*, B^*) = g_P(\nabla_{X^*}A^*, B^*) = 0,$$

(iii) $$\begin{aligned} g_P(\nabla_{A^*}X^*, Y^*) &= g_P(\nabla_{X^*}A^*, Y^*) \\ &= -g_P(\nabla_{X^*}Y^*, A^*) = \frac{c^2}{2}\langle \Omega(X^*, Y^*), A\rangle, \end{aligned}$$

(iv) $$g_P(\nabla_{X^*}Y^*, Z^*) = g(D_X Y, Z).$$

We denote by R and R_P the curvature tensor of (M, g) and (P, g_P), respectively. Moreover, we denote by f^* the adjoint of a linear transformation $f\colon \mathfrak{g} \to \mathfrak{g}$ with respect to $\langle\ ,\ \rangle$. Taking account of Proposition 2.2.1, Jacobi identity and 2nd Bianchi identity, by direct calculation, we have the following.

Proposition 2.2.2. *For $A, B, C, D \in \mathfrak{g}$ and $X, Y, Z, W \in \mathfrak{X}(M)$, we have*

(i) $$\begin{aligned} &g_P(R_P(A^*, B^*)C^*, D^*) \\ &= \frac{c^2}{2}\langle [A,B], [C,D]\rangle + \frac{c^2}{4}\langle [A,C], [B,D]\rangle - \frac{c^2}{4}\langle [A,D], [B,C]\rangle \\ &\quad - \frac{c^2}{4}\{\langle [[A,B],C], D\rangle - \langle [[A,B],D], C\rangle \\ &\qquad + \langle [[C,D],A], B\rangle - \langle [[C,D],B], A\rangle\} \\ &\quad + \frac{c^2}{4}\{\langle ad(A)^*C + ad(C)^*A, ad(B)^*D + ad(D)^*B\rangle \\ &\qquad - \langle ad(A)^*D + ad(D)^*A, ad(B)^*C + ad(C)^*B\rangle\}, \end{aligned}$$

(ii) $$g_P(R_P(X^*, A^*)B^*, C^*) = 0,$$

(iii) $$\begin{aligned} &g_P(R_P(X^*, A^*)Y^*, B^*) \\ &= -\frac{c^4}{4}\sum_{k=1}^{n}\langle \Omega(X^*, E_k^*), B\rangle\langle \Omega(Y^*, E_k^*), A\rangle - \frac{c^2}{4}\langle \Omega(X^*, Y^*), [A,B]\rangle \\ &\quad + \frac{c^2}{4}\langle [\Omega(X^*, Y^*), A], B\rangle + \frac{c^2}{4}\langle [B, \Omega(X^*, Y^*)], A\rangle \end{aligned}$$

(iv) $\quad g_P(R_P(X^*,Y^*)A^*,Z^*) = -\dfrac{c^2}{2}\langle(\nabla_{Z^*}\Omega)(X^*,Y^*),A\rangle,$

(v) $$g_P(R_P(X^*,Y^*)Z^*,W^*)$$
$$= g(R(X,Y)Z,W) + \frac{c^2}{2}\langle\Omega(X^*,Y^*),\Omega(Z^*,W^*)\rangle$$
$$+ \frac{c^2}{4}\langle\Omega(X^*,Z^*),\Omega(Y^*,W^*)\rangle - \frac{c^2}{4}\langle\Omega(X^*,W^*),\Omega(Y^*,Z^*)\rangle,$$

where $\{E_k\}_{k=1,\cdots,n}$ is a local orthonormal frame field of M.

We denote by R_E the curvature tensor of (E, g_E). Applying the formulas due to B. O'Neill ([31]) to the Riemannian submersion $j\colon (P, g_P) \to (E, g_E)$, we obtain the following.

Proposition 2.2.3. *For A, B, C, $D \in \mathfrak{m}$ and X, Y, Z, $W \in \mathfrak{X}(M)$, we have*

(i) $$g_E(R_E(j_*(A^*),j_*(B^*))j_*(C^*),j_*(D^*))$$
$$= \frac{c^2}{2}\langle[A,B]_\mathfrak{m},[C,D]_\mathfrak{m}\rangle + \frac{c^2}{4}\langle[A,C]_\mathfrak{m},[B,D]_\mathfrak{m}\rangle$$
$$- \frac{c^2}{4}\langle[A,D]_\mathfrak{m},[B,C]_\mathfrak{m}\rangle$$
$$- \frac{c^2}{4}\{\langle[[A,B],C],D\rangle - \langle[[A,B],D],C\rangle$$
$$+ \langle[[C,D],A],B\rangle - \langle[[C,D],B],A\rangle\}$$
$$+ \frac{c^2}{4}\{\langle ad(A)^*C + ad(C)^*A, ad(B)^*D + ad(D)^*B\rangle$$
$$- \langle ad(A)^*D + ad(D)^*A, ad(B)^*C + ad(C)^*B\rangle\},$$

(ii) $\quad g_E(R_E(j_*(X^*),j_*(A^*))j_*(B^*),j_*(C^*)) = 0,$

(iii) $$g_E(R_E(j_*(X^*),j_*(A^*))j_*(Y^*),j_*(B^*))$$
$$= -\frac{c^4}{4}\sum_{k=1}^{n}\langle\Omega(X^*,E_k^*),B\rangle\langle\Omega(Y^*,E_k^*),A\rangle$$
$$- \frac{c^2}{4}\langle\Omega(X_u^*,Y_u^*)_\mathfrak{m},[A,B]_\mathfrak{m}\rangle + \frac{c^2}{4}\langle[\Omega(X^*,Y^*),A],B\rangle$$
$$+ \frac{c^2}{4}\langle[B,\Omega(X^*,Y^*)],A\rangle$$

(iv) $\quad g_E(R_E(j_*(X^*),j_*(Y^*))j_*(A^*),j_*(Z^*)) = -\dfrac{c^2}{2}\langle(\nabla_{Z^*}\Omega)(X^*,Y^*),A\rangle$

(v) $\quad g_E(R_E(j_*(X^*), j_*(Y^*))j_*(Z^*), j_*(W^*))$

$$= g(R(X,Y)Z,W) + \frac{c^2}{2}\langle \Omega(X^*,Y^*)_{\mathfrak{m}}, \Omega(Z^*,W^*)_{\mathfrak{m}}\rangle$$
$$+ \frac{c^2}{4}\langle \Omega(X^*,Z^*)_{\mathfrak{m}}, \Omega(Y^*,W^*)_{\mathfrak{m}}\rangle$$
$$- \frac{c^2}{4}\langle \Omega(X^*,W^*)_{\mathfrak{m}}, \Omega(Y^*,Z^*)_{\mathfrak{m}}\rangle,$$

where $\{E_k\}_{k=1,\cdots,n}$ is a local orthonormal frame field of M.

We put $\|A\|^2 = \langle A, A\rangle$ for $A \in \mathfrak{g}$. We denote by K and K_E the sectional curvature of (M,g) and (E,g_E), respectively. Then, by direct computation, we have the following.

Corollary 2.2.4. *Let A, $B \in \mathfrak{m}$ and X, $Y \in \mathfrak{X}(M)$ be orthogonal unit vectors with respect to $\langle\ ,\ \rangle$ and g, respectively. Then, we have*

(i) $\quad K_E(j_*(A^*), j_*(B^*))$

$$= \frac{1}{4c^2}\|ad(A)^*B + ad(B)^*A\|^2 - \frac{1}{c^2}\langle ad(A)^*A, ad(B)^*B\rangle$$
$$- \frac{3}{4c^2}\|[A,B]_{\mathfrak{m}}\|^2 - \frac{1}{2c^2}\{\langle[[A,B],B],A\rangle - \langle[[A,B],A],B\rangle\},$$

(ii) $$K_E(j_*(X^*), j_*(A^*)) = \frac{c^2}{4}\sum_{k=1}^{n}\langle \Omega(X^*,E_k^*), A\rangle^2,$$

(iii) $$K_E(j_*(X^*), j_*(Y^*)) = K(X,Y) - \frac{3c^2}{4}\|\Omega(X^*,Y^*)_{\mathfrak{m}}\|^2,$$

where $\{E_k\}_{k=1,\cdots,n}$ is a local orthonormal frame field of M.

We denote by ρ and ρ_E the Ricci tensor of (M,g) and (E,g_E), respectively. We put $m = \dim\mathfrak{m}$ and let $\{F_k\}_{k=1,\cdots,m}$ be an orthonormal basis of $\mathfrak{m}$ with respect to $\langle\ ,\ \rangle$. For the brevity, we put

$$\langle\langle A,B\rangle\rangle_\Omega = \sum_{k,l=1}^{n}\langle \Omega(E_k^*,E_l^*), A\rangle\langle \Omega(E_k^*,E_l^*), B\rangle$$

and

$$\langle\langle X^*,Y^*\rangle\rangle_{\Omega,\mathfrak{m}} = \sum_{k=1}^{n}\langle \Omega(X^*,E_k^*)_{\mathfrak{m}}, \Omega(Y^*,E_k^*)_{\mathfrak{m}}\rangle$$

for A, $B \in \mathfrak{m}$ and X, $Y \in \mathfrak{X}(M)$, where $\{E_k\}_{k=1,\cdots,n}$ is a local orthonormal frame field of M. Then, by direct calculation, we have the following.

Corollary 2.2.5. *For A, $B \in \mathfrak{m}$ and X, $Y \in \mathfrak{X}(M)$, we have*

(i)
$$\begin{aligned}
&\rho_E(j_*(A^*), j_*(B^*)) \\
&= \frac{c^4}{4}\langle\!\langle A, B\rangle\!\rangle_\Omega - \frac{3}{4}\sum_{k=1}^{m}\langle [A, F_k]_\mathfrak{m}, [B, F_k]_\mathfrak{m}\rangle \\
&\quad - \frac{1}{4}\sum_{k=1}^{m}\{\langle [[[F_k, A], B], F_k\rangle - \langle [[F_k, A], F_k], B\rangle \\
&\qquad + \langle [[B, F_k], F_k], A\rangle - \langle [[B, F_k], A], F_k\rangle\} \\
&\quad + \frac{1}{4}\sum_{k=1}^{m}\{\langle ad(A)^*F_k + ad(F_k)^*A, ad(B)^*F_k + ad(F_k)^*B\rangle \\
&\qquad - 2\langle ad(A)^*B + ad(B)^*A, ad(F_k)^*F_k\rangle\}
\end{aligned}$$

(ii)
$$\rho_E(j_*(X^*), j_*(A^*)) = -\frac{c^2}{2}\sum_{k=1}^{n}\langle (\nabla_{E_k^*}\Omega)(E_k^*, X^*), A\rangle,$$

(iii)
$$\rho_E(j_*(X^*), j_*(Y^*)) = \rho(X, Y) - \frac{c^2}{2}\langle\!\langle X^*, Y^*\rangle\!\rangle_{\Omega,\mathfrak{m}}$$

where $\{E_k\}_{k=1,\cdots,n}$ is a local orthonormal frame field of M.

3. The case of symmetric fiber

In this section, we consider the special case where the standard fiber $F = G/K$ of E is a symmetric space. In this case, it is well-known that

$$[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}$$

is valid with respect to the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. It is also well-known that the curvature tensor R_F of F is given by

$$R_F(A, B)C = -[[A, B], C]$$

for A, B, $C \in \mathfrak{m}$. Taking account of these facts, from Proposition 2.2.3, we have the following.

Proposition 2.3.1. *For A, B, C, $D \in \mathfrak{m}$ and X, Y, Z, $W \in \mathfrak{X}(M)$, we have*

(i)
$$g_E(R_E(j_*(A^*), j_*(B^*))j_*(C^*), j_*(D^*)) = c^2\langle R_F(A, B)C, D\rangle,$$

(ii) $$g_E(R_E(j_*(X^*), j_*(A^*))j_*(B^*), j_*(C^*)) = 0,$$

(iii) $$\begin{aligned} g_E(R_E(j_*(X^*), j_*(A^*))j_*(Y^*), j_*(B^*)) \\ = -\frac{c^4}{4}\sum_{k=1}^{n}\langle\Omega(Y^*, E_k^*), A\rangle\langle\Omega(X^*, E_k^*), B\rangle \\ - \frac{c^2}{2}\langle[A, \Omega(X^*, Y^*)_{\mathfrak{k}}], B\rangle, \end{aligned}$$

(iv) $$\begin{aligned} g_E(R_E(j_*(X^*), j_*(Y^*))j_*(A^*), j_*(Z^*)) \\ = -\frac{c^2}{2}\langle(\nabla_{Z^*}\Omega)(X^*, Y^*), A\rangle, \end{aligned}$$

(v) $$\begin{aligned} g_E(R_E(j_*(X^*), j_*(Y^*))j_*(Z^*), j_*(W^*)) \\ = g(R(X, Y)Z, W) + \frac{c^2}{2}\langle\Omega(X^*, Y^*)_{\mathfrak{m}}, \Omega(X^*, W^*)_{\mathfrak{m}}\rangle \\ + \frac{c^2}{4}\langle\Omega(X^*, Z^*)_{\mathfrak{m}}, \Omega(Y^*, W^*)_{\mathfrak{m}}\rangle \\ - \frac{c^2}{4}\langle\Omega(X^*, W^*)_{\mathfrak{m}}, \Omega(Y^*, Z^*)_{\mathfrak{m}}\rangle, \end{aligned}$$

where $\{E_k\}_{k=1,\cdots,n}$ is a local orthonormal frame filed of M.

We denote by K_F the sectional curvature of the fiber $(F, \langle\ ,\ \rangle)$ of E. From Corollary 2.2.4, we have the following.

Corollary 2.3.2. *Let A, $B \in \mathfrak{m}$ and X, $Y \in \mathfrak{X}(M)$ be orthogonal unit vectors with respect to $\langle\ ,\ \rangle$ and g respectively. Then, we have*

(i) $$K_E(j_*(A^*), j_*(B^*)) = \frac{1}{c^2}K_F(A, B),$$

(ii) $$K_E(j_*(X^*), j_*(A^*)) = \frac{c^2}{4}\sum_{k=1}^{n}\langle\Omega(X^*, E_k^*), A\rangle^2,$$

(iii) $$K_E(j_*(X^*), j_*(Y^*)) = K(X, Y) - \frac{3c^2}{4}\|\Omega(X^*, Y^*)_{\mathfrak{m}}\|^2,$$

where $\{E_k\}_{k=1,\cdots,n}$ is a local orthonormal frame field of M.

We denote by ρ_F the Ricci tensor of $(F, \langle\ ,\ \rangle)$. From Corollary 2.2.5, we have the following.

Corollary 2.3.3. *For A, $B \in \mathfrak{m}$ and X, $Y \in \mathfrak{X}(M)$, we have*

(i) $$\rho_E(j_*(A^*), j_*(B^*)) = \rho_F(A, B) + \frac{c^4}{4}\langle\langle A, B\rangle\rangle_\Omega,$$

(ii) $$\rho_E(j_*(X^*), j_*(A^*)) = -\frac{c^2}{2}\sum_{k=1}^{n}\langle(\nabla_{E_k^*}\Omega)(E_k^*, X^*), A\rangle,$$

(iii) $$\rho_E(j_*(X^*), j_*(Y^*)) = \rho(X, Y) - \frac{c^2}{2}\langle\!\langle X^*, Y^*\rangle\!\rangle_{\Omega,\mathfrak{m}}$$

where $\{E_k\}_{k=1,\cdots,n}$ is a local orthonormal frame field of M.

From this corollary, we obtain a sufficient condition for the bundle E to be an Einstein space. A connection Γ on P is called a Yang-Mills connection if the corresponding curvature form Ω is co-closed, namely, $\delta\Omega = 0$ identically on P, where δ is the coderivative of the exterior covariant derivative with respect to g_P.

Corollary 2.3.4. *If the connection Γ on P is a Yang-Mills connection and its curvature form Ω satisfies*

$$\langle\!\langle A, B\rangle\!\rangle_\Omega = \lambda\langle A, B\rangle, \qquad \langle\!\langle X^*, Y^*\rangle\!\rangle_{\Omega,\mathfrak{m}} = \mu g(X, Y)$$

for all A, $B \in \mathfrak{m}$ and X, $Y \in \mathfrak{X}(M)$. Further, M and F are Einstein spaces with constant κ and ν, respectively. If the positive constant c in (2.1.7) satisfies

$$(\lambda + 2\mu)c^4 - 4\kappa c^2 + 4\nu = 0,$$

then (E, g_E) is an Einstein space.

4. Examples

4.1. Metric twistor bundle

We apply the formula obtained in Section 3 to the metric twistor bundle. We refer [4], [38] and so on for more details about the metric twistor bundle.

Let (M, g) be a $2n$-dimensional oriented Riemannian manifold and $\pi\colon F(M) \to M$ the oriented orthonormal frame bundle over M. The structure group of $F(M)$ is $SO(2n)$, the special orthogonal group of order $2n$. Let $\mathbb{R}^{2n} = (\mathbb{R}^{2n}, \cdot)$ be a $2n$-dimensional Euclidean space with canonical inner product. We define a linear endomorphism J_0 of $\mathbb{R}^{2n}$ represented by

$$J_0 = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$$

with respect to the canonical orthonormal basis $\{e\} = \{e_1, \cdots, e_{2n}\}$ of $\mathbb{R}^{2n}$, where I_n denotes the identity matrix of degree n. For each $u \in F(M)$, we define a linear endomorphism $j(u)$ of $T_{\pi(u)}M$ by

$$j(u) = uJ_0u^{-1}.$$

We see immediately that $j(u)^2 = -id$ and $j(u)$ is a linear isometry of $T_{\pi(u)}M$. Thus, $j(u)$ is an orthogonal complex structure at $\pi(u)$. If we put $J_p(M) = \{\, j(u) \mid u \in F(M), \pi(u) = p\,\}$, then $J_p(M)$ is diffeomorphic to the homogeneous space $SO(2n)/U(n)$, where $U(n) = \{\, a \in SO(n) \mid aJ_0 = J_0a\,\}$ is the unitary group of order n. We put $J(M) = \bigcup_{p\in M} J_p(M)$. Then, $F(M)$ is a principal fiber bundle over $J(M)$ with structure group $U(n)$, and hence $J(M)$ is a fiber bundle over M associated with $F(M)$ with standard fiber $SO(2n)/U(n)$. The fiber bundle $\pi_1 \colon J(M) \to M$ is called the metric twistor bundle over M.

$$\begin{tikzcd}
F(M) \arrow[r, "j"] \arrow[d, "\pi"'] & J(M) \arrow[d, "\pi_1"] \\
M \arrow[r, equal] & M
\end{tikzcd}$$

Now, for each $a \in SO(2n)$, we define an involutive automorphism σ of $SO(2n)$ by

$$\sigma(a) = -J_0aJ_0.$$

Then the fixed point set of σ is isomorphic to $U(n)$, and furthermore we have the corresponding Cartan decomposition of the Lie algebra $\mathfrak{so}(2n)$ of $SO(2n)$, namely,

$$\mathfrak{so}(2n) = \mathfrak{u}(n) \oplus \mathfrak{m}, \tag{2.4.1}$$

where $\mathfrak{u}(n)$ is the Lie algebra of $U(n)$.

We see that $J_0A \in \mathfrak{m}$ for any $A \in \mathfrak{m}$ and $Ad(a)J_0 = J_0Ad(a)$ on $\mathfrak{m}$ for all $a \in U(n)$. Thus J_0 gives rise to $SO(2n)$-invariant almost complex structure on $SO(2n)/U(n)$. For $A, B \in \mathfrak{so}(2n)$, we define an inner product $\langle\ ,\ \rangle$ on $\mathfrak{so}(2n)$ by

$$\langle A, B\rangle = \sum_{i=1}^{2n} (Ae_i)\cdot(Be_i) = -\mathrm{trace}(AB).$$

Then, we see that the inner product $\langle\ ,\ \rangle$ gives rise to a bi-invariant Riemannian metric on $SO(2n)$ and hence a $SO(2n)$-invariant Riemannian metric on $SO(2n)/U(n)$. Further, we see that $(SO(2n)/U(n), J_0, \langle\ ,\ \rangle)$ is a compact Hermitian symmetric space.

From the arguments above, we can apply the formulas in the preceding section to $J(M)$. Note that the curvature form Ω on $F(M)$ is given by

$$\Omega(X_u^*, Y_u^*) = u^{-1}R(x,y)u$$

for $u \in F(M)$ and $X, Y \in \mathfrak{X}(M)$ with $X_{\pi(u)} = x$, $Y_{\pi(u)} = y$, where R denotes the curvature tensor of M ([20]). Taking account of this, the curvature tensor $R_{J(M)}$ of $J(M)$ is given by

$$\begin{aligned}
&g_{J(M)}(R_{J(M)}(j_*(A_u^*), j_*(B_u^*))j_*(C_u^*), j_*(D_u^*)) = -c^2\langle [A,B],[C,D]\rangle,\\
&g_{J(M)}(R_{J(M)}(j_*(X_u^*), j_*(A_u^*))j_*(B_u^*), j_*(C_u^*)) = 0,\\
&g_{J(M)}(R_{J(M)}(j_*(X_u^*), j_*(A_u^*))j_*(Y_u^*), j_*(B_u^*))\\
&\quad = -\frac{c^4}{4}\sum_{k=1}^{2n}\langle u^{-1}R(x,e_k)u, B\rangle\langle u^{-1}R(y,e_k)u, A\rangle\\
&\qquad + \frac{c^2}{2}\langle u^{-1}R(x,y)u, [A,B]\rangle,\\
&g_{J(M)}(R_{J(M)}(j_*(X_u^*), j_*(Y_u^*))j_*(A_u^*), j_*(Z_u^*))\\
&\quad = -\frac{c^2}{2}\langle u^{-1}(\nabla_Z R)(X,Y)u, A\rangle,\\
&g_{J(M)}(R_{J(M)}(j_*(X_u^*), j_*(Y_u^*))j_*(Z_u^*), j_*(W_u^*))\\
&\quad = g(R(x,y)z,w) + \frac{c^2}{2}\langle (u^{-1}R(x,y)u)_{\mathfrak{m}}, (u^{-1}R(z,w)u)_{\mathfrak{m}}\rangle\\
&\qquad + \frac{c^2}{4}\langle (u^{-1}R(x,z)u)_{\mathfrak{m}}, (u^{-1}R(y,w)u)_{\mathfrak{m}}\rangle\\
&\qquad - \frac{c^2}{4}\langle (u^{-1}R(x,w)u)_{\mathfrak{m}}, (u^{-1}R(y,z)u)_{\mathfrak{m}}\rangle,
\end{aligned}$$

at $j(u) \in J(M)$ ($u = (\pi(u); e_1, \cdots, e_{2n}) \in F(M)$), where A, B, C, $D \in \mathfrak{m}$ and X, Y, Z, $W \in \mathfrak{X}(M)$ with $X_{\pi(u)} = x$, $Y_{\pi(u)} = y$, $Z_{\pi(u)} = z$, $W_{\pi(u)} = w$.

The Ricci tensor $\rho_{J(M)}$ of $J(M)$ is given by

$$\rho_{J(M)}(j_*(A_u^*), j_*(B_u^*))$$

$$= \rho_F(A,B) + \frac{c^4}{4}\sum_{k,l=1}^{2n}\langle u^{-1}R(e_k,e_l)u, A\rangle\langle u^{-1}R(e_k,e_l)u, B\rangle,$$

$$\rho_{J(M)}(j_*(X_u^*), j_*(A_u^*))$$

$$= -\frac{c^2}{2}\sum_{k=1}^{2n}\langle u^{-1}(\nabla_{e_k}R)(e_k,x)u, A\rangle,$$

$$\rho_{J(M)}(j_*(X_u^*), j_*(Y_u^*))$$

$$= \rho(x,y) - \frac{c^2}{2}\sum_{k=1}^{2n}\langle (u^{-1}R(x,e_k)u)_{\mathfrak{m}}, (u^{-1}R(y,e_k)u)_{\mathfrak{m}}\rangle,$$

at $j(u) \in J(M)$ $(u = (\pi(u); e_1, \cdots, e_{2n}) \in F(M))$, where A, $B \in \mathfrak{m}$ and X, $Y \in \mathfrak{X}(M)$ with $X_{\pi(u)} = x$, $Y_{\pi(u)} = y$.

As a special case, let $M = S^4$, the standard unit 4-sphere with canonical metric. Then $J(S^4)$ is the fiber bundle over S^4 associated with $F(S^4)$ with standard fiber $S^2 = SO(4)/U(2)$. Then, the assumptions of Corollary 2.3.4 are satisfied by $\lambda = 4$, $\mu = 2$, $\kappa = 3$ and $\nu = 1$. Thus, we see that $J(S^4)$ is a Einstein space if $c = 1$ or $c = \sqrt{1/2}$.

4.2. Tangent bundle

Let (M, g) be an n-dimensional Riemannian manifold. An affine frame at $p \in M$ is a pair (p, X, v) of a linear frame X at p and a tangent vector v at p. We denote by $A_p(M)$ the set of all affine frames at p and put $A(M) = \bigcup_{p\in M} A_p(M)$. Let π be the mapping of $A(M)$ onto M which maps an affine frame at p into p. The group $G = GL(n;\mathbb{R}) \ltimes \mathbb{R}^n$, semi-direct product of the general linear group $GL(n;\mathbb{R})$ and $\mathbb{R}^n$, acts freely on $A_p(M)$ for each $p \in M$ by

$$(p, X, v)\cdot(A, a) := (p, XA, Xa + v)$$

where $(p, X, v) \in A_p(M)$, $A \in GL(n;\mathbb{R})$ and $a \in \mathbb{R}^n$. The differentiable structure in $A(M)$ is given as follows. Let $(x^1, \cdots, x^n)$ be a local coordinate system of a coordinate neighborhood U of $p \in M$. Every frame $X = (X_1, \cdots, X_n)$ and tangent vector $v \in T_p(M)$ can be express uniquely in the form $X_i = X_i^j \dfrac{\partial}{\partial x_j}$ and $v = v^i \dfrac{\partial}{\partial x_i}$, respectively. This shows that $A(M)$ admits a differentiable structure by taking (x^i, X_i^j, v^i) as a local coordinate system in $\pi^{-1}(U)$. It is now easy to see that $A(M)$ is a principal fiber bundle over M with structure group G.

The tangent bundle $T(M)$ of M is identified with the fiber bundle $A(M) \times_G \mathbb{R}^n$. We define a map $j\colon A(M) \to T(M)$ by $j(p, X, v) = (p, v)$.

$$\begin{tikzcd} A(M) \arrow[r, "j"] \arrow[d, "\pi"'] & T(M) \arrow[d, "\pi_1"] \\ M \arrow[r, equal] & M \end{tikzcd}$$

A connection on $A(M)$ is called a generalized affine connection. It is known that there is a one to one correspondence between the set of all generalized affine connections and the set of all pairs of a linear connection on M and a tensor field of type (1,1) on M ([20]).

Now, we define a generalized affine connection by using a liner connection Γ on M and a tensor field K of type (1,1). Let $E^i_j \in \mathfrak{gl}(n;\mathbb{R})$ be the $n \times n$ matrix such that the element of i-th column and the j-th row is 1 and other elements are all zero, and $\{e_1, \cdots, e_n\}$ the canonical basis of $\mathbb{R}^n$. Then $\{E^i_j, e_k\}_{i,j,k=1,\dots,n}$ is a basis of $\mathfrak{g} = \mathfrak{gl}(n;\mathbb{R}) \oplus \mathbb{R}^n$ (direct sum), the Lie algebra of G. We define a $\mathfrak{g}$-valued 1-form $\tilde{\omega} = \omega^i_j E^j_i + \eta^i e_i$ on a coordinate neighborhood $(x_i, X^i_j, v_i) \subset A(M)$ by

$$\omega^i_j = Y^i_k(dX^k_j + \Gamma^k_{hl} X^l_j \, dx^h)$$
$$\eta^i = Y^i_k(dv^k + K^k_j \, dx^j + \Gamma^k_{hl} v^l \, dx^h)$$

where (Y^i_j) is the inverse matrix of (X^i_j) and Γ^k_{ij} are the components of Γ with respect to $\left\{ \dfrac{\partial}{\partial x^i} \right\}_{i=1,\dots,n}$. It is easy to verify that the $\tilde{\omega}$ is a connection form on $A(M)$. On the coordinate neighborhood with coordinate system (x^i, X^i_j, v^i), the horizontal distribution with respect to $\tilde{\omega}$ is spanned by the vectors of the form

$$V_i = \frac{\partial}{\partial x^i} + \xi^j_k \frac{\partial}{\partial X^j_k} + \zeta^j \frac{\partial}{\partial v^j} \qquad (i = 1, \cdots, n).$$

Since $\tilde{\omega}(V_i) = 0$, we have

$$\xi^j_k = -\Gamma^j_{ih} X^h_k, \qquad \zeta^j = -(K^j_i + \Gamma^j_{ik} v^k)$$

for each $i = 1, \dots, n$. Thus, by definition of the map j, we see that the vectors

$$(2.4.2) \qquad j_*(V_i) = \frac{\partial}{\partial x^i} - (K^j_i + \Gamma^j_{ik} v^k)\frac{\partial}{\partial v^j} \qquad (i = 1, \cdots, n)$$

span the horizontal distribution on the coordinate neighborhood (x^i, v^i) of $T(M)$.

S. Sasaki introduced a Riemannian metric on $T(M)$ ([35]). On the coordinate neighborhood (x^i, v^i) of $T(M)$, we put $dx^{n+i} = dv^i$ for $i = 1, \ldots, n$, and define a Riemannian metric $ds^2 = G_{ij}\, dx^i\, dx^j$ on $T(M)$ by

$$G_{ij} = g_{ij} + g_{st}\begin{Bmatrix} s \\ ki \end{Bmatrix}\begin{Bmatrix} t \\ lj \end{Bmatrix} v^k v^l,$$

$$G_{i\, n+j} = [ki, j] v^k$$

$$G_{n+i\, n+j} = g_{ij}$$

for $i, j = 1, \ldots, n$, where $[ki, j]$, $\begin{Bmatrix} i \\ jk \end{Bmatrix}$ is the first and second kind of Christoffel's symbols of M, respectively. With respect to his metric, the vectors

$$\frac{\partial}{\partial x^i} - \begin{Bmatrix} j \\ ik \end{Bmatrix} v^k \frac{\partial}{\partial v^j} \qquad (i = 1, \cdots, n)$$

form a basis of the horizontal distribution. Therefore, in particular, if we take $\Gamma^i_{jk} = \begin{Bmatrix} i \\ jk \end{Bmatrix}$ and $K \equiv 0$ in (2.4.2), then our connection on $T(M)$ induced from $\tilde{\omega}$ coincides with his connection. In this point of view, our connection on $T(M)$ is a generalization of the Sasaki metric.

Chapter 3

Integrability of almost Kähler manifolds

1. Almost Kähler manifold

Let $M = (M, J, g)$ be an almost Kähler manifold. Then, it is known that the curvature tensor R satisfies the following identity ([12]).

$$
\begin{aligned}
(3.1.1) \qquad & R(w,x,y,z) - R(w,x,Jy,Jz) - R(Jw,Jx,y,z) \\
& + R(Jw,Jx,Jy,Jz) + R(Jw,x,Jy,z) - R(Jw,x,Jz,y) \\
& - R(Jx,w,Jy,z) + R(Jx,w,Jz,y) \\
& = 2g((\nabla_w J)x - (\nabla_x J)w, (\nabla_y J)z - (\nabla_z J)y)
\end{aligned}
$$

for w, x, y, $z \in T_pM$, $p \in M$. For an almost Kähler manifold, the following equality is also well-known ([46]).

$$
(3.1.2) \qquad \|\nabla J\|^2 = 2(\tau^* - \tau).
$$

This implies that an almost Kähler manifold is integrable if and only if the $*$-scalar curvature coincides with the scalar curvature.

Next, we define a complex structure on the vector bundle $\wedge^r M$ of all r-forms on an almost Hermitian manifold M and recall the decomposition theorem for $\wedge^2 M$ in terms of the Kähler form Ω. Let $M = (M, J, g)$ be an almost Hermitian manifold. On $\wedge^r M$, the induced scalar product, also denote by g, in each fiber $\wedge^r T_pM$ ($p \in M$) is defined by

$$
g(\alpha, \beta) = \frac{1}{r!} \sum_{i_1,\dots,i_r} \alpha_{i_1\dots i_r} \beta_{i_1\dots i_r}
$$

for $\alpha,\ \beta \in \wedge^r T_pM$, where $\alpha_{i_1\dots i_r} = \alpha(e_{i_1}, \dots, e_{i_r})$, $\beta_{i_1\dots i_r} = \beta(e_{i_1}, \dots, e_{i_r})$ and $\{e_i\}_{1,\dots,2n}$ is an orthonormal basis of the tangent space T_pM at $p \in M$. Each vector bundle $\wedge^r M$ is endowed with a natural complex structure, also denote by J, which is defined by

$$(J\alpha)(X_1, X_2, \dots, X_r) = -\alpha(JX_1, X_2, \dots, X_r),$$

for $\alpha \in \wedge^r M$ and $X_1, \dots, X_r \in \mathfrak{X}(M)$. Now, we recall the following decomposition theorem for the vector bundle $\wedge^2 M$ ([22]).

$$\wedge^2 M = \mathbb{R}\Omega \oplus LM \oplus \wedge_0^{1,1} M,$$

where LM denotes the vector bundle of J-skew-invariant 2-forms on M and $\wedge_0^{1,1}M$ the vector bundle of J-invariant 2-forms on M with vanishing trace.

$$LM = \left\{ \alpha \in \wedge^2 M \;\middle|\; \begin{array}{l} \alpha(JX, JY) = -\alpha(X,Y) \text{ for all } X, Y \in \mathfrak{X}(M) \\ g(\alpha, \Omega) = 0 \end{array} \right\},$$

$$\wedge_0^{1,1} M = \left\{ \alpha \in \wedge^2 M \;\middle|\; \begin{array}{l} \alpha(JX, JY) = \alpha(X,Y) \text{ for all } X, Y \in \mathfrak{X}(M) \\ g(\alpha, \Omega) = 0 \end{array} \right\}.$$

This decomposition is orthogonal with respect to the scalar product g on $\wedge^2 M$. We note that the dimension of each fiber of LM is equal to $n(n-1)$. Let $\{e_i, e_{n+i} = Je_i\}_{i=1,\dots,n}$ be an orthonormal basis of T_pM, $p \in M$ and $\{e^i, e^{n+i} = Je^i\}_{i=1,\dots,n}$ its dual basis. We define 2-forms $\Phi_{[ab]}$ $(1 \le a < b \le n)$ by

$$\Phi_{[ab]} = \frac{1}{\sqrt{2}}(e^a \wedge e^b - Je^a \wedge Je^b).$$

Then, it is easily verified that

$$J\Phi_{[ab]} = \frac{1}{\sqrt{2}}(e^a \wedge Je^b + Je^a \wedge e^b)$$

and $\{\Phi_{[ab]}, J\Phi_{[ab]}\}_{1 \le a < b \le n}$ forms an orthonormal basis of the fiber of LM over $p \in M$. Since an almost Kähler manifold is necessarily an quasi-Kähler manifold, by definition, the equality

$$(\nabla_X J)Y + (\nabla_{JX} J)JY = 0$$

holds for all $X, Y \in \mathfrak{X}(M)$. Further, since the 2-form $\nabla_X \Omega$ belongs to LM for each $X \in \mathfrak{X}(M)$, locally, we can write

$$\nabla\Omega = \sum_{1\le a<b\le n} \varphi_{[ab]} \otimes \Phi_{[ab]} - \sum_{1\le a<b\le n} J\varphi_{[ab]} \otimes J\Phi_{[ab]} \tag{3.1.3}$$

for some local sections $\varphi_{[ab]}$ $(1 \le a < b \le n)$ of $\wedge^1 M$.

At the end of this section, we define the curvature operator. The curvature operator $\mathcal{R}$ is a symmetric endomorphism of the bundle $\wedge^2 M$ defined by

$$g(\mathcal{R}(\iota(x) \wedge \iota(y)), \iota(z) \wedge \iota(w)) = -g(R(x,y)z, w)$$

for x, y, z, $w \in T_pM$, $p \in M$, where ι denotes the duality $T_pM \to T_p^*M$ defined by means of the Riemannian metric g. If $M = (M, J, g)$ is an almost Kähler manifold, the curvature identity (3.1.1) can be written as

$$\begin{aligned} &g(\mathcal{R}(\iota(w) \wedge \iota(x) - J\iota(w) \wedge J\iota(x)), \iota(y) \wedge \iota(z) - J\iota(y) \wedge J\iota(z)) \\ &+ g(\mathcal{R}(J\iota(w) \wedge \iota(x) + \iota(w) \wedge J\iota(x)), J\iota(y) \wedge \iota(z) + \iota(y) \wedge J\iota(z)) \\ &= -2g((\nabla_w J)x - (\nabla_x J)w, (\nabla_y J)z - (\nabla_z J)y) \end{aligned} \tag{3.1.4}$$

for w, x, y, $z \in T_pM$, $p \in M$.

2. Some examples of strictly almost Kähler manifold

2.1. The Thurston manifold

The first example of compact symplectic manifold which does not admit Kähler structure is given by Thurston ([43]). His construction is as follows. Let $T^2 = S^1 \times S^1$ be the 2-torus. The universal covering space $\pi\colon \mathbb{R}^2 \to T^2$ can be regarded as a principal fiber bundle over T^2 with the structure group $\mathbb{Z}\oplus\mathbb{Z}$. Now, we demote by $\bar{\mu}$ the representation of $\mathbb{Z} \oplus \mathbb{Z}$ into $GL(2, \mathbb{R})$ defined by

$$\bar{\mu}(m,n) = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$$

for $(m, n) \in \mathbb{Z} \oplus \mathbb{Z}$. Then, we may easily see that $\bar{\mu}$ induces a representation μ of $\mathbb{Z} \oplus \mathbb{Z}$ into the diffeomorphism group of T^2 in the natural way. The Thurston manifold W^4 is defined as the fiber bundle $W^4 = \mathbb{R}^2 \times_{\mathbb{Z}\oplus\mathbb{Z}} T^2$ associated with the principal fiber bundle $\pi\colon \mathbb{R}^2 \to T^2$ with the standard fiber T^2. Thus, W^4 is a T^2-bundle over T^2. Afterwards, L. A. Cordero, M. Fernandez and M. de Leon generalize the Thurston manifold for every dimension $2n \geq 4$ ([5]).

2.2. The Abbena manifold

E. Abbena ([1]) showed that the Thurston manifold W^4 can be represented as $(\Gamma\backslash H) \times S^1$, where Γ is a discrete subgroup of the Heisenberg group H and S^1 is the unit circle. We shall review the Abbena manifold A^4. The manifold A^4 is defined by $A^4 = (\Gamma\backslash H) \times S^1$, where H is the Heisenberg group

$$H = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \middle| \, x, y, z \in \mathbb{R} \right\},$$

and Γ is the discrete subgroup of H with integer entries. Let $\Phi\colon \mathbb{R}^4 \to H \times \mathbb{R}$ be a diffeomorphism defined by

$$\Phi\colon (t, v, w, u) \mapsto \left(\begin{pmatrix} 1 & v & w \\ 0 & 1 & -u \\ 0 & 0 & 1 \end{pmatrix}, t \right).$$

Then, we may easily show that the Φ induces a diffeomorphism φ from the Thurston manifold W^4 onto the Abbena manifold A^4.

$$\begin{tikzcd}
\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2 \arrow[r, "\Phi"] \arrow[d, "\mathbb{Z}\oplus\mathbb{Z}"] & H \times \mathbb{R} \arrow[dd, "\Gamma \times \mathbb{Z}"] \\
\mathbb{R}^2 \times T^2 \arrow[d] & \\
W^4 = \mathbb{R}^2 \times_{\mathbb{Z}\oplus\mathbb{Z}} T^2 \arrow[r, "\varphi"] & A^4 = (\Gamma\backslash H) \times S^1
\end{tikzcd}$$

Let $\{X_1, X_2, X_3, X_4\}$ be the vector fields on $H \times \mathbb{R}$ defined by

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = -\frac{\partial}{\partial t}, \quad X_3 = -\frac{\partial}{\partial z}, \quad X_4 = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}. \tag{3.2.1}$$

Then, $\{X_1, X_2, X_3, X_4\}$ is a $H \times \mathbb{R}$-invariant frame field on $H \times \mathbb{R}$. We define a $H \times \mathbb{R}$-invariant Riemannian metric $\bar{g}$ on $H \times \mathbb{R}$ by $\bar{g}(X_i, X_j) = \delta_{ij}$, $1 \leq i, j \leq 4$. Further, we define a $H \times \mathbb{R}$-invariant almost complex structure $\bar{J}$ on $H \times \mathbb{R}$ by

$$\bar{J}X_1 = X_2, \quad \bar{J}X_2 = -X_1, \quad \bar{J}X_3 = X_4, \quad \bar{J}X_4 = -X_3. \tag{3.2.2}$$

Then, from (3.2.1) and (3.2.2), we find that $(\bar{J}, \bar{g})$ is a $H \times \mathbb{R}$-invariant almost Hermitian structure on $H \times \mathbb{R}$ and that the corresponding Kähler form $\bar{\Omega}$ is expressed as

$$\bar{\Omega} = dx \wedge dt + dz \wedge dy. \tag{3.2.3}$$

It is easily checked that $(H \times \mathbb{R}, \bar{J}, \bar{g})$ is a strictly almost Kähler manifold. By the above argument, we can see furthermore that the strictly almost Kähler structure $(\bar{J}, \bar{g})$ on $H \times \mathbb{R}$ induces a strictly almost Kähler structure (J, g) on the Abbena manifold A^4. Pulling back the above almost Kähler structure (J, g) on A^4 to W^4 by the diffeomorphism φ, we obtain a strictly almost Kähler structure on W^4. We may show that the first Betti number $b_1(A^4)$ of A^4 is equal to 3. By virtue of the fact that odd-dimensional Betti numbers of a Kähler manifold are even, we find that A^4 (and hence W^4) cannot admit Kähler structure.

2.3. Uncountably many strictly almost Kähler structures on T^6

Recently, W. Jelonek ([15]) constructed uncountably many strictly almost Kähler structures on $M \times T^2_\Gamma$, where M is an almost Kähler manifold and T^2_Γ is the 2-torus $\mathbb{C}/\mathbb{Z}\gamma_1 \oplus \mathbb{Z}\gamma_2$, γ_1, $\gamma_2 \in \mathbb{C}$, linearly independent over $\mathbb{R}$. In particular, if we choose the Thurston manifold W^4 as M, then we obtain uncountably many strictly almost Kähler structures on $W^4 \times T^2_\Gamma$. Iterating his construction, we obtain such structures on every manifolds $W^4 \times T^2_{\Gamma_1} \times \cdots \times T^2_{\Gamma_r}$ $(r \geq 1)$. Moreover, he prove that the 6-torus T^6 admits uncountably many strictly almost Kähler structures by constructing them explicitly. His construction of strictly almost Kähler manifold (T^6, J, g) is as follows. Let Φ, Ψ be any smooth, real-valued functions on $\mathbb{R}$ satisfying

$$(\Phi')^2 + (\Psi')^2 = 1, \quad \Phi(1+t) = \Phi(t), \quad \Psi(1+t) = \Psi(t). \tag{3.2.4}$$

It can be proved that there exist uncountably many functions Φ and Ψ satisfying the condition (3.2.4) ([15]). We define smooth functions $\bar{u}$, $\bar{v}$ on $\mathbb{C}$ by

$$\bar{u}(z) = c\,\Phi(mx + ny), \qquad \bar{v}(z) = c\,\Psi(mx + ny),$$

where $z = x + y\,i$, $c \in \mathbb{R} - \{0\}$ and $m, n \in \mathbb{Z}$ with $m^2 + n^2 > 0$. The functions $\bar{u}$, $\bar{v}$ induce smooth functions u, v on $T^2 = \mathbb{C}/\mathbb{Z}1 \oplus \mathbb{Z}i$. We put $f = \exp u$ and $h = \exp v$. Now, let $M = (T^2 = \mathbb{C}/\mathbb{Z}1 \oplus \mathbb{Z}i, J_0, g_0)$ be the standard flat Kähler 2-torus, and we consider the 6-torus

$$T^6 = M \times T^2 \times T^2 = M \times_{f^2} S^1 \times_{f^{-2}} S^1 \times_{h^2} S^1 \times_{h^{-2}} S^1.$$

with warped-product metric g, namely, g is defined by

$$g = g_0 + f^2\theta_1 \otimes \theta_1 + f^{-2}\theta_2 \otimes \theta_2 + h^2\theta_3 \otimes \theta_3 + h^{-2}\theta_4 \otimes \theta_4,$$

where $\theta_i \otimes \theta_i$ $(i = 1, 2, 3, 4)$ is the canonical Riemannian metric on S^1. Let ξ_i be the dual vectors of θ_i $(i = 1, 2, 3, 4)$. We define an almost complex structure J on T^6 by

$$J\xi_i = \xi_{i+1}, \qquad J\xi_{i+1} = -\xi_i \qquad \text{for} \quad i = 1, 3,$$
$$JX = J_0 X \qquad \text{for} \quad X \in \mathfrak{X}(M).$$

Then, we can prove that $T^6 = (T^6, J, g)$ is a strictly almost Kähler manifold with negative constant scalar curvature $\tau = -c^2(m^2 + n^2)$. Further, we observe that the $*$-scalar curvature τ^* of T^6 is positive constant $\tau^* = -\tau$ and the first Chern class $c_1(T^6)$ of T^6 vanishes. K. Sekigawa and L. Vanhecke ([41]) proved the following.

Theorem 3.2.1. *A compact almost Kähler manifold with vanishing first Chern class and non-positive $*$-scalar curvature is a Kähler manifold with zero scalar curvature.*

Therefore, the above example $T^6 = (T^6, J, g)$ shows that the hypothesis $\tau^* \leq 0$ in Theorem 3.2.1 is essential. We note that, iterating the construction of the almost Kähler structure on T^6, we can obtain uncountably many strictly almost Kähler structures on every T^{4r+2} $(r \in \mathbb{N})$.

2.4. Metric twistor bundle

First of all, we shall recall the definition of self-duality of a four-dimensional Riemannian manifold. Let $M = (M, g)$ be a n-dimensional oriented Riemannian manifold. On $\wedge^r M$, the Hodge $*$-operator $*\colon \wedge^r M \to \wedge^{n-r} M$ is defined by

$$\alpha \wedge *\beta = g(\alpha, \beta)\, dM,$$

for α, $\beta \in \wedge^r M$. The Weyl conformal curvature tensor W is defined by

$$W(x,y)z = R(x,y)z - \frac{1}{n-2}(Qx \wedge y + x \wedge Qy)z + \frac{\tau}{(n-1)(n-2)}(x \wedge y)z,$$

where $x \wedge y$ denotes the endomorphism $z \mapsto g(y,z)x - g(x,z)y$ on T_pM for x, y, $z \in T_pM$. A Riemannian manifold M is called conformally flat if $W = 0$. The Weyl tensor field W defines a symmetric endomorphism $\mathcal{W}$ on $\wedge^2 M$ by

$$g(\mathcal{W}(x \wedge y), z \wedge w) = -g(\mathcal{W}(x,y)z, w).$$

Here, we assume that $\dim M = 4$. On $\wedge^2 M$, the $*$-operator satisfies

$$** = -id.$$

Therefore, we obtain a orthogonal decomposition

$$\wedge^2 M = \wedge^2_+ M \oplus \wedge^2_- M,$$

where $\wedge^2_\pm M$ is the subbundle of $\wedge^2 M$ corresponding to the (± 1)-eigenvalues of $*$. Let $\{e_i\}_{i=1,\dots,4}$ be an arbitrary orthonormal of M and $\{e^i\}_{i=1,\dots,4}$ its dual basis. We put

$$\begin{aligned} f_1^+ &= e^1 \wedge e^2 + e^3 \wedge e^4, & f_1^- &= e^1 \wedge e^2 - e^3 \wedge e^4, \\ f_2^+ &= e^1 \wedge e^3 + e^4 \wedge e^2, & f_2^- &= e^1 \wedge e^3 - e^4 \wedge e^2, \\ f_3^+ &= e^1 \wedge e^4 + e^2 \wedge e^3, & f_3^- &= e^1 \wedge e^4 - e^2 \wedge e^3. \end{aligned}$$

Then $\{f_i^\pm\}_{i=1,2,3}$ is an orthonormal basis of $\wedge^2_\pm M$. With respect to this basis, the curvature operator $\mathcal{R}$ can be expressed as

$$\mathcal{R} = \begin{pmatrix} \frac{\tau}{6} I_3 + \mathcal{W}_+ & \mathcal{B} \\ {}^t\mathcal{B} & \frac{\tau}{6} I_3 + \mathcal{W}_- \end{pmatrix}, \tag{3.2.5}$$

where τ is the scalar curvature of M, I_3 is the identity matrix of order 3, $\mathcal{W}_+$ (resp. $\mathcal{W}_-$) is the restriction of $\mathcal{W}$ to $\wedge^2_+ M$ (resp. $\wedge^2_- M$), $\mathcal{B}\colon \wedge^2_+ M \to \wedge^2_- M$ is a bundle map and ${}^t\mathcal{B}$ denotes the adjoint of $\mathcal{B}$ ([6], [9]). A Riemannian manifold M is said to be self-dual (resp. anti-self-dual) if $\mathcal{W}_- = 0$ (resp. $\mathcal{W}_+ = 0$). It known that M is an Einstein space if and only if $\mathcal{B} = 0$.

Now, we shall consider the metric twistor bundle. Let $M = (M, g)$ be a $2n$-dimensional oriented Riemannian manifold. As we have seen in Section 4.1 of Chapter 2, the metric twistor bundle $J(M)$ over M is a fiber bundle associated with the oriented orthonormal frame bundle $F(M)$ with standard fiber $SO(2n)/U(n)$.

$$\begin{array}{ccc} F(M) & \xrightarrow{\ j\ } & J(M) \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi_1} \\ M & = & M \end{array}$$

We use the same notations as in Section 4.1 of Chapter 2. Let θ and ω be the canonical form and the connection form on $F(M)$ with respect to the Riemannian connection ∇ on M. Corresponding to the Cartan decomposition (2.4.1) of the Lie algebra $\mathfrak{so}(2n)$, we may write

$$\omega = \omega_1 + \omega_2,$$

where ω_1 (resp. ω_2) is the $\mathfrak{u}(n)$-component (resp. $\mathfrak{m}$-component) of ω. Then, we see that there exists a linear isomorphism $\lambda(u)\colon T_{j(u)}J(M) \to \mathfrak{m} \oplus \mathbb{R}^{2n}$ for each $u \in F(M)$ such that

$$\lambda(ua) = (Ad(a^{-1}) \oplus a^{-1})\lambda(u)$$

for all $a \in U(n)$ and the diagram

$$\begin{array}{ccc} T_uF(M) & \xrightarrow{(j_*)_u} & T_{j(u)}J(M) \\ {\scriptstyle \omega_2+\theta}\downarrow & & \downarrow{\scriptstyle \lambda(u)} \\ \mathfrak{m} \oplus \mathbb{R}^{2n} & = & \mathfrak{m} \oplus \mathbb{R}^{2n} \end{array}$$

is commutative for all $u \in F(M)$.

We now define two almost complex structures on $J(M)$. We denote by $B(\xi)$ the standard horizontal vector field corresponding to $\xi \in \mathbb{R}^{2n}$, namely, $B(\xi)$ is the unique horizontal vector field such that $\pi_*(B(\xi)_u) = u(\xi)$ for each $u \in F(M)$. First, we define almost complex structures J_1' and J_2' on $F(M)$ by the following.

$$\begin{aligned} J_1' A^* &= 0 && \text{for} \quad A \in \mathfrak{u}(n), \\ J_1' A^* &= (J_0 A)^* && \text{for} \quad A \in \mathfrak{m}, \\ J_1' B(\xi) &= B(J_0 \xi) && \text{for} \quad \xi \in \mathbb{R}^{2n}, \end{aligned}$$

and

$$\begin{aligned} J_2' A^* &= 0 && \text{for} \quad A \in \mathfrak{u}(n), \\ J_2' A^* &= -(J_0 A)^* && \text{for} \quad A \in \mathfrak{m}, \\ J_2' B(\xi) &= B(J_0 \xi) && \text{for} \quad \xi \in \mathbb{R}^{2n}, \end{aligned}$$

where J_0 is a linear endomorphism of $\mathbb{R}^{2n}$ represented by

$$J_0 = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$$

with respect to the canonical orthonormal basis. Next, we define almost complex structures J_1 and J_2 on $J(M)$ by the following.

$$\begin{aligned} (J_1)_{j(u)}(j_*(A_u^*)) &= \lambda(u)^{-1}(J_0 A) && \text{for} \quad A \in \mathfrak{m}, \\ (J_1)_{j(u)}(j_*(B(\xi)_u)) &= \lambda(u)^{-1}(J_0 \xi) && \text{for} \quad \xi \in \mathbb{R}^{2n}, \end{aligned}$$

and

$$\begin{aligned} (J_2)_{j(u)}(j_*(A_u^*)) &= -\lambda(u)^{-1}(J_0 A) && \text{for} \quad A \in \mathfrak{m}, \\ (J_2)_{j(u)}(j_*(B(\xi)_u)) &= \lambda(u)^{-1}(J_0 \xi) && \text{for} \quad \xi \in \mathbb{R}^{2n}. \end{aligned}$$

It is well-known that J_1 is integrable if and only if M is conformally flat ($n \geq 3$) or self-dual ($n = 2$) ([2], [34]). Unlike J_1, the almost complex structure J_2 is never integrable ([8]). These almost complex structures are compatible with the Riemannian metric $g_{J(M)}$ on $J(M)$ constructed in Chapter 2. We shall write g_c instead of $g_{J(M)}$. Let D be the Riemannian connection on $J(M)$ with respect to the Riemannian metric g_c. Applying the formula due to B. O'Neil ([31]), we have the following.

$$g_c((D_{j_*(A^*_u)}J_1)j_*(B^*_u), j_*(C^*_u)) = 0,$$

$$g_c((D_{j_*(A^*_u)}J_2)j_*(B^*_u), j_*(C^*_u)) = 0,$$

$$g_c((D_{j_*(A^*_u)}J_1)j_*(B^*_u), j_*(B(\xi)_u)) = 0,$$

$$g_c((D_{j_*(A^*_u)}J_2)j_*(B^*_u), j_*(B(\xi)_u)) = 0,$$

$$\begin{aligned} &g_c((D_{j_*(A^*_u)}J_1)j_*(B(\xi)_u), j_*(B(\eta)_u)) \\ &= g_c((D_{j_*(A^*_u)}J_2)j_*(B(\xi)_u), j_*(B(\eta)_u)) \\ &= \frac{c^2}{2}\{\langle A, u^{-1}R(j(u)x, y)u\rangle + \langle A, u^{-1}R(x, j(u)y)u\rangle\} \\ &\qquad + 2g(uAu^{-1}j(u)x, y), \end{aligned}$$

$$g_c((D_{j_*(B(\xi)_u)}J_1)j_*(A^*_u), j_*(B^*_u)) = 0,$$

$$g_c((D_{j_*(B(\xi)_u)}J_2)j_*(A^*_u), j_*(B^*_u)) = 0,$$

$$\begin{aligned} &g_c((D_{j_*(B(\xi)_u)}J_1)j_*(A^*_u), j_*(B(\eta)_u)) \\ &= -\frac{c^2}{2}\{\langle A, u^{-1}j(u)R(x, y)u\rangle - \langle A, u^{-1}R(x, j(u)y)u\rangle\}, \end{aligned}$$

$$\begin{aligned} &g_c((D_{j_*(B(\xi)_u)}J_2)j_*(A^*_u), j_*(B(\eta)_u)) \\ &= \frac{c^2}{2}\{\langle A, u^{-1}j(u)R(x, y)u\rangle + \langle A, u^{-1}R(x, j(u)y)u\rangle\}, \end{aligned}$$

$$g_c((D_{j_*(B(\xi)_u)}J_1)j_*(B(\eta)_u), j_*(B(\zeta)_u)) = 0,$$

$$g_c((D_{j_*(B(\xi)_u)}J_2)j_*(B(\eta)_u), j_*(B(\zeta)_u)) = 0,$$

for A, B, $C \in \mathfrak{m}$, ξ, η, $\zeta \in \mathbb{R}^{2n}$, with $\pi_*(B(\xi)_u) = x$, $\pi_*(B(\eta)_u) = y$, $u \in F(M)$ ([38]). Therefore, $(J(M), J_1, g_c)$ is an almost Kähler manifold if and only if

$$2g(uAu^{-1}j(u)x, y) + c^2\langle A, u^{-1}j(u)R(x, y)u\rangle = 0, \tag{3.2.6}$$

and $(J(M), J_2, g_c)$ is an almost Kähler manifold if and only if

$$2g(uAu^{-1}j(u)x, y) - c^2\langle A, u^{-1}j(u)R(x, y)u\rangle = 0. \tag{3.2.7}$$

We shall introduce the example by J. Davidov and O. Muškarov ([6]). Let $M = (M, g)$ be a four-dimensional oriented self-dual Einstein manifold with negative scalar curvature τ. Then, from (3.2.5) and (3.2.7), we have $c^2 = -\dfrac{12}{\tau}$. Since J_2 is never integrable, $(J(M), J_2, g_c)$ is a strictly almost Kähler manifold.

3. Almost Kähler manifolds of constant curvature

A space of constant curvature is necessarily an Einstein space. Thus, concerning the Goldberg conjecture, the following problem is naturally posed.

Problem 1. *Is an almost Kähler manifold of constant curvature integrable?*

In this section, we study the integrability of almost Kähler manifolds of constant sectional curvature.

3.1. Almost Kähler structure on a space of constant curvature

Let $M = (M, J, g)$ be a $2n$-dimensional almost Kähler manifold of constant sectional curvature c. Then, (3.1.1) reduces to the following.

$$\begin{aligned} 2c\{&g(x,y)g(w,z) - g(x,z)g(w,y) \\ &- g(x,Jy)g(w,Jz) + g(x,Jz)g(w,Jy)\} \\ &\qquad = g((\nabla_w J)x - (\nabla_x J)w, (\nabla_y J)z - (\nabla_z J)y). \end{aligned}$$

Thus, by direct calculation, we obtain

$$\|\nabla J\|^2 = -8cn(n-1). \tag{3.3.1}$$

From this equality, we observe that c is non-positive and that $c = 0$ if and only if M is a flat Kähler manifold. Therefore, any almost Kähler structure on a $2n(\geq 4)$-dimensional space of constant negative sectional curvature cannot be integrable.

Z. Olszak ([30]) gave a partial affirmative answer to the Problem 1, namely, he proved the following.

Theorem 3.3.1. *A $2n(\geq 8)$-dimensional almost Kähler manifold of constant sectional curvature is a flat Kähler manifold.*

In dimension 4, D. E. Blair claimed that the same assertion is valid by making use of quaternionic analysis. However, there is a gap in the final step of his proof. The statement "each $a_i = 0$" is not correct ([3], p. 1038).

Now, we shall prove the following Theorem 3.3.2 and Theorem 3.3.4.

Theorem 3.3.2. *The hyperbolic space $\mathbb{H}^{2n}$ of dimension $2n(\geq 4)$ cannot admit an compatible almost Kähler structure.*

Since a $2n$-dimensional space of constant negative sectional curvature is locally homothetic to the hyperbolic space $\mathbb{H}^{2n}$ of constant sectional curvature -1, the following corollary is a immediate consequence of the above arguments and Theorem 3.3.2.

Corollary 3.3.3. *Let $M = (M, J, g)$ be a $2n(\geq 4)$-dimensional complete almost Kähler manifold of constant sectional curvature. Then M is a flat Kähler manifold.*

Recently, the author proved that the hypothesis of completeness in the above Corollary 3.3.3 is needless in dimensions ≥ 8, namely, we can prove the following.

Theorem 3.3.4. *In dimensions ≥ 8, there are no almost Kähler manifolds of constant curvature unless the constant is 0, in which case the manifold is Kählerian.*

Above Theorem 3.3.4 is nothing but the result of Z. Olszak ([30]), but we will give an another short proof.

3.2. Proofs

In this section, we shall prove Theorem 3.3.2 and Theorem 3.3.4

Proof of Theorem 3.3.2

Let $\mathbb{H}^{2n} = (\mathbb{R}^{2n}_{+}, g)$ be a $2n(\geq 4)$-dimensional hyperbolic space of constant sectional curvature -1, where $\mathbb{R}^{2n}_{+} = \left\{ (x_1, x_2, \cdots, x_{2n}) \in \mathbb{R}^{2n} \;\middle|\; x_1 > 0 \right\}$ and g is the Riemannian metric on $\mathbb{R}^{2n}_{+}$ defined by

$$g = \frac{1}{x_1^2} \sum_{j=1}^{2n} dx_j \otimes dx_j.$$

We put $X_j = x_1 \dfrac{\partial}{\partial x_j}$ for $j = 1, 2, \ldots, 2n$. Then, $\{X_j\} = \{X_1, X_2, \cdots, X_{2n}\}$ is an global orthonormal frame field on $\mathbb{H}^{2n}$. Direct calculation implies

$$[X_1, X_j] = -[X_j, X_1] = X_j \tag{3.3.2}$$

for $j = 2, \ldots, 2n$, and are otherwise zero. We put $\nabla_{X_j} X_k = \sum_{a=1}^{2n} \Gamma_{jka} X_a$ for j, $k = 1, 2, \ldots, 2n$, where ∇ denotes the Riemannian connection on $\mathbb{H}^{2n}$ with respect to g. Then, we have

$$\Gamma_{jj1} = -\Gamma_{j1j} = 1 \tag{3.3.3}$$

for $j = 2, \ldots, 2n$, and are otherwise zero.

Now, we assume that $\mathbb{H}^{2n} = (\mathbb{H}^{2n}, g)$ admits an compatible almost Kähler structure (J, g). We put $JX_j = \sum_{a=1}^{2n} J_{ja}X_a$ for $j = 1, 2, \ldots, 2n$. Then the smooth functions $\{J_{jk}\}_{j,k=1,\cdots,2n}$ satisfy the following conditions.

$$J_{jk} = -J_{kj}$$

and

$$\sum_{a=1}^{2n} J_{ja}J_{ka} = \delta_{jk}$$

for $j, k = 1, 2, \ldots, 2n$.

Let Δ be the Laplace-Beltrami operator corresponding to the metric g. First, we shall compute the Laplacian ΔJ_{ij} of J_{ij}.

Lemma 3.3.5.

$$\Delta J_{1k} = -6(n-1)J_{1k} + 2X_1J_{1k}, \tag{3.3.4}$$

$$\Delta J_{jk} = -2(2n-1)J_{jk} + 2X_1J_{jk}, \tag{3.3.5}$$

for $j, k = 2, \ldots, 2n$.

Proof. Taking account of (3.3.3), we have

$$\Delta J_{jk} = \sum_{a\geq 1}^{2n} X_aX_aJ_{jk} - (2n-1)X_1J_{jk}. \tag{3.3.6}$$

It is well-known that an almost Kähler manifold necessarily a semi-Kähler manifold ($\delta\Omega = 0$), thus we have

$$\sum_{a\geq 1} \nabla_a J_{aj} = 0$$

for all $j \geq 1$. Using above equality and the Ricci identity

$$\nabla^2_{ij}J_{kl} - \nabla^2_{ji}J_{kl} = -\sum_{a\geq 1} R_{ijka}J_{al} - \sum_{a\geq 1} R_{ijla}J_{ka},$$

we have

$$\sum_{a\geq 1}\nabla^2_{aa}J_{jk} = -\sum_{a\geq 1}\nabla^2_{aj}J_{ka} - \sum_{a\geq 1}\nabla^2_{ak}J_{aj} \tag{3.3.7}$$
$$= \sum_{a,b\geq 1}(R_{ajkb}J_{ba} + R_{ajab}J_{kb})$$
$$+ \sum_{a,b\geq 1}(R_{akab}J_{bj} + R_{akjb}J_{ab})$$
$$= -4(n-1)J_{jk}.$$

On the other hand, taking account of (3.3.3), we have

$$\sum_{a\geq 1}\nabla^2_{aa}J_{jk} = \sum_{a\geq 1}X_aX_aJ_{jk} - (2n-1)X_1J_{jk} \tag{3.3.8}$$
$$- 2\sum_{a,b\geq 1}\Gamma_{ajb}X_aJ_{bk} - 2\sum_{a,b\geq 1}\Gamma_{akb}X_aJ_{jb}$$
$$+ \sum_{a,b,c\geq 1}(\Gamma_{ajc}\Gamma_{acb}J_{bk} + \Gamma_{ajc}\Gamma_{akb}J_{cb}$$
$$+ \Gamma_{akc}\Gamma_{ajb}J_{bc} + \Gamma_{akc}\Gamma_{acb}J_{jb}).$$

From (3.3.8), we have, in particular

$$\sum_{a\geq 1}\nabla^2_{aa}J_{1k} = \sum_{a\geq 1}X_a(X_aJ_{1k}) - (2n-1)X_1J_{1k} + 2\sum_{a\geq 2}X_aJ_{ak} - 2(n-1)J_{1k}. \tag{3.3.9}$$

By semi-Kähler condition, we have

$$0 = \sum_{a\geq 1}\nabla_aJ_{aj} = X_1J_{1j} + \sum_{a\geq 2}X_aJ_{aj} - 2(n-1)J_{1j},$$

and hence

$$\sum_{a\geq 2}X_aJ_{aj} = -X_1J_{1j} + 2(n-1)J_{1j} \tag{3.3.10}$$

for $j \geq 2$. Thus, from (3.3.9) and (3.3.10), we have

$$\sum_{a\geq 1}\nabla^2_{aa}J_{1k} = \sum_{a\geq 1}X_aX_aJ_{1k} - (2n+1)X_1J_{1k} + 2(n-1)J_{1k}. \tag{3.3.11}$$

Therefore, from (3.3.6), (3.3.7) and (3.3.11), we obtain

$$\Delta J_{1k} = -6(n-1)J_{1k} + 2X_1J_{1k}.$$

Next, from (3.3.8), we have for $j,\ k \geq 2$

$$\sum_{a\geq 1} \nabla^2_{aa} J_{jk} = \sum_{a\geq 1} X_a X_a J_{jk} - (2n+1)X_1 J_{jk} + 2J_{jk}. \tag{3.3.12}$$

Thus, from (3.3.6), (3.3.7) and (3.3.12), we obtain

$$\Delta J_{jk} = -2(2n-1)J_{jk} + 2X_1 J_{jk}$$

for $j,\ k \geq 2$. □

Next, we shall prove the following.

Lemma 3.3.6. *$\sum_{a,b,c=1}^{2n} (X_a J_{bc})^2$ and $\sum_{a,b,c,d=1}^{2n} (X_a X_b J_{cd})^2$ are both bounded.*

Proof. First, we prove the boundedness of the former. From (3.3.4), we have

$$\sum_{c\geq 1} J_{1c} \Delta J_{1c} = -6(n-1),$$

and hence

$$\sum_{a,c\geq 1} (X_a J_{1c})^2 = 6(n-1). \tag{3.3.13}$$

Further, from (3.3.5), we have

$$\sum_{b,c\geq 2} J_{bc} \Delta J_{bc} = -2(2n-1) \sum_{b,c\geq 2} J^2_{bc},$$

and hence

$$\sum_{\substack{b,c\geq 2\\ a\geq 1}} (X_a J_{bc})^2 = 2(2n-1) \sum_{b,c\geq 2} J^2_{bc}. \tag{3.3.14}$$

On one hand, we have

$$2n = \sum_{b,c\geq 1} J^2_{bc} = 2\sum_{c\geq 1} J^2_{1c} + \sum_{b,c\geq 2} J^2_{bc} = 2 + \sum_{b,c\geq 2} J^2_{bc}.$$

From this and (3.3.14), we have

$$\sum_{\substack{b,c\geq 2\\ a\geq 1}} (X_a J_{bc})^2 = 4(n-1)(2n-1). \tag{3.3.15}$$

Thus, from (3.3.13) and (3.3.15), we obtain

$$(3.3.16) \qquad \sum_{a,b,c\geq 1}(X_aJ_{bc})^2 = 2\sum_{a,c\geq 1}(X_aJ_{1c})^2 + \sum_{\substack{b,c\geq 2\\ a\geq 1}}(X_aJ_{bc})^2$$
$$= 8(n^2-1).$$

Next, we prove the boundedness of the latter. Taking account of (3.3.16), we have

$$\begin{aligned}
&\sum_{a,b,c,d\geq 1}(X_aX_bJ_{cd})^2 \\
&= \frac{1}{2}\sum_{a,b,c,d\geq 1}X_a(X_a(X_bJ_{cd})^2) - \sum_{a,b,c,d\geq 1}(X_aX_aX_bJ_{cd})X_bJ_{cd} \\
&= -\sum_{a,b,c,d\geq 1}(X_aX_aX_bJ_{cd})X_bJ_{cd}.
\end{aligned}$$

So, we calculate $\sum_{a,b,c,d\geq 1}(X_aX_aX_bJ_{cd})X_bJ_{cd}$. Using (3.3.2), (3.3.4), (3.3.5), (3.3.6) and (3.3.16), we can derive

$$\begin{aligned}
&\sum_{a,b,c,d\geq 1}(X_aX_aX_bJ_{cd})X_bJ_{cd} \\
&= -12(n-1)\sum_{b,d\geq 1}(X_bJ_{1d})^2 - 2(2n-1)\sum_{\substack{c,d\geq 2\\ b\geq 1}}(X_bJ_{cd})^2 \\
&\quad -(2n+1)\sum_{\substack{b\geq 2\\ c,d\geq 1}}(X_bJ_{cd})^2 - 2(2n+1)\sum_{c,d\geq 1}(X_1J_{cd})^2 - \sum_{\substack{b\geq 2\\ c,d\geq 1}}(X_bJ_{cd})^2
\end{aligned}$$

Thus, from (3.3.16), (3.3.13) and (3.3.15), we finally obtain

$$\begin{aligned}
&\sum_{a,b,c,d\geq 1}(X_aX_aX_bJ_{cd})X_bJ_{cd} \\
&= -24(n-1)(n+2)(2n-1) - 2n\sum_{c,d\geq 1}(X_1J_{cd})^2
\end{aligned}$$

Therefore, lemma follows. □

Now, we shall begin to prove Theorem 3.3.2.

For G, $h>0$, we put

$$I_{ij} = \int_G^{G+h} x_1\frac{\partial^2 J_{ij}}{\partial x_1^2}(x_1,x_2,\cdots,x_{2n})dx_1.$$

Integrating I_{ij} by parts, we have

$$(3.3.17)\qquad \begin{aligned} I_{ij} &= \left[x_1 \frac{\partial J_{ij}}{\partial x_1}(x_1, x_2, \cdots, x_{2n})\right]_G^{G+h} \\ &\quad - \int_G^{G+h} \frac{\partial J_{ij}}{\partial x_1}(x_1, x_2, \cdots, x_{2n})dx_1 \\ &= (X_1 J_{ij})(G+h, x_2, \cdots, x_{2n}) - (X_1 J_{ij})(G, x_2, \cdots, x_{2n}) \\ &\quad - h\frac{\partial J_{ij}}{\partial x_1}(w_1, x_2, \cdots, x_{2n}) \end{aligned}$$

for some w_1 with $G < w_1 < G + h$. On the other hand,

$$(3.3.18)\qquad I_{ij} = hz_1 \frac{\partial^2 J_{ij}}{\partial x_1^2}(z_1, x_2, \cdots, x_{2n})$$

for some z_1 with $G < z_1 < G+h$. By Lemma 3.3.6, we see that $X_1 J_{ij}$ and $X_1 X_1 J_{ij}$ are both bounded. Thus, from the equalities

$$X_1 J_{ij} = x_1 \frac{\partial J_{ij}}{\partial x_1}, \qquad \frac{1}{x_1} X_1 X_1 J_{ij} = \frac{\partial J_{ij}}{\partial x_1} + x_1 \frac{\partial^2 J_{ij}}{\partial x_1^2},$$

we have

$$\lim_{x_1 \to +\infty} \frac{\partial J_{ij}}{\partial x_1} = 0, \qquad \lim_{x_1 \to +\infty} x_1 \frac{\partial^2 J_{ij}}{\partial x_1^2} = 0.$$

Thus, from (3.3.17) and (3.3.18), we obtain

$$(3.3.19)\qquad \lim_{G \to +\infty} \{(X_1 J_{ij})(G+h, x) - (X_1 J_{ij})(G, x)\} = 0$$

for any h and $x = (x_2, \cdots, x_{2n}) \in \mathbb{R}^{2n-1}$. Moreover, by direct calculation, we have

$$\begin{aligned} &(X_1 J_{ij})(G+h, x_2, \cdots, x_{2n}) - (X_1 J_{ij})(G, x_2, \cdots, x_{2n}) \\ &\quad = \int_G^{G+h} \frac{\partial}{\partial x_1}(X_1 J_{ij})(x_1, x_2, \cdots, x_{2n})dx_1 \\ &\quad = \int_G^{G+h} \frac{\partial J_{ij}}{\partial x_1}(x_1, x_2, \cdots, x_{2n})dx_1 + \int_G^{G+h} x_1 \frac{\partial^2 J_{ij}}{\partial x_1^2}(x_1, x_2, \cdots, x_{2n})dx_1 \\ &\quad = J_{ij}(G+h, x_2, \cdots, x_{2n}) - J_{ij}(G, x_2, \cdots, x_{2n}) \\ &\qquad + h\left[u_1 \frac{\partial^2 J_{ij}}{\partial x_1^2}(u_1, x_2, \cdots, x_{2n})\right] \end{aligned}$$

for some u_1 with $G < u_1 < G + h$. Thus, we have

$$(3.3.20)\qquad \lim_{G \to +\infty} \{J_{ij}(G+h, x) - J_{ij}(G, x)\} = 0$$

for any $h > 0$ and $x = (x_2, \cdots, x_{2n}) \in \mathbb{R}^{2n-1}$.

We choose $x \in \mathbb{R}^{2n-1}$ arbitrary and fix it. For any sequence $\{h_m\}$ ($h_m > 0$) which diverges to infinity, we can chose a subsequence $\{h_{m'}\}$ of $\{h_m\}$ such that the sequence $\{J_{ij}(h_{m'}, x)\}$ converges because of the boundedness of $\{J_{ij}(h_m, x)\}$. Let $\{h_m\}$ and $\{h'_m\}$ ($h_m > 0$, $h'_m > 0$) be any divergent sequences to infinity such that the sequences $\{J_{ij}(h_m, x)\}$ and $\{J_{ij}(h'_m, x)\}$ both converge, say

$$\lim_{m\to\infty} J_{ij}(h_m, x) = \alpha_{ij}, \quad \lim_{m\to\infty} J_{ij}(h'_m, x) = \alpha'_{ij}, \qquad (-1 \le \alpha_{ij}, \alpha'_{ij} \le 1).$$

Then, for any $\varepsilon > 0$, there exists an integer $m_0 = m_0(\varepsilon) > 0$ such that

$$|J_{ij}(h_m, x) - \alpha_{ij}| < \varepsilon, \qquad |J_{ij}(h'_m, x) - \alpha'_{ij}| < \varepsilon. \tag{3.3.21}$$

for $m \ge m_0$. Taking account of (3.3.20), there are positive numbers $G_0 = G_0(\varepsilon, h_m)$ and $G'_0 = G'_0(\varepsilon, h'_m)$ such that

$$|J_{ij}(G + h_m, x) - J_{ij}(G, x)| < \varepsilon \qquad \text{whenever} \quad G \ge G_0$$

and

$$|J_{ij}(G + h'_m, x) - J_{ij}(G, x)| < \varepsilon \qquad \text{whenever} \quad G \ge G'_0.$$

From the last two inequalities, we have immediately

$$|J_{ij}(G + h_m, x) - J_{ij}(G + h'_m, x)| < 2\varepsilon \tag{3.3.22}$$

for any $G \ge \max\{G_0, G'_0\}$. Again, from (3.3.20), for $G_1 \ge \max\{G_0, G'_0\}$, there exists an integer $m_1 (\ge m_0)$ such that

$$|J_{ij}(h_m + G_1, x) - J_{ij}(h_m, x)| < \varepsilon, \quad |J_{ij}(h'_m + G_1, x) - J_{ij}(h'_m, x)| < \varepsilon \tag{3.3.23}$$

for any $m \ge m_1$, and therefore

$$\begin{aligned}
&|\alpha_{ij} - \alpha'_{ij}| \\
&< |\alpha_{ij} - J_{ij}(h_m, x)| + |J_{ij}(h_m, x) - J_{ij}(h_m + G_1, x)| \\
&\qquad + |J_{ij}(h_m + G_1, x) - J_{ij}(h'_m + G_1, x)| + |J_{ij}(h'_m + G_1, x) - J_{ij}(h'_m, x)| \\
&\qquad + |J_{ij}(h'_m, x) - \alpha'_{ij}| \\
&< 6\varepsilon,
\end{aligned}$$

by virtue of (3.3.21) $\sim$ (3.3.23). Thus, the limit $\lim_{G\to+\infty} J_{ij}(G,x)$ exists for every $x \in \mathbb{R}^{2n-1}$, so we set

$$\lim_{G\to+\infty} J_{ij}(G,x) = A_{ij}(x).$$

From (3.3.19), applying the same argument as above, we observe that the limit $\lim_{G\to+\infty}(X_1 J_{ij})(G,x)$ exists for all $x \in \mathbb{R}^{2n-1}$, so we set

$$\lim_{G\to+\infty}(X_1 J_{ij})(G,x) = B_{ij}(x).$$

Now, form (3.3.3), we see that each trajectory of the vector field X_1 is a geodesic. We denote by s the arc-length parameter measured from the point $(1, x_2, \cdots, x_{2n}) \in \mathbb{H}^{2n}$ along x_1-axis, namely

$$s = \int_1^t \frac{1}{t}dt = \log t, \qquad t > 0.$$

For $x = (1, x_2, \cdots, x_{2n})$, we define functions f_{ij}^x by

$$f_{ij}^x(s) = J_{ij}(e^s, x_2, \cdots, x_n),$$

where i, $j = 1, \ldots, 2n$. Then, for $s_1 > s_2$, we have

$$\begin{aligned}
&J_{ij}(e^{s_1}, x_2, \cdots, x_{2n}) - J_{ij}(e^{s_2}, x_2, \cdots, x_{2n}) \\
&= f_{ij}^x(s_1) - f_{ij}^x(s_2) \\
&= (s_1 - s_2)\frac{df_{ij}^x}{ds}(s_{ij}) \\
&= (s_1 - s_2)e^{s_{ij}}\frac{\partial J_{ij}}{\partial x_1}(e^{s_{ij}}, x_2, \cdots, x_{2n}) \\
&= (s_1 - s_2)(X_1 J_{ij})(e^{s_{ij}}, x_2, \cdots, x_{2n})
\end{aligned}$$

for some s_{ij} with $s_1 > s_{ij} > s_2$. And hence, for $G > 0$ and $a > 1$, we have

$$J_{ij}(aG, x_2, \cdots, x_{2n}) - J_{ij}(G, x_2, \cdots, x_{2n}) = (\log a)(X_1 J_{ij})(e^{s_{ij}}, x_2, \cdots, x_{2n})$$

where $G < e^{s_{ij}} = e^{s_{ij}(a,G)} < aG$. Therefore, we have

$$\begin{aligned}
0 &= \lim_{G\to+\infty}\{J_{ij}(aG, x_2, \cdots, x_{2n}) - J_{ij}(G, x_2, \cdots, x_{2n})\} \\
&= (\log a)\lim_{G\to\infty}(X_1 J_{ij})(e^{s_{ij}}, x_2, \cdots, x_{2n}) \\
&= (\log a)B_{ij}(x_2, \cdots, x_{2n}),
\end{aligned}$$

and hence

$$\lim_{G\to+\infty}(\nabla_1 J_{ij})(G, x_2, \cdots, x_{2n}) = \lim_{G\to+\infty}(X_1 J_{ij})(G, x_2, \cdots, x_{2n}) = 0$$

for any $(x_2, \cdots, x_{2n}) \in \mathbb{R}^{2n-1}$.

Now, we denote by ψ_G $(G > 0)$ the isometry of $\mathbb{H}^{2n}$ defined by

$$\psi_G(x_1, x_2, \ldots, x_{2n}) = (Gx_1, Gx_2, \ldots, Gx_{2n}). \tag{3.3.24}$$

From (3.3.3), we see that each integral curve of X_1 is geodesic and X_i $(i = 1, 2, \ldots, 2n)$ are all parallel along the geodesic. From (3.3.24), we have

$$(\psi_G)_* X_i = X_i, \tag{3.3.25}$$

for $i = 1, 2, \ldots, 2n$. We denote by γ the integral curve of X_1 through the point $x_0 = (1, 0, \ldots, 0) \in \mathbb{H}^{2n}$. Furthermore, we denote by φ_a $(a = 2, \ldots, 2n)$ the isometries of $\mathbb{H}^{2n}$ such that $\varphi_a(x_0) = x_0$ and $(\varphi_a)_*(X_1) = X_a$ at the point x_0. Then, we may easily observe that

$$(\psi_G \circ \gamma)(x_1) = \gamma_1(Gx_1) \qquad \text{for} \quad G, x_1 > 0.$$

Since X_i $(i = 1, 2, \ldots, 2n)$ are parallel along γ, taking account of (3.3.25), we have

$$\nabla_{X_1}\big((\psi_G^{-1})_* \circ J \circ (\psi_G)_*\big) = \nabla_{X_1} J, \tag{3.3.26}$$

at x_0. We put

$$\psi_G^{(a)} = \varphi_a \circ \psi_G \circ \varphi_a^{-1} \tag{3.3.27}$$

and

$$J_{(a)} = (\varphi_a^{-1})_* \circ J \circ (\varphi_a)_*,$$

for $a = 2, \ldots, 2n$. Then, we may easily observe that $(\mathbb{H}^3 \times \mathbb{R}, J_{(a)}, g)$ $(a = 2, \ldots, 2n)$ are all almost Kähler manifolds. Corresponding to (3.3.25), we have easily

$$\nabla_{X_a}\big((\psi_G^{(a)-1})_* \circ J \circ (\psi_G^{(a)})_*\big) = \nabla_{X_a} J, \tag{3.3.28}$$

at x_0 for $a = 2, \dots, 2n$. From (3.3.26) $\sim$ (3.3.28), we have

$$
\begin{aligned}
(3.3.29)\qquad & \Big((\nabla_{X_1} J_{(a)})X_j\Big)(\gamma(G)) \\
&=\Big((\nabla_{X_1}((\varphi_a^{-1})_* \circ J \circ (\varphi_a)_*))X_j\Big)(\gamma(G)) \\
&=\Big((\nabla_{(\psi_G)_*(X_1)_{x_0}}((\varphi_a^{-1})_* \circ J \circ (\varphi_a)_*))(\psi_G)_*(X_j)_{x_0}\Big)(\varphi_G(x_0)) \\
&=\Big((\nabla_{(X_1)_{x_0}}((\psi_G^{-1} \circ \varphi_a^{-1})_* \circ J \circ (\varphi_a \circ \psi_G)_*))(X_j)_{x_0}\Big)(x_0) \\
&=\Big((\nabla_{(\varphi_a)_*(X_1)_{x_0}}((\psi_G^{(a)-1})_* \circ J \circ (\psi_G^{(a)})_*))(\varphi_a)_*(X_j)_{x_0}\Big)(x_0) \\
&=\Big((\nabla_{(X_a)_{x_0}}((\psi_G^{(a)-1})_* \circ J \circ (\psi_G^{(a)})_*))(\varphi_a)_*(X_j)_{x_0}\Big)(x_0) \\
&=\Big((\nabla_{(X_a)_{x_0}} J)(\varphi_a)_*(X_j)_{x_0}\Big)(x_0)
\end{aligned}
$$

for $a = 2, \dots, 2n$, $j = 1, 2, \dots, 2n$. Therefore, from (3.3.27) $\sim$ (3.3.29), we have

$$
\begin{aligned}
0 &= \lim_{G \to +\infty} \sum_{j,k=1}^{2n} \left\{ (\nabla_1 J_{jk})^2(\gamma(G)) + \sum_{a=2}^{2n} (\nabla_a J_{(a)jk})^2(\gamma(G)) \right\} \\
&= \lim_{G \to +\infty} \sum_{j,k=1}^{2n} (\nabla_i J_{jk})^2(x_0).
\end{aligned}
$$

But, this contradicts (3.3.1), which completes the proof of the Theorem 3.3.2.

Next, we shall prove Theorem 3.3.4

Proof of Theorem 3.3.4

Taking account of the argument in Section 3.1, if there exist a strictly almost Kähler structure on a space of constant sectional curvature, then we find that locally hyperbolic space must carry such a structure. Therefore, our aim is to prove that this is absurd.

We denote by $\mathbb{H}^{2n} = (\mathbb{H}^{2n}, g)$ the hyperbolic space of constant sectional curvature -1 with $n \geq 2$. For a connected open set U about an arbitrary point p in $\mathbb{H}^{2n}$, we assume that there exists an compatible almost Kähler structure (J, g) on U. First, we have

$$
\begin{aligned}
(3.3.30)\qquad & g((\nabla_w J)x - (\nabla_x J)w, (\nabla_y J)z - (\nabla_z J)y) \\
&= \sum_{i=1}^{2n} g((\nabla_w J)x - (\nabla_x J)w, e_i) g((\nabla_y J)z - (\nabla_z J)y, e_i)
\end{aligned}
$$

$$= \sum_i g((\nabla_{e_i} J)w, x) g((\nabla_{e_i} J)y, z)$$

$$= \sum_i (\nabla_{e_i} \Omega)(w, x)(\nabla_{e_i} \Omega)(y, z)$$

for any w, x, y, $z \in T_pM$, where $\{e_i\} = \{e_i, e_{n+i} = Je_i\}_{1 \leq i \leq n}$ is an orthonormal basis of T_pU. By (3.1.3) and (3.3.30), we have further

$$\begin{aligned}
&g((\nabla_w J)x - (\nabla_x J)w, (\nabla_y J)z - (\nabla_z J)y) \\
&= \sum_{\substack{a<b \\ c<d}} g((\varphi_{[ab]})_p, (\varphi_{[cd]})_p) \Phi_{[ab]}(w, x) \Phi_{[cd]}(y, z) \\
&\quad + \sum_{\substack{a<b \\ c<d}} g((\varphi_{[ab]})_p, (\varphi_{[cd]})_p)(J\Phi)_{[ab]}(w, x)(J\Phi)_{[cd]}(y, z) \\
&\quad - \sum_{\substack{a<b \\ c<d}} g((J\varphi_{[ab]})_p, (\varphi_{[cd]})_p)(J\Phi_{[ab]})(w, x) \Phi_{[cd]}(y, z) \\
&\quad - \sum_{\substack{a<b \\ c<d}} g((\varphi_{[ab]})_p, (J\varphi_{[cd]})_p) \Phi_{[ab]}(w, x)(J\Phi_{[cd]})(y, z).
\end{aligned}$$

Thus, in particular, we have

$$\begin{aligned}
\text{(3.3.31)} \qquad &g((\nabla_{e_i} J)e_j - (\nabla_{e_j} J)e_i, (\nabla_{e_k} J)e_l - (\nabla_{e_l} J)e_k) \\
&= \sum_{\substack{a<b \\ c<d}} g((\varphi_{[ab]})_p, (\varphi_{[cd]})_p) \Phi_{[ab]}(e_i, e_j) \Phi_{[cd]}(e_k, e_l) \\
&= \frac{1}{2} g((\varphi_{[ij]})_p, (\varphi_{[kl]})_p)
\end{aligned}$$

and

$$\begin{aligned}
\text{(3.3.32)} \qquad &g((\nabla_{e_i} J)e_j - (\nabla_{e_j} J)e_i, (\nabla_{Je_k} J)e_l - (\nabla_{e_l} J)Je_k) \\
&= -\sum_{\substack{a<b \\ c<d}} g((\varphi_{[ab]})_p, (J\varphi_{[cd]})_p) \Phi_{[ab]}(e_i, e_j)(J\Phi_{[cd]})(Je_k, e_l) \\
&= -\frac{1}{2} g((\varphi_{[ij]})_p, (J\varphi_{[kl]})_p).
\end{aligned}$$

for $1 \leq i < j \leq n$, $1 \leq k < l \leq n$.

On one hand, from (3.1.4), we have

$$\begin{aligned}
&g((\nabla_{e_i} J)e_j - (\nabla_{e_j} J)e_i, (\nabla_{e_k} J)e_l - (\nabla_{e_l} J)e_k) \\
&= -g(\mathcal{R}((\Phi_{[ij]})_p), (\Phi_{[kl]})_p) - g(\mathcal{R}((J\Phi_{[ij]})_p), (J\Phi_{[kl]})_p)
\end{aligned}$$

and

$$g((\nabla_{e_i}J)e_j - (\nabla_{e_j}J)e_i, (\nabla_{Je_k}J)e_l - (\nabla_{e_l}J)Je_k)$$
$$= -g(\mathcal{R}((\Phi_{[ij]})_p), (J\Phi_{[kl]})_p) + g(\mathcal{R}((J\Phi_{[ij]})_p), (\Phi_{[kl]})_p)$$

for $1 \leq i < j \leq n$, $1 \leq k < l \leq n$. Since (U, g) is a space of constant sectional curvature -1, we have further

$$\begin{aligned} &g(\mathcal{R}((\Phi_{[ij]})_p), (\Phi_{[kl]})_p) + g(\mathcal{R}((J\Phi_{[ij]})_p), (J\Phi_{[kl]})_p) \\ &= -2g((\Phi_{[ij]})_p, (\Phi_{[kl]})_p) \\ &= -2\delta_{ik}\delta_{jl} \end{aligned} \tag{3.3.33}$$

and

$$g(\mathcal{R}((\Phi_{[ij]})_p), (J\Phi_{[kl]})_p) - g(\mathcal{R}((J\Phi_{[ij]})_p), (\Phi_{[kl]})_p) = 0. \tag{3.3.34}$$

for $1 \leq i < j \leq n$, $1 \leq k < l \leq n$.

Thus, by (3.3.31), (3.3.32), (3.3.33) and (3.3.34), we obtain

$$g((\varphi_{[ij]})_p, (\varphi_{[kl]})_p) = 0, \qquad g((\varphi_{[ij]})_p, (J\varphi_{[kl]})_p) = 0$$

for $1 \leq i < j \leq n$, $1 \leq k < l \leq n$, $(i, j) \neq (k, l)$, and

$$g((\varphi_{[ij]})_p, (\varphi_{[ij]})_p) = g((J\varphi_{[ij]})_p, (J\varphi_{[ij]})_p) = 4$$

for $1 \leq i < j \leq n$. Therefore, the $n(n-1)$ vectors $\{(\varphi_{[ij]})_p, (J\varphi_{[ij]})_p\}_{1 \leq i < j \leq n}$ are orthogonal vectors with length 2 in the dual space T_p^*U of T_pU. Therefore, it must follows that

$$n(n-1) \leq \dim T_p^*U = 2n,$$

namely $n = 2$ or 3.

This completes the proof of Theorem 3.3.4.

4. Four-dimensional almost Kähler locally symmetric space

It is well-known that an irreducible symmetric space is an Einstein space. Concerning the Goldberg conjecture, it is interesting to study the integrability of almost Kähler structure on locally symmetric spaces, namely, we set the following problem.

Problem 2. *Does there exist a strictly almost Kähler locally symmetric space?*

In this section, we study the integrability of the almost Kähler structure on four-dimensional locally symmetric space.

4.1. Four-dimensional almost Kähler manifold

Let $M = (M, J, g)$ be a four-dimensional almost Kähler manifold. For a arbitrary fixed local orthonormal frame field $\{X_1, X_2, X_3, X_4\}$ of M, we put

$$JX_i = \sum_{j=1}^{4} J_{ij} X_j$$

for $i = 1, 2, 3, 4$. Since

$$J_{ij} = -J_{ji}, \quad \sum_{k=1}^{2n} J_{ik} J_{jk} = \delta_{ij}$$

holds for $1 \le i, j \le 4$, the 4×4 matrix (J_{ij}) is a skew-symmetric and orthogonal one. Therefore, the matrix (J_{ij}) must be locally of the form either

$$\begin{pmatrix} 0 & J_{12} & J_{13} & J_{14} \\ -J_{12} & 0 & J_{14} & -J_{13} \\ -J_{13} & -J_{14} & 0 & J_{12} \\ -J_{14} & J_{13} & -J_{12} & 0 \end{pmatrix} \tag{3.4.1}$$

or

$$\begin{pmatrix} 0 & J_{12} & J_{13} & J_{14} \\ -J_{12} & 0 & -J_{14} & J_{13} \\ -J_{13} & J_{14} & 0 & -J_{12} \\ -J_{14} & -J_{13} & J_{12} & 0 \end{pmatrix} \tag{3.4.2}$$

with $J_{12}^2 + J_{13}^2 + J_{14}^2 = 1$. Since M is connected, for an oriented orthonormal frame field compatible with a given orientation of M, the corresponding 4×4 matrix (J_{ij}) takes one of the forms (3.4.1), (3.4.2) whole on M.

By (local) orthonormal frames, we shall mean always the oriented ones which are compatible with a given orientation.

4.2. An example of strictly almost Kähler structure on $\mathbb{H}^3 \times \mathbb{R}$

We give an affirmative answer to Problem 2 by constructing a compatible strictly almost Kähler structure on the symmetric space $\mathbb{H}^3 \times \mathbb{R}$ (the Riemannian product).

The Riemannian product $\mathbb{H}^3 \times \mathbb{R}$ can be regarded as a Riemannian manifold $(\mathbb{R}^4_+, g)$, namely, $\mathbb{R}^4_+ = \{ (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1 > 0 \}$ equipped with the Riemannian metric

$$g = \frac{1}{x_1^2} \sum_{i=1}^{3} dx_i \otimes dx_i + dx_4 \otimes dx_4.$$

We put $X_i = x_1 \dfrac{\partial}{\partial x_i}$ $(i = 1, 2, 3)$ and $X_4 = \dfrac{\partial}{\partial x_4}$. Then, $\{X_1, X_2, X_3, X_4\}$ is a global orthonormal frame field on $\mathbb{H}^3 \times \mathbb{R}$. It is easily verified that

$$[X_1, X_i] = -[X_i, X_1] = X_i \tag{3.4.3}$$

for $i = 2, 3$, and are otherwise zero. Furthermore, we have

$$\Gamma_{ii1} = \Gamma_{i1i} = 1 \tag{3.4.4}$$

for $i = 2, 3$, and are otherwise zero, where we put $\Gamma_{ijk} = g(\nabla_{X_i} X_j, X_k)$.

Now, we write down an example of strictly almost Kähler structure (J, g) on $\mathbb{H}^3 \times \mathbb{R}$. To my knowledge, this is the first example of strictly almost Kähler symmetric space.

Example. We define an almost complex structure J by

$$(J_{ij}) = \begin{pmatrix} 0 & \cos x_4 & \sin x_4 & 0 \\ -\cos x_4 & 0 & 0 & -\sin x_4 \\ -\sin x_4 & 0 & 0 & \cos x_4 \\ 0 & \sin x_4 & -\cos x_4 & 0 \end{pmatrix}$$

with respect to the orthonormal frame field $\{X_i\}_{i=1,2,3,4}$ defined above. Then, it is easy to verify that (J, g) is a strictly almost Kähler structure on $\mathbb{H}^3 \times \mathbb{R}$.

A differentiable transformation φ on an almost Hermitian manifold (M, J, g) is called an automorphism if φ is an isometry and pseudo-holomorphic transformation, that is, φ satisfies

$$\varphi^* g = g \quad \text{and} \quad \varphi_* \circ J = J \circ \varphi_*,$$

where φ_* denotes the differential map of φ and φ^* its dual map. We denote by $\mathrm{Aut}_M(J, g)$ the set of all automorphisms of (M, J, g). It is obvious that the set $\mathrm{Aut}_M(J, g)$ is a group under the composition of maps, and we call it the automorphism group of (M, J, g).

We shall determine the automorphism group $\mathrm{Aut}_{\mathbb{H}^3\times\mathbb{R}}(J,g)$ of the example of strictly almost Kähler manifold $(\mathbb{H}^3\times\mathbb{R},J,g)$ constructed above.

For $\varphi\in\mathrm{Aut}_{\mathbb{H}^3\times\mathbb{R}}(J,g)$, we put $\varphi_*(X_i)=\sum_{j=1}^{4}\varphi_{ij}X_j$ for $i=1,\,2,\,3,\,4$. Since φ is an orientation-preserving isometry, we see that 4×4 matrix (φ_{ij}) must be of the form either

$$\text{(3.4.5)}\qquad \begin{pmatrix}\varphi_{11}&\varphi_{12}&\varphi_{13}&0\\ \varphi_{21}&\varphi_{22}&\varphi_{23}&0\\ \varphi_{31}&\varphi_{32}&\varphi_{33}&0\\ 0&0&0&1\end{pmatrix}\quad\text{with}\quad (\varphi_{ij})_{1\le i,j\le 3}\in SO(3),$$

or

$$\text{(3.4.6)}\qquad \begin{pmatrix}\varphi_{11}&\varphi_{12}&\varphi_{13}&0\\ \varphi_{21}&\varphi_{22}&\varphi_{23}&0\\ \varphi_{31}&\varphi_{32}&\varphi_{33}&0\\ 0&0&0&-1\end{pmatrix}\quad\text{with}\quad -(\varphi_{ij})_{1\le i,j\le 3}\in SO(3).$$

We notice that φ_{ij} $(1\le i,j\le 3)$ are independent on the variable x_4. Since φ satisfies $\varphi_*^{-1}\circ J\circ\varphi_*=J$, we have, in particular,

$$(\varphi_*^{-1}\circ J\circ\varphi_*)X_1=JX_1.$$

We now suppose that the matrix (φ_{ij}) is of the form (3.4.5). Then, by direct calculation, we have

$$\begin{aligned}&(\varphi_*^{-1}\circ J\circ\varphi_*)X_1\\ =&\{(\varphi_{11}\varphi_{22}-\varphi_{12}\varphi_{21})J_{12}+(\varphi_{11}\varphi_{23}-\varphi_{13}\varphi_{21})J_{13}\}X_2\\ &+\{(\varphi_{11}\varphi_{32}-\varphi_{12}\varphi_{31})J_{12}+(\varphi_{11}\varphi_{33}-\varphi_{13}\varphi_{31})J_{13}\}X_3\\ &+(\varphi_{13}J_{12}-\varphi_{12}J_{13})X_4\\ =&(\varphi_{33}\cos(x_4+c_4)-\varphi_{32}\sin(x_4+c_4))X_2\\ &+(-\varphi_{23}\cos(x_4+c_4)+\varphi_{22}\sin(x_4+c_4))X_3\\ &+(\varphi_{13}\cos(x_4+c_4)-\varphi_{12}\sin(x_4+c_4))X_4,\end{aligned}$$

for some constant $c_4\in\mathbb{R}$, and hence

$$\text{(3.4.7)}\qquad \begin{pmatrix}\varphi_{33}&-\varphi_{32}&\varphi_{31}\\ -\varphi_{23}&\varphi_{22}&-\varphi_{21}\\ \varphi_{13}&-\varphi_{12}&\varphi_{11}\end{pmatrix}\begin{pmatrix}\cos(x_4+c_4)\\ \sin(x_4+c_4)\\ 0\end{pmatrix}=\begin{pmatrix}\cos x_4\\ \sin x_4\\ 0\end{pmatrix}.$$

Similarly, by the conditions $(\varphi_*^{-1} \circ J \circ \varphi_*)X_i = JX_i$, $(i = 2, 3, 4)$, we obtain the same equality (3.4.7). From (3.4.7), it follows that

$$(\varphi_{ij}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos c_4 & \sin c_4 & 0 \\ 0 & -\sin c_4 & \cos c_4 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Therefore, the automorphism $\varphi = (\varphi_1, \varphi_2, \varphi_3, \varphi_4)$ satisfies the following system of first order partial differential equations.

$$\begin{aligned}
&\frac{\partial \varphi_1}{\partial x_1} = \frac{1}{x_1}\varphi_1, && \frac{\partial \varphi_1}{\partial x_2} = \frac{\partial \varphi_1}{\partial x_3} = \frac{\partial \varphi_1}{\partial x_4} = 0, && \\
&\frac{\partial \varphi_2}{\partial x_2} = \frac{1}{x_1}\varphi_1 \cos c_4, && \frac{\partial \varphi_2}{\partial x_3} = -\frac{1}{x_1}\varphi_1 \sin c_4, && \frac{\partial \varphi_2}{\partial x_1} = \frac{\partial \varphi_2}{\partial x_4} = 0, \\
&\frac{\partial \varphi_3}{\partial x_2} = \frac{1}{x_1}\varphi_1 \sin c_4, && \frac{\partial \varphi_3}{\partial x_3} = \frac{1}{x_1}\varphi_1 \cos c_4, && \frac{\partial \varphi_3}{\partial x_1} = \frac{\partial \varphi_3}{\partial x_4} = 0, \\
&\frac{\partial \varphi_4}{\partial x_4} = 1, && \frac{\partial \varphi_4}{\partial x_1} = \frac{\partial \varphi_4}{\partial x_2} = \frac{\partial \varphi_4}{\partial x_3} = 0. &&
\end{aligned}$$

Solving this system, we find that the automorphism φ can be express as the form

$$\begin{aligned}
\varphi(x_1, x_2, x_3, x_4) = (&e^{c_1} x_1, e^{c_1}((\cos c_4)x_2 - (\sin c_4)x_3) + c_2, \\
&e^{c_1}((\sin c_4)x_2 + (\cos c_4)x_3) + c_3, x_4 + c_4)
\end{aligned}$$

for $c_i \in \mathbb{R}$, $i = 1, 2, 3, 4$.

Next, we suppose that the matrix (φ_{ij}) is of the form (3.4.2). Then, in exactly the same way, we have

$$\begin{pmatrix} -\varphi_{33} & \varphi_{32} & -\varphi_{31} \\ \varphi_{23} & -\varphi_{22} & \varphi_{21} \\ -\varphi_{13} & \varphi_{12} & -\varphi_{11} \end{pmatrix} \begin{pmatrix} \cos(-x_4 + c_4) \\ \sin(-x_4 + c_4) \\ 0 \end{pmatrix} = \begin{pmatrix} \cos x_4 \\ \sin x_4 \\ 0 \end{pmatrix},$$

which implies

$$(\varphi_{ij}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos c_4 & \sin c_4 & 0 \\ 0 & \sin c_4 & -\cos c_4 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

and hence, we have

$$\begin{aligned}
\varphi(x_1, x_2, x_3, x_4) = (&e^{c_1} x_1, e^{c_1}((\cos c_4)x_2 + (\sin c_4)x_3) + c_2, \\
&e^{c_1}((\sin c_4)x_2 - (\cos c_4)x_3) + c_3, -x_4 + c_4)
\end{aligned}$$

for $c_i \in \mathbb{R}$, $i = 1, 2, 3, 4$.

We can summarize the above arguments as follows.

Proposition 3.4.1. *The automorphism group* $\mathrm{Aut}_{\mathbb{H}^3\times\mathbb{R}}(J,g)$ *of the example of strictly almost Kähler manifold* $(\mathbb{H}^3\times\mathbb{R},J,g)$ *constructed above is isomorphic to a solvable subgroup of affine transformation group* $GL(4,\mathbb{R})\ltimes\mathbb{R}^4$ *(embedded in* $GL(5,\mathbb{R})$*), which consists of the elements*

$$\begin{pmatrix} e^{c_1} & 0 & 0 & 0 & 0 \\ 0 & e^{c_1}\cos c_4 & -e^{c_1}\sin c_4 & 0 & c_2 \\ 0 & e^{c_1}\sin c_4 & e^{c_1}\cos c_4 & 0 & c_3 \\ 0 & 0 & 0 & 1 & c_4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.4.8}$$

and

$$\begin{pmatrix} e^{c_1} & 0 & 0 & 0 & 0 \\ 0 & e^{c_1}\cos c_4 & e^{c_1}\sin c_4 & 0 & c_2 \\ 0 & e^{c_1}\sin c_4 & -e^{c_1}\cos c_4 & 0 & c_3 \\ 0 & 0 & 0 & -1 & c_4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3.4.9}$$

where $c_i\in\mathbb{R}$, $i=1,2,3,4$.

Remark 1. We may easily observe that the system of differential equations $\nabla_{X_i}J=0$ $(i=1,2,3,4)$ has no solution, and thus the Riemannian product $(\mathbb{H}^3\times\mathbb{R},g)$ can not admit a compatible Kähler structure.

Remark 2. Let ψ be an isometry on $(\mathbb{H}^3\times\mathbb{R},g)$ and (J,g) be the almost Kähler structure constructed above. Then, almost Hermitian structure $(\psi(J),g)$ is also an almost Kähler structure on $(\mathbb{H}^3\times\mathbb{R},g)$, where $\psi(J)$ is defined by $\psi_*^{-1}\circ J\circ\psi_*$. The automorphism group $\mathrm{Aut}_{\mathbb{H}^3\times\mathbb{R}}(\psi(J),g)$ is determined by the automorphism group $\mathrm{Aut}_{\mathbb{H}^3\times\mathbb{R}}(J,g)$. Indeed, the map $\mathrm{Aut}_{\mathbb{H}^3\times\mathbb{R}}(J,g)\ni\varphi\mapsto\psi^{-1}\circ\varphi\circ\psi\in\mathrm{Aut}_{\mathbb{H}^3\times\mathbb{R}}(\psi(J),g)$ is an isomorphism.

4.3. Almost Kähler structure on $\mathbb{H}^3\times\mathbb{R}$

From Proposition 3.4.1, we may easily observe that the group $\mathrm{Aut}_{\mathbb{H}^3\times\mathbb{R}}(J,g)$ acts transitively on $\mathbb{H}^3\times\mathbb{R}$ and that the identity component $\mathrm{Aut}_{\mathbb{H}^3\times\mathbb{R}}(J,g)_0$ of $\mathrm{Aut}_{\mathbb{H}^3\times\mathbb{R}}(J,g)$ is a subgroup consists of the elements of the form (3.4.8) and acts simply transitively on $\mathbb{H}^3\times\mathbb{R}$.

Concerning the Goldberg conjecture, it is worthwhile to consider whether the almost Kähler manifold $(\mathbb{H}^3\times\mathbb{R},J,g)$ has a compact quotient, namely, whether there exists a uniform discrete subgroup of $\mathrm{Aut}_{\mathbb{H}^3\times\mathbb{R}}(J,g)_0$.

We assume that (J, g) is an almost Kähler structure on the Riemannian product $(\mathbb{H}^3 \times \mathbb{R}, g)$. Then the almost Kähler condition $\mathfrak{S}_{i,j,k}\, g((\nabla_{X_i} J)X_j, X_k) = 0$ yields the following system of first order partial differential equations:

$$
(3.4.10) \qquad \begin{cases} X_1 J_{23} - X_2 J_{13} + X_3 J_{12} - 2J_{23} = 0, \\ X_1 J_{24} - X_2 J_{14} + X_4 J_{12} - J_{24} = 0, \\ X_1 J_{34} - X_3 J_{14} + X_4 J_{13} - J_{34} = 0, \\ X_2 J_{34} - X_3 J_{24} + X_4 J_{23} = 0. \end{cases}
$$

We may regard the triple (J_{12}, J_{13}, J_{14}) as a unit vector in the 3-dimensional Euclidean space $\mathbb{R}^3$. First of all, we may observe that the unit vector (J_{12}, J_{13}, J_{14}) has the following property.

Proposition 3.4.2. *The vector (J_{12}, J_{13}, J_{14}) varies with the variable x_4 on an open subset of $\mathbb{H}^3 \times \mathbb{R}$.*

Proof. We assume that the vector (J_{12}, J_{13}, J_{14}) is independent on the variable x_4. Then the system of partial differential equations (3.4.10) reduces to the following.

$$
(3.4.11) \qquad \begin{cases} X_1 J_{23} - X_2 J_{13} + X_3 J_{12} - 2J_{23} = 0, \\ X_1 J_{24} - X_2 J_{14} - J_{24} = 0, \\ X_1 J_{34} - X_3 J_{14} - J_{34} = 0, \\ X_2 J_{34} - X_3 J_{24} = 0. \end{cases}
$$

Now, we suppose that the matrix (J_{ij}) is of the form (3.4.1). Then, from (3.4.4), (3.4.3) and (3.4.11), we have

$$
(3.4.12) \qquad \begin{cases} \Delta J_{12} - 2X_1 J_{12} + 3J_{12} = 0, \\ \Delta J_{13} - 2X_1 J_{13} + 3J_{13} = 0, \\ \Delta J_{14} - 2X_1 J_{14} + 4J_{14} = 0. \end{cases}
$$

From (3.4.12), we have

$$
J_{12}\Delta J_{12} + J_{13}\Delta J_{13} + J_{14}\Delta J_{14} + 3 + J_{14}^2 = 0.
$$

Since $J_{12}^2 + J_{13}^2 + J_{14}^2 = 1$, we have

$$
\sum_{i=1}^{4} \{(X_i J_{12})^2 + (X_i J_{13})^2 + (X_i J_{14})^2\} = 3 + J_{14}^2,
$$

and hence

$$\sum_{i,j,k=1}^{4} (X_i J_{jk})^2 = 4(3 + J_{14}^2). \tag{3.4.13}$$

Next, from the equality above, we have

$$\sum_{i,j,k=1}^{4} (X_i J_{jk}) X_l X_i J_{jk} = 4 J_{14} X_l J_{14},$$

for each X_l. Thus, by direct calculation, we obtain

$$\begin{aligned} &\sum_{l,i,j,k=1}^{4} (X_l X_i J_{jk})^2 \\ &= 8 \sum_i (X_i J_{14})^2 + 12 \sum_i (X_1 J_{1i})^2 - 16 J_{14} X_1 J_{14} + 16 J_{14}^2 + 96. \end{aligned} \tag{3.4.14}$$

From (3.4.13) and (3.4.14), we find that $\sum_{i,j,k}(X_i J_{jk})^2$ and $\sum_{i,j,k,l}(X_l X_i J_{jk})^2$ are both bounded. Applying the similar argument in the proof of Theorem 3.3.2 (section 3.2 of chapter 3) along x_1-curve, we can deduce a contradiction. More precisely, let γ_1 be any integral curve of X_1. Then, we obtain

$$\lim_{x_1 \to \infty} X_1 J_{ij} = 0 \quad (1 \le i, j \le 4). \tag{3.4.15}$$

along the geodesic γ_1. We denote by $\bar{\varphi}_a$ ($a = 2, 3$) isometries of $\mathbb{H}^3$ such that $(\bar{\varphi}_a)_* X_1 = X_a$ ($a = 2, 3$) Let $\varphi_a(x_1, x_2, x_3, x_4) = (\bar{\varphi}_a(x_1, x_2, x_3), x_4)$ ($a = 2, 3$) be the naturally induced isometries of $\mathbb{H}^3 \times \mathbb{R}$, and we define almost complex structures $J_{(a)}$ ($a = 2, 3$) on $\mathbb{H}^3 \times \mathbb{R}$ by $J_{(a)} = (\varphi_a)_*^{-1} \circ J \circ (\varphi_a)_*$. Because J is independent on x_4, so are $J_{(a)}$. We may easily check that $(J_{(a)}, g)$ are almost Kähler structures on $\mathbb{H}^3 \times \mathbb{R}$. Thus, by similar argument as above, we obtain

$$\lim_{x_1 \to \infty} X_1 J_{(a)ij} = 0 \quad (1 \le i, j \le 4, \ a = 2, 3). \tag{3.4.16}$$

along the geodesic γ_1. Moreover, by the semi-Kähler condition, we can derive

$$\sum_{i=1}^{4} \{(\nabla_i J_{12})^2 + (\nabla_i J_{13})^2 + (\nabla_i J_{14})^2\} = 2,$$

and hence, we have

$$\|\nabla J\|^2 = \sum_{i,j,k=1}^{4} (\nabla_i J_{jk})^2 = 8, \tag{3.4.17}$$

where $\nabla_i J_{jk} = g((\nabla_{X_i} J)X_j, X_k)$. From (3.4.15), (3.4.16) and (3.4.17), we can derive a contradiction.

In the case where (J_{ij}) is of the form (3.4.2), in exactly the same way, we also arrive at a contradiction. $\square$

Now, we shall prove the following Theorem 3.4.3

First of all, we recall an integral formula on a 4-dimensional compact almost Kähler manifold. Let $\bar{M} = (\bar{M}, \bar{J}, \bar{g})$ be a 4-dimensional compact almost Kähler manifold. Then, it is known that the square of the first Chern class $c_1(\bar{M})$ is given by the following formula ([7], [39]):

$$(3.4.18) \qquad c_1(\bar{M})^2 = \frac{1}{16\pi^2}\int_{\bar{M}}\Big\{(\bar{\tau}^*)^2 - 2\|\bar{\rho}^{*\mathrm{sym}}\|^2 + 2\|\bar{\rho}^{*\mathrm{skew}}\|^2 - \frac{1}{4}(\bar{\tau}^* + \bar{\tau})\|\bar{\nabla}\bar{J}\|^2 + (\bar{\rho}, \bar{D})\Big\}\, d\bar{M},$$

where $\bar{\rho}^*$, $\bar{\tau}^*$, $\bar{\tau}$, $\bar{\nabla}$, $d\bar{M}$ denote the Ricci $*$-tensor, the $*$-scalar curvature, the scalar curvature, the Levi-Civita connection on $\bar{M}$ and the volume element of $\bar{M}$ respectively, and $\bar{\rho}^{*\mathrm{sym}}$ (resp. $\bar{\rho}^{*\mathrm{skew}}$) is the symmetric (resp. skew-symmetric) part of $\bar{\rho}^*$, and $(\bar{\rho}, \bar{D}) = \sum_{a,b,i,j=1}^{4} \bar{\rho}_{ab}(\bar{\nabla}_a \bar{J}_{ij})\bar{\nabla}_b \bar{J}_{ij}$. Here we put $\bar{\nabla}_a \bar{J}_{ij} = \bar{g}((\bar{\nabla}_{\bar{X}_a}\bar{J})\bar{X}_i, \bar{X}_j)$ and $\bar{\rho}_{ab} = \bar{\rho}(\bar{X}_a, \bar{X}_b)$ for a local orthonormal frame field $\{\bar{X}_1, \bar{X}_2, \bar{X}_3, \bar{X}_4\}$ on $\bar{M}$.

Theorem 3.4.3. *Let (g, J) be a compatible almost Kähler structure on the Riemannian product $\mathbb{H}^3 \times \mathbb{R}$. Then, the almost Kähler manifold $(\mathbb{H}^3 \times \mathbb{R}, J, g)$ cannot be a universal almost Hermitian covering of any compact almost Kähler manifold.*

Proof. Let (g, J) be a compatible almost Kähler structure on the Riemannian product $\mathbb{H}^3 \times \mathbb{R}$. We assume that there exists a compact almost Kähler manifold $(\bar{M}, \bar{J}, \bar{g})$ whose universal almost Hermitian covering is the almost Kähler manifold $(\mathbb{H}^3 \times \mathbb{R}, J, g)$. We denote by $\pi\colon \mathbb{H}^3 \times \mathbb{R} \to \bar{M}$ the covering projection. For any point $\bar{p} \in \bar{M}$, we may choose a local orthonormal frame field $\{\bar{X}_1, \bar{X}_2, \bar{X}_3, \bar{X}_4\}$ near $\bar{p}$ in such a way that $\pi_*(X_i) = \bar{X}_i$, $(i = 1, 2, 3, 4)$. We set $\bar{J}\bar{X}_i = \sum_{j=1}^{4} \bar{J}_{ij}\bar{X}_j$ $(i = 1, 2, 3, 4)$. For the proof, without loss of essentially, it is sufficient to consider the case where (J_{ij}) (and hence $(\bar{J}_{ij})$) is of the form (3.4.1). We may easily observe that $\sum_{i,j=1}^{4} (\bar{X}_4 \bar{J}_{ij})^2$ gives rise to a differentiable function on $\bar{M}$. Since $\bar{M}$ is locally the product Riemannian manifold of 3-dimensional hyperbolic space and a real line, it follows that the Euler class $\chi(\bar{M})$ of $\bar{M}$ vanishes. Further, since $\bar{M}$ is conformally flat, it follows that the first Pontrjagin class $p_1(\bar{M})$ of $\bar{M}$ also vanishes. Thus, by the Wu's theorem ([45]), we have $c_1(\bar{M})^2 = 0$. On one hand, by direct calculation, it follows from (3.4.18)

$$\begin{aligned} c_1(\bar{M})^2 &= \frac{1}{8\pi^2}\int_{\bar{M}}\Big(\|\bar{\nabla}\bar{J}\|^2 - \sum_{i,j=1}^{4}\sum_{a=1}^{3}(\bar{\nabla}_a\bar{J}_{ij})^2\Big)\,d\bar{M} \\ &= \frac{1}{8\pi^2}\int_{\bar{M}}\sum_{i,j=1}^{4}(\bar{\nabla}_4\bar{J}_{ij})^2\,d\bar{M}. \end{aligned}$$

Thus, it must follows that $\bar{\nabla}_4\bar{J}_{ij} = 0$ $(1 \leq i,j \leq 4)$ everywhere on $\bar{M}$, and hence, $\nabla_4 J_{ij} = X_4 J_{ij} = 0$ $(1 \leq i,j \leq 4)$ everywhere on M. But this contradicts Proposition 3.4.2, which completes the proof of theorem. □

Taking account of this theorem, we see that there does not exist a discrete uniform subgroup of the identity component $\mathrm{Aut}_{\mathbb{H}^3\times\mathbb{R}}(J,g)_0$ of $\mathrm{Aut}_{\mathbb{H}^3\times\mathbb{R}}(J,g)$ of the Example, namely, there does not exist discrete subgroup Γ of $\mathrm{Aut}_{\mathbb{H}^3\times\mathbb{R}}(J,g)_0$ such that the orbit space $\Gamma\backslash\mathbb{H}^3\times\mathbb{R}$ is compact.

5. Four-dimensional compact almost Kähler locally symmetric space

The arguments in the preceding section naturally lead us to the following problem.

Problem 3. *Does there exist a compact strictly almost Kähler locally symmetric space?*

In this section, we shall prove the following theorem, which is a negative answer to the above Problem 3.

Theorem 3.5.1. *A four-dimensional compact almost Kähler locally symmetric space is a Kähler manifold.*

We denote by $S^n(c)$, $\mathbb{H}^n(-c)$, $\mathbb{R}^n$, $\mathbb{C}P^2(c)$, $\mathbb{C}H^2(-c)$ the n-dimensional sphere of constant sectional curvature c, the n-dimensional hyperbolic space of constant sectional curvature $-c$, the n-dimensional Euclidean space, the complex projective plane with the Fubini-Study metric of constant holomorphic sectional curvature c and the complex hyperbolic plane with the Bergman metric of constant holomorphic sectional curvature $-c$, respectively, where c is a positive constant.

Let $\bar{M} = (\bar{M}, \bar{J}, \bar{g})$ be a compact four-dimensional almost Kähler locally symmetric space and $M = (M, J, g)$ its universal almost Hermitian covering. Then, taking account of the result due to G. R. Jensen ([16]) and of the well-known fact that $S^4(1)$ does not admit a almost complex structure, we see that the covering space M must be homothetic to one of the following spaces.

(i) $\mathbb{R}^4$,	(ii) $\mathbb{H}^4(-1)$,	(iii) $\mathbb{C}P^2(4)$,
(iv) $\mathbb{C}H^2(-4)$,	(v) $S^2(c^2) \times S^2(1)$,	(vi) $S^2(c^2) \times \mathbb{H}^2(-1)$,
(vii) $S^2(1) \times \mathbb{R}^2$,	(viii) $\mathbb{H}^2(-c^2) \times \mathbb{H}^2(-1)$,	(ix) $\mathbb{H}^2(-1) \times \mathbb{R}^2$,
(x) $S^3(1) \times \mathbb{R}$,	(xi) $\mathbb{H}^3(-1) \times \mathbb{R}^1$.	

Since the almost Kähler structures on $\mathbb{R}^4$ and $\mathbb{C}P^2(4)$ are integrable ([37]), an almost Kähler manifold $\bar{M}$ whose universal almost Hermitian covering is homothetic to $\mathbb{R}^4$ or $\mathbb{C}P^2(4)$ is a Kähler manifold. Since $\mathbb{H}^4(-1)$ cannot admit a compatible almost Kähler structure (Theorem 3.3.2), we see that there is no four-dimensional almost Kähler manifold whose universal almost Hermitian covering is homothetic to $\mathbb{H}^4(-1)$. Further, $\mathbb{H}^3(-1)\times\mathbb{R}^1$ itself admits an compatible almost Kähler structure, but $\mathbb{H}^3(-1) \times \mathbb{R}^1$ cannot be homothetic to almost Hermitian covering space of any four-dimensional compact almost Kähler manifold (Theorem 3.4.3).

Thus, the Theorem 3.5.1 immediately follows from the above argument and following series of Propositions 3.5.2 $\sim$ 3.5.8.

Proposition 3.5.2. *A compact almost Kähler manifold $\bar{M} = (\bar{M}, \bar{J}, \bar{g})$ whose universal almost Hermitian covering is homothetic to $M = (\mathbb{C}H^2(-4), J, g)$ is a Kähler manifold.*

Proposition 3.5.3. *An almost Kähler manifold $(S^2(c^2) \times S^2(1), J, g)$ is a Kähler manifold.*

Proposition 3.5.4. *A compact almost Kähler manifold $\bar{M} = (\bar{M}, \bar{J}, \bar{g})$ whose* *universal almost Hermitian covering is homothetic to* $M = (S^2(c^2) \times \mathbb{H}^2(-1), J, g)$ *is a Kähler manifold.*

Proposition 3.5.5. *An almost Kähler manifold* $(S^2(1) \times \mathbb{R}^2, J, g)$ *is a Kähler manifold.*

Proposition 3.5.6. *A compact almost Kähler manifold* $\bar{M} = (\bar{M}, \bar{J}, \bar{g})$ *whose universal almost Hermitian covering is homothetic to* $M = (\mathbb{H}^2(-c^2) \times \mathbb{H}^2(-1), J, g)$ *is a Kähler manifold.*

Proposition 3.5.7. *A compact almost Kähler manifold* $\bar{M} = (\bar{M}, \bar{J}, \bar{g})$ *whose universal almost Hermitian covering is homothetic to* $M = (\mathbb{H}^2(-1) \times \mathbb{R}^2, J, g)$ *is a Kähler manifold.*

Proposition 3.5.8. $S^3(1) \times \mathbb{R}$ *cannot admit a compatible almost Kähler structure.*

5.1. Proofs

We shall prove the above propositions 3.5.2 $\sim$ 3.5.8.

Let $M = (M, J, g)$ be a four-dimensional almost Kähler manifold. We denote by ∇, R, ρ, τ the Riemannian connection, the curvature tensor, the Ricci tensor and the scalar curvature of M, respectively. Through the present section, we shall use the following notational conventions.

$$\Gamma_{ijk} = g(\nabla_{X_i} X_j, X_k),$$

$$R_{ijkl} = g(R(X_i, X_j)X_k, X_l),$$

$$J_{ij} = g(JX_i, X_j), \quad \nabla_i J_{jk} = g((\nabla_{X_i} J)X_j, X_k),$$

and so on for a arbitrary fixed local orthonormal frame field $\{X_1, \ldots, X_{2n}\}$ of M.

Proof of Proposition 3.5.2.

Let G be a connected, simply connected Lie group with the Lie algebra spanned over $\mathbb{R}$ by $\{X_1, X_2, X_3, X_4\}$ with multiplication table

$$\begin{aligned} &[X_1, X_2] = 2X_2, \quad [X_1, X_3] = X_3, \quad [X_1, X_4] = X_4, \\ &[X_2, X_3] = 0, \qquad [X_2, X_4] = 0, \qquad [X_3, X_4] = 2X_2. \end{aligned} \tag{3.5.1}$$

Then, it is known that $\mathbb{C}H^2(-4)$ is isometric to the Lie group G provided with the left invariant Riemannian metric g defined by $g(X_i, X_j) = \delta_{ij}$, $1 \leq i, j \leq 4$ ([16]). From (3.5.1), we have

$$\Gamma_{212} = -\Gamma_{221} = -2, \quad \Gamma_{234} = -\Gamma_{243} = -1, \quad \Gamma_{313} = -\Gamma_{331} = -1,$$
$$\Gamma_{324} = -\Gamma_{342} = -1, \quad \Gamma_{414} = -\Gamma_{441} = -1, \quad \Gamma_{423} = -\Gamma_{432} = 1,$$

and are otherwise zero. Moreover, we see that

$$R_{1212} = 4, \quad R_{3434} = 4, \quad R_{1313} = 1, \quad R_{1414} = 1, \quad R_{2323} = 1,$$
$$R_{2424} = 1, \quad R_{1324} = 1, \quad R_{1423} = -1, \quad R_{1234} = 2,$$

and otherwise $R_{ijkl} = 0$ (up to sign).

For the Proposition 3.5.2, it suffices to prove that the almost Kähler structure (J, g) on $M = (\mathbb{C}H^2(-4), g)$ is a Kähler structure, namely, $\nabla J = 0$.

First, we assume that the matrix (J_{ij}) is of the form (3.4.1). In this case, we can prove that J_{12} is a constant function as follows. By direct calculation, we have

$$\begin{aligned}
\sum \nabla^2_{aa} J_{12} &= \sum X_a X_a J_{12} - 4X_1 J_{12} - X_3 J_{23} + X_3 J_{14} - X_4 J_{24} - X_4 J_{13} \\
&\qquad - 4J_{12} + 4J_{34} + J_{14} - J_{23} - J_{13} - J_{24} \\
&= \sum X_a X_a J_{12} - 4X_1 J_{12} \\
&= \Delta J_{12},
\end{aligned}$$

where Δ denotes the Laplace-Beltrami operator of M. On one hand, by the semi-Kähler condition, we have

$$\begin{aligned}
\sum \nabla^2_{aa} J_{12} &= -\sum \nabla^2_{a1} J_{2a} - \sum \nabla^2_{a2} J_{a1} \\
&= \sum R_{a12b} J_{ba} + \sum R_{a1ab} J_{2b} + \sum R_{a2ab} J_{b1} + \sum R_{a21b} J_{ab} \\
&= -4J_{12} + 4J_{34} \\
&= 0.
\end{aligned}$$

Thus, the function J_{12} is harmonic. Since

$$\tau^* = -\frac{1}{2} \sum J_{ab} R_{abij} J_{ij} = -24 J_{12}^2,$$

it follows that

$$\Delta\tau^* = -48(J_{12}\Delta J_{12} + \|\mathrm{grad}\, J_{12}\|^2) = -48\|\mathrm{grad}\, J_{12}\|^2 \leq 0.$$

Let $\pi\colon M \to \bar{M}$ be the covering projection and $\bar{\tau}$, $\bar{\Delta}$ the $*$-scalar curvature and Laplace-Beltrami operator of $\bar{M}$, respectively. Then, the above inequality implies $\bar{\Delta}\bar{\tau}^* \leq 0$ on $\bar{M}$ because of $\Delta\tau^* = \bar{\Delta}\bar{\tau}^* \circ \pi$, and hence $\bar{\tau}^*$ is constant by compactness of $\bar{M}$. Since $\tau^* = \bar{\tau}^* \circ \pi$, the $*$-scalar curvature τ^* is constant on M. Thus, it must follow that J_{12} is constant.

Now, the almost Kähler condition $\underset{ijk}{\mathfrak{S}}\, \nabla_i J_{jk} = 0$ is equivalent to the following system of first order partial differential equations.

$$\begin{cases} X_1 J_{23} + X_2 J_{31} + X_3 J_{12} - 3J_{23} = 0, \\ X_1 J_{24} + X_2 J_{41} + X_4 J_{12} - 3J_{24} = 0, \\ X_1 J_{34} + X_3 J_{41} + X_4 J_{13} + 2J_{12} - 2J_{34} = 0, \\ X_2 J_{34} + X_3 J_{42} + X_4 J_{23} = 0. \end{cases}$$

Taking account of $J_{12} = J_{34} = \text{constant}$, $J_{14} = J_{23}$, $J_{13} = -J_{24}$, the above system reduces to the following.

$$\begin{cases} X_1 J_{14} - X_2 J_{13} = 3J_{14}, \\ X_1 J_{13} + X_2 J_{14} = 3J_{13}, \\ X_3 J_{14} - X_4 J_{13} = 0, \\ X_3 J_{13} + X_4 J_{14} = 0. \end{cases} \tag{3.5.2}$$

Since $J_{13}^2 + J_{14}^2 = 1 - J_{12}^2 = \text{constant}$, we have also

$$J_{13} X_i J_{13} + J_{14} X_i J_{14} = 0 \tag{3.5.3}$$

for $i = 1, 2, 3, 4$. If $J_{12}^2 = 1$ (and hence $J_{13} = J_{14} = 0$), then we see immediately that $\nabla J = 0$. So, we suppose that $J_{12}^2 \neq 1$. Then, from (3.5.2) and (3.5.3), we have

$$X_1 J_{13} = 0, \quad X_1 J_{14} = 0, \quad X_2 J_{13} = -3J_{14}, \quad X_2 J_{14} = 3J_{13},$$

$$X_3 J_{13} = 0, \quad X_3 J_{14} = 0, \quad X_4 J_{13} = 0, \quad X_4 J_{14} = 0,$$

and hence$\nabla_i J_{13} = 0$ and $\nabla_i J_{14} = 0$ for $i = 1, 2, 3, 4$. We see immediately that $\nabla_i J_{12} = 0$ for $i = 1, 2, 3, 4$. Therefore, we can conclude that $\nabla J = 0$.

Next, we suppose that (J_{ij}) is of the form (3.4.2). In this case, it is straightforward to verify that $\mathbb{C}H^2(-4)$ is Einstein and $*$-Einstein, and hence $\bar{M}$ is also Einstein and $*$-Einstein. Since $\bar{M}$ is compact, $\bar{M}$ is a Kähler manifold ([40]). Thus, form (3.1.2), $*$-scalar curvature $\bar{\tau}^*$ must coincide with the scalar curvature of $\bar{M}$. This implies $\tau^* = \tau$. But, because of $\tau^* = -8$ and $\tau = -24$, this is absurd. Therefore, the matrix (J_{ij}) can not be of the form (3.4.2). This completes the proof of Proposition 3.5.2. □

Proof of Proposition 3.5.3.

Let $\{X_1, X_2\}$, $\{X_3, X_4\}$ be a local orthonormal frame field of $S^2(c^2)$ and $S^2(1)$, respectively. Then, it follows immediately that all the components of the curvature tensor R_{ijkl} are equal to zero except for

$$R_{1212} = -c^2, \qquad R_{3434} = -1 \qquad \text{(up to sign)}.$$

By direct calculation, we have

$$\tau = 2(c^2+1), \qquad \tau^* = 2(c^2 J_{12}^2 + J_{34}^2) = 2(c^2+1)J_{12}^2.$$

Thus, taking account of (3.1.2), we have

$$0 \le \tau^* - \tau = 2(c^2+1)(J_{12}^2 - 1) \le 0.$$

Then, it follows that $J_{12}^2 = 1$ (and hence $J_{13} = J_{14} = 0$) everywhere on M, and therefore, we have $\nabla J = 0$. □

Proof of Proposition 3.5.4.

Let $\{X_1, X_2\}$ be a local orthonormal frame field of $S^2(c^2)$ and $\{X_3 = x_3(\partial/\partial x_3),$ $X_4 = x_3(\partial/\partial x_4)\}$ be a global orthonormal frame field of $\mathbb{H}^2(-1) = \{\,(x_3, x_4) \in \mathbb{R}^2 \mid x_3 > 0\,\}$. Then, we see easily that the connection coefficients Γ_{ijk} are all equal to zero except for

$$\Gamma_{121} = -\Gamma_{112}, \qquad \Gamma_{212} = -\Gamma_{221}, \qquad \Gamma_{434} = -\Gamma_{443} = -1.$$

Moreover, it follows immediately that $R_{ijkl} = 0$ except for

$$R_{1212} = -c^2, \qquad R_{3434} = 1 \qquad \text{(up to sign)}.$$

By direct calculation, we obtain

$$\tau = 2(c^2 - 1), \qquad \tau^* = 2(c^2 J_{12}^2 - J_{34}^2) = 2(c^2 - 1)J_{12}^2.$$

Thus, from (3.1.2), we have

$$0 \leq \tau^* - \tau = 2(c^2 - 1)(J_{12}^2 - 1).$$

If $c > 1$, then the above inequality implies that $J_{12}^2 = 1$ (and hence $J_{13} = J_{14} = 0$) and then we have $\nabla J = 0$. We suppose $(0 \leq)\ c \leq 1$. By the same argument as in the proof of Proposition 3.5.2, we can show that the function J_{12} is harmonic, and hence we obtain

$$\Delta\tau^* = 4(c^2 - 1)\left\{J_{12}\Delta J_{12} + \|\mathrm{grad}\, J_{12}\|^2\right\} = 4(c^2 - 1)\|\mathrm{grad}\, J_{12}\|^2 \leq 0.$$

Again, by the same argument as in the proof of Proposition 3.5.2, we can prove that J_{12} is constant.

First, we suppose that the matrix (J_{ij}) is of the form (3.4.1). The almost Kähler condition $\underset{ijk}{\mathfrak{S}}\, \nabla_i J_{jk} = 0$ together with $J_{12} = J_{34} = \text{constant}$, $J_{23} = J_{14}$ and $J_{24} = -J_{13}$, we have the following system of partial differential equations

$$\text{(3.5.4)} \qquad \begin{cases} X_1 J_{14} - X_2 J_{13} = \Gamma_{121} J_{13} - \Gamma_{212} J_{14} \\ X_1 J_{13} + X_2 J_{14} = -\Gamma_{121} J_{14} - \Gamma_{212} J_{13} \\ X_3 J_{14} - X_4 J_{13} = J_{14} \\ X_3 J_{13} + X_4 J_{14} = J_{13} \end{cases}$$

The equality (3.5.3) is valid in this case. If $J_{12}^2 = 1$ (and hence $J_{13} = J_{14} = 0$), then it is straightforward to check that $\nabla J = 0$. So, we suppose $J_{12}^2 \neq 1$. Then, from (3.5.3) and (3.5.4), we have

$$X_1 J_{13} = -\Gamma_{121} J_{14}, \quad X_1 J_{14} = \Gamma_{121} J_{13}, \quad X_2 J_{13} = \Gamma_{212} J_{14}, \quad X_2 J_{14} = -\Gamma_{212} J_{13},$$

$$X_3 J_{13} = 0, \qquad X_3 J_{14} = 0, \qquad X_4 J_{13} = -J_{14}, \qquad X_4 J_{14} = J_{13},$$

and hence, by straightforward calculation, we get $\nabla_i J_{13} = 0$ and $\nabla_i J_{14} = 0$ for $i = 1, 2, 3, 4$. We see immediately that $\nabla_i J_{12} = 0$ for $i = 1, 2, 3, 4$. Therefore, in this case, we can conclude $\nabla J = 0$.

Next, we assume that (J_{ij}) is of the form (3.4.2), namely, $J_{12} = -J_{34} =$ constant, $J_{23} = -J_{14}$ and $J_{24} = J_{13}$. In this case, the almost Kähler condition $\underset{ijk}{\mathfrak{S}} \nabla_i J_{jk} = 0$ is equivalent to the following system of partial differential equations

$$\begin{cases} X_1 J_{14} + X_2 J_{13} = -\Gamma_{121} J_{13} - \Gamma_{212} J_{14} \\ X_1 J_{13} - X_2 J_{14} = \Gamma_{121} J_{14} - \Gamma_{212} J_{13} \\ X_3 J_{14} - X_4 J_{13} = J_{14} \\ X_3 J_{13} + X_4 J_{14} = J_{13}. \end{cases}$$

By the similar way as in the case (3.4.1), we can also conclude $\nabla J = 0$. This completes the proof of Proposition 3.5.4. □

Proof of Proposition 3.5.5.

Let $\{X_1, X_2\}$ be a local orthonormal frame field of $S^2(1)$ and $\{X_3, X_4\}$ a global orthonormal frame field of $\mathbb{R}^2$. Then, all components of the curvature tensor R_{ijkl} $(i \leq j,\ k \leq l)$ vanish except for

$$R_{1212} = -1 \qquad \text{(up to sign)}.$$

By a direct calculation, we get

$$\tau^* = 2J_{12}^2, \qquad \tau = 2.$$

Thus, from (3.1.2), we have

$$0 \leq \tau^* - \tau = 2(J_{12}^2 - 1) \leq 0.$$

Then, it follows that $J_{12}^2 = 1$ (and hence $J_{13} = J_{14} = 0$) everywhere on M. Therefore, by direct calculation, we have $\nabla J = 0$. □

Proof of Proposition 3.5.6.

Let $\{X_1, X_2\}$ and $\{X_3, X_4\}$ be a global orthonormal frame field of $\mathbb{H}^2(-c^2)$ and $\mathbb{H}^2(-1)$, respectively. By the similar argument as in the proof of Proposition 3.5.2, we can prove that J_{12} is a constant function. Thus, we see easily that the almost Kähler condition $\mathfrak{S}_{ijk} \nabla_i J_{jk} = 0$ is equivalent to the following system of partial differential equations

$$\begin{cases} X_1 J_{14} - X_2 J_{13} = cJ_{14} \\ X_1 J_{13} + X_2 J_{14} = cJ_{13} \\ X_3 J_{14} - X_4 J_{13} = J_{14} \\ X_3 J_{13} + X_4 J_{14} = J_{13} \end{cases} \quad \text{if } (J_{ij}) \text{ is of the form (3.4.1),}$$

and

$$\begin{cases} X_1 J_{14} + X_2 J_{13} = cJ_{14} \\ X_1 J_{13} - X_2 J_{14} = cJ_{13} \\ X_3 J_{14} - X_4 J_{13} = J_{14} \\ X_3 J_{13} + X_4 J_{14} = J_{13} \end{cases} \quad \text{if } (J_{ij}) \text{ is of the form (3.4.2).}$$

In both case, by the same way as in the proof of Proposition 3.5.4, we can conclude $\nabla J = 0$. □

Proof of Proposition 3.5.7.

Let $\{X_1, X_2\}$ and $\{X_3, X_4\}$ be a global orthonormal frame field of $\mathbb{H}^2(-1)$ and a global orthonormal frame field of $\mathbb{R}^2$, respectively. By the similar argument as in the proof of Proposition 3.5.2 we can prove that J_{12} is a constant function. Thus, we see easily that the almost Kähler condition $\mathfrak{S}_{ijk} \nabla_i J_{jk} = 0$ is equivalent to the following system of partial differential equations

$$\begin{cases} X_1 J_{14} - X_2 J_{13} = J_{14} \\ X_1 J_{13} + X_2 J_{14} = J_{13} \\ X_3 J_{14} - X_4 J_{13} = 0 \\ X_3 J_{13} + X_4 J_{14} = 0 \end{cases} \quad \text{if } (J_{ij}) \text{ is of the form (3.4.1),}$$

and

$$\begin{cases} X_1 J_{14} + X_2 J_{13} = J_{14} \\ X_1 J_{13} - X_2 J_{14} = J_{13} \\ X_3 J_{14} - X_4 J_{13} = 0 \\ X_3 J_{13} + X_4 J_{14} = 0 \end{cases} \quad \text{if } (J_{ij}) \text{ is of the form (3.4.2).}$$

In both case, by the same way as in the proof of Proposition 3.5.4, we can conclude $\nabla J = 0$. $\square$

Proof of Proposition 3.5.8.

We suppose that $S^3(1) \times \mathbb{R}$ admit a compatible almost Kähler structure (J, g). Let $\{X_1, X_2, X_3\}$ and $\{X_4\}$ be a global orthonormal frame field of $S^3(1) = SU(2)$ and a global unit vector field of $\mathbb{R}$, respectively. Since

$$R_{1212} = R_{1313} = R_{2323} = -1$$

and otherwise $R_{ijkl} = 0$ (up to sign), by a straightforward computation, we have

$$\tau = 6, \qquad \tau^* = 2(J_{12}^2 + J_{13}^2 + J_{23}^2) = 2(J_{12}^2 + J_{13}^2 + J_{14}^2) = 2.$$

Thus, from (3.1.2), we have

$$0 \leq \tau^* - \tau = -4 < 0.$$

But, this is a contradiction. $\square$

This completes the proof of the Theorem 3.5.1.

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