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Geometry of Almost Complex Manifolds

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CONTENTS

INTRODUCTION

For any complex manifold M a tensor field J of type $(1,1)$ which has the property $J^2 = -I$ may be naturally defined. Generalizing the notion of complex manifolds, C. Ehresmann introduced the notion of almost complex manifolds. A tensor field J of type $(1,1)$ on a differentiable manifold M is called an almost complex structure on M if $J^2 = -I$, and the pair (M, J) is called an almost complex manifold. An almost complex manifold (M, J) equipped with a Riemannian metric g on M satisfying $g(JX, JY) = g(X, Y)$ for any $X, Y \in \mathfrak{X}(M)$ is called an almost Hermitian manifold and denoted by (M, J, g). Observing the fundamental properties of the covariant derivative ∇J of an almost complex structure J, A. Gray and L. Hervella [17] divided the space of almost Hermitian structures on a differentiable manifold M into sixteen classes which are invariant irreducible subspaces under the action of the unitary group $U(n)$. One of these classes is the class of nearly Kähler structures, which is defined by $(\nabla_X J)X = 0$ for any $X \in \mathfrak{X}(M)$. The unit sphere S^6 in the space of purely imaginary Cayley numbers is a well-known example of non-Kähler nearly Kähler manifolds. And later we have come to know many such examples — Riemannian 3-symmetric spaces.

The notion of Riemannian 3-symmetric spaces is introduced by A. Gray [15] as a special case of s-regular manifolds [34], and it has a deep connection with almost complex structures. For any almost complex structure J on a C^∞ manifold M and any integer $k \geqq 3$, there exists a family of local diffeomorphisms of order k on M. Conversely, when is given a family $\{\theta_p\}_{p\in M}$ of local diffeomorphisms of order k on M, if the order $k = 3$, the family $\{\theta_p\}_{p\in M}$ always induces an almost complex structure on M. Hence there is a one-to-one correspondence between the set of almost complex structures on a C^∞ manifold M and the set of families of local diffeomorphisms of order 3 on M. So, in order to investigate properties of an almost complex Hermitian manifold (M, J, g), it will be one approach to do those of a C^∞ manifold admitting a family of local diffeomorphisms of order 3 with some geometrical properties.

On one hand, from a Riemannian geometrical viewpoint, it is natural to consider the case where all local diffeomorphism θ_p of order 3 is a local isometry. On the other hand, from an analytical viewpoint, it is natural to consider the case where each θ_p preserves the canonical almost complex structure J. An almost Hermitian manifold (M, J, g) which admits a family $\{\theta_p\}_{p\in M}$ of local diffeomorphisms of order 3 such that each θ_p is a holomorphic isometry is called a Riemannian 3-symmetric space. Riemannian 3-symmetric spaces become homogeneous almost Hermitian manifolds, and include Hermitian symmetric spaces.

To any even-dimensional oriented Riemannian manifold N, we may associate an almost complex manifold $\mathcal{J}(N)$ which is a bundle over N with typical fiber $SO(2n)/U(n)$ ($2n =$ dim$N$), each fiber over $x \in N$ is formed by orthogonal complex structures of T_xN. $\mathcal{J}(N)$ is called a (metric) twistor space over N. The manifold $\mathcal{J}(N)$ comes equipped with two almost complex structures J_1 and J_2.

The former is occasionally integrable. M. F. Atiyah, N. J. Hitchin and I. M. Singer [1] showed that for a 4-dimensional oriented Riemannian manifold N, J_1 is integrable if and only if N is self-dual. B. Y. Chen[11], J. P. Bourguignon[7] and A. Derdzinski[12] gave the classification of compact self-dual Kähler surfaces, independently. In chapter I, we shall consider the problem to classify (anti-) self-dual (almost) Hermitian manifolds of dimension 4.

On the other hand J_2 is never integrable, but does play an important role in the theory of harmonic maps in the folowing sense: any J_2-holomorphic map $\psi : M \longrightarrow \mathcal{J}(N)$ from a quasi-Kähler manifold M projects to a harmonic map $\pi \circ \psi : M \longrightarrow N$ (cf.

[43]). It is known ([10]) that many Riemannian 3-symmetric spaces appear as holomorphic twistor spaces over even-dimensional Riemannian symmetric spaces. So it is meaningful to investigate geometrical or topological properties of Riemannian 3-symmetric spaces.

Characteristic classes of differentiable manifolds are important invariants of the manifolds. It is known that the Chern class $c(M, J)$ of an almost complex manifold (M, J) is an invariant with respect to the almost complex structure J, and the Pontrjagin class $p(M)$ of a differentiable manifold M is an invariant with respect to the differentiable structure on M but not a topological invariant. It is known that the Stiefel-Whitney classes are topological invariants.

For a principal G-bundle $\xi = (E_\xi, B_\xi, G)$ (G is a compact connected Lie group, K is a closed subgroup of G and S is a maximal torus of K), A. Borel and F. Hirzebruch [6] gave the method to calculate the characteristic classes of the bundle along the fibers of the bundle $(E_\xi, B_\xi, G/K)$ in terms of the roots of G relative to S, and concretely calculated some characteristic classes of compact Hermitian symmetric spaces. In Chapter II, we shall calculate the characteristic classes of connected simply connected irreducible compact Riemannian 3-symmetric spaces by the method of Borel and Hirzebruch. In order to calculate the Stiefel-Whitney classes, we have to know the second cohomology groups $H^2(G/K, \mathbb{Z}_2)$. And we may show that a connected simply connected irreducible compact Riemannian 3-symmetric space is Einstein if and only if its first Chern class vanishes. In the course of the calculation of $H^2(G/K, \mathbb{Z}_2)$, we determine explicitly the connected subgroup K of G, and give a modification of the classification tables listed in [62].

The present thesis consists of two chapters. In Chapter I, we shall consider the problem to classify self-dual (or anti-self-dual) almost Hermitian manifolds of dimension 4. In section 1.1, we recall the definition of (anti-)self-dual Riemannian 4-manifolds and some known results. In section 1.2, we recall the classification of compact self-dual Kähler surfaces which has been shown by B. Y. Chen [11], J. P. Bourguignon [7] and A. Derdzinski [12] independently, and the results about compact anti-self-dual Kähler surfaces due to

M. Itoh [23]. Though Kähler manifolds are not conformal invariant, the notion of (anti-) self-duality is conformal invariant. So it is natural to consider the problem to classify (anti-) self-dual (almost) Hermitian manifolds of dimension 4. We may easily see that any almost Hermitian 4-manifold (M, J, g) of pointwise constant holomorphic sectional curvature is self-dual. In section 1.3, We shall show that the converse is true if in addition (M, J, g) is Einstein. R. Schoen[50] showed that for any Riemannian manifold (M, g) there exists a Riemannian metric g' on M which is conformal to g such that the scalar curvature of g' is constant. Hence instead of considering all Riemannian metrics in the conformal class, we need only to consider the Riemannian metrics of constant scalar curvature. In section 1.4, we shall give a characterization of compact anti-self-dual Hermitian surfaces.

In Chapter II, we shall calculate the characteristic classes of compact Riemannian 3-symmetric spaces by the method of Borel and Hirzebruch. In section 2.1, we recall the definition of Riemannian 3-symmetric spaces and some of their basic properties. In section 2.2, we shall calculate the second homotopy groups and second cohomology groups of coefficient $\mathbb{Z}_2$ of Riemannian 3-symmetric spaces, and determine the connected Lie subgroup K of G by the information of the roots of $\mathfrak{k}$. In section 2.3, we recall the definitions of Chern classes, Pontrjagin classes, Stiefel-Whitney classes and Euler classes and the method of calculating the characteristic classes of compact homogeneous spaces shown by A. Borel and F. Hirzebruch [6]. In section 2.4, we shall calculate the characteristic classes of compact Riemannian 3-symmetric spaces by the method of A. Borel and F. Hirzebruch.

Throughout this thesis, all manifolds are assumed to be connected and of class C^∞ unless otherwise specified.

PRELIMINARIES

In this section, we prepare some fundamental notations and definitions in an almost Hermitian manifold.

Definition. A tensor field J of type $(1,1)$ on a differentiable manifold M of dimension $m(=2n)$ satisfying $J^2 = -I$ is called an almost complex structure on M. Then the pair (M, J) is called an almost complex manifold.

An almost complex manifold (M, J) equipped with a Riemannian metric g such that

$$g(JX, JY) = g(X, Y)$$

for any $X, Y \in \mathfrak{X}(M)$ ($\mathfrak{X}(M)$ denotes the Lie algebra of all differentiable vector fields on M) is called an almost Hermitian manifold and denoted by (M, J, g). The 2-form Ω on $M = (M, J, g)$ defined by $\Omega(X, Y) = g(X, JY)$ for $X, Y \in \mathfrak{X}(M)$ is called the Kähler form of M. We assume that the almost Hermitian manifold (M, J, g) is always oriented by the volume form $dM = \dfrac{(-1)^n}{n!}\Omega^n$.

Definition. An almost complex structure on a differentiable manifold M is called integrable if there exists a complex structure on M whose natural almost complex structure coincides with J.

The following theorem gives a geometrical characterization for the integrability of an almost complex structure.

Theorem 0.1. [41] *An almost complex structure J on a differentiable manifold M is integrable if and only if the Nijenhuis tensor N of (M, J) defined by*

$$N(X,Y) = [JX, JY] - [X,Y] - J[JX,Y] - J[X,JY], \quad \text{for } X, Y \in \mathfrak{X}(M)$$

vanishes.

Definition. An almost Hermitian manifold (M, J, g) with an integrable almost complex structure J is called a Hermitian manifold.

Here we recall the definitions of some classes of almost Hermitian manifolds which will appear in the present thesis.

Definition. Let $M = (M, J, g)$ be an almost Hermitian manifold.

(1) M is a Kähler manifold if $\nabla J = 0$, or equivalently $d\Omega = 0$ and J is integrable.

(2) M is a nearly Kähler manifold if $(\nabla_X J)X = 0$ for any $X \in \mathfrak{X}(M)$.

(3) M is a quasi-Kähler manifold if $(\nabla_{JX} J)JY + (\nabla_X J)Y = 0$ for any $X, Y \in \mathfrak{X}(M)$.

Let $M = (M, g)$ be in general an m-dimensional Riemannian manifold. We denote by ∇ and R the Riemannian connection and the Riemannian curvature tensor of M, respectively. R is defined by

$$R(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z, \tag{0.1}$$

for $X, Y, Z \in \mathfrak{X}(M)$. We denote by ρ, ρ^1, τ and W the Ricci tensor (of type $(0,2)$), the Ricci tensor of type $(1,1)$ and the scalar curvature of M defined respectively by

$$\rho(x,y) = \text{trace of } (z \longmapsto R(z,x)y), \tag{0.2}$$

$$g(\rho^1 x, y) = \rho(x,y), \tag{0.3}$$

$$\tau = \text{trace of } \rho^1, \tag{0.4}$$

for $x, y, z \in T_pM (p \in M)$. The Weyl's conformal curvature tensor W is defined by

$$\begin{aligned} W(x,y)z = R(x,y)z &- \frac{1}{m-2}\left(\rho^1 x \wedge y + x \wedge \rho^1 y\right) z \\ &+ \frac{\tau}{(m-1)(m-2)}(x \wedge y)z, \end{aligned} \tag{0.5}$$

where $x \wedge y$ denotes the endomorphism which maps z upon $g(y,z)x - g(x,z)y$, for $x, y, z \in T_pM$ $(p \in M)$. The Weyl's conformal curvature operator (also denoted by the same symbol W) is defined by the symmetric endomorphism of the vector bundle $\Lambda^2 M$ of 2-forms over M defined by

$$g(W(\iota(x) \wedge \iota(y)), \iota(z) \wedge \iota(w)) = -g(W(x,y)z, w) \tag{0.6}$$

for $x, y, z, w \in T_pM$ $(p \in M)$, where ι denotes the duality $TM \longrightarrow T^*M$ (the cotangent bundle of M) defined by means of the Riemannian metric g.

Definition. Two Riemannian metrics g and g' on a manifold M are said to be conformal if there exists a C^∞ function f on M such that $g' = e^{2f} g$.

The Weyl's conformal curvature tensor W is invariant under the conformal change of the metric.

Definition. A Riemannian manifold (M, g) is conformally flat if, for any $p \in M$, there exists a neighbourhood U of p and a C^∞ function f on U such that $(U, e^{2f} g)$ is flat.

It is known that if the dimension m of the manifold M is greater than 3, the Weyl's conformal curvature tensor W is the only obstruction to conformal flatness.

Theorem 0.1. [see e.g., 2] *If $m \geqq 4$, an m-dimensional Riemannian manifold (M, g) is conformally flat if and only if the Weyl's conformal curvature tensor W of (M, g) vanishes.*

Next, we recall the definition of Einstein manifolds.

Definition. A Riemannian manifold (M, g) is called an Einstein manifold if $\rho = \lambda g$ for some constant λ.

For an almost Hermitian manifold $M = (M, J, g)$, we may define the $*$-Ricci tensor ρ^* and the $*$-scalar curvature τ^*. They are defined respectively by

$$\rho^*(x, y) = \frac{1}{2} \text{ trace of } (z \longmapsto R(x, Jy)Jz), \tag{0.7}$$

$$\tau^* = \sum_{j=1}^{2n} \rho^*(e_j, e_j), \tag{0.8}$$

for $x, y, z \in T_pM$ $(p \in M)$ where $\{e_j\}$ is an orthonormal basis of T_pM.

Remark. If (M, J, g) is a Kähler manifold then $R(X,Y) \circ J = J \circ R(X,Y)$, and it follows that $\rho^* = \rho$ and $\tau^* = \tau$.

Let (M, J, g) be a $m(=2n)$-dimensional almost Hermitian manifold. We denote by $\Lambda^k M$ the space of k-forms on M. Each fiber $\Lambda^k(T_pM)$ of the vector bundle $\Lambda^k M$ over $p \in M$ inherits an inner product $(\ ,\ )$ from the Riemannian metric g, and so $\Lambda^k M$ becomes a Riemannian vector bundle. The star operator $*$ is the map $* : \Lambda^k M \longrightarrow \Lambda^{m-k} M$ defined by

$$\alpha \wedge *\beta = (\alpha, \beta)(dM)_p$$

for $\alpha, \beta \in \Lambda^k(T_pM)$ $(p \in M)$. We have an orthogonal decomposition of the vector bundle $\Lambda^2 M$ ([38]).

$$\Lambda^2 M = \mathbb{R}\Omega \oplus \Lambda_0^{1,1} M \oplus LM, \tag{0.9}$$

where $\Lambda_0^{1,1} M$ denotes the vector bundle of real primitive J-invariant 2-forms and LM denotes the vector bundle of real primitive J-skew-invariant 2-forms over M.

$$\Lambda_0^{1,1} M = \{\alpha \in \Lambda^2 M \mid \alpha(Jx, Jy) = \alpha(x, y), \quad (\alpha, \Omega) = 0\},$$
$$LM = \{\alpha \in \Lambda^2 M \mid \alpha(Jx, Jy) = -\alpha(x, y), \quad (\alpha, \Omega) = 0\}.$$

If $2n = 4$, the 3-form $d\Omega$ cannot be primitive, and it is always written as $d\Omega = \omega \wedge \Omega, \quad \omega = \delta\Omega \circ J$. The 1-form ω is called the Lee form of (M, J, g).

Definition. [57] Let (M, g) be a Hermitian manifold of complex dimension $n(\geqq 2)$. The metric g is called a locally conformal Kähler metric if for each $p \in M$ there exists a neighbourhood U of p and a C^∞ function σ_U on U such that $e^{-\sigma_U} g$ is a Kähler metric on U. We call (M, g) a locally conformal Kähler manifold.

Let (M, g) be a locally conformal Kähler manifold. Then we have $d\sigma_U = d\sigma_V$ on $U \cap V$, and the 1-form ω defined by $\omega = d\sigma_U$ on U becomes a closed 1-form defined globally on

M. We call ω the *Lee form* of (M, g). ω satisfies the following equalities (see [44], [47]).

$$J^{ij}\nabla_i\omega_j = 0, \tag{0.10}$$

$$2\nabla_i J_j{}^k = \omega_a J_j{}^a \delta_i^k - \omega_a J^{ka} g_{ij} - \omega_j J_i{}^k + \omega^k J_{ij}, \tag{0.11}$$

$$\tau - \tau^* = 2\delta\omega + \|\omega\|^2. \tag{0.12}$$

CHAPTER I

SELF-DUAL HERMITIAN SURFACES

1.1. Self-dual Riemannian manifolds

In this section, we recall the definition of self-dual Riemannian 4-manifolds.

Let (M, g) be a 4-dimensional oriented Riemannian manifold. The star operator $*$ on the space of 2-forms $\Lambda^2 M$ satisfies $* \circ * = id$ and hence has eigenvalues ± 1.

Definition. A 2-form $\alpha \in \Lambda^2 M$ is calles self-dual (resp. anti-self-dual) if $*\alpha = \alpha$ (resp. $*\alpha = -\alpha$).

We denote by $\Lambda^2_+ M$ (resp. $\Lambda^2_- M$) the space of self-dual (resp. anti-self-dual) 2-forms.

$$\Lambda^2_+ M = \{\alpha \in \Lambda^2 M \mid * \alpha = \alpha\},$$
$$\Lambda^2_- M = \{\alpha \in \Lambda^2 M \mid * \alpha = -\alpha\}.$$

The bundle $\Lambda^2 M$ splits into two vector subbundles $\Lambda^2_\pm M : \Lambda^2 M = \Lambda^2_+ M \oplus \Lambda^2_- M$. This decomposition corresponds to that of the Lie algebra $\mathfrak{so}(4) = \mathfrak{so}(3) \oplus \mathfrak{so}(3)$. And we may see that

$$\Lambda^2_+ M = \mathbb{R}\Omega \oplus \Lambda^{1,1}_0 M,$$
$$\Lambda^2_- M = LM.$$

We denote by W_+ (resp. W_-) the restriction of the Weyl's conformal curvature operator to $\Lambda^2_+ M$ (resp. $\Lambda^2_- M$).

Definition. An oriented Riemannian manifold (M, g) of dimension 4 is called self-dual (resp. anti-self-dual) if $W_- = 0$ (resp. $W_+ = 0$).

Remark. Since the Weyl's conformal curvature tensor W and the star operator $*$ are conformal invariants, the notion of self-duality and anti-self-duality are properties of the underlying conformal structure, and the choice of orientation. If the orientation of M is reversed, the self-duality is altered into the anti-self-duality.

There are topological restrictions on manifolds which admits a self-dual conformal structures, in particular their signatures τ have to be non-negative, for τ is equal to $\frac{1}{3}p_1(M)[M]$ and the first Pontrjagin number $p_1(M)[M]$ may be calculated by

$$p_1(M)[M] = \frac{1}{4\pi^2}\int_M (\|W_+\|^2 - \|W_-\|^2)dM.$$

Examples. (1) Any conformally flat oriented Riemannian 4-manifold is self-dual and anti-self-dual. Hence the 4-sphere S^4, the 4-torus T^4 and the Hopf surface $S^1 \times S^3$ with natural metrics are self-dual and anti-self-dual.

(2) The complex projective plane $\mathbb{C}P^2$ with Fubini–Study metric and the natural orientation is self-dual.

(3) Any Kähler surface with vanishing Ricci tensor is anti-self-dual. So any K3-surface with the Calabi–Yau metric is anti-self-dual.

(4) The manifold $\widetilde{\mathbb{C}}^2$ obtained by the blowing up at the origin of $\mathbb{C}^2$ is defined as a submanifold of $\mathbb{C}^2 \times \mathbb{C}P^1$. Burns [see 36] and LeBrun [36] have shown that the Riemannian metric g on $\widetilde{\mathbb{C}}^2$ induced by the product metric $g_0 + h_c$ on $\mathbb{C}^2 \times \mathbb{C}P^1$ is a Kähler metric with vanishing scalar curvature, where g_0 is a flat metric on $\mathbb{C}^2$ and h_c is the Fubini–Study metric on $\mathbb{C}P^1$ with constant holomorphic sectional curvature $c(> 0)$. So we see that $(\widetilde{\mathbb{C}}^2, g)$ is anti-self-dual but not conformally flat.

(5) Donaldson and Friedman [13] have shown the existence of self-dual metrics on the connected sum $\mathbb{C}P^2 \# \ldots \# \mathbb{C}P^2$ of any number of copies of the complex projective plane. Remark that $\mathbb{C}P^2 \# \mathbb{C}P^2$ does not admit any almost complex structure.

The self-duality of oriented Riemannian 4-manifolds is deeply concerned with the integrability of the almost complex structures on their twistor spaces.

Definition. Let (M,g) be a $2n$-dimensional Riemannian manifold. The bundle $\mathcal{J}(M)$ over M defined by

$$\mathcal{J}(M) = \{j_p : T_pM \longrightarrow T_pM : \text{linear isometry} \mid j_p^2 = -I_p, p \in M\}$$

is called the twistor space of (M,g).

The standard fiber of $\mathcal{J}(M)$ is a Hermitian symmetric space $SO(2n)/U(n)$ with the natural almost complex structure J_0. Using the Riemannian connection of (M,g), we may split the tangent bundle of $\mathcal{J}(M)$:

$$T\mathcal{J}(M) = TF \oplus \pi^{-1}TM,$$

where TF denotes the bundle along the fibers of $\mathcal{J}(M)$ and $\pi^{-1}TM$ denotes the bundle over $\mathcal{J}(M)$ induced by the projection $\pi : \mathcal{J}(M) \longrightarrow M$. We may define two almost complex structures J_1 and J_2 as follows.

$$J_1 = J_0 + j_p,$$
$$J_2 = -J_0 + j_p$$

on each fiber $T_zF \oplus (\pi^{-1}TM)_z$ over $z = j_p \in \mathcal{J}(M)$. Though J_2 is never integrable, J_1 is occasionally integrable.

Theorem 1.1.1. [1] *Let (M,g) be an oriented Riemannian 4-manifold. Then the almost complex structure J_1 is integrable if and only if (M,g) is self-dual.*

The following theorem assures for us to restrict ourselves to (anti-)self-dual metrics of constant scalar curvature.

Theorem 1.1.2. [50] *Let (M,g) be an m-dimensional Riemannian manifold. Then there exist a positive function $u \in C^\infty(M)$ such that the scalar curvature of the metric $u^{\frac{4}{m-2}}g$ is constant.*

An Einstein metric is one of such metrics whose scalar curvatures are constant. About compact self-dual Einstein 4-manifolds, N. J. Hitchin showed the following.

Theorem 1.1.3. [2] *Let $M = (M, g)$ be a compact self-dual Einstein manifold. Then*

(1) *If $\tau > 0$, M is isometric to S^4 or $\mathbb{C}P^2$ with their canonical metrics.*

(2) *If $\tau = 0$, M is either flat or its universal covering is a $K3$ surface with the Calabi-Yau metric.*

1.2 Self-dual and anti-self-dual Kähler surfaces

In this section a Kähler manifold of complex dimension 2 is called a Kähler surface. A classification of compact self-dual Kähler surfaces was given by B. Y. Chen [11] in terms of Bochner-Kähler metrics and thereafter by J. P. Bourguignon [7] by the aid of theorems with respect to harmonic curvature tensors, and by A. Derdzinski [12] using the following theorem.

Theorem 1.2.1. [12] *Every self-dual Kähler surface with constant scalar curvature is locally symmetric.*

The complete classification of compact self-dual Kähler surfaces is obtained by the following theorem.

Theorem 1.2.2. [11, 7, 12] *Let (M, g) be a compact self-dual Kähler surface. Then (M, g) is one of the following spaces.*

(1) *the complex projective plane $\mathbb{C}P^2$ with the standard Fubini-Study metric,*

(2) *a compact quotient of unit disk D^2 with the Bergman metric,*

(3) *a Kählerian flat torus T^2,*

(4) *a compact quotient of a product space $\mathbb{C}P^1 \times D^1$ of the complex projective line $\mathbb{C}P^1$ and the Poincaré disk D^1 with metrics of opposite curvature.*

If a Kähler surface is of constant holomorphic sectional curvature, then it is self-dual. About the converse M. Itoh has showed the following.

Theorem 1.2.3. [23] *Let (M, g) be a Kähler surface. If it is self-dual with respect to the canonical orientation and it is Einstein, then it is of constant holomorphic sectional curvature.*

About anti-self-dual Kähler surfaces the following characterization has been given by M. Itoh [23].

Theorem 1.2.4. *Let (M, g) be a Kähler surface. Then it is anti-self-dual if and only if its scalar curvature vanishes everywhere.*

M. Itoh has given the possibility of the compact complex surfaces which admit anti-self-dual Kähler structures.

Theorem 1.2.5. [23] *Let (M, g) be a compact anti-self-dual Kähler surface. Then (M, g) is necessarily one of the following*

(1) *a Kählerian flat torus,*

(2) *a Kähler surface covered by a $K3$ surface with a Ricci flat metric,*

(3) *a Kählerian ruled surface of genus $k(\geqq 2)$ and*

(4) *a Kähler surface which is obtained by blowing up either $\mathbb{C}P^2$ at least 10 times, a ruled surface of genus 0 at least 9 times or a ruled surface of genus $k(\geqq 1)$ at least once.*

Remark. It is not yet known whether there is an anti-self-dual Kähler metric on a complex surface of type (4) of Theorem 1.2.5.

The notions of self-duality and anti-self-duality are invariant under the conformal change $e^{2f}g$ ($f \in C^\infty(M)$) of the Riemannian metric g. But the metric $e^{2f}g$ is not necessarily a Kähler metric on M. If (M, J, g) is an almost Hermitian manifold (resp. a Hermitian manifold), then for any $f \in C^\infty(M)$ $(M, J, e^{2f}g)$ is also an almost Hermitian manifold (resp. a Hermitian manifold). So it is natural to classify (anti-)self-dual manifolds which are Hermitian surfaces (or almost Hermitian manifolds) including Kähler surfaces.

1.3 Self-dual Hermitian surfaces

In this section, we consider self-dual almost Hermitian 4-manifolds and compact self-dual Einstein Hermitian surfaces.

Let (M, J, g) be an almost Hermitian manifold of dimension 4 with the canonical orientation, $\{e_j\} = \{e_1, e_2 = Je_1, e_3, e_4 = Je_3\}$ a positively oriented orthonormal basis of the tangent space T_pM at a point $p \in M$, and $\{e^j\}$ the dual basis. We denote by $T_p^{\mathbb{C}}M = T_pM \otimes_{\mathbb{R}} \mathbb{C}$ the complexification of T_pM. Put

$$\begin{aligned} f_1 &= \frac{1}{\sqrt{2}}(e_1 - \sqrt{-1}e_2), \\ f_{\bar{1}} &= \frac{1}{\sqrt{2}}(e_1 + \sqrt{-1}e_2), \\ f_2 &= \frac{1}{\sqrt{2}}(e_3 - \sqrt{-1}e_4), \\ f_{\bar{2}} &= \frac{1}{\sqrt{2}}(e_3 + \sqrt{-1}e_4). \end{aligned} \tag{1.3.1}$$

Then $\{f_1, f_2\}$ forms a unitary basis of $T_p^{\mathbb{C}}M$, and its dual basis $\{f^A\}$ is given by

$$\begin{aligned} f^1 &= \frac{1}{\sqrt{2}}(e^1 + \sqrt{-1}e^2), \\ f^{\bar{1}} &= \frac{1}{\sqrt{2}}(e^1 - \sqrt{-1}e^2), \\ f^2 &= \frac{1}{\sqrt{2}}(e^3 + \sqrt{-1}e^4), \\ f^{\bar{2}} &= \frac{1}{\sqrt{2}}(e^3 - \sqrt{-1}e^4). \end{aligned}$$

In the sequel we shall adopt the following conventions.

$$\begin{aligned} R_{ijkl} &= g(R(e_i . e_j)e_k, e_l), \\ K_{ABCD} &= g(R(f_A, f_B)f_C, f_D), \\ K_{AB} &= \rho(f_A, f_B), \\ W_{ABCD} &= g(W(f_A, f_B)f_C, f_D), \end{aligned}$$

where $i, j, k, l \in \{1, 2, 3, 4\}$ and $A, B, C, D \in \{1, 2, \bar{1}, \bar{2}\}$. By the definition we have

Proposition 1.3.1. [26] *A 4-dimensional almost Hermitian manifold (M, J, g) is self-dual with respect to the canonical orientation if and only if*

$$12K_{1\bar{2}2\bar{1}} = \tau, \quad K_{1\bar{1}1\bar{2}} - K_{2\bar{2}1\bar{2}} = 0, \quad K_{1\bar{2}1\bar{2}} = 0$$

for any basis $\{f_A\}$ of $T_p^{\mathbb{C}}M$ of the form (1.3.1) at each point $p \in M$.

A sectional curvature of a plane invariant under J in T_pM is called a holomorphic sectional curvature.

Proposition 1.3.2. [54] *An almost Hermitian manifold (M^m, J, g) is of constant holomorphic sectional curvature H at $p \in M$ if and only if*

$$\begin{aligned}
&R(x, Jy, Jz, w) + R(x, Jz, Jy, w) + R(x, Jy, Jw, z)\\
&\quad + R(x, Jw, Jy, z) + R(x, Jz, Jw, y) + R(x, Jw, Jz, y)\\
&\quad + R(y, Jx, Jz, w) + R(y, Jz, Jx, w) + R(y, Jx, Jw, z)\\
&\quad + R(y, Jw, Jx, z) + R(z, Jy, Jx, w) + R(z, Jx, Jy, w)\\
&\quad = 4H(g(y, z)g(x, w) + g(z, w)g(x, y) + g(y, w)g(x, z))
\end{aligned}$$

for any $x, y, z, w \in T_pM$, where $R(x, y, z, w) = g(R(x, y)z, w)$.

Lemma 1.3.3. [26] *A 4-dimensional almost Hermitian manifold (M, J, g) is of constant holomorphic sectional curvature H at a point $p \in M$ if and only if*

$$\begin{aligned}
&K_{1\bar{1}1\bar{1}} = K_{2\bar{2}2\bar{2}} = H,\\
&K_{1\bar{1}1\bar{2}} = K_{2\bar{2}1\bar{2}} = 0,\\
&K_{1\bar{1}2\bar{2}} + K_{1\bar{2}2\bar{1}} = H,\\
&K_{1\bar{2}1\bar{2}} = 0
\end{aligned}$$

for any basis $\{f_A\}$ of $T_p^{\mathbb{C}}M$ of the form (1.3.1).

From Proposition 1.3.1 and Lemma 1.3.3, we may easily see that any 4-dimensional almost Hermitian manifold of pointwise constant holomorphic sectional curvature is self-dual. About its converse we have

Theorem 1.3.4. [26] *Let (M, J, g) be a 4-dimensional almost Hermitian manifold. If it is self-dual and Einstein, then it is of pointwise constant holomorphic sectional curvature.*

Let (M, J, g) be a Hermitian surface of pointwise constant holomorphic sectional curvature $c = c(p)$ $(p \in M)$. Then its Riemannian curvature tensor may be expressed in terms of g, J and the Lee form ω ([44]).

$$
\begin{aligned}
(1.3.1) \qquad g(R(x,y)z,w) = &-\frac{c}{4}(\pi_1 + \pi_2)(x,y,z,w) \\
&+ \frac{1}{96}\{-3(2\delta\omega + \|\omega\|^2)(3\pi_1 - \pi_2) \\
&+ (3\varphi - \psi)(A) + 12\varphi(T) - 4\psi(T^*)\}(x,y,z,w),
\end{aligned}
$$

where

$$
\begin{aligned}
\pi_1(x,y,z,w) &= g(x,z)g(y,w) - g(y,z)g(x,w), \\
\pi_2(x,y,z,w) &= 2g(Jx,y)g(Jz,w) \\
&\quad + g(Jx,z)g(Jy,w) - g(Jy,z)g(Jx,w), \\
A(x,y) &= 3\{(\nabla_x\omega)y + (\nabla_y\omega)x + \omega(x)\omega(y) \\
&\quad + (\nabla_{Jx}\omega)Jy + (\nabla_{Jy}\omega)Jx + \omega(Jx)\omega(Jy)\} \\
&\quad - \frac{3}{2}(-2\delta\omega + \|\omega\|^2)g(x,y), \\
T(x,y) &= (\nabla_x\omega)y + (\nabla_y\omega)x + \omega(x)\omega(y) \\
&\quad - (\nabla_{Jx}\omega)Jy - (\nabla_{Jy}\omega)Jx - \omega(Jx)\omega(Jy), \\
T^*(x,y) &= (\nabla_x\omega)y + (\nabla_{Jy}\omega)Jx - (\nabla_y\omega)x - (\nabla_{Jx}\omega)Jy, \\
\varphi(B)(x,y,z,w) &= g(x,z)B(y,w) + g(y,w)B(x,z) \\
&\quad - g(x,w)B(y,z) - g(y,z)B(x,w), \\
\psi(B)(x,y,z,w) &= 2g(x,Jy)B(z,Jw) + 2g(z,Jw)B(x,Jy) \\
&\quad + g(x,Jz)B(y,Jw) + g(y,Jw)B(x,Jz) \\
&\quad - g(x,Jw)B(y,Jz) - g(y,Jz)B(x,Jw).
\end{aligned}
$$

From (1.3.1) we have

$$\rho(x,y) = \left\{\frac{3}{2}c + \frac{3}{16}(\tau - \tau^*)\right\} g(x,y) - \frac{1}{4}T(x,y),$$
$$\rho^*(x,y) = \left\{\frac{3}{2}c - \frac{3}{16}(\tau - \tau^*)\right\} g(x,y) + \frac{1}{4}T^*(x,y).$$

Let S be a tensor field of type $(0,2)$ defined by

$$S(x,y) = (\nabla_x \omega)y - (\nabla_{Jx}\omega)Jy + \frac{1}{2}\{\omega(x)\omega(y) - \omega(Jx)\omega(Jy)\}.$$

Calculating the global inner product $\int_M (T, \omega \otimes \omega) dM$, we have

Proposition 1.3.5. [31] *Let $M = (M,J,g)$ be a compact Einstein Hermitian surface of pointwise constant holomorphic sectional curvature. Then M is a locally conformal Kähler surface and the tensor field S vanishes on M.*

From this proposition and (0.12), taking account of $S = 0$, we may get the differential equation

$$\nabla_X(\tau - 3\tau^*) + \frac{3}{2}(\tau - 3\tau^*)\omega(X) = 0 \quad \text{for any } X \in \mathfrak{X}(M). \tag{1.3.2}$$

About compact Einstein Hermitian surfaces, the following theorem has been shown by K. Sekigawa [46].

Theorem 1.3.6. [46] *If a compact Einstein Hermitian surface $M = (M,J,g)$ has a negative $*$-scalar curvature τ^*, then M is a Kähler surface.*

Then from this theorem, (1.3.2), Theorem 1.3.4, Proposition 1.3.5, Theorem 1.4.6 and the classification of compact complex surfaces (see [3]), we may show the following theorem.

Theorem 1.3.7. [31] *Let $M = (M,J,g)$ be a compact self-dual Einstein Hermitian surface. Then M is a Kähler surface of constant holomorphic sectional curvature, i.e., M is one of the following*

(1) *flat,*

(2) $\mathbb{C}P^2$ *with the standard Fubini-Study metric and*

(3) *a compact quotient of unit disk D^2 with the Bergman metric.*

Remark. C. P. Boyer [9] has asserted the above result without detailed proof.

1.4 Anti-self-dual Hermitian surfaces

In this section we shall consider anti-self-dual Hermitian surfaces. Though if we reverse the orientation of a Riemannian 4-manifold (M, g) the self-duality turns to the anti-self-duality, it is meaningful to consider the anti-self-duality of an (almost) Hermitian manifold of dimension 4 because it has the natural orientation.

Proposition 1.4.1. [26] *A 4-dimensional almost Hermitian manifold (M, J, g) is anti-self-dual with respect to the canonical orientation if and only if*

$$\tau = 3\tau^*, \quad K_{1\bar{1}12} + K_{2\bar{2}12} = 0, \quad K_{1212} = 0$$

for any basis $\{f_A\}$ of $T_p^{\mathbb{C}}M$ of the form (1.3.1) at each point $p \in M$.

Proposition 1.4.2. [26] *A Hermitian surface (M, J, g) is anti-self-dual with respect to the canonical orientation if and only if*

$$\tau = 3\tau^*, \quad K_{1\bar{1}12} + K_{2\bar{2}12} = 0$$

for any basis $\{f_A\}$ of $T_p^{\mathbb{C}}M$ of the form (1.3.1) at each point $p \in M$.

Lemma 1.4.3. [26] *Let (M, J, g) be a Hermitian surface. Then the condition $K_{1\bar{1}12} + K_{2\bar{2}12} = 0$ holds for any basis $\{f_A\}$ of $T_p^{\mathbb{C}}M$ of the form (1.3.1) at each point $p \in M$ if and only if the 2-form $d\omega$ is anti-self-dual.*

Hence we have

Proposition 1.4.4. [26] *A Hermitian surface (M, J, g) is anti-self-dual with respect to the canonical orientation if and only if*

$$\tau = 3\tau^*, \quad \text{and}$$
$$d\omega \text{ is an anti-self-dual 2-form.}$$

From this proposition we may derive the condition for Kähler surfaces to be anti-self-dual shown by M. Itoh.

Corollary 1.4.5. [23] *A Kähler surface M is anti-self-dual with respect to the canonical orientation if and only if its scalar curvature τ vanishes on M.*

If furthermore M is compact, we have

Theorem 1.4.6. [26] *A compact Hermitian surface (M, J, g) is anti-self-dual with respect to the canonical orientation if and only if (M, J, g) is a locally conformal Kähler manifold with $\tau = 3\tau^*$.*

For a Hermitian surface (M, J, g) we have

$$\tau - \tau^* = 2\delta\omega + \|\omega\|^2, \tag{1.4.1}$$

hence by integrating the scalar curvature we may see the following.

Corollary 1.4.7. *Let $M = (M, J, g)$ be an anti-self-dual compact Hermitian surface. If its total scalar curvature $\int_M \tau dM$ is nonpositive, then M is a Kähler surface.*

I. Vaisman [56] has shown the following.

Theorem 1.4.8. [56] *A compact locally conformal Kähler manifold (M, J, g) is a globally conformal Kähler manifold if and only if (M, J) bears some Kähler metric γ.*

The following theorem is well-known.

Theorem 1.4.9. [40], [51] *A compact complex surface has a Kähler metric if and only if its first Betti number is even.*

From these theorems, we may easily see that if an anti-self-dual Hermitian surface M is in class I, II, III, IV or V of the classification of compact complex surfaces, then it is a Kähler surface, and by Theorem 1.2.5 the possibility of M is known.

Computing the plurigenura and using the classification of compact complex surfaces, C.P. Boyer [9] has shown the following.

Theorem 1.4.10. [9] *Let M be a minimal compact anti-self-dual Hermitian surface. Then M is conformally equivalent to one of the following:*

(1) *a Kählerian flat torus T^2,*

(2) *a hyperelliptic surface with a flat metric,*

(3) *a $K3$ surface with a Calabi–Yau metric,*

(4) *an Enriques surface with a Calabi–Yau metric,*

(5) *a ruled surface $\mathbb{C}P^1 \times S_k$ of genus $k(\geqq 2)$,*

(6) *a Hopf surface with a conformally flat metric,*

(7) *a class VII_0 surface with positive second Betti number.*

Remark. It remains open to classify minimal compact anti-self-dual Hermitian surfaces of type VII_0.

For non-minimal complex surfaces, C.P. Boyer has shown the following.

Theorem 1.4.11. [9] *Let M be a non-minimal complex surface which admits an anti-self-dual Hermitian metric. Then M must be obtained by blowing-up one of the following minimal surfaces k-times:*

(1) *a minimal surface of class VII_0, $k \geqq 1$,*

(2) *a ruled surface of genus $g \geqq 1$, $k \geqq 1$,*

(3) *a rational ruled surface, $k \geqq 9$,*

(4) $\mathbb{C}P^2$, $k \geqq 10$.

Remark. It is not known whether there exists an anti-self-dual Hermitian metric on the complex surfaces listed in Theorem 1.4.11.

CHAPTER II

CHARACTERISTIC CLASSES OF RIEMANNIAN 3-SYMMETRIC SPACES

We may see that any 2-dimensional oriented C^∞ manifold admits an almost complex structure. But it is not always true for higher dimension. For example, it is well-known that the unit spheres S^m which admit almost complex structures are S^2 and S^6. Wu [63] gave the characterizations for 4- or 6-dimensional oriented C^∞ manifolds to admit almost complex structures in terms of their cohomology groups.

Theorem 2.0.1. [63] *A 4-dimensional oriented C^∞ manifold M admits an almost complex structure if and only if there exists an integral cohomology class $c \in H^2(M, \mathbb{Z})$ such that*

(1) $c \equiv w_2(M) \mod 2,$ *and*

(2) $p_1(M) + 2e(M) = c^2.$

Theorem 2.0.2. [63] *A 6-dimensional oriented C^∞ manifold M admits an almost complex structure if and only if there exists an integral cohomology class $c \in H^2(M, \mathbb{Z})$ such that $c \equiv w_2(M) \mod 2$.*

Heaps [18] has given the characterizations for 8- and 10-dimensional C^∞ manifolds.

The Pontrjagin classes are sometimes used in the theory of isometric immersions. The Stiefel–Whitney classes play an important role in the theory of embeddings and the Stiefel–Whitney numbers do in the cobordism theory (see [39] and [42]). The characteristic classes are very important quantities in geometry. Borel and Hirzebruch [6] have calculated explicitly the characteristic classes of compact Hermitian symmetric spaces, and Takeuchi [53] has calculated the Pontrjagin classes of compact symmetric spaces. In this chapter, we shall calculate some characteristic classes of compact Riemannian 3-symmetric spaces which contain Hermitian symmetric spaces.

2.1. Riemannian 3-symmetric spaces

In this section, we shall recall the definition and basic properties of Riemannian 3-symmetric spaces.

First we shall review a relationship between almost complex structures and families of local diffeomorphisms of order 3. Let M be a differentiable manifold. Assume that M possesses a C^∞ almost complex structure J and let $k \geqq 3$ be any integer. Then we may define a C^∞ tensor field Θ of type $(1,1)$ by

$$\Theta = \cos\frac{2\pi}{k} I + \sin\frac{2\pi}{k} J.$$

Making use of Θ and a normal coordinate neighbourhood with respect to any chosen affine connection, we may see the following.

Proposition 2.1.1. [15] *Let (M, J) be a C^∞ almost complex manifold and $k \geqq 3$ be any integer. Then for each $p \in M$ there exists a neighbourhood $U(p)$ and a diffeomorphism $\theta_p : U(p) \longrightarrow U(p)$ such that*

(1) ${\theta_p}^k = 1$,

(2) *p is a unique fixed point of θ_p.*

This fact suggests that it is natural to define the following notion.

Definition. A family of local diffeomorphisms of order k on a C^∞ manifold M is a differentiable function $p \longmapsto \theta_p$ which assigns to each point $p \in M$ a diffeomorphism θ_p on a neighbourhood $U(p)$ of p which satisfies (1) and (2) of Proposition 2.1.

We have thus seen that any almost complex manifold (M, J) admits a family of local diffeomorphisms of any order $k \geqq 3$.

The converse is also true for $k = 3$: let $p \longmapsto \theta_p$ be a family of local diffeomorphisms of order 3 on a C^∞ manifold M. Then the C^∞ tensor field J of type $(1,1)$ on M defined by

$$(\theta_p)_{*p} = -\frac{1}{2}I_p + \frac{\sqrt{3}}{2}J_p, \quad p \in M$$

becomes an almost complex structure on M. J is called the canonical almost complex structure of the family $\{\theta_p\}_{p\in M}$.

Thus we have seen that there is a one-to-one correspondence between the set of differentiable almost complex structures on a differentiable manifold M and the set of families of local diffeomorphisms of order 3 on M.

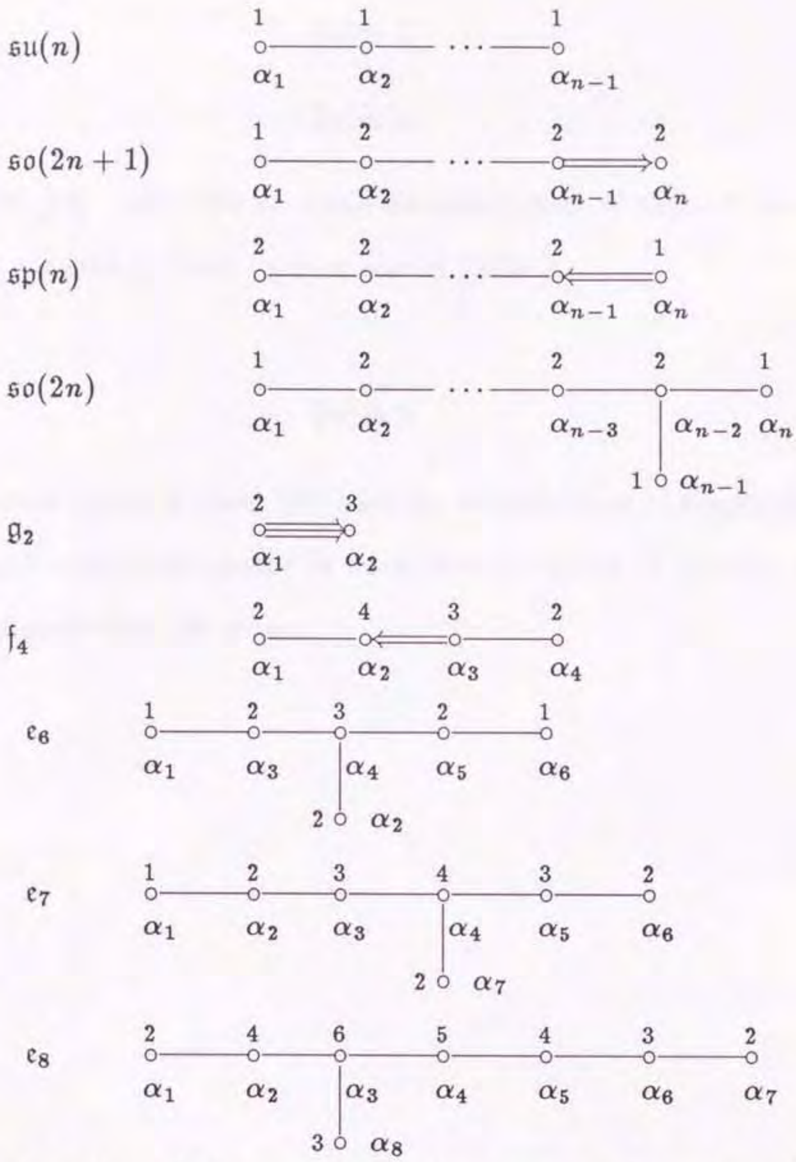
Although each θ_p preserves the canonical almost complex structure J and the Riemannian metric g at p, it is in general false that θ_p is a holomorphic map or an isometry in a neighbourhood $U(p)$.

Definition. A Riemannian locally 3-symmetric space M is a C^∞ Riemannian manifold M together with a family of local diffeomorphisms $p \longmapsto \theta_p$ of order 3 such that each θ_p is a holomorphic isometry in a neighbourhood of p with respect to the canonical almost complex structure of the family. And then θ_p is called a local cubic holomorphic isometry.

A connected Riemannian locally 3-symmetric space in which the domain of definition of each local cubic isometry is M is called a *Riemannian 3-symmetric space*. A.Gray [15] has shown that the group $\mathcal{I}(M)$ of holomorphic isometries of a Riemannian 3-symmetric space M acts transitively on M, and M is characterized by the triple (G, σ, H) of a Lie group G, an automorphism σ of order 3 of G and a closed subgroup $H \subset G$ such that $G_0^\sigma \subset H \subset G^\sigma$.

Definition. Let M be a Riemannian 3-symmetric space. M is called irreducible (or sometimes, indecompossable) if M is not flat and whenever M is a Riemannian product of two Riemannian 3-symmetric spaces M_1 and M_2, then either $M = M_1$ or $M = M_2$.

In this thesis, the simple roots of simple Lie algebras are numbered as follows:



J. A. Wolf and A. Gray [62] gave the classification of such triples (G, σ, H) so that G is a simple Lie group. First they classified the pair $(\mathfrak{g}, \sigma)$ where $\mathfrak{g}$ is a simple Lie algebra and σ is an automorphism of $\mathfrak{g}$ of order 3.

Theorem 2.1.6. [62] *Let φ be an inner automorphism of order 3 on a compact or complex simple Lie algebra $\mathfrak{g}$. Choose a Cartan subalgebra $\mathfrak{t}$ and let $\Psi = \{\alpha_1, \cdots, \alpha_l\}$ be a simple root system of $\mathfrak{g}$ with respect to $\mathfrak{t}$. Then φ is conjugate (up to inner automorphism of $\mathfrak{g}$) to some $\theta = Ad(exp 2\pi\sqrt{-1}x)$ where $x = \frac{1}{3}m_i v_i$ with $1 \leq m_i \leq 3$ or $x = \frac{1}{3}(v_i + v_j)$ with $m_i = m_j = 1$. A complete list of the possibilities for x is listed in the table below.*

Table 1

Table 2

Theorem 2.1.7. [62] *Let θ be an outer automorphism of order 3 on a compact or complex simple Lie algebra $\mathfrak{g}$. Then $(\mathfrak{g}, \mathfrak{k})$ is one of Table 3.*

Table 3

Using this classification table, A. Gray [15] gave the classification of simply connected irreducible Riemannian 3-symmetric spaces M such that the group of (pseudo-) holomorphic isometries of M is a reductive Lie group.

Table 1

$\mathfrak{g}$	x	Ψ_x	$\mathfrak{g}^\theta$
$\mathfrak{su}(2)$	$\frac{1}{3}v_1$	empty	$\mathfrak{t}^1$
$\mathfrak{su}(n)$ $n \geq 3$	$\frac{1}{3}v_i$	$\{\alpha_1, \cdots, \alpha_{i-1},$ $\alpha_{i+1}, \cdots, \alpha_{n-1}\}$	$\mathfrak{su}(i) \oplus \mathfrak{su}(n-i) \oplus \mathfrak{t}^1$
	$\frac{1}{3}(v_i + v_j)$ $i < j$	$\{\alpha_1, \cdots, \alpha_{i-1},$ $\alpha_{i+1}, \cdots, \alpha_{j-1},$ $\alpha_{j+1}, \cdots, \alpha_{n-1}\}$	$\mathfrak{su}(i) \oplus \mathfrak{su}(j-i)$ $\oplus \mathfrak{su}(n-j) \oplus \mathfrak{t}^2$
$\mathfrak{so}(2n+1)$ $n \geq 2$	$\frac{1}{3}v_1$	$\{\alpha_2, \cdots, \alpha_n\}$	$\mathfrak{so}(2n-1) \oplus \mathfrak{t}^1$
	$\frac{2}{3}v_i$ $2 \leq i \leq n$	$\{\alpha_1, \cdots, \alpha_{i-1},$ $\alpha_{i+1}, \cdots, \alpha_n\}$	$\mathfrak{su}(i) \oplus \mathfrak{so}(2(n-i)+1)$ $\oplus \mathfrak{t}^1$
$\mathfrak{sp}(n)$ $n \geq 2$	$\frac{2}{3}v_i$ $1 \leq i \leq n-1$	$\{\alpha_1, \cdots, \alpha_{i-1},$ $\alpha_{i+1}, \cdots, \alpha_n\}$	$\mathfrak{su}(i) \oplus \mathfrak{sp}(n-i)$ $\oplus \mathfrak{t}^1$
	$\frac{1}{3}v_n$	$\{\alpha_1, \cdots, \alpha_{n-1}\}$	$\mathfrak{su}(n) \oplus \mathfrak{t}^1$
$\mathfrak{so}(8)$	$\frac{1}{3}v_1$	$\{\alpha_2, \alpha_3, \alpha_4\}$	$\mathfrak{su}(4) \oplus \mathfrak{t}^1$
	$\frac{2}{3}v_2$	$\{\alpha_1, \alpha_3, \alpha_4\}$	$\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ $\oplus \mathfrak{su}(2) \oplus \mathfrak{t}^1$
	$\frac{1}{3}(v_1 + v_3)$	$\{\alpha_2, \alpha_4\}$	$\mathfrak{su}(3) \oplus \mathfrak{t}^2$
$\mathfrak{so}(2n)$ $n \geq 5$	$\frac{1}{3}v_1$	$\{\alpha_2, \alpha_3, \cdots, \alpha_n\}$	$\mathfrak{so}(2n-2) \oplus \mathfrak{t}^1$
	$\frac{1}{3}v_n$	$\{\alpha_1, \alpha_2, \cdots, \alpha_{n-1}\}$	$\mathfrak{su}(n) \oplus \mathfrak{t}^1$
	$\frac{2}{3}v_i$ $2 \leq i \leq n-3$	$\{\alpha_1, \cdots, \alpha_{i-1},$ $\alpha_{i+1}, \cdots, \alpha_n\}$	$\mathfrak{su}(i) \oplus \mathfrak{so}(2n-2i)$ $\oplus \mathfrak{t}^1$
	$\frac{1}{3}(v_{n-1} + v_n)$	$\{\alpha_1, \alpha_2, \cdots, \alpha_{n-2}\}$	$\mathfrak{su}(n-1) \oplus \mathfrak{t}^2$

Table 2

$\mathfrak{g}$	x	Ψ_x	$\mathfrak{g}^\theta$
$\mathfrak{g}_2$	v_1	$\{\alpha_2, -\mu\}$	$\mathfrak{su}(3)$
	$\frac{2}{3}v_2$	$\{\alpha_1\}$	$\mathfrak{su}(2) \oplus \mathfrak{t}^1$
$\mathfrak{f}_4$	$\frac{2}{3}v_1$	$\{\alpha_2, \alpha_3, \alpha_4\}$	$\mathfrak{so}(7) \oplus \mathfrak{t}^1$
	v_3	$\{\alpha_1, \alpha_2, \alpha_4, -\mu\}$	$\mathfrak{su}(3) \oplus \mathfrak{su}(3)$
	$\frac{2}{3}v_4$	$\{\alpha_1, \alpha_2, \alpha_3\}$	$\mathfrak{sp}(3) \oplus \mathfrak{t}^1$
$\mathfrak{e}_6$	$\frac{1}{3}v_1$	$\{\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$	$\mathfrak{so}(10) \oplus \mathfrak{t}^1$
	$\frac{2}{3}v_3$	$\{\alpha_1, \alpha_2, \alpha_4, \alpha_5, \alpha_6\}$	$\mathfrak{su}(2) \oplus \mathfrak{su}(5) \oplus \mathfrak{t}^1$
	$\frac{2}{3}v_2$	$\{\alpha_1, \alpha_3, \cdots, \alpha_6\}$	$\mathfrak{su}(6) \oplus \mathfrak{t}^1$
	v_4	$\{\alpha_1, \alpha_2, \alpha_3, \alpha_5, \alpha_6, -\mu\}$	$\mathfrak{su}(3) \oplus \mathfrak{su}(3) \oplus \mathfrak{su}(3)$
	$\frac{1}{3}(v_1 + v_6)$	$\{\alpha_2, \alpha_3, \alpha_4, \alpha_5\}$	$\mathfrak{so}(8) \oplus \mathfrak{t}^2$
$\mathfrak{e}_7$	$\frac{1}{3}v_1$	$\{\alpha_2, \cdots, \alpha_7\}$	$\mathfrak{e}_6 \oplus \mathfrak{t}^1$
	$\frac{2}{3}v_2$	$\{\alpha_1, \alpha_3, \cdots, \alpha_7\}$	$\mathfrak{su}(2) \oplus \mathfrak{so}(10) \oplus \mathfrak{t}^1$
	$\frac{2}{3}v_6$	$\{\alpha_1, \cdots, \alpha_5, \alpha_7\}$	$\mathfrak{so}(12) \oplus \mathfrak{t}^1$
	$\frac{2}{3}v_7$	$\{\alpha_1, \cdots, \alpha_6\}$	$\mathfrak{su}(7) \oplus \mathfrak{t}^1$
	v_3	$\{\alpha_1, \alpha_2, \alpha_4, \alpha_5,$ $\alpha_6, \alpha_7, -\mu\}$	$\mathfrak{su}(3) \oplus \mathfrak{su}(6)$
$\mathfrak{e}_8$	$\frac{2}{3}v_1$	$\{\alpha_2, \cdots, \alpha_8\}$	$\mathfrak{so}(14) \oplus \mathfrak{t}^1$
	$\frac{2}{3}v_7$	$\{\alpha_1, \cdots, \alpha_6, \alpha_8\}$	$\mathfrak{e}_7 \oplus \mathfrak{t}^1$
	v_6	$\{\alpha_7, -\mu,$ $\alpha_1, \cdots, \alpha_5, \alpha_8\}$	$\mathfrak{su}(3) \oplus \mathfrak{e}_6$
	v_8	$\{\alpha_1, \cdots, \alpha_7, -\mu\}$	$\mathfrak{su}(9)$

Table 3

$\mathfrak{g}$	$\mathfrak{k} = \mathfrak{g}^\theta$
$\mathfrak{so}(8)$	$\mathfrak{g}_2$
	$\mathfrak{su}(3)$

2.2 Second homotopy groups of compact Riemannian 3-symmetric spaces

In order to calculate the Stiefel-Whitney classes of compact homogeneous spaces G/K by the method of A. Borel and F. Hirzebruch, we have to know explicitly the cohomology ring $H^*(G/K, \mathbb{Z}_2)$ in terms of the roots of G. In this section, for the purpose to determine which Riemannian 3-symmetric spaces admit a spin structure, we shall calculate the second cohomology groups of compact Riemannian 3-symmetric spaces G/K in terms of the roots of G, namely prove the following.

Theorem 2.2.1. *Let $M = G/K$ be a connected simply connected irreducible compact Riemannian 3-symmetric space with a left-invariant Riemannian metric, where G is a compact connected centerless simple Lie group and K is the connected Lie subgroup of G with Lie algebra $\mathfrak{k} = \mathfrak{g}^\theta$ for some automorphism θ of $\mathfrak{g}$ of order 3. Then K, the second homotopy group $\pi_2(M)$ and the second cohomology group $H^2(M, \mathbb{Z}_2)$ are given by the following Table4, Table5 and Table6.*

In the sequel, we shall prove this theorem.

By the universal coefficient theorem, we have an exact sequence

$$0 \longrightarrow Ext(H_1(M,\mathbb{Z}),\mathbb{Z}_2) \longrightarrow H^2(M,\mathbb{Z}_2) \longrightarrow Hom(H_2(M,\mathbb{Z}),\mathbb{Z}_2) \longrightarrow 0.$$

Since M is simply connected, we have $H_1(M,\mathbb{Z}) = 0$. Hence we have

$$H^2(M,\mathbb{Z}_2) \cong Hom(H_2(M,\mathbb{Z}),\mathbb{Z}_2).$$

Since M is 1-connected, by Hurewicz Theorem (cf. Whitehead [60], p.169), we have

$$H_2(M,\mathbb{Z}) \cong \pi_2(M).$$

So, in order to prove Theorem 2.2.1, we have only to calculate the second homotopy group $\pi_2(M)$.

The homotopy exact sequence of the principal K-bundle $(G, K, M = G/K)$ is as follows:

$$(2.2.1) \qquad \pi_2(G) \longrightarrow \pi_2(G/K) \xrightarrow{f} \pi_1(K) \xrightarrow{h} \pi_1(G) \longrightarrow \pi_1(G/K) \longrightarrow \pi_0(K).$$

Table 4

G	K	$\pi_2(G/K)$	$H^2(G/K, \mathbb{Z}_2)$
$SU(n)/\mathbb{Z}_n$ $(n \geq 2)$	$S\{U(r_1) \times U(r_2) \times U(r_3)\}/\mathbb{Z}_n$ $0 \leq r_1 \leq r_2 \leq r_3$, $0 < r_2$, $r_1 + r_2 + r_3 = n$	$\mathbb{Z} \times \mathbb{Z}$ if $r_1 = 0,\ n = 2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
		$\mathbb{Z}$ if $r_1 = 0, n \geq 3$	$\mathbb{Z}_2$
		$\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ if $r_1 > 0, n = 3$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
		$\mathbb{Z} \times \mathbb{Z}$ if $r_1 > 0, n \geq 4$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
$SO(2n+1)$ $(n \geq 1)$	$U(r) \times SO(2n-2r+1)$ $(1 \leq r \leq n)$	$\mathbb{Z}$	$\mathbb{Z}_2$
$Sp(n)/\mathbb{Z}_2$ $(n \geq 1)$	$\{U(r) \times Sp(n-r)\}/\mathbb{Z}_2$ $(1 \leq r \leq n)$	$\mathbb{Z}$	$\mathbb{Z}_2$
$SO(2n)/\mathbb{Z}_2$ $(n \geq 3)$	$\{U(r) \times SO(2n-2r)\}/\mathbb{Z}_2$ $(1 \leq r \leq n)$	$\mathbb{Z} \times \mathbb{Z}$ if $r = n-1$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
		$\mathbb{Z}$ if $1 \leq r < n-1$	$\mathbb{Z}_2$
		$\mathbb{Z}$ if $r = n$	$\mathbb{Z}_2$

Table 5

G	K	$\pi_2(G/K)$	$H^2(G/K,\mathbb{Z}_2)$
G_2	$U(2)$	$\mathbb{Z}$	$\mathbb{Z}_2$
F_4	$\{Spin(7)\times T^1\}/\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}_2$
	$\{Sp(3)\times T^1\}/\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}_2$
$E_6/\mathbb{Z}_3$	$\{Spin(10)\times SO(2)\}/\mathbb{Z}_4$	$\mathbb{Z}_4\times\mathbb{Z}$	$\mathbb{Z}_2\times\mathbb{Z}_2$
	$\{[S(U(5)\times U(1))/\mathbb{Z}_3]\times SU(2)\}/\mathbb{Z}_2$	$\mathbb{Z}_2\times\mathbb{Z}_5\times\mathbb{Z}$	$\mathbb{Z}_2\times\mathbb{Z}_2$
	$\{[SU(6)/\mathbb{Z}_3]\times T^1\}/\mathbb{Z}_2$	$\mathbb{Z}_2\times\mathbb{Z}$	$\mathbb{Z}_2\times\mathbb{Z}_2$
	$\{[Spin(8)\times SO(2)]/\mathbb{Z}_2\times SO(2)\}/\mathbb{Z}_2$	$\mathbb{Z}_3\times\mathbb{Z}_2\times\mathbb{Z}_2$ $\times\mathbb{Z}\times\mathbb{Z}$	$\mathbb{Z}_2\times\mathbb{Z}_2$ $\times\mathbb{Z}_2\times\mathbb{Z}_2$
$E_7/\mathbb{Z}_2$	$\{E_6\times T^1\}/\mathbb{Z}_3$	$\mathbb{Z}_3\times\mathbb{Z}$	$\mathbb{Z}_2$
	$\{[SU(2)\times(Spin(10)\times SO(2))/\mathbb{Z}_2]/\mathbb{Z}_2\}/\mathbb{Z}_2$	$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}$	$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$
	$\{[SO(2)\times Spin(12)]/\mathbb{Z}_2\}/\mathbb{Z}_2$	$\mathbb{Z}_2\times\mathbb{Z}$	$\mathbb{Z}_2\times\mathbb{Z}_2$
	$S\{U(7)\times U(1)\}/\mathbb{Z}_2$	$\mathbb{Z}_2\times\mathbb{Z}$	$\mathbb{Z}_2\times\mathbb{Z}_2$
E_8	$SO(14)\times SO(2)$	$\mathbb{Z}_2\times\mathbb{Z}$	$\mathbb{Z}_2\times\mathbb{Z}_2$
	$\{E_7\times T^1\}/\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}_2$
G_2	$SU(3)$	0	0
F_4	$\{SU(3)\times SU(3)\}/\mathbb{Z}_3$	$\mathbb{Z}_3$	0
$E_6/\mathbb{Z}_3$	$\{SU(3)\times SU(3)\times SU(3)\}/\{\mathbb{Z}_3\times\mathbb{Z}_3\}$	$\mathbb{Z}_3$	0
$E_7/\mathbb{Z}_2$	$\{SU(3)\times[SU(6)/\mathbb{Z}_2]\}/\mathbb{Z}_3$	$\mathbb{Z}_3$	0
E_8	$\{SU(3)\times E_6\}/\mathbb{Z}_3$	$\mathbb{Z}_3$	0
	$SU(9)/\mathbb{Z}_3$	$\mathbb{Z}_3$	0

Table 6

G	K	$\pi_2(G/K)$	$H^2(G/K,\mathbb{Z}_2)$
$Spin(8)$	$SU(3)/\mathbb{Z}_3$	$\mathbb{Z}_3$	0
	G_2	0	0
$\{L\times L\times L\}/Z$ where L is compact simple and simply connected and Z is its center embedded diagonally.	L/Z where L is embedded diagonally in $L\times L\times L$ and Z is its center.	0	0

Let $\widetilde{G}$ and $Z(\widetilde{G})$ be the universal covering group of G and the center of $\widetilde{G}$, respectively. Then G is isomorphic to the quotient group $\widetilde{G}/Z(\widetilde{G})$. Since the second homotopy group of a simply connected compact simple Lie group $\widetilde{G}$ is trivial and $\pi_2(G) \cong \pi_2(\widetilde{G})$, the homomorphism f is injective and $\pi_2(G/K) \cong \mathrm{Im} f = \ker h$. So we shall calculate the kernel of the homomorphism h.

Now we shall express $\pi_1(G) \cong Z(\widetilde{G})$ in terms of the roots of $\widetilde{G}$. Let T and $\mathfrak{t}$ be a maximal torus of $\widetilde{G}$ and the Lie algebra of T, respectively. We denote by $\Psi = \{\alpha_1, \cdots, \alpha_l\}$ the simple root system of $\mathfrak{g}$ with respect to $\mathfrak{t}$, and by $\exp : \mathfrak{g} \longrightarrow \widetilde{G}$ the exponential map. The central lattice Λ_1 and the unit lattice $\Lambda(\widetilde{G})$ of $\widetilde{G}$ are defined by

$$\Lambda_1(\widetilde{G}) = \exp^{-1}(Z(\widetilde{G})),$$
$$\Lambda(\widetilde{G}) = \exp^{-1}(e),$$

respectively, where e denotes the identity element of $\widetilde{G}$. We choose an $\mathrm{Ad}(\widetilde{G})$-invariant inner product $(\ ,\ )$ on $\mathfrak{g}$. For each linear form $a \in \mathfrak{t}^*$, the element $\vec{a} \in \mathfrak{t}$ is defined by

$$(\vec{a}, v) = a(v) \quad \text{for any} \quad v \in \mathfrak{t},$$

and for each root α, we define $\alpha^* \in \mathfrak{t}$ by

$$\alpha^* = \frac{2\vec{\alpha}}{(\alpha, \alpha)},$$

where the inner product (a, b) of two linear forms a and b is defined by $(a, b) = (\vec{a}, \vec{b})$. Then we have the following proposition (cf. [19] p.479).

Proposition 2.2.2. *Let $\widetilde{G}$ be a compact semisimple Lie group and $\Psi = \{\alpha_1, \cdots, \alpha_l\}$ the simple root system of $\widetilde{G}$ with respect to a maximal torus T of $\widetilde{G}$. Then*

(1) $Z(\widetilde{G}) \cong \Lambda_1(\widetilde{G})/\Lambda(\widetilde{G})$.

(2) $\Lambda_1(\widetilde{G}) = \{v \in \mathfrak{t} \mid \alpha_j(v) \in \mathbb{Z}, \text{ for any } j = 1, \ldots, l\ \}$.

(3) Furthermore, if $\widetilde{G}$ is simply connected, then $\Lambda(\widetilde{G}) = \mathbb{Z}{\alpha_1}^* + \cdots + \mathbb{Z}{\alpha_l}^*$.

By a straightforward calculation, we have

Proposition 2.2.3. *The centers of $SU(n), Spin(n), Sp(n), G_2, F_4, E_6, E_7$ and E_8 are given as follows;*

$$\begin{aligned}
Z(SU(n)) &= \{\exp\left(\frac{j}{n}\sum_{i=1}^{n-1} i\alpha_i{}^*\right) \mid j = 0, 1, \cdots, n-1\},\\
Z(Spin(2n+1)) &= Z(Spin(2n))\\
&= \{\exp\left(\frac{j}{2}\sum_{i=1}^{n-2} i\alpha_i{}^* + \frac{j}{4}\left(n\alpha_{n-1}{}^* + (n-2)\alpha_n{}^*\right)\right.\\
&\quad \left. + \frac{k(n-1)}{2}\left(\alpha_{n-1}{}^* + \alpha_n{}^*\right)\right) \mid j = 0, 1, 2, 3, \quad k = 0, 1\},\\
Z(Sp(n)) &= \{e\},\\
Z(G_2) &= \{e\},\\
Z(F_4) &= \{e\},\\
Z(E_6) &= \{\exp\left(\frac{j}{3}\left(\alpha_1{}^* + 2\alpha_3{}^* + \alpha_5{}^* + 2\alpha_6{}^*\right)\right) \mid j = 0, 1, 2\},\\
Z(E_7) &= \{\exp\left(\frac{j}{2}\left(\alpha_1{}^* + \alpha_3{}^* + \alpha_7{}^*\right)\right) \mid j = 0, 1\},\\
Z(E_8) &= \{e\}.
\end{aligned}$$

In the case where $\widetilde{G}$ is a classical Lie group or $Z(\widetilde{G}) = 1$, then we may easily calculate $\pi_2(G/K)$. So we shall deal with the case where $\widetilde{G} = E_6$ or E_7.

First we shall show the following lemma.

Lemma 2.2.4. *Let $\mathfrak{k}$ be the Lie algebra of a connected Lie group $\widetilde{K}$. Suppose $\mathfrak{k}$ is a direct sum $\mathfrak{k}_1 \oplus \mathfrak{k}_2$ of two ideals $\mathfrak{k}_1$ and $\mathfrak{k}_2$. We denote by $\widetilde{K}_i$ the connected Lie subgroup of $\widetilde{K}$ of Lie algebra $\mathfrak{k}_i$ $(i = 1, 2)$. Then $\widetilde{K}$ is isomorphic to the quotient group $\widetilde{K}_1 \times \widetilde{K}_2 / \widetilde{K}_1 \cap \widetilde{K}_2$.*

Proof. For any $X \in \mathfrak{k}_1, Y \in \mathfrak{k}_2$,

$$\begin{aligned}
\exp Y \exp X (\exp Y)^{-1} &= \exp(Ad(\exp Y)X)\\
&= \exp(e^{ad(Y)}X)\\
&= \exp X.
\end{aligned}$$

Hence we have $k_1k_2 = k_2k_1$, for any $k_1 \in \widetilde{K}_1, k_2 \in \widetilde{K}_2$. We consider the homomorphism $\pi : \widetilde{K}_1 \times \widetilde{K}_2 \longrightarrow \widetilde{K}$ defined by $\pi(k_1, k_2) = k_1k_2$. Since

$$\begin{aligned}
\ker \pi &= \{(k_1, k_2) \in \widetilde{K}_1 \times \widetilde{K}_2 \mid k_1k_2 = e\} \\
&= \{(k, k^{-1}) \in \widetilde{K}_1 \times \widetilde{K}_2 \mid k \in \widetilde{K}_1 \cap \widetilde{K}_2\} \\
&\cong \widetilde{K}_1 \cap \widetilde{K}_2,
\end{aligned}$$

we obtain the lemma.

In the sequel, we shall adopt the following notation. Let $p : \widetilde{G} \longrightarrow G$ be the universal covering group of compact Lie group G, and $\widetilde{K}$ (resp. K) the connected Lie subgroup of $\widetilde{G}$ (resp. G) generated by the Lie subalgebra $\mathfrak{k}$. We denote by $\pi : \bar{K} \longrightarrow \widetilde{K}$ the universal covering group of $\widetilde{K}$. Let $\bar{\gamma} : I \longrightarrow \bar{K}$ be a path with $\bar{\gamma}(1) \in (p \circ \pi)^{-1}(e)$. We define a loop γ at e in K by $\gamma = p \circ \pi \circ \bar{\gamma}$. By the unique lifting property, the curve $\widetilde{\gamma} := \pi \circ \bar{\gamma}$ is the lifting of γ starting at the identity of $\widetilde{K}$.

Case (E6-1) $\mathfrak{g} = \mathfrak{e}_6$, $x = \frac{1}{3}v_1$.

Take a direct sum decomposition of $\mathfrak{k}$ by the following two ideals;

$$\begin{aligned}
\mathfrak{k}_1 &= [\mathfrak{k}, \mathfrak{k}] \cong \mathfrak{so}(10), \\
\mathfrak{k}_2 &= \mathbb{R}(4\alpha_1{}^* + 3\alpha_2{}^* + 5\alpha_3{}^* + 6\alpha_4{}^* + 4\alpha_5{}^* + 2\alpha_6{}^*).
\end{aligned}$$

Put

$$\begin{aligned}
v_1 &= \frac{1}{2}(\alpha_2{}^* + \alpha_3{}^*), \\
w_1 &= \frac{1}{4}(3\alpha_2{}^* + 5\alpha_3{}^* + 2\alpha_4{}^* + 2\alpha_6{}^*), \\
v_2 &= 4\alpha_1{}^* + 3\alpha_2{}^* + 5\alpha_3{}^* + 6\alpha_4{}^* + 4\alpha_5{}^* + 2\alpha_6{}^*.
\end{aligned}$$

Then $\{w_1\}$ forms a basis of $\Lambda_1(\widetilde{K}_1)$. We have

$$\begin{aligned}
Z(\widetilde{K}_1) &= \{\exp(kw_1) \mid k = 0, 1, 2, 3\} \cong \mathbb{Z}_4, \\
\widetilde{K}_1 &= Spin(10).
\end{aligned}$$

Since the intersection $\widetilde{K}_1 \cap \widetilde{K}_2$ is equal to $\{\exp \frac{k}{4} v_2 \mid k = 0, 1, 2, 3\}$, we have

$$\widetilde{K} = \{Spin(10) \times SO(2)\}/\mathbb{Z}_4.$$

If we put $\Gamma = Z(\widetilde{G}) \cap \widetilde{K}$, then K is isomorphic to $\widetilde{K}/\Gamma$. In our case,

$$\begin{aligned} K &\cong \{[Spin(10) \times SO(2)]/\mathbb{Z}_4\}/\mathbb{Z}_3 \\ &= \{Spin(10) \times [SO(2)/\mathbb{Z}_3]\}/\mathbb{Z}_4 \\ &= \{Spin(10) \times SO(2)\}/\mathbb{Z}_4. \end{aligned}$$

Thus we have $\pi_1(K) = \mathbb{Z}_3 \times \mathbb{Z}_4 \times \mathbb{Z}$. We define paths $\bar{\gamma}_j (j = 1, 2, 3)$ in $\bar{K} = Spin(10) \times \mathbb{R}$ by

$$\begin{aligned} \bar{\gamma}_1(t) &= (e, \frac{t}{3} v_2), \\ \bar{\gamma}_2(t) &= (\exp(t w_1), 0), \\ \bar{\gamma}_3(t) &= (e, t v_2), \end{aligned}$$

so that the corresponding paths $\widetilde{\gamma}_1, \widetilde{\gamma}_2$ and $\widetilde{\gamma}_3$ represent the generators $(1,0,0), (0,1,0)$ and $(0,0,1)$ of $\pi_1(\widetilde{K})$ respectively. It is easily seen that γ_2 and γ_3 are null-homotopic and γ_1 is not. Therefore we have $\pi_2(G/K) \cong \ker h = \mathbb{Z}_4 \times \mathbb{Z}$.

Case (E6-2) $\mathfrak{g} = \mathfrak{e}_6, \quad x = \dfrac{2}{3} v_3.$

Take a direct sum decomposition of $\mathfrak{k}$ by the following two ideals;

$$\begin{aligned} \mathfrak{k}_1 &= [\mathfrak{k}, \mathfrak{k}] \cong \mathfrak{su}(2) \oplus \mathfrak{su}(5), \\ \mathfrak{k}_2 &= \mathbb{R}(5{\alpha_1}^* + 6{\alpha_2}^* + 10{\alpha_3}^* + 12{\alpha_4}^* + 8{\alpha_5}^* + 4{\alpha_6}^*). \end{aligned}$$

Put

$$\begin{aligned} v_1 &= \frac{1}{2}{\alpha_1}^*, \\ w_1 &= \frac{1}{5}(4{\alpha_2}^* + 3{\alpha_4}^* + 2{\alpha_5}^* + {\alpha_6}^*), \\ v_2 &= 5{\alpha_1}^* + 6{\alpha_2}^* + 10{\alpha_3}^* + 12{\alpha_4}^* + 8{\alpha_5}^* + 4{\alpha_6}^*. \end{aligned}$$

Then $\{v_1, w_1\}$ forms a basis of $\Lambda_1(\widetilde{K}_1)$. We have

$$\begin{aligned} Z(\widetilde{K}_1) &= \{\exp(jv_1) \mid j = 0,1\} \times \{\exp(kw_1) \mid k = 0,1,2,3,4\} \\ &\cong \mathbb{Z}_2 \times \mathbb{Z}_5 \\ &\cong Z(SU(2) \times SU(5)), \\ \widetilde{K}_1 &\cong SU(2) \times SU(5). \end{aligned}$$

Since the intersection $\widetilde{K}_1 \cap \widetilde{K}_2$ is equal to $\{\exp \frac{k}{10} v_2 \mid k = 0, 1, \cdots, 9\} = \{\exp \frac{j}{5} v_2 \mid j = 0, 1, 2, 3, 4\} \times \{\exp \frac{k}{2} v_2 \mid k = 0, 1\}$, we have

$$\begin{aligned} \widetilde{K} &\cong \{SU(2) \times [SU(5) \times U(1)]/\mathbb{Z}_5\}/\mathbb{Z}_2 \\ &\cong \{SU(2) \times S(U(5) \times U(1))\}/\mathbb{Z}_2. \end{aligned}$$

If we put $\Gamma = Z(\widetilde{G}) \cap \widetilde{K}$, then K is isomorphic to $\widetilde{K}/\Gamma$. In our case,

$$\begin{aligned} K &\cong \{[SU(2) \times S(U(5) \times U(1))]/\mathbb{Z}_2\}/\mathbb{Z}_3 \\ &= \{SU(2) \times [S(U(5) \times U(1))/\mathbb{Z}_3]\}/\mathbb{Z}_2. \end{aligned}$$

Thus we have $\pi_1(K) = \mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_5 \times \mathbb{Z}$. We define paths $\bar{\gamma}_j (j = 1, 2, 3, 4)$ in $\bar{K} = \{SU(2) \times SU(5)\} \times \mathbb{R}$ by

$$\begin{aligned} \bar{\gamma}_1(t) &= (e, \frac{2t}{3} v_2), \\ \bar{\gamma}_2(t) &= (\exp \frac{1}{2} v_2, \frac{-t}{2} v_2), \\ \bar{\gamma}_3(t) &= (\exp \frac{1}{5} v_2, \frac{-t}{5} v_2), \\ \bar{\gamma}_4(t) &= (e, t v_2), \end{aligned}$$

so that the corresponding paths $\widetilde{\gamma}_1$, $\widetilde{\gamma}_2$, $\widetilde{\gamma}_3$ and $\widetilde{\gamma}_4$ represent the generators $(1,0,0,0)$, $(0,1,0,0)$, $(0,0,1,0)$ and $(0,0,0,1)$ of $\pi_1(\widetilde{K})$ respectively. It is easily seen that γ_2, γ_3 and γ_4 are null-homotopic and γ_1 is not. Therefore we have $\pi_2(G/K) \cong \ker h = \mathbb{Z}_2 \times \mathbb{Z}_5 \times \mathbb{Z}$.

Case (E6-3) $\mathfrak{g} = \mathfrak{e}_6, \quad x = \dfrac{2}{3} v_2.$

Take a direct sum decomposition of $\mathfrak{k}$ by the following two ideals:

$$\mathfrak{k}_1 = [\mathfrak{k}, \mathfrak{k}] \cong \mathfrak{su}(6),$$

$$\mathfrak{k}_2 = \mathbb{R}\left(\alpha_1{}^* + 2\alpha_2{}^* + 2\alpha_3{}^* + 3\alpha_4{}^* + 2\alpha_5{}^* + \alpha_6{}^*\right).$$

Put

$$v_1 = \frac{1}{6}(5\alpha_1{}^* + 4\alpha_3{}^* + 3\alpha_4{}^* + 2\alpha_5{}^* + \alpha_6{}^*) \in \mathfrak{k}_1,$$
$$v_2 = \alpha_1{}^* + 2\alpha_2{}^* + 2\alpha_3{}^* + 3\alpha_4{}^* + 2\alpha_5{}^* + \alpha_6{}^* \in \mathfrak{k}_2.$$

Then $\{v_1\}$ forms a basis of $\Lambda_1(\widetilde{K}_1)$. We have

$$\begin{aligned} Z(\widetilde{K}_1) &= \exp \Lambda_1(\widetilde{K}_1) \\ &= \{\exp(jv_1) \mid j = 0, 1, \cdots, 5\} \\ &\cong \mathbb{Z}_6 \cong Z(SU(6)), \\ \widetilde{K}_1 &\cong SU(6). \end{aligned}$$

Since the intersection $\widetilde{K}_1 \cap \widetilde{K}_2$ is equal to $\{\exp(\frac{j}{2}v_2) \mid j = 0, 1\} \cong \mathbb{Z}_2$, we have

$$\widetilde{K} \cong \{SU(6) \times T^1\}/\mathbb{Z}_2.$$

If we put $\Gamma = Z(\widetilde{G}) \cap \widetilde{K}$, then K is isomorphic to $\widetilde{K}/\Gamma$. In our case,

$$K \cong \{[SU(6)/\mathbb{Z}_3] \times T^1\}/\mathbb{Z}_2.$$

Thus we have $\pi_1(K) = \mathbb{Z} \times \mathbb{Z}_3 \times \mathbb{Z}_2$. We define paths $\bar{\gamma}_j \quad (j = 1, 2, 3)$ in $\bar{K} = SU(6) \times \mathbb{R}$ by

$$\begin{aligned} \bar{\gamma}_1(t) &= (e, tv_2), \\ \bar{\gamma}_2(t) &= (\exp(2tv_1), 0), \\ \bar{\gamma}_3(t) &= (\exp \frac{1}{2}v_2, -\frac{t}{2}v_2), \end{aligned}$$

so that the corresponding paths $\widetilde{\gamma}_1, \widetilde{\gamma}_2$ and $\widetilde{\gamma}_3$ represent the generators $(1,0,0),(0,1,0)$ and $(0,0,1)$ of $\pi_1(\widetilde{K})$ respectively. It is easily seen that γ_1 and γ_3 are null-homotopic and γ_2 is not. Therefore we have $\pi_2(G/K) \cong \ker h = \mathbb{Z} \times \mathbb{Z}_2$.

Case (E6-4) $\quad \mathfrak{g} = \mathfrak{e}_6, \quad x = v_4$.

The center of $\mathfrak{k}$ is 0, and $\mathfrak{k}$ is semisimple. We denote by $\alpha_0 = -\mu$ the negative of the maximal root. Then we have

$$\begin{aligned} Z(\widetilde{K}) &= \{\exp \frac{j}{3}({\alpha_1}^* + 2{\alpha_3}^*) \mid j = 0,1,2\} \times \{\exp \frac{k}{3}({\alpha_5}^* + 2{\alpha_6}^*) \mid k = 0,1,2\} \\ &\cong \mathbb{Z}_3 \times \mathbb{Z}_3, \\ \widetilde{K} &\cong \{SU(3) \times SU(3) \times SU(3)\}/\mathbb{Z}_3. \end{aligned}$$

If we put $\Gamma = Z(\widetilde{G}) \cap \widetilde{K}$, then K is isomorphic to $\widetilde{K}/\Gamma$. In our case

$$K \cong \{SU(3) \times SU(3) \times SU(3)\}/\{\mathbb{Z}_3 \times \mathbb{Z}_3\}.$$

Thus we have $\pi_1(K) \cong \mathbb{Z}_3 \times \mathbb{Z}_3$. We define paths $\bar{\gamma}_j$ $(j = 1,2)$ in $\bar{K} = SU(3) \times SU(3) \times SU(3)$ by

$$\begin{aligned} \bar{\gamma}_1(t) &= (\exp \frac{t}{3}({\alpha_1}^* + 2{\alpha_3}^*), \exp \frac{t}{3}({\alpha_0}^* + 2{\alpha_2}^*), \exp \frac{2t}{3}({\alpha_5}^* + 2{\alpha_6}^*)), \\ \bar{\gamma}_2(t) &= (\exp \frac{t}{3}({\alpha_1}^* + 2{\alpha_3}^*), e, \exp \frac{t}{3}({\alpha_5}^* + 2{\alpha_6}^*)), \end{aligned}$$

so that the corresponding paths $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ represent the generators $(1,0)$ and $(0,1)$ of $\pi_1(\widetilde{K})$ respectively. It is easily seen that γ_1 is null-homotopic and γ_2 is not. Therefore we have $\pi_2(G/K) \cong \ker h = \mathbb{Z}_3$.

Case (E6-5) $\quad \mathfrak{g} = \mathfrak{e}_6, \quad x = \frac{1}{3}(v_1 + v_6)$.

Take a direct sum decomposition of $\mathfrak{k}$ by the following two ideals:

$$\begin{aligned} \mathfrak{k}_1 &= [\mathfrak{k}, \mathfrak{k}] \cong \mathfrak{so}(8), \\ \mathfrak{k}_2 &= \mathbb{R}(4{\alpha_1}^* + {\alpha_2}^* + 3{\alpha_3}^* + 2{\alpha_4}^* - 2{\alpha_6}^*) \\ &\quad \oplus \mathbb{R}(-2{\alpha_1}^* - {\alpha_3}^* + {\alpha_5}^* + 2{\alpha_6}^*). \end{aligned}$$

Put

$$v_1 = \frac{1}{2}({\alpha_2}^* + {\alpha_3}^*),$$
$$w_1 = \frac{1}{2}({\alpha_2}^* + {\alpha_5}^*),$$
$$v_2 = 4{\alpha_1}^* + {\alpha_2}^* + 3{\alpha_3}^* + 2{\alpha_4}^* - 2{\alpha_6}^*,$$
$$w_2 = -2{\alpha_1}^* - {\alpha_3}^* + {\alpha_5}^* + 2{\alpha_6}^*.$$

Then $\{v_1, w_1\}$ forms a basis of $\Lambda_1(\widetilde{K}_1)$. We have

$$\begin{aligned} Z(\widetilde{K}_1) &= \{\exp(jv_1) \mid j = 0,1\} \times \{\exp(kw_1) \mid k = 0,1\} \\ &\cong \mathbb{Z}_2 \times \mathbb{Z}_2 \\ &\cong Z(Spin(8)), \\ \widetilde{K}_1 &\cong Spin(8). \end{aligned}$$

Since the intersection $\widetilde{K}_1 \cap \widetilde{K}_2$ is equal to $\{\exp \frac{j}{2}v_2 \mid j = 0,1\} \times \{\exp \frac{k}{2}(v_2 + w_2) \mid k = 0,1\}$, we have

$$\widetilde{K} \cong \{[Spin(8) \times SO(2)]/\mathbb{Z}_2 \times SO(2)\}/\mathbb{Z}_2.$$

If we put $\Gamma = Z(\widetilde{G}) \cap \widetilde{K}$, then K is isomorphic to $\widetilde{K}/\Gamma$. In our case,

$$\begin{aligned} K &\cong \{\{[Spin(8) \times SO(2)]/\mathbb{Z}_2 \times SO(2)\}/\mathbb{Z}_2\}/\mathbb{Z}_3 \\ &= \{[Spin(8) \times SO(2)]/\mathbb{Z}_2 \times [SO(2)/\mathbb{Z}_3]\}/\mathbb{Z}_2 \\ &= \{[Spin(8) \times SO(2)]/\mathbb{Z}_2 \times SO(2)\}/\mathbb{Z}_2. \end{aligned}$$

Thus we have $\pi_1(K) \cong \mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z} \times \mathbb{Z}$. We define paths $\bar{\gamma}_j (j = 1, \cdots, 5)$ in $\bar{K} = Spin(8) \times \mathbb{R} \times \mathbb{R}$ by

$$\begin{aligned} \bar{\gamma}_1(t) &= (\exp(v_1 + w_1), 0, -\frac{t}{6}w_2), \\ \bar{\gamma}_2(t) &= (\exp v_1, -\frac{t}{2}v_2, 0), \\ \bar{\gamma}_3(t) &= (\exp w_1, -\frac{t}{2}v_2, -\frac{t}{2}w_2), \\ \bar{\gamma}_4(t) &= (e, tv_2, 0), \\ \bar{\gamma}_5(t) &= (e, 0, tw_2), \end{aligned}$$

so that the corresponding paths $\widetilde{\gamma}_1$, $\widetilde{\gamma}_2$, $\widetilde{\gamma}_3$, $\widetilde{\gamma}_4$ and $\widetilde{\gamma}_5$ represent the generators $(1,0,0,0,0)$, $(0,1,0,0,0)$, $(0,0,1,0,0)$, $(0,0,0,1,0)$ and $(0,0,0,0,1)$ of $\pi_1(\widetilde{K})$ respectively. It is easily seen that $\gamma_2, \gamma_3, \gamma_4$ and γ_5 are null-homotopic and γ_1 is not. Therefore we have $\pi_2(G/K) \cong \ker h = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z} \times \mathbb{Z}$.

Case (E7-1) $\quad \mathfrak{g} = \mathfrak{e}_7, \quad x = \dfrac{1}{3} v_1.$

Take a direct sum decomposition of $\mathfrak{k}$ by the following two ideals:

$$\mathfrak{k}_1 = [\mathfrak{k}, \mathfrak{k}] \cong \mathfrak{e}_6,$$
$$\mathfrak{k}_2 = \mathbb{R}(3\alpha_1{}^* + 4\alpha_2{}^* + 5\alpha_3{}^* + 6\alpha_4{}^* + 4\alpha_5{}^* + 2\alpha_6{}^* + 3\alpha_7{}^*).$$

Put

$$v_1 = \frac{1}{3}(\alpha_2{}^* + 2\alpha_3{}^* + \alpha_5{}^* + 2\alpha_6{}^*),$$
$$v_2 = (3\alpha_1{}^* + 4\alpha_2{}^* + 5\alpha_3{}^* + 6\alpha_4{}^* + 4\alpha_5{}^* + 2\alpha_6{}^* + 3\alpha_7{}^*).$$

Then $\{v_1\}$ forms a basis of $\Lambda_1(\widetilde{K}_1)$. We have

$$Z(\widetilde{K}_1) = \{\exp(jv_1) \mid j = 0,1,2\} \cong \mathbb{Z}_3 \cong Z(E_6),$$
$$\widetilde{K}_1 \cong E_6.$$

Since the intersection $\widetilde{K}_1 \cap \widetilde{K}_2$ is equal to $\{\exp \frac{k}{3} v_2 \mid k = 0,1,2\ \}$, we have

$$\widetilde{K} \cong \{E_6 \times T^1\}/\mathbb{Z}_3.$$

If we put $\Gamma = Z(\widetilde{G}) \cap \widetilde{K}$, then K is isomorphic to $\widetilde{K}/\Gamma$. In our case,

$$\begin{aligned} K &\cong \{[E_6 \times T^1]/\mathbb{Z}_3\}/\mathbb{Z}_2 \\ &= \{E_6 \times [T^1/\mathbb{Z}_2]\}/\mathbb{Z}_3 \\ &\cong \{E_6 \times T^1\}/\mathbb{Z}_3. \end{aligned}$$

Thus we have $\pi_1(K) \cong \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}$. We define paths $\bar{\gamma}_j (j = 1, 2, 3)$ in $\bar{K} = E_6 \times \mathbb{R}$ by

$$\begin{aligned}
\bar{\gamma}_1(t) &= (\exp \frac{1}{3}(\alpha_2{}^* + 2\alpha_3{}^* + \alpha_5{}^* + 2\alpha_6{}^*), \frac{t}{6} v_2), \\
\bar{\gamma}_2(t) &= (\exp \frac{1}{3}(\alpha_2{}^* + 2\alpha_3{}^* + \alpha_5{}^* + 2\alpha_6{}^*), -\frac{t}{3} v_2), \\
\bar{\gamma}_3(t) &= (e, t v_2),
\end{aligned}$$

so that the corresponding paths $\widetilde{\gamma}_1, \widetilde{\gamma}_2$ and $\widetilde{\gamma}_3$ represent the generators $(1,0,0), (0,1,0)$ and $(0,0,1)$ of $\pi_1(\widetilde{K})$ respectively. It is easily seen that γ_2 and γ_3 are null-homotopic and γ_1 is not. Therefore we have $\pi_2(G/K) \cong \ker h = \mathbb{Z}_3 \times \mathbb{Z}$.

Case (E7-2) $\quad \mathfrak{g} = \mathfrak{e}_7, \quad x = \frac{2}{3} v_2$.

Take a direct sum decomposition of $\mathfrak{k}$ by the following two ideals:

$$\begin{aligned}
\mathfrak{k}_1 &= [\mathfrak{k}, \mathfrak{k}] \cong \mathfrak{su}(2) \oplus \mathfrak{so}(10), \\
\mathfrak{k}_2 &= \mathbb{R}(2\alpha_1{}^* + 4\alpha_2{}^* + 5\alpha_3{}^* + 6\alpha_4{}^* + 4\alpha_5{}^* + 2\alpha_6{}^* + 3\alpha_7{}^*).
\end{aligned}$$

Put

$$\begin{aligned}
v_1 &= \frac{1}{2}\alpha_1{}^*, \\
w_1 &= \frac{1}{4}(\alpha_3{}^* + 2\alpha_4{}^* + 2\alpha_6{}^* + 3\alpha_7{}^*), \\
v_2 &= 2\alpha_1{}^* + 4\alpha_2{}^* + 5\alpha_3{}^* + 6\alpha_4{}^* + 4\alpha_5{}^* + 2\alpha_6{}^* + 3\alpha_7{}^*.
\end{aligned}$$

Then $\{v_1, w_1\}$ forms a basis of $\Lambda_1(\widetilde{K}_1)$. We have

$$\begin{aligned}
Z(\widetilde{K}_1) &= \{\exp(j v_1) \mid j = 0, 1\} \times \{\exp(k w_1) \mid k = 0, 1, 2, 3\} \\
&\cong \mathbb{Z}_2 \times \mathbb{Z}_4 \\
&\cong Z(SU(2) \times Spin(10)), \\
\widetilde{K}_1 &\cong SU(2) \times Spin(10).
\end{aligned}$$

Since the intersection $\widetilde{K}_1 \cap \widetilde{K}_2$ is equal to $\{\exp \frac{k}{4} v_2 \mid k = 0, 1, 2, 3\}$, we have

$$\begin{aligned}
\widetilde{K} &\cong \{[SU(2) \times Spin(10)] \times T^1\}/\mathbb{Z}_4 \\
&\cong \{SU(2) \times [Spin(10) \times T^1]/\mathbb{Z}_2\}/\mathbb{Z}_2.
\end{aligned}$$

If we put $\Gamma = Z(\widetilde{G}) \cap \widetilde{K}$, then K is isomorphic to $\widetilde{K}/\Gamma$. In our case,

$$K \cong \{[SU(2) \times (Spin(10) \times SO(2))/\mathbb{Z}_2]/\mathbb{Z}_2\}/\mathbb{Z}_2.$$

Thus we have $\pi_1(K) \cong \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}$. We define paths $\gamma_j (j = 1, 2, 3)$ in $\bar{K} = SU(2) \times Spin(10) \times \mathbb{R}$ by

$$\bar{\gamma}_1(t) = (\exp(v_1), \frac{t}{2} v_2),$$
$$\bar{\gamma}_2(t) = (\exp(v_1 + w_1), -\frac{t}{4} v_2),$$
$$\bar{\gamma}_3(t) = (e, t v_2),$$

so that the corresponding paths $\widetilde{\gamma}_1, \widetilde{\gamma}_2$ and $\widetilde{\gamma}_3$ represent the generators $(1,0,0), (0,1,0)$ and $(0,0,1)$ of $\pi_1(\widetilde{K})$ respectively. It is easily seen that γ_2 and γ_3 are null-homotopic and γ_1 is not. Therefore we have $\pi_2(G/K) \cong \ker h = \mathbb{Z}_4 \times \mathbb{Z}$.

Case (E7-3) $\quad \mathfrak{g} = \mathfrak{e}_7, \quad x = \frac{2}{3} v_6.$

Take a direct sum decomposition of $\mathfrak{k}$ by the following two ideals:

$$\mathfrak{k}_1 = [\mathfrak{k}, \mathfrak{k}] \cong \mathfrak{so}(12),$$
$$\mathfrak{k}_2 = \mathbb{R}(\alpha_1{}^* + 2\alpha_2{}^* + 3\alpha_3{}^* + 4\alpha_4{}^* + 3\alpha_5{}^* + 2\alpha_6{}^* + 2\alpha_7{}^*).$$

Put

$$v_1 = \frac{1}{2}(\alpha_1{}^* + 3\alpha_3{}^* + 3\alpha_5{}^*),$$
$$w_1 = \frac{1}{2}(\alpha_5{}^* + \alpha_7{}^*),$$
$$v_2 = \alpha_1{}^* + 2\alpha_2{}^* + 3\alpha_3{}^* + 4\alpha_4{}^* + 3\alpha_5{}^* + 2\alpha_6{}^* + 2\alpha_7{}^*.$$

Then $\{v_1, w_1\}$ forms a basis of $\Lambda_1(\widetilde{K}_1)$. We have

$$\begin{aligned} Z(\widetilde{K}_1) &= \{\exp(jv_1) \mid j = 0,1\} \times \{\exp(kw_1) \mid k = 0,1\} \\ &\cong \mathbb{Z}_2 \times \mathbb{Z}_2 \\ &\cong Z(Spin(12)), \end{aligned}$$
$$\widetilde{K}_1 \cong Spin(12).$$

Since the intersection $\widetilde{K}_1 \cap \widetilde{K}_2$ is equal to $\{\exp \frac{k}{2} v_2 \mid k = 0, 1\}$, we have

$$\widetilde{K} \cong \{Spin(12) \times T^1\}/\mathbb{Z}_2.$$

If we put $\Gamma = Z(\widetilde{G}) \cap \widetilde{K}$, then K is isomorphic to $\widetilde{K}/\Gamma$. In our case,

$$K \cong \{[Spin(12) \times SO(2)]/\mathbb{Z}_2\}/\mathbb{Z}_2.$$

Thus we have $\pi_1(k) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}$. We define paths $\bar{\gamma}_j (j = 1, 2, 3)$ in $\bar{K} = Spin(12) \times \mathbb{R}$ by

$$\begin{aligned}
\bar{\gamma}_1(t) &= (\exp \frac{t}{2}({\alpha_1}^* + {\alpha_3}^* + {\alpha_7}^*), 0),\\
\bar{\gamma}_2(t) &= (\exp \frac{1}{2} v_2, -\frac{t}{2} v_2),\\
\bar{\gamma}_3(t) &= (e, t v_2),
\end{aligned}$$

so that the corresponding paths $\widetilde{\gamma}_1, \widetilde{\gamma}_2$ and $\widetilde{\gamma}_3$ represent the generators $(1, 0, 0), (0, 1, 0)$ and $(0, 0, 1)$ of $\pi_1(\widetilde{K})$ respectively. It is easily seen that γ_2 and γ_3 are null-homotopic and γ_1 is not. Therefore we have $\pi_2(G/K) \cong \ker h = \mathbb{Z}_2 \times \mathbb{Z}$.

Case (E7-4) $\quad \mathfrak{g} = \mathfrak{e}_7, \quad x = \frac{2}{3} v_7.$

Take a direct sum decomposition of $\mathfrak{k}$ by the following two ideals:

$$\begin{aligned}
\mathfrak{k}_1 &= [\mathfrak{k}, \mathfrak{k}] \cong \mathfrak{su}(7),\\
\mathfrak{k}_2 &= \mathbb{R}(3{\alpha_1}^* + 6{\alpha_2}^* + 9{\alpha_3}^* + 12{\alpha_4}^* + 8{\alpha_5}^* + 4{\alpha_6}^* + 7{\alpha_7}^*).
\end{aligned}$$

Put

$$\begin{aligned}
v_1 &= \frac{1}{7}({\alpha_1}^* + 2{\alpha_2}^* + 3{\alpha_3}^* + 4{\alpha_4}^* + 5{\alpha_5}^* + 6{\alpha_6}^*),\\
v_2 &= (3{\alpha_1}^* + 6{\alpha_2}^* + 9{\alpha_3}^* + 12{\alpha_4}^* + 8{\alpha_5}^* + 4{\alpha_6}^* + 7{\alpha_7}^*).
\end{aligned}$$

Then $\{v_1\}$ forms a basis of $\Lambda_1(\widetilde{K}_1)$. We have

$$\begin{aligned}
Z(\widetilde{K}_1) &= \{\exp(j v_1) \mid j = 0, 1, \cdots, 6\} \cong \mathbb{Z}_7 \cong Z(SU(7)),\\
\widetilde{K}_1 &\cong SU(7).
\end{aligned}$$

Since the intersection $\widetilde{K}_1 \cap \widetilde{K}_2$ is equal to $\{\exp \frac{k}{7} v_2 \mid k = 0, 1, \cdots 6\}$, we have

$$\widetilde{K} \cong \{SU(7) \times T^1\}/\mathbb{Z}_7 \cong S\{U(7) \times U(1)\}.$$

If we put $\Gamma = Z(\widetilde{G}) \cap \widetilde{K}$, then K is isomorphic to $\widetilde{K}/\Gamma$. In our case,

$$K \cong S(U(7) \times U(1))/\mathbb{Z}_2.$$

Thus we have $\pi_1(K) \cong \mathbb{Z}_2 \times \mathbb{Z}_7 \times \mathbb{Z}$. We define paths $\bar{\gamma}_j (j = 1, 2, 3)$ in $\bar{K} = SU(7) \times \mathbb{R}$ by

$$\begin{aligned}
\bar{\gamma}_1(t) &= (e, \frac{t}{2} v_2), \\
\bar{\gamma}_2(t) &= (\exp(3v_1), -\frac{1}{7} v_2), \\
\bar{\gamma}_3(t) &= (e, t v_2),
\end{aligned}$$

so that the corresponding paths $\widetilde{\gamma}_1, \widetilde{\gamma}_2$ and $\widetilde{\gamma}_3$ represent the generators $(1,0,0), (0,1,0)$ and $(0,0,1)$ of $\pi_1(\widetilde{K})$ respectively. It is easily seen that γ_2 and γ_3 are null-homotopic and γ_1 is not. Therefore we have $\pi_2(G/K) \cong \ker h = \mathbb{Z}_2 \times \mathbb{Z}$.

Case (E7-5) $\quad \mathfrak{g} = \mathfrak{e}_7, \quad x = v_3.$

The center of $\mathfrak{k}$ is 0, and $\mathfrak{k}$ is semisimple. We denote by $\mu = -\alpha_0$ the maximal root $\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + 2\alpha_7$ of $\mathfrak{g}$. Put

$$\begin{aligned}
v_1 &= \frac{1}{3}(\alpha_1{}^* + 2\alpha_2{}^*), \\
w_1 &= \frac{1}{6}(\alpha_0{}^* + 2\alpha_6{}^* + 3\alpha_5{}^* + 4\alpha_4{}^* + 5\alpha_7{}^*) \\
&= \frac{1}{6}(-\alpha_1{}^* - 2\alpha_2{}^* - 3\alpha_3{}^* + 3\alpha_7{}^*).
\end{aligned}$$

Then $\{w_1\}$ forms a basis of $\Lambda_1(\widetilde{K})$. We have

$$\begin{aligned}
Z(\widetilde{K}) &= \{\exp(k w_1) \mid k = 0, 1, \cdots, 5\} \cong \mathbb{Z}_6, \\
\widetilde{K} &\cong \{SU(3) \times SU(6)\}/\mathbb{Z}_3,
\end{aligned}$$

If we put $\Gamma = Z(\widetilde{G}) \cap \widetilde{K}$, then K is isomorphic to $\widetilde{K}/\Gamma$. In our case,

$$\begin{aligned} K &\cong \{[SU(3) \times SU(6)]/\mathbb{Z}_3\}/\mathbb{Z}_2 \\ &= \{SU(3) \times [SU(6)/\mathbb{Z}_2]\}/\mathbb{Z}_3. \end{aligned}$$

Thus we have $\pi_1(K) \cong \mathbb{Z}_2 \times \mathbb{Z}_3$. We define paths $\bar{\gamma}_j (j = 1, 2)$ in $\bar{K} = SU(3) \times SU(6)$ by

$$\begin{aligned} \bar{\gamma}_1(t) &= (e, \exp(3tw_1)), \\ \bar{\gamma}_2(t) &= (\exp(tv_1), \exp(2tw_1)), \end{aligned}$$

so that the corresponding paths $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ represent the generators $(1, 0)$ and $(0, 1)$ of $\pi_1(\widetilde{K})$ respectively. It is easily seen that γ_2 is null-homotopic and γ_1 is not. Therefore we have $\pi_2(G/K) \cong \ker h = \mathbb{Z}_3$.

2.3. Characteristic classes of homogeneous spaces

In this section we shall recall the definitions of Chern, Pontrjagin, Stiefel-Whitney and Euler classes, and the method to calculate the characteristic classes of compact homogeneous spaces in terms of the roots, which has been shown by Borel and Hirzebruch [6].

We consider the category of differentiable principal bundles over differentiable manifolds. The Chern classes of a principal $U(n)$-bundle ξ over a differentiable manifold X is defined by the unique cohomology classes $c_i(\xi)$ satisfying the following four axioms.

Axiom I. For every principal $U(n)$-bundle ξ over a differentiable manifold X and every integer $i \geqq 0$ there is a Chern class $c_i(\xi) \in H^{2i}(X, \mathbb{Z})$. The class $c_0(\xi) = 1$ is the unit element. (The sum $c(\xi) = \sum c_i(\xi)$ is called the total Chern class of ξ).

Axiom II (Naturality). For every principal $U(n)$-bundle ξ over a differentiable manifold X and any differentiable map $f : Y \longrightarrow X$ from a differentiable manifold Y,

$$c(f^*\xi) = f^*c(\xi).$$

Axiom III (Whitney sum formula). If $\xi_1, \cdots, \xi_r$ are principal $U(1)$-bundles over a differentiable manifold X, then

$$c(\xi_1 \oplus \cdots \oplus \xi_r) = c(\xi_1) \cdots c(\xi_r).$$

Axiom IV (Normalization). Let η_n be the canonical principal $U(1)$-bundle over $\mathbb{C}P^n$, then

$$c(\eta_n) = 1 + h_n,$$

where h_n denotes the generator of $H^2(\mathbb{C}P^n, \mathbb{Z})$ determined by the hyperplane $z^0 = 0$ in $\mathbb{C}P^n$.

The existence and the uniqueness of Chern classes may be shown (cf., [20]). We shall recall a construction of Chern classes shown by Borel and Hirzebruch [6]. Let $\xi = (E_\xi, B_\xi, U(n))$

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX be a principal $U(n)$-bundle, and $T \subset U(n)$ be the maximal torus formed by all diagonal matrices. For the sake of simplicity, we shall denote by $[\lambda_1, \cdots, \lambda_n] \in \mathfrak{u}(n)$ the diagonal matrix whose (j,j)-element is $2\pi\sqrt{-1}\lambda_j$ ($\lambda_j \in \mathbb{R}, 1 \leqq j \leqq n$). Let $x_j \in \mathfrak{t}^*$ ($1 \leqq j \leqq n$) be a linear form defined by

$$x_j([\lambda_1, \cdots, \lambda_n]) = \lambda_j.$$

The restriction of x_j to the unit lattice $\Lambda(U(n)) = \exp^{-1}(e)$ (e denotes the identity element of $U(n)$) defines a basis of $Hom(H_1(T,\mathbb{Z}),\mathbb{Z})$ and thus a basis of $H^1(T,\mathbb{Z})$. They are also denoted by the same symbol x_j. We denote by τ the transgression in the fiber bundle $(E_\xi, E_\xi/T, T)$, and by $\rho : E_\xi/T \longrightarrow B_\xi$ the projection from E_ξ/T onto B_ξ. Then $\tau(x_j) \in H^2(E_\xi/T, \mathbb{Z})$ and the total Chern class $c(\xi)$ is given by

$$\rho^*(c(\xi)) = \prod_{j=1}^{n}(1 - \tau(x_j)).$$

Remark. For principal $GL(n,\mathbb{C})$-bundles, Chern classes are also defined similarly as for principal $U(n)$-bundles, and we may see that if ξ' is a reduction of the structural group of a principal $GL(n,\mathbb{C})$-bundle ξ to $U(n)$, then the Chern classes of ξ' are equal to those of ξ. If E is a complex vector bundle over X associated to a principal $U(n)$- (or $GL(n,\mathbb{C})$-) bundle ξ, we call $c(\xi)$ the (total) Chern class of E and write $c(E) = c(\xi)$.

Now we consider a $2n$-dimensional almost complex manifold (M, J). We may regard the tangent bundle TM as a complex vector bundle TM_J by means of the almost complex structure J.

Definition. The Chern class $c(M,J)$ of an almost complex manifold (M,J) is defined by $c(M,J) = c(TM_J)$.

The principal $GL(n,\mathbb{C})$-bundle associated to TM_J is the bundle $\mathcal{C}(M)$ of complex linear frames:

$$\mathcal{C}(M) := \{u : \mathbb{R}^{2n} \longrightarrow T_xM : \text{linear isomorphism} \mid u \circ J_0 = J_x \circ u,\ x \in M\},$$

where J_0 is the natural complex structure of $\mathbb{R}^{2n}$. $\mathcal{C}(M)$ is a subbundle of the bundle $L(M)$ of linear frames over M, so it is a reduction of the structural group of $L(M)$ to $GL(n, \mathbb{C})$. About a relationship between almost complex structures and reductions of the structural group of $L(M)$ to $GL(n, \mathbb{C})$, we have the following.

Proposition 2.3.1. [25] *Given a $2n$-dimensional manifold M, there is a natural one-to-one correspondence between the almost complex structures and the reductions of the structural group of $L(M)$ to $GL(n, \mathbb{C})$.*

From this proposition, we may see that our definition of the Chern classes of an almost complex manifold (M, J) coincides with one in [20], that is, the Chern class $c(M, J)$ is equal to the Chern class of its associated principal $GL(n, \mathbb{C})$-bundle.

The Pontrjagin classes of a principal $O(n)$-bundle ξ over a differentiable manifold X are defined in terms of the Chern classes of unitary bundles.

Let $\{U_j\}$ and $\{\varphi_{ij}\}$ be an open covering of X such that $\pi^{-1}(U_j) \cong U_j \times O(n)$ and the transition functions of ξ, respectively. We denote by $\iota : O(n) \hookrightarrow U(n)$ the inclusion. Then we have a principal $U(n)$-bundle over X defined by $\{U_j\}$ and $\{\iota \circ \varphi_{ij}\}$. This principal $U(n)$-bundle $\iota(\xi)$ is called the ι-extension of ξ.

Definition. The i-th Pontrjagin class of a principal $O(n)$-bundle ξ over a differential manifold X is defined by

$$p_i(\xi) = (-1)^i c_{2i}(\iota(\xi)) \in H^{4i}(X, \mathbb{Z}).$$

As the Chern class of an almost complex manifold, we may define Pontrjagin class of a differentiable manifold M by

$$p(M) := p(TM).$$

For an almost complex manifold (M, J), we have a relationship between the Chern class of (M,J) and the Pontrjagin class of M: by the definition we have

$$c_i(M,J) = c_i(TM_J),$$
$$p_i(M) = (-1)^i c_{2i}(T^{\mathbb{C}}M),$$

where $T^{\mathbb{C}}M = TM \otimes_{\mathbb{R}} \mathbb{C}$ denotes the complexification of the tangent bundle TM. The almost complex structure J extends to a $\mathbb{C}$-linear bundle isomorphism on $T^{\mathbb{C}}M$ and has (fiberwise) eigenvalues $\pm\sqrt{-1}$. Let $T^{1,0}M$ (resp. $T^{0,1}M$) be the bundle of eigenspaces of J corresponding to the eigenvalue $\sqrt{-1}$ (resp. $-\sqrt{-1}$). Then we have the decomposition

$$T^{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M,$$

and $\mathbb{C}$-linear isomorphisms

$$T^{1,0}M \cong TM_J,$$
$$T^{0,1}M \cong TM_{-J} = \overline{TM_J} \quad \text{(the conjugate bundle).}$$

By virtue of Axiom III for Chern class, we have a relation

$$\check{p}(M) = \sum_{i=0}^{\infty}(-1)^i p_i(M) = \sum_{i=0}^{\infty} c_i(M,J) \sum_{j=0}^{\infty}(-1)^j c_j(M,J).$$

We recall the axiomatic definition of Stiefel-Whitney classes of a principal $O(n)$-bundle over a differentiable manifold X.

Axiom I. To each principal $O(n)$-bundle ξ over a differentiable manifold X there corresponds a sequence of cohomology classes

$$w_i(\xi) \in H^i(X, \mathbb{Z}_2), \quad i = 0, 1, 2, \cdots,$$

called the Stiefel-Whitney classes of ξ. The class $w_0(\xi) = 1$ is the unit element.

Axiom II (Naturality). For any differentiable map $f : Y \longrightarrow X$ from a differentiable manifold Y,

$$w(f^*\xi) = f^*w(\xi).$$

Axiom III (Whitney sum formula). If ξ and η are principal $O(n)$- and $O(m)$-bundles over the same base space, then

$$w(\xi \oplus \eta) = w(\xi)w(\eta).$$

Axiom IV (Normalization). $w(\eta_n) = 1 + h_n$, where η_n is the canonical principal $O(1)$-bundle over $\mathbb{R}P^n$ and h_n is the non-zero element of $H^1(\mathbb{R}P^n, \mathbb{Z}_2)$.

We shall recall the definition of Euler classes. Let ξ be an oriented real vector bundle of rank $2n$ with structural group $SO(2n)$, and η the associated bundle of unit spheres.

Definition. The Euler class $e(\xi)$ of ξ is defined by

$$e(\xi) = -\tau(x),$$

where x is the generator of $H^{2n-1}(S^{2n-1}, \mathbb{Z})$ defined by the positive orientation of S^{2n-1} and τ is the transgression in η.

We define the Euler class of a principal $SO(2n)$-bundle ξ to be the Euler class of the real vector bundle $E_\xi \times_{SO(2n)} \mathbb{R}^{2n}$ associated to ξ. The embedding $SO(2n) \hookrightarrow O(2n)$ defines the Stiefel-Whitney classes and Pontrjagin classes for a principal $SO(2n)$-bundle ξ. In this case

$$w_{2n}(\xi) \equiv e(\xi) \mod 2,$$

and

$$p_{2n}(\xi) = (e(\xi))^2.$$

Let $\iota : U(n) \hookrightarrow SO(2n)$ be the inclusion map. For a principal $U(n)$-bundle ξ we have

$$c_n(\xi) = e(\iota(\xi)).$$

In this case

$$w_{2i}(\iota(\xi)) \equiv c_i(\xi) \mod 2,$$
$$w_{2i+1}(\iota(\xi)) = 0.$$

Now we shall recall the method to calculate the characteristic classes of compact homogeneous spaces which has been shown by A. Borel and F. Hirzebruch [6].

Theorem 2.3.2. [6] *Let G, K and S be a compact connected Lie group, a closed subgroup of G and a maximal torus of K, respectively, and $\pm b_j$ $(1 \leqq j \leqq k)$ the roots of G relative to S complementary to those of K. Let ξ be a principal G-bundle, $\rho : E_\xi/S \longrightarrow E_\xi/K$ the projection, τ the transgression in the principal S-bundle $(E_\xi, E_\xi/S, S)$, and η the bundle along the fibers of the fiber bundle $(E_\xi/S, E_\xi/K, G/K)$.*

(1) *The Pontrjagin classes of η are given by*

$$\rho^*(\widetilde{p}(\eta)) = \prod_{j=1}^{k}(1 + (\tau(b_j))^2).$$

(2) *If, moreover, K is connected and the dimension of G/K is even, then the Euler class is given by*

$$\rho^*(e(\eta)) = \pm \prod_{j=1}^{k}(-\tau(b_j)),$$

here the sign is determined once η has been oriented.

(3) *If, moreover, G/K has an invariant almost complex structure J, and we denote by η' the complex vector bundle structure of η associated to J, then*

$$\rho^*(c(\eta')) = \prod_{j \in J}(1 - \tau(\epsilon_j b_j)), \quad (\epsilon = \pm 1),$$

where $\epsilon_j b_j$ runs through the weights of the complex isotropy representation $\iota_{\mathbb{C}}$ defining J.

If we consider the special case where ξ is a trivial G-bundle over one point, then the bundle along the fibers η coincides with the tangent bundle $T(G/K)$ of the homogeneous space G/K. Hence by Theorem 2.2.1, we may calculate the characteristic classes of a compact homogeneous space G/K.

In order to calculate the Chern classes of a compact homogeneous almost complex manifold $(G/K, J)$ with an invariant almost complex structure J, we have to determine the sign ϵ_j. If the closed connected subgroup K is of maximal rank, we may determine ϵ_j as follows. Let $\pm b_j$ $(1 \leqq j \leqq n)$ be the complementary roots of G relative to S. Since K has maximal rank, denoting by $\mathfrak{k}$ the Lie algebra of K, we get a decomposition of $\mathfrak{g}^{\mathbb{C}}$.

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{k}^{\mathbb{C}} \oplus \sum_{j=1}^{n} \mathfrak{g}_{b_j} \oplus \mathfrak{g}_{-b_j}.$$

Put

$$\begin{aligned} \mathfrak{b}_j &= (\mathfrak{g}_{b_j} \oplus \mathfrak{g}_{-b_j}) \cap \mathfrak{g}, \\ \mathfrak{m} &= \mathfrak{b}_1 \oplus \cdots \mathfrak{b}_n. \end{aligned}$$

Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ becomes an $Ad_{\mathfrak{g}}(K)$-invariant decomposition. The invariant almost complex structure J induces a linear endomorphism (which we denote by the same letter J) of the subspace $\mathfrak{m}$ such that

$$\begin{aligned} &J^2 = -id_{\mathfrak{m}}, \\ &J \circ Ad_{\mathfrak{g}}a|_{\mathfrak{m}} = Ad_{\mathfrak{g}}a|_{\mathfrak{m}} \circ J \quad \text{for any} \quad a \in K. \end{aligned}$$

So each $\mathfrak{b}_j$ is invariant under J. Take any non-zero element $e_j \in \mathfrak{b}_j$. Then $e_j - \sqrt{-1}Je_j$ is a non-zero element of ${\mathfrak{b}_j}^{\mathbb{C}} = \mathfrak{g}_{b_j} \oplus \mathfrak{g}_{-b_j}$, and is an eigenvector corresponding to either the eigenvalue $e^{2\pi\sqrt{-1}b_j(H)}$ or the eigenvalue $e^{-2\pi\sqrt{-1}b_j(H)}$ of $Ad_{\mathfrak{g}}(\exp(H))|_{\mathfrak{m}}$ for any $H \in \mathfrak{t}$. In the former case, we define $\epsilon_j = +1$, and in the latter case, $\epsilon_j = -1$. That is to say, we determine the sign ϵ_j so that the equality

$$Ad_{\mathfrak{g}}(\exp(H))\{e_j - \sqrt{-1}Je_j\} = e^{2\pi\sqrt{-1}\epsilon_j b_j(H)}\{e_j - \sqrt{-1}Je_j\}$$

holds for any $H \in \mathfrak{t}$.

Definition. Linear forms $\epsilon_j b_j$ $(1 \leqq j \leqq n)$ are called the roots of the invariant almost complex structure J.

Corollary 2.3.3. *Let $M = (G/K, J)$ be a compact homogeneous almost complex manifold, where G is a compact connected Lie group, K is a connected closed subgroup of G of maximal rank and J is an invariant almost complex structure on G/K. Let $T \subset K \subset G$ be a maximal torus, $\pm b_j (1 \leqq j \leqq n)$ the complementary roots, $\epsilon_j b_j$ the roots of J, ρ the projection from G/T onto G/K, and τ the transgression in the principal T-bundle $(G, G/T, T)$. Then the Chern classes of the almost complex manifold M is given by*

$$\rho^*(c(G/K, J)) = \prod_{j=1}^{n}(1 - \tau(\epsilon_j b_j)).$$

In order to calculate the characteristic classes of a compact homogeneous space G/K from Theorem 2.2.2, we need to know the cohomology ring of G/K. The cohomology ring of real coefficient of G/K is completely calculated ([4]): let G be a compact Lie group and K a closed subgroup of G. We denote by B_G and $\rho(K, G)$ the classifying space of G and the projection from B_K onto B_G respectively. A. Borel [4] has proved the followings.

Theorem 2.3.4. [4] *Let G be a compact Lie group and K a closed subgroup of same rank. Then*

(1) *$\rho^*(K, G) : H^*(B_G, \mathbb{R}) \longrightarrow H^*(B_K, \mathbb{R})$, is injective.*

(2) *If G is connected, then*

$$H^*(G/K, \mathbb{R}) \cong H^*(B_K, \mathbb{R})/\rho^*(K, G)(H^+(B_G, \mathbb{R})),$$

where $H^+(B_G, \mathbb{R})$ denotes the subalgebra formed by the elements of positive degrees of $H^*(B_G, \mathbb{R})$. Furthermore, if we denote by $\phi : G/K \longrightarrow G_K$ the characteristic map, then

$$H^*(G/K, \mathbb{R}) = \phi^*(H^*(B_K, \mathbb{R})).$$

Theorem 2.3.5. [4] *Let G be a compact Lie group and T a maximal torus of G. Then the subalgebra $\rho^*(T,G)(H^*(B_G,\mathbb{R}))$ of $H^*(B_T,\mathbb{R})$ is the subalgebra I_G formed by the elements of $H^*(B_T,\mathbb{R})$ which are invariant by the action of the Weyl group $W(G)$ of G.*

From the above theorems, we have

$$H^*(G/K,\mathbb{R}) \cong I_K/I_G^+,$$

where I_G^+ denotes the set of all elements in I_G which have positive degrees.

Remark. Let $\{x_1,\cdots,x_l\}$ be a basis of $H^1(T,\mathbb{Z}) \subset H^1(T,\mathbb{R})$, and τ' the transgression in the universal bundle (E_T,B_T,T) for T. We denote by the same letter x_j, the corresponding integral linear form on $\mathfrak{t}$ and $-\tau'(x_j) \in H^2(B_T,\mathbb{R})$. Then we may see that $H^*(B_T,\mathbb{R}) \cong \mathbb{R}[x_1,\cdots,x_l]$, the ring of all polynomials in the x_j's (see [4]).

In the sequel, we denote by $S\{x_1,\cdots,x_l\}$ the ring of all symmetric formal power series in the x_j's, with respect to a ring of coefficients which the context will make precise, and by $\mathbb{R}\{x\}$ the ring of all formal power series in x.

2.4. Characteristic classes of Riemannian 3-symmetric spaces

In this section we shall calculate characteristic classes of compact irreducible Riemannian 3-symmetric spaces, in particular their first Chern classes and second Stiefel-Whitney classes. It is known that the first and the second Stiefel-Whitney classes have following geometrical meanings (cf., [39]).

Proposition 2.4.1. *A real vector bundle ξ is orientable if and only if the first Stiefel-Whitney class $w_1(\xi)$ vanishes.*

Proposition 2.4.2. *A principal $SO(n)$-bundle ξ admits a spin structure if and only if its second Stiefel-Whitney class $w_2(\xi)$ vanishes. And then the number of distinct spin structures on ξ is equal to the number of elements in $H^1(B_\xi, \mathbb{Z}_2)$.*

Here a spin structure on a principal $SO(n)$-bundle ξ is defined as follows.

Definition. A spin structure on a principal $SO(n)$-bundle ξ is a pair (η, f) consisting of

(1) A principal $Spin(n)$-bundle η over the same base space B as ξ ; and

(2) A map $f : E_\eta \longrightarrow E_\xi$ such that the following diagram is commutative.

$$\begin{array}{ccccc} E_\eta \times Spin(n) & \xrightarrow{\text{right translation}} & E_\eta & \xrightarrow{\pi_\eta} & B \\ {\scriptstyle f\times\chi}\downarrow & & {\scriptstyle f}\downarrow & & \Vert \\ E_\xi \times SO(n) & \xrightarrow{\text{right translation}} & E_\xi & \xrightarrow{\pi_\xi} & B \end{array}$$

Here χ denotes the standard homomorphism from $Spin(n)$ onto $SO(n)$.

For an oriented Riemannian manifold (M, g), a spin structure on the bundle of oriented orthonormal frames of (M, g) is called a spin structure on (M, g). From Proposition 2.4.2 it follows that (M, g) admits a spin structure if and only if the second Stiefel-Whitney class $w_2(M)$ vanishes. An oriented Riemannian manifold admitting a spin structure is called a spin manifold.

Examples of spin manifolds.

(1) completely parallelizable oriented Riemannian manifolds.

(2) oriented Riemannian manifolds with vanishing second cohomology group, e.g., n-dimensional unit sphere S^n ($n \geqq 3$).

(3) Hermitian symmetric space $SO(2n)/U(n)$, ($n \geqq 2$).

(4) $G_2/SO(4)$.

(5) Cayley plane $F_4/Spin(9)$.

(6) Riemannian homogeneous space G/S, where G is a compact connected Lie group and S is any toral subgroup of G.

In order to calculate the Stiefel-Whitney classes of compact homogeneous spaces G/K by the method of Borel–Hirzebruch, we have to know explicitly the cohomology ring $H^*(G/K, \mathbb{Z}_2)$. The following proposition gives an explicit description of cohomology groups of some compact homogeneous spaces. In the sequel a manifold M is called to be *without torsion* if $ax = 0$ ($a \in \mathbb{Z}$, $x \in H^*(M, \mathbb{Z})$) implies $a = 0$ or $x = 0$. And M is called to be *without k-torsion* if there is no non-zero element in $H^i(M, \mathbb{Z})$ ($i \geqq 0$) that has finite order divisible by k.

Proposition 2.4.3. [4] *Let K be a connected closed subgroup of maximal rank in a compact connected Lie group G, T a common maximal torus. If G, K, G/T and K/T are without p-torsion (resp. without torsion), the homomorphism $\rho_p^*(K, G)$, (resp. $\rho_{\mathbb{Z}}^*(K, G)$) is bijective, $H(G/K, \mathbb{Z}_p)$, (resp. $H(G/K, \mathbb{Z})$) is the quotient of $H(B_K, \mathbb{Z}_p)$, (resp. $H(B_K, \mathbb{Z})$) by the ideal $\rho_p^*(H^+(B_G, \mathbb{Z}_p))$, (resp. $\rho_{\mathbb{Z}}^*(H^+(B_G, \mathbb{Z}))$), and is equal to its characteristic subalgebra.*

The compact Lie groups $U(n)$, $SU(n)$ and $Sp(n)$ ($n \geqq 1$) are without torsion, $SO(n)$ and G_2 are without p-torsion for $p \neq 2$, F_4, E_6 and E_7 are without p-torsion for $p \neq 2, 3$ and E_8 is without p-torsion for $p \neq 2, 3, 5$.

In the sequel, we denote by $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ the set of all real numbers, complex numbers and quaternionic numbers, respectively, and furthermore by $\mathfrak{gl}(N,\mathbb{R})$, $\mathfrak{gl}(N,\mathbb{C})$ and $\mathfrak{gl}(N,\mathbb{H})$ the set of all $N \times N$ real matrices, complex matrices and quaternionic matrices, respectively. We denote by $E_{\lambda,\mu} \in \mathfrak{gl}(N,\mathbb{R}) \subset \mathfrak{gl}(N,\mathbb{C}) \subset \mathfrak{gl}(N,\mathbb{H})$ $(1 \leqq \lambda, \mu \leqq N)$ the matrix whose r-th row and s-th column is given by $\delta_{r\lambda}\delta_{s\mu}$.

(A_n) The Lie algebra of the Lie group $SU(n+1)$ is given by

$$\mathfrak{g} = \mathfrak{su}(n+1) = \{X \in \mathfrak{gl}(n+1,\mathbb{C}) \mid X + \overline{{}^tX} = 0,\ trace X = 0\}.$$

Let T be a maximal torus of $SU(n+1)$ defined by

$$T = \left\{ \sum_{\alpha=1}^{n+1} e^{2\pi i\theta_\alpha} E_{\alpha,\alpha} \in SU(n+1) \;\middle|\; \theta_\alpha \in \mathbb{R}\ (1 \leqq \alpha \leqq n+1), \sum_{\alpha=1}^{n+1} \theta_\alpha = 0 \right\}.$$

We define linear forms x_λ $(1 \leqq \lambda \leqq n+1)$ on the Lie algebra $\mathfrak{t}$ of T by

$$x_\lambda \left(2\pi i \sum_{\alpha=1}^{n+1} \theta_\alpha E_{\alpha,\alpha} \right) = \theta_\alpha,$$

for $2\pi i \sum_{\alpha=1}^{n+1} \theta_\alpha E_{\alpha,\alpha} \in \mathfrak{t}$, then x_λ's are integral linear forms. The roots of G relative to T are

$$\pm(x_\lambda - x_\mu) \quad (1 \leqq \lambda < \mu \leqq n+1).$$

If we take a lexicographic ordering $x_1 > x_2 > \cdots > x_{n+1} > 0$, the simple roots $\alpha_j, 1 \leqq j \leqq n$ are given by

$$\alpha_j = x_j - x_{j+1}.$$

The Weyl group $W(SU(n+1))$ of $SU(n+1)$ is isomorphic to the group of all permutations of $x_1, \cdots, x_{n+1}$, hence the set of $W(SU(n+1))$-invariant formal power series in $H^*(B_T, \mathbb{R})$ is $I_{SU(n+1)} = S\{x_1, \cdots, x_{n+1}\}$.

(B_n) The Lie algebra of $G = SO(2n+1)$ is given by

$$\mathfrak{so}(2n+1) = \{X \in \mathfrak{gl}(2n+1,\mathbb{R}) \mid X +^t X = 0\}.$$

Let T be a maximal torus of $SO(2n+1)$ defined by

$$T = \{\sum_{\alpha=1}^{n}(\cos 2\pi\theta_\alpha(E_{2\alpha-1,2\alpha-1} + E_{2\alpha,2\alpha}) - \sin 2\pi\theta_\alpha(E_{2\alpha-1,2\alpha} - E_{2\alpha,2\alpha-1})) \\ + E_{2n+1,2n+1} \mid \theta_\alpha \in \mathbb{R}\ (1 \leqq \alpha \leqq n)\}.$$

We define linear forms x_λ $(1 \leqq \lambda \leqq n)$ on the Lie algebra $\mathfrak{t}$ of T by

$$x_\lambda(\sum_{\alpha=1}^{n} 2\pi\theta_\alpha(E_{2\alpha-1,2\alpha} - E_{2\alpha,2\alpha-1})) = \theta_\lambda,$$

for $\sum_{\alpha=1}^{n} 2\pi\theta_\alpha(E_{2\alpha-1,2\alpha} - E_{2\alpha,2\alpha-1}) \in \mathfrak{t}$, then x_λ's are integral linear forms. The roots of G relative to T are

$$\pm x_\lambda \pm x_\mu, \quad 1 \leqq \lambda < \mu \leqq n,$$
$$\pm x_\nu, \quad 1 \leqq \nu \leqq n,$$

where $\pm$ signs are taken independently. We take a lexicographic ordering $x_1 > x_2 > \cdots > x_n > 0$, then the simple roots $\alpha_j, 1 \leqq j \leqq n$ are given by

$$\alpha_j = x_j - x_{j+1} \quad (1 \leqq j \leqq n-1), \quad \alpha_n = x_n.$$

(C_n) The Lie algebra of the Lie group $Sp(n)$ is given by

$$\mathfrak{sp}(n) = \{X \in \mathfrak{gl}\,(n, \mathbb{H}) \mid X + {}^t\overline{X} = 0\}.$$

Let T be a maximal torus of $Sp(n)$ defined by

$$T = \left\{\sum_{\alpha=1}^{n} e^{2\pi i\theta_\alpha} E_{\alpha,\alpha} \in Sp(n) \mid \theta_\alpha \in \mathbb{R}\ (1 \leqq \alpha \leqq n)\right\}.$$

We define linear forms x_λ $(1 \leqq \lambda \leqq n)$ on the Lie algebra $\mathfrak{t}$ of T by

$$x_\lambda(\sum_{\alpha=1}^{n} 2\pi i\theta_\alpha E_{\alpha,\alpha}) = \theta_\lambda,$$

for $\sum_{\alpha=1}^{n} 2\pi i\theta_\alpha E_{\alpha,\alpha} \in \mathfrak{t}$, then x_λ's are integral linear forms. We may easily see that the roots of $Sp(n)$ relative to T are

$$\pm x_\lambda \pm x_\mu, \quad 1 \leqq \lambda < \mu \leqq n,$$
$$\pm 2x_\nu, \quad 1 \leqq \nu \leqq n,$$

where $\pm$ signs are taken independently. We take a lexicographic ordering $x_1 > x_2 > \cdots > x_n > 0$, then the simple roots $\alpha_j, 1 \leqq j \leqq n$ are given by

$$\alpha_j = x_j - x_{j+1} \quad (1 \leqq j \leqq n-1), \quad \alpha_n = 2x_n.$$

(D_n) The Lie algebra of $G = SO(2n)$ is given by

$$\mathfrak{so}(2n) = \{X \in \mathfrak{gl}(2n+1, \mathbb{R}) \mid X +^t X = 0\}.$$

Let T be a maximal torus of $SO(2n)$ defined by

$$T = \{\sum_{\alpha=1}^{n} (\cos 2\pi\theta_\alpha (E_{2\alpha-1,2\alpha-1} + E_{2\alpha,2\alpha}) - \sin 2\pi\theta_\alpha (E_{2\alpha-1,2\alpha} - E_{2\alpha,2\alpha-1}))$$
$$\mid \theta_\alpha \in \mathbb{R} \ (1 \leqq \alpha \leqq n)\}.$$

We define linear forms x_λ $(1 \leqq \lambda \leqq n)$ on the Lie algebra $\mathfrak{t}$ of T by

$$x_\lambda(\sum_{\alpha=1}^{n} 2\pi\theta_\alpha (E_{2\alpha-1,2\alpha} - E_{2\alpha,2\alpha-1})) = \theta_\lambda,$$

for $\sum_{\alpha=1}^{n} 2\pi\theta_\alpha (E_{2\alpha-1,2\alpha} - E_{2\alpha,2\alpha-1}) \in \mathfrak{t}$, then x_λ's are integral linear forms. The roots of G relative to T are

$$\pm x_\lambda \pm x_\mu, \quad 1 \leqq \lambda < \mu \leqq n,$$

where $\pm$ signs are taken independently. We take a lexicographic ordering $x_1 > x_2 > \cdots > x_n > 0$, then the simple roots $\alpha_j, 1 \leqq j \leqq n$ are given by

$$\alpha_j = x_j - x_{j+1} \quad (1 \leqq j \leqq n-1), \quad \alpha_n = x_n.$$

(G_2) Let $\mathfrak{C} = \mathbb{H} \oplus \mathbb{H}e$ be the division Cayley algebra with the multiplication

$$(m + ae)(n + be) = (mn - \bar{b}a) + (a\bar{n} + bm)e,$$

the conjugation $\overline{m + ae} = \overline{m} - ae$ and the inner product $(m+ae, n+be) = (m, n)+(a, b)(= \frac{1}{2}((m\overline{n} + n\overline{m}) + (a\bar{b} + b\overline{a})))$. The connected simply connected compact simple Lie group G_2 is obtained as the automorphism group of the Cayley algebra ([64]).

$$G_2 = \{A \in Iso_{\mathbb{R}}(\mathfrak{C}) \mid A(xy) = (Ax)(Ay)\}.$$

The roots of G_2 relative to some maximal torus T are

$$\begin{aligned} &\pm (x_1 - x_2), \quad \pm(2x_1 + x_2), \quad \pm(x_1 + 2x_2), \\ &\pm x_1, \quad \pm x_2, \quad \pm(x_1 + x_2), \end{aligned}$$

where $x_1, x_2 \in \mathfrak{t}^*$ are an appropriate basis such that $(x_i, x_j) = -\frac{1}{3} + \delta_{ij},\ 1 \leqq i, j \leqq 2$. We take a lexicographic ordering $x_1 > x_2 > 0$, then the simple roots α_i $(i = 1, 2)$ and the maximal root μ are given respectively by

$$\begin{aligned} &\alpha_1 = x_1 - x_2, \quad \alpha_2 = x_2, \\ &\mu = 2x_1 + x_2 = 2\alpha_1 + 3\alpha_2. \end{aligned}$$

Furthermore $I_2 = {x_1}^2 + x_1 x_2 + {x_2}^2$ is a generator of the ring of $W(G_2)$-invariant polynomials in $H^*(B_T, \mathbb{R})$.

(F_4) Let k be $\mathbb{H}, \mathbb{H}^{\mathbb{C}}, \mathfrak{C}$ or $\mathfrak{C}^{\mathbb{C}}$. We denote by $\mathfrak{J}(3, k)$ the Jordan algebra

$$\mathfrak{J}(3, k) = \{X \in M(3, k) \mid \overline{{}^tX} = X\}$$

with the Jordan multiplication $X \circ Y$ and the inner product (X, Y):

$$X \circ Y = \frac{1}{2}(XY + YX), \quad (X, Y) = tr(X \circ Y).$$

In $\mathfrak{J}(3,k)$ we define another multiplication $X \times Y$, called the Freudenthal multiplication, by

$$X \times Y = \frac{1}{2}(2X \circ Y - tr(X)Y - tr(Y)X + (tr(X)tr(Y) - (X,Y))E)$$

and the trilinear form (X,Y,Z), the determinant $\det X$ by

$$(X,Y,Z) = (X, Y \times Z), \quad \det X = \frac{1}{3}(X,X,X).$$

Then the connected, simply connected compact simple Lie group F_4 is obtained by

$$F_4 = \{A \in Iso_{\mathbb{R}}(\mathfrak{J}(3,\mathfrak{C})) \mid A(X \circ Y) = AX \circ AY\}.$$

The Lie algebra $\mathfrak{f}_4$ of the Lie group F_4 of automorphisms of $\mathfrak{J}$ is given by

$$\mathfrak{f}_4 = \{A \in Hom_{\mathbb{R}}(\mathfrak{J}) \mid A(X \circ Y) = AX \circ Y + X \circ AY\}.$$

The roots of F_4 relative to some maximal torus T are

$$\begin{aligned}
&\pm x_i \pm x_j \quad ,(1 \leqq i < j \leqq 4),\\
&\pm x_i, \quad (1 \leqq i \leqq 4),\\
&\pm \frac{1}{2}(x_1 \pm x_2 \pm x_3 \pm x_4),
\end{aligned}$$

where $\pm$ signs are taken independently, and $x_i \in \mathfrak{t}^*$ satisfies $(x_i, x_j) = \frac{1}{18}\delta_{ij}$ for the inner product $(,)$ on $\mathfrak{t}$ induced by the Killing form of $\mathfrak{f}_4$. We take a lexicographic ordering $x_1 > x_2 > x_3 > x_4 > 0$, then the simple roots α_i $(1 \leqq i \leqq 4)$ and the maximal root μ are given respectively by

$$\begin{aligned}
&\alpha_1 = x_2 - x_3, \quad \alpha_2 = x_3 - x_4, \quad \alpha_3 = x_4,\\
&\alpha_4 = \frac{1}{2}(x_1 - x_2 - x_3 - x_4),\\
&\mu = x_1 + x_2 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4.
\end{aligned}$$

For positive integer k, $I_k = \frac{1}{2}\sum_{\alpha\in\Delta}\alpha^k$ is $W(F_4)$-invariant, and $\{I_2, I_6, I_8, I_{12}\}$ gives a system of generators of the ring of $W(F_4)$-invariant polynomials in $H^*(B_T,\mathbb{R})$.

(E_6) We denote by τ the complex conjugation in $\mathfrak{J}^{\mathbb{C}} = \mathfrak{J}(3, \mathfrak{C}^{\mathbb{C}})$ with respect to $\mathfrak{J}(3, \mathfrak{C})$. Then the connected, simply connected compact simple Lie group E_6 is obtained as

$$E_6 = \{A \in Iso_{\mathbb{C}}(\mathfrak{J}(3, \mathfrak{C}^{\mathbb{C}})) \mid \det AX = \det X, \ < AX, AY >=< X, Y >\},$$

where $< X, Y >= (\tau X, Y)$. The roots of E_6 relative to some maximal torus T may be represented as follows.

$$\begin{aligned}
&\pm (x_i + x_j), \quad \pm(x_i - x_j), \quad (1 \leqq i < j \leqq 5),\\
&\pm \frac{1}{2}(\sqrt{3}x_6 + \sum_{i=1}^{5}(-1)^{\nu(i)}x_i), \quad (\nu(i) = 0 \text{ or } 1, \quad \sum_{i=1}^{5}\nu(i) \text{ is even}),
\end{aligned}$$

where $\{x_1, \cdots, x_6\}$ is some appropriate orthogonal basis of $\mathfrak{t}^*$ such that $(x_j, x_j) = \frac{1}{24}, (1 \leqq j \leqq 6)$ with respect to the inner product $(,)$ on $\mathfrak{t}^*$ induced by the Killing form of $\mathfrak{e}_6$. If we consider a lexicographic ordering $x_6 > x_5 > x_4 > x_3 > x_2 > x_1 > 0$, the simple roots are

$$\begin{aligned}
&\alpha_1 = \frac{1}{2}(x_1 - x_2 - x_3 - x_4 - x_5 + \sqrt{3}x_6),\\
&\alpha_2 = x_1 + x_2,\\
&\alpha_j = x_{j-1} - x_{j-2}, \quad (j = 3, 4, 5, 6).
\end{aligned}$$

(E_7) The connected simply connected compact simple Lie group E_7 of type E_7 has been realized as a subgroup of the linear transformation group of some vector space ([65]). The roots of E_7 relative to some maximal torus T may be represented as follows.

$$\begin{aligned}
&\pm (x_i - x_j), \quad \pm(x_i + x_j), \quad (1 \leqq i < j \leqq 6),\\
&\pm \sqrt{2}x_7,\\
&\pm \frac{1}{2}(\sqrt{2}x_7 + \sum_{i=1}^{6}(-1)^{\nu(i)}x_i),\\
&\quad (\nu(i) = 0 \text{ or } 1, \quad \sum_{i=1}^{6}\nu(i) \text{ is odd}),
\end{aligned}$$

where $\{x_1, \cdots, x_7\}$ is some appropriate orthogonal basis of $\mathfrak{t}^*$ such that $(x_j, x_j) = \frac{1}{36}$ with respect to the inner product $(,)$ on $\mathfrak{t}^*$ induced by the Killing form of $\mathfrak{e}_7$. If we take a lexicographic ordering $x_7 > x_6 > x_5 > x_4 > x_3 > x_2 > x_1 > 0$, then the simple roots $\alpha_j (1 \leqq j \leqq 7)$ and the maximal root μ are given respectively by

$$\begin{aligned}
&\alpha_1 = \frac{1}{2}(\sqrt{2}x_7 - x_1 - x_2 - x_3 - x_4 - x_5 - x_6),\\
&\alpha_i = x_{i+1} - x_i, \quad (2 \leqq i \leqq 6),\\
&\alpha_7 = x_2 + x_1,\\
&\quad \mu = \sqrt{2}x_7 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 + 2\alpha_7.
\end{aligned}$$

The degrees of generators of the ring of $W(E_7)$-invariant polynomials on $\mathfrak{t}$ are 2,6,8,10,12,14 and 18.

(E_8) The roots of E_8 relative to some maximal torus T may be expressed as follows.

$$\begin{aligned}
&\pm(x_i - x_j), \quad \pm(x_i + x_j), \quad (1 \leqq i < j \leqq 8),\\
&\pm\frac{1}{2}\sum_{i=1}^{8}(-1)^{\nu(i)}x_i,\\
&(\nu(i) = 0 \text{ or } 1, \quad \sum_{i=1}^{8}\nu(i) \text{ is odd}),
\end{aligned}$$

where $\{x_1, \cdots, x_8\}$ is an orthogonal basis of $\mathfrak{t}^*$ such that $(x_j, x_j) = \frac{1}{60}$ $(1 \leqq j \leqq 8)$ with respect to the inner product $(,)$ on $\mathfrak{t}^*$ induced by the Killing form of $\mathfrak{e}_8$. If we take a lexicographic ordering $x_1 > x_2 > x_3 > x_4 > x_5 > x_6 > x_7 > x_8 > 0$, then the simple roots α_i $(1 \leqq i \leqq 8)$ and the maximal root μ are given respectively by

$$\begin{aligned}
&\alpha_1 = \frac{1}{2}(x_1 - x_2 - x_3 - x_4 - x_5 - x_6 - x_7 - x_8),\\
&\alpha_i = x_i - x_{i+1}, \quad (2 \leqq i \leqq 7),\\
&\alpha_8 = x_7 + x_8,\\
&\quad \mu = x_1 + x_2 = 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 5\alpha_5 + 6\alpha_6 + 3\alpha_7 + 4\alpha_8.
\end{aligned}$$

The degrees of generators of the ring of $W(E_8)$-invariant polynomials on $\mathfrak{t}$ are 2, 8, 12, 14, 18, 20, 24 and 30. For example, $I_2 = \sum_{i=1}^{8} x_i{}^2$ is $W(E_8)$-invariant.

We shall calculate the Chern classes of compact irreducible (non-Kähler) Riemannian 3-symmetric spaces G/K with respect to each invariant almost complex structure, especially the first Chern classes with respect to the canonical almost complex structure J, and show directly the following theorem.

Theorem 2.4.4. [27, 28] *Let $(G/K, J, <,>)$ be a simply connected irreducible (non-Kähler) compact Riemannian 3-symmetric space. Then $(G/K, J, <,>)$ is Einstein if and only if the first Chern class of $(G/K, J)$ vanishes, where J is the canonical almost complex structure and $<,>$ is the G-invariant Riemannian metric on G/K induced by a biinvariant Riemannian metric on G.*

To calculate the Ricci tensor of Riemannian homogeneous spaces, the following facts are very useful.

Theorem 2.4.5. [61] *Let M be an effective coset space G/K of a connected Lie group G by a compact subgroup K, where K is $\mathbb{R}$-irreducible on the tangent space. Choose a G-invariant Riemannian metric $<\ ,\ >$ on M. Then $(M, <\ ,\ >)$ is an Einstein space.*

Proposition 2.4.6. [27] *The Ricci tensor ρ of a naturally reductive homogeneous space $(G/K, <\ ,\ >)$ with a Riemannian metric $<\ ,\ >= cB(\ ,\ )$ is given by*

$$\rho(Y_\beta, Z) = -\sum_{\alpha\in\Phi}\Big(\frac{\chi_\Theta(\alpha+\beta)}{2} + \frac{\chi_\Phi(\alpha+\beta)}{8}\Big)q_\beta(\alpha)(1+p_\beta(\alpha))(\alpha,\alpha) < Y_\beta, Z > /c$$

for $Y_\beta \in \mathfrak{g}_\beta (\beta \in \Phi), Z \in \mathfrak{m}^{\mathbb{C}}$, where $p_\beta(\alpha)$ and $q_\beta(\alpha)$ are the nonnegative integers such that the sequence

$$\beta - p_\beta(\alpha)\alpha, \ldots, \beta-\alpha, \beta, \beta+\alpha, \ldots, \beta+q_\beta(\alpha)$$

becomes the α-sequence of β, $(\ ,\ )$ is the inner product on $\mathfrak{t}^*$ induced by the Killing form B of G, Θ is the root system of K, Φ is the set of the complementary roots, and χ_Φ is the characteristic function.

Case (1) $G/K = \{SU(n+1)/\mathbb{Z}_{n+1}\}/S(U(h) \times U(m-h) \times U(n-m+1))/\mathbb{Z}_{n+1}$

The roots of the canonical almost complex structure J are given by

$$\begin{aligned}
&x_\lambda - x_\mu, \quad 1 \leqq \lambda \leqq h < \mu \leqq m, \\
&-(x_\lambda - x_\nu), \quad 1 \leqq \lambda \leqq h, \ m+1 \leqq \nu \leqq n+1, \\
&x_\mu - x_\nu, \quad h+1 \leqq \mu \leqq m < \nu \leqq n+1.
\end{aligned}$$

Besides J, there are invariant almost complex structures J_1 and J_2. J_1 is an integrable almost complex structure whose roots are given by

$$\begin{aligned}
&x_\lambda - x_\mu, \quad 1 \leqq \lambda \leqq h < \mu \leqq m, \\
&x_\lambda - x_\nu, \quad 1 \leqq \lambda \leqq h, \ m+1 \leqq \nu \leqq n+1, \\
&x_\mu - x_\nu, \quad h+1 \leqq \mu \leqq m < \nu \leqq n+1,
\end{aligned}$$

and J_2 is an almost complex structure whose roots are given by

$$\begin{aligned}
&x_\lambda - x_\mu, \quad 1 \leqq \lambda \leqq h < \mu \leqq m, \\
&-(x_\lambda - x_\nu), \quad 1 \leqq \lambda \leqq h, \ m+1 \leqq \nu \leqq n+1, \\
&-(x_\mu - x_\nu), \quad h+1 \leqq \mu \leqq m < \nu \leqq n+1.
\end{aligned}$$

From Proposition 2.4.3, we may see that

$$H^*(G/K, \mathbb{Z}) \cong S\{x_1, \cdots, x_h\} \otimes S\{x_{h+1}, \cdots, x_m\} \otimes S\{x_{m+1}, \cdots, x_{n+1}\}/S\{x_1, \cdots, x_{n+1}\}.$$

Hence we have

$$\begin{aligned}
c(G/K, J) = &\prod_{\substack{1 \leqq \lambda \leqq h \\ h+1 \leqq \mu \leqq m}} (1 + x_\lambda - x_\mu) \\
&\times \prod_{\substack{1 \leqq \lambda \leqq h \\ m+1 \leqq \nu \leqq n+1}} (1 - x_\lambda + x_\nu) \prod_{\substack{h+1 \leqq \mu \leqq m \\ m+1 \leqq \nu \leqq n+1}} (1 + x_\mu - x_\nu) \quad \bmod I_G^+.
\end{aligned}$$

The first Chern class $c_1(G/K, J)$ is

$$c_1(G/K,J) = (3(m-h)-(n+1))\sum_{\lambda=1}^{h} x_\lambda + (n+1-3h)\sum_{\mu=h+1}^{m} x_\mu \qquad \mathrm{mod}\ I_G^+.$$

We see that $c_1(G/K, J) = 0$ if and only if

$$G/K = \{SU(3h)/\mathbb{Z}_{3h}\}/S(U(h)\times U(h)\times U(h))/\mathbb{Z}_{3h}.$$

Case (2) $G/K = SO(2n+1)/\{U(r)\times SO(2n-2r+1)\}$

The roots of the canonical almost complex structure J are given by

$$\begin{aligned}
&x_\lambda \pm x_\mu, \quad 1 \leqq \lambda \leqq m < \mu \leqq n,\\
&x_\nu, \quad 1 \leqq \nu \leqq m,\\
&-(x_\lambda + x_\mu), \quad 1 \leqq \lambda < \mu \leqq m.
\end{aligned}$$

There is another invariant integrable almost complex structure J_1 whose roots are given by

$$\begin{aligned}
&x_\lambda \pm x_\mu, \quad 1 \leqq \lambda \leqq m < \mu \leqq n,\\
&x_\nu, \quad 1 \leqq \nu \leqq m,\\
&x_\lambda + x_\mu, \quad 1 \leqq \lambda < \mu \leqq m.
\end{aligned}$$

We denote by $\sigma_r \in H^{2r}(G/K,\mathbb{Z})$ the r-th Chern class of the canonical principal $U(m)$-bundle $(SO(2n+1)/SO(2n-2m+1), G/K, U(m))$, then the first Chern class $c_1(G/K, J)$ is given by

$$c_1(G/K,J) = (2n-3m+2)\sigma_1.$$

Since $\sigma_1 \neq 0$, we see that $c_1(G/K, J) = 0$ if and only if

$$G/K = SO(3m-1)/\{U(m)\times SO(m-1)\} \quad (m \geqq 2 : \mathrm{odd}).$$

Case (3) $G/K = \{Sp(n)/\mathbb{Z}_2\}/\{U(r)\times Sp(n-r)\}/\mathbb{Z}_2$

The roots of the canonical almost complex structure J are given by

$$
\begin{aligned}
&-2x_\nu, \quad 1 \leqq \nu \leqq m, \\
&-(x_\lambda + x_\mu), \quad 1 \leqq \lambda < \mu \leqq m, \\
&x_\lambda \pm x_\mu, \quad 1 \leqq \lambda \leqq m < \mu \leqq n.
\end{aligned}
$$

We may see that $I_G = S\{{x_1}^2, \ldots, {x_n}^2\}$ and $I_K = S\{x_1, \cdots, x_m\} \otimes S\{{x_{m+1}}^2, \cdots, {x_n}^2\}$. Therefore we have

$$
H^*(G/K, \mathbb{R}) \cong S\{x_1, \cdots, x_m\} \otimes S\{{x_{m+1}}^2, \cdots, {x_n}^2\}/I_G^+,
$$

and any elements of $H^*(G/K, \mathbb{R})$ may be expressed by the elements of $S\{x_1, \cdots, x_m\}$. If we denote by $\sigma_r \in H^{2r}(G/K, \mathbb{R})$ the r-th Chern class of the canonical principal $U(m)$-bundle $(Sp(n)/Sp(n-m), G/K, U(m))$, then

$$
\sum_{r=0}^{m} \sigma_r = \prod_{\lambda=1}^{m} (1 + x_\lambda) \quad \text{mod } I_G^+.
$$

Then we have

$$
\begin{aligned}
c(G/K, J) = & \prod_{\lambda=1}^{m} (1 - 2x_\lambda) \prod_{1 \leqq \lambda < \mu \leqq m} (1 - x_\lambda - x_\mu) \\
& \times \prod_{1 \leqq \lambda \leqq m < \mu \leqq n} (1 + x_\lambda + x_\mu)(1 + x_\lambda - x_\mu) \qquad \text{mod } I_G^+.
\end{aligned}
$$

In particular, the first Chern class $c_1(G/K, J)$ is given by

$$
c_1(G/K, J) = (2n - 3m - 1)\sigma_1.
$$

Since $\sigma_1 \neq 0$, we see that $c_1(G/K, J) = 0$ if and only if

$$
G/K = Sp(3m-1)/(U(2m-1) \times Sp(m)) \quad (m \geqq 1).
$$

Case (4) $\quad G/K = \{SO(2n)/\mathbb{Z}_2\}/\{U(r) \times SO(2n-2r)\}/\mathbb{Z}_2$

The roots of the canonical almost complex structure J are given by

$$x_\lambda \pm x_\mu, \quad 1 \leqq \lambda \leqq m < \mu \leqq n,$$
$$-(x_\lambda + x_\mu), \quad 1 \leqq \lambda < \mu \leqq m.$$

We may see that $I_G = S\{x_1{}^2, \ldots, x_n{}^2\}$ and $I_K = S\{x_1, \ldots, x_m\} \otimes S\{x_{m+1}{}^2, \ldots, x_n{}^2\}$. Therefore we have

$$H^*(G/K, \mathbb{R}) \cong S\{x_1, \ldots, x_m\} \otimes S\{x_{m+1}{}^2, \ldots, x_n{}^2\}/I_G^+,$$

and any elements of $H^*(G/K, \mathbb{R})$ may be expressed by the elements of $S\{x_1, \ldots, x_m\}$. If we denote by $\sigma_r \in H^{2r}(G/K, \mathbb{R})$ the r-th Chern class of the canonical principal $U(m)$-bundle $(SO(2n)/SO(2n-2m), G/K, U(m))$, then

$$\sum_{r=0}^{m} \sigma_r = \prod_{\lambda=1}^{m}(1 + x_\lambda) \mod I_G^+.$$

Then we have

$$\begin{aligned} c(G/K, J) = &\prod_{1 \leqq \lambda < \mu \leqq m} (1 - x_\lambda - x_\mu) \\ &\times \prod_{1 \leqq \lambda \leqq m < \nu \leqq n} (1 + x_\lambda - x_\nu)(1 + x_\lambda + x_\nu) \qquad \mod I_G^+. \end{aligned}$$

In particular, the first Chern class $c_1(G/K, J)$ is given by

$$c_1(G/K, J) = (2n - 3m + 1)\sigma_1.$$

Since $\sigma_1 \neq 0$, we see that $c_1(G/K, J) = 0$ if and only if

$$G/K = SO(3m-1)/(U(m) \times SO(m-1)) \quad (m \geqq 2; \text{odd}).$$

Case (5) $\quad G/K = G_2/U(2)$

The roots of the canonical almost complex structure J are given by

$$x_1 - x_2, \quad x_1 + 2x_2, \quad x_1, \quad x_1 + x_2,$$
$$-(2x_1 + x_2).$$

So the first Chern class is given by

$$c_1(G/K, J) = 2x_1 + x_2 \qquad \bmod I_G^+,$$

which does not vanish. If we put

$$\Phi_1 = \{\pm(x_1 - x_2), \pm(x_1 + 2x_2), \pm x_1, \pm(x_1 + x_2)\},$$
$$\Phi_2 = \{\pm(2x_1 + x_2)\},$$
$$\mathfrak{m}_i^{\mathbb{C}} = \sum_{\alpha \in \Phi_i} \mathfrak{g}_\alpha,$$

then the Ricci tensor ρ is given by

$$\rho(X, Y) = -\frac{7}{4}(\alpha_1, \alpha_1) < X, Y > /c \quad \text{for } X \in \mathfrak{m}_1, Y \in \mathfrak{m},$$
$$\rho(X, Y) = -\frac{3}{2}(\alpha_1, \alpha_1) < X, Y > /c \quad \text{for } X \in \mathfrak{m}_2, Y \in \mathfrak{m}.$$

Hence $(G/K, < \ , \ >)$ is not Einstein.

Case (6) $\quad G/K = F_4/\{Spin(7) \times T^1\}/\mathbb{Z}_2$

The roots of the canonical almost complex structure J are given by

$$x_1 \pm x_i, \quad 2 \leqq i \leqq 4,$$
$$x_1,$$
$$(x_1 \pm x_2 \pm x_3 \pm x_4)/2.$$

So the first Chern class is given by

$$c_1(G/K, J) = 11x_1 \qquad \bmod I_G^+,$$

which does not vanish. If we put

$$\Phi_1 = \{\pm(x_1 \pm x_i) \quad (2 \leqq i \leqq 4), \pm x_1\},$$
$$\Phi_2 = \{\pm(x_1 \pm x_2 \pm x_3 \pm x_4)/2\},$$
$$\mathfrak{m}_i^{\mathbb{C}} = \sum_{\alpha \in \Phi_i} \mathfrak{g}_\alpha,$$

then the Ricci tensor ρ is given by

$$\rho(X,Y) = -4(\alpha_4,\alpha_4) < X,Y > /c \quad \text{for } X \in \mathfrak{m}_1, Y \in \mathfrak{m},$$
$$\rho(X,Y) = -\frac{29}{4}(\alpha_4,\alpha_4) < X,Y > /c \quad \text{for } X \in \mathfrak{m}_2, Y \in \mathfrak{m}.$$

Hence $(G/K, < , >)$ is not Einstein.

Case (7) $G/K = F_4/\{Sp(3) \times T^1\}/\mathbb{Z}_2$

The roots of the canonical almost complex structure J are given by

$$-(\boldsymbol{x}_1 + \boldsymbol{x}_2),$$
$$\boldsymbol{x}_i \pm \boldsymbol{x}_j, \quad 1 \leqq i \leqq 2 < j \leqq 4,$$
$$\boldsymbol{x}_1, \boldsymbol{x}_2,$$
$$(\boldsymbol{x}_1 + \boldsymbol{x}_2 \pm \boldsymbol{x}_3 \pm \boldsymbol{x}_4)/2.$$

So the first Chern class is given by

$$c_1(G/K, J) = 6(\boldsymbol{x}_1 + \boldsymbol{x}_2) \qquad \mathrm{mod}\ I_G^+,$$

which does not vanish. If we put

$$\Phi_1 = \{\pm(\boldsymbol{x}_1 + \boldsymbol{x}_2)\},$$
$$\Phi_2 = \{\pm(\boldsymbol{x}_i \pm \boldsymbol{x}_j)(1 \leqq i \leqq 2 < j \leqq 4), \pm\boldsymbol{x}_1, \pm\boldsymbol{x}_2,$$
$$\pm(\boldsymbol{x}_1 + \boldsymbol{x}_2 \pm \boldsymbol{x}_3 \pm \boldsymbol{x}_4)/2\},$$
$$\mathfrak{m}_i^{\mathbb{C}} = \sum_{\alpha \in \Phi_i} \mathfrak{g}_\alpha,$$

then the Ricci tensor ρ is given by

$$\rho(X,Y) = -\frac{11}{4}(\alpha_2,\alpha_2) < X,Y > /c \quad \text{for } X \in \mathfrak{m}_1, Y \in \mathfrak{m},$$
$$\rho(X,Y) = -\frac{17}{4}(\alpha_2,\alpha_2) < X,Y > /c \quad \text{for } X \in \mathfrak{m}_2, Y \in \mathfrak{m}.$$

Hence $(G/K, < , >)$ is not Einstein.

Case (8) $G/K = \{E_6/\mathbb{Z}_3\}/\{[S(U(5)\times U(1))/\mathbb{Z}_3]\times SU(2)\}/\mathbb{Z}_2$

The roots of the canonical almost complex structure J are given by

$$
\begin{aligned}
&-x_1+x_i, \quad 2\leqq i\leqq 5,\\
&x_i+x_j, \quad 2\leqq i<j\leqq 5,\\
&(-x_1+\sum_{i=2}^{5}(-1)^{\nu(i)}x_i+\sqrt{3}x_6)/2 \quad (\sum_{i=2}^{5}\nu(i)=3),\\
&-(-x_1+\sum_{i=2}^{5}(-1)^{\nu(i)}x_i+\sqrt{3}x_6)/2 \quad (\sum_{i=2}^{5}\nu(i)=1),\\
&(x_1+\sum_{i=2}^{5}(-1)^{\nu(i)}x_i+\sqrt{3}x_6)/2 \quad (\sum_{i=2}^{5}\nu(i)=2),\\
&(x_1+x_2+x_3+x_4+x_5+\sqrt{3}x_6)/2.
\end{aligned}
$$

So the first Chern class is given by

$$
c_1(G/K,J)=(-3x_1+7\sum_{i=2}^{5}x_1-5\sqrt{3}x_6)/2,
$$

which does not vanish. If we put

$$
\begin{aligned}
\Phi_1=\{&\pm(-x_1+x_i) \quad (2\leqq i\leqq 5),\\
&\pm(x_i+x_j) \quad (2\leqq i<j\leqq 5),\\
&\pm(-x_1+\sum_{i=2}^{5}(-1)^{\nu(i)}x_i+\sqrt{3}x_6)/2 \quad (\sum_{i=2}^{5}\nu(i)=3),\\
&\pm(x_1+\sum_{i=2}^{5}(-1)^{\nu(i)}x_i+\sqrt{3}x_6)/2 \quad (\sum_{i=2}^{5}\nu(i)=2)\},\\
\Phi_2=\{&\pm(-x_1+\sum_{i=2}^{5}(-1)^{\nu(i)}x_i+\sqrt{3}x_6)/2 \quad (\sum_{i=2}^{5}\nu(i)=1),\\
&\pm(x_1+x_2+x_3+x_4+x_5+\sqrt{3}x_6)/2\}.
\end{aligned}
$$

then the Ricci tensor ρ is given by

$$
\begin{aligned}
\rho(X,Y)&=-\frac{21}{4}(\alpha_1,\alpha_1)<X,Y>/c \quad \text{for } X\in\mathfrak{m}_1, Y\in\mathfrak{m},\\
\rho(X,Y)&=-\frac{9}{2}(\alpha_1,\alpha_1)<X,Y>/c \quad \text{for } X\in\mathfrak{m}_2, Y\in\mathfrak{m}.
\end{aligned}
$$

Hence $(G/K, < \, , \, >)$ is not Einstein.

Case (9) $G/K = \{E_6/\mathbb{Z}_3\}/\{[SU(6)/\mathbb{Z}_3] \times T^1\}/\mathbb{Z}_2$

The roots of the canonical almost complex structure J are given by

$$\begin{aligned}
&(\sum_{i=1}^{5}(-1)^{\nu(i)}x_i + \sqrt{3}x_6)/2 \quad (\sum_{i=1}^{5}\nu(i) = 2),\\
&x_i + x_j, \quad 1 \leqq i < j \leqq 5,\\
&-(x_1 + x_2 + x_3 + x_4 + x_5 + \sqrt{3}x_6)/2.
\end{aligned}$$

So the first Chern class is given by

$$c_1(G/K, J) = 9(x_1 + x_2 + x_3 + x_4 + x_5 + \sqrt{3}x_6)/2,$$

which does not vanish. If we put

$$\begin{aligned}
\Phi_1 &= \{\pm(\sum_{i=1}^{5}(-1)^{\nu(i)}x_i + \sqrt{3}x_6)/2 \quad (\sum_{i=1}^{5}\nu(i) = 2),\\
&\quad \pm(x_i + x_j) \quad (1 \leqq i < j \leqq 5)\},\\
\Phi_2 &= \{\pm(x_1 + x_2 + x_3 + x_4 + x_5 + \sqrt{3}x_6)/2\}.
\end{aligned}$$

then the Ricci tensor ρ is given by

$$\begin{aligned}
\rho(X,Y) &= -\frac{23}{4}(\alpha_1, \alpha_1) < X, Y > /c \quad \text{for } X \in \mathfrak{m}_1, Y \in \mathfrak{m},\\
\rho(X,Y) &= -\frac{7}{2}(\alpha_1, \alpha_1) < X, Y > /c \quad \text{for } X \in \mathfrak{m}_2, Y \in \mathfrak{m}.
\end{aligned}$$

Hence $(G/K, < \, , \, >)$ is not Einstein.

Case (10) $G/K = \{E_6/\mathbb{Z}_3\}/\{[Spin(8) \times SO(2)]/\mathbb{Z}_2 \times SO(2)\}/\mathbb{Z}_2$

The roots of the canonical almost complex structure J are given by

$$(\sum_{i=1}^{4}(-1)^{\nu(i)}x_i - x_5 + \sqrt{3}x_6)/2 \quad (\sum_{i=1}^{4}\nu(i) = 3),$$
$$-(-x_1 - x_2 - x_3 - x_4 + x_5 + \sqrt{3}x_6)/2,$$
$$-x_i + x_5, \quad 1 \leqq i \leqq 4,$$
$$-(-x_1 + \sum_{i=2}^{4}(-1)^{\nu(i)}x_i + x_5 + \sqrt{3}x_6)/2 \quad (\sum_{i=2}^{4}\nu(i) = 1),$$
$$(\sum_{i=1}^{4}(-1)^{\nu(i)}x_i - x_5 + \sqrt{3}x_6)/2 \quad (\sum_{i=1}^{4}\nu(i) = 1),$$
$$x_i + x_5, \quad 2 \leqq i \leqq 4,$$
$$-(x_1 + x_2 + x_3 + x_4 + x_5 + \sqrt{3}x_6)/2.$$

Hence $c_1(G/K, J) = 0$. And by the straightforward calculation, the Ricci tensor ρ is given by

$$\rho(X, Y) = -5(\alpha_1, \alpha_1) < X, Y > /c \quad \text{for } X, Y \in \mathfrak{m}.$$

Therefore $(G/K, < \, , \, >)$ is Einstein.

Case (11) $G/K = \{E_7/\mathbb{Z}_2\}/\{[SU(2) \times (Spin(10) \times SO(2))/\mathbb{Z}_2]\mathbb{Z}_2\}/\mathbb{Z}_2$

The roots of the canonical almost complex structure J are given by

$$x_j \pm x_i, \quad 1 \leqq i \leqq 4 < j \leqq 6,$$
$$-\sqrt{2}x_7, \quad -(x_6 + x_5),$$
$$-x_7/\sqrt{2} - (x_6 + x_5 + \sum_{i=1}^{4}(-1)^{\nu(i)}x_i)/2 \quad (\sum_{i=1}^{4}\nu(i) \text{ is odd}),$$
$$x_7/\sqrt{2} + (x_6 - x_5 + \sum_{i=1}^{4}(-1)^{\nu(i)}x_i)/2 \quad (\sum_{i=1}^{4}\nu(i) \text{ is even}),$$
$$x_7/\sqrt{2} + (-x_6 + x_5 + \sum_{i=1}^{4}(-1)^{\nu(i)}x_i)/2 \quad (\sum_{i=1}^{4}\nu(i) \text{ is even}).$$

So the first Chern class is given by

$$c_1(G/K, J) = 6x_7/\sqrt{2} + 3(x_6 + x_5) \qquad \mathrm{mod}\ I_G^+,$$

which does not vanish. If we put

$$\Phi_1 = \{\pm(x_j \pm x_i) \quad (1 \leqq i \leqq 4 < j \leqq 6),$$

$$\pm (x_7/\sqrt{2} + (x_6 - x_5 + \sum_{i=1}^{4}(-1)^{\nu(i)} x_i)/2),$$

$$\pm (x_7/\sqrt{2} + (-x_6 + x_5 + \sum_{i=1}^{4}(-1)^{\nu(i)} x_i)/2) \quad (\sum_{i=1}^{4} \nu(i) \text{ is even})\},$$

$$\Phi_2 = \{\pm\sqrt{2} x_7, \pm(x_6 + x_5),$$

$$\pm (x_7/\sqrt{2} + (x_6 + x_5 + \sum_{i=1}^{4}(-1)^{\nu(i)} x_i)/2)(\sum_{i=1}^{4} \nu(i) \text{ is odd})\},$$

$$\mathfrak{m}_i^{\mathbb{C}} = \sum_{\alpha \in \Phi_i} \mathfrak{g}_\alpha,$$

then the Ricci tensor ρ is given by

$$\rho(X,Y) = -\frac{31}{4}(\alpha_1, \alpha_1) < X, Y > /c \quad \text{for } X \in \mathfrak{m}_1, Y \in \mathfrak{m},$$

$$\rho(X,Y) = -7(\alpha_1, \alpha_1) < X, Y > /c \quad \text{for } X \in \mathfrak{m}_2, Y \in \mathfrak{m}.$$

Hence $(G/K, < \ , \ >)$ is not Einstein.

Case (12) $G/K = \{E_7/\mathbb{Z}_2\}/\{[SO(2) \times Spin(12)]/\mathbb{Z}_2\}/\mathbb{Z}_2$

The roots of the canonical almost complex structure J are given by

$$-\sqrt{2} x_7,$$

$$x_7/\sqrt{2} + (\sum_{i=1}^{6}(-1)^{\nu(i)} x_i)/2 \quad (\sum_{i=1}^{6} \nu(i) \text{ is odd}).$$

So the first Chern class is given by

$$c_1(G/K, J) = 30 x_7/\sqrt{2} \qquad \text{mod } I_G^+,$$

which does not vanish. If we put

$$\Phi_1 = \{\pm\sqrt{2} x_7\},$$

$$\Phi_2 = \{\pm(x_7/\sqrt{2} + (\sum_{i=1}^{6}(-1)^{\nu(i)} x_i)/2) \quad (\sum_{i=1}^{6} \nu(i) \text{ is odd})\},$$

$$\mathfrak{m}_i^{\mathbb{C}} = \sum_{\alpha \in \Phi_i} \mathfrak{g}_\alpha,$$

then the Ricci tensor ρ is given by

$$\rho(X,Y) = -5(\alpha_1,\alpha_1) < X,Y > /c \quad \text{for } X \in \mathfrak{m}_1, Y \in \mathfrak{m},$$
$$\rho(X,Y) = -\frac{35}{4}(\alpha_1,\alpha_1) < X,Y > /c \quad \text{for } X \in \mathfrak{m}_2, Y \in \mathfrak{m}.$$

Hence $(G/K, < \ , \ >)$ is not Einstein.

Case (13) $\quad G/K = \{E_7/\mathbb{Z}_2\}/S\{U(7) \times U(1)\}/\mathbb{Z}_2$

The roots of the canonical almost complex structure J are given by

$$\begin{aligned} &-\sqrt{2}x_7, \\ &x_j + x_i, \quad 1 \leqq i < j \leqq 6, \\ &-x_7/\sqrt{2} - (\sum_{i=1}^{6}(-1)^{\nu(i)}x_i)/2 \quad (\sum_{i=1}^{6}\nu(i) = 1), \\ &x_7/\sqrt{2} + (\sum_{i=1}^{6}(-1)^{\nu(i)}x_i)/2 \quad (\sum_{i=1}^{6}\nu(i) = 3). \end{aligned}$$

So the first Chern class is given by

$$c_1(G/K,J) = 12x_7/\sqrt{2} + 3(x_6 + x_5 + x_4 + x_3 + x_2 + x_1) \qquad \mathrm{mod}\ I_G^+,$$

which does not vanish. If we put

$$\begin{aligned} \Phi_1 = \{&\pm\sqrt{2}x_7, \\ &\pm(x_7/\sqrt{2} + (\sum_{i=1}^{6}(-1)^{\nu(i)}x_i)/2) \quad (\sum_{i=1}^{6}\nu(i) = 1)\}, \\ \Phi_2 = \{&\pm(x_j + x_i) \quad (1 \leqq i < j \leqq 6), \\ &\pm(x_7/\sqrt{2} + (\sum_{i=1}^{6}(-1)^{\nu(i)}x_i)/2) \quad (\sum_{i=1}^{6}\nu(i) = 3)\}, \\ \mathfrak{m}_i^{\mathbb{C}} = &\sum_{\alpha\in\Phi_i}\mathfrak{g}_\alpha, \end{aligned}$$

then the Ricci tensor ρ is given by

$$\rho(X,Y) = -\frac{13}{2}(\alpha_1,\alpha_1) < X,Y > /c \quad \text{for } X \in \mathfrak{m}_1, Y \in \mathfrak{m},$$
$$\rho(X,Y) = -8(\alpha_1,\alpha_1) < X,Y > /c \quad \text{for } X \in \mathfrak{m}_2, Y \in \mathfrak{m}.$$

Hence $(G/K, <\ ,\ >)$ is not Einstein.

Case (14) $G/K = E_8/\{SO(14) \times SO(2)\}$

The roots of the canonical almost complex structure J are given by

$$-(x_1 \pm x_i), \quad 2 \leqq i \leqq 8,$$
$$(x_1 + \sum_{i=2}^{8}(-1)^{\nu(i)}x_i)/2 \quad (\sum_{i=2}^{8}\nu(i) \text{ is odd}).$$

So the first Chern class is given by

$$c_1(G/K, J) = 18x_1 \qquad \mod I_G^+,$$

which does not vanish. If we put

$$\Phi_1 = \{\pm(x_1 \pm x_i) \quad (2 \leqq i \leqq 8)\},$$
$$\Phi_2 = \{\pm(x_1 + \sum_{i=2}^{8}(-1)^{\nu(i)}x_i)/2 \quad (\sum_{i=2}^{8}\nu(i) \text{ is odd})\},$$
$$\mathfrak{m}_i^{\mathbb{C}} = \sum_{\alpha \in \Phi_i} \mathfrak{g}_\alpha,$$

then the Ricci tensor ρ is given by

$$\rho(X,Y) = -11(\alpha_1, \alpha_1) < X, Y > /c \quad \text{for } X \in \mathfrak{m}_1, Y \in \mathfrak{m},$$
$$\rho(X,Y) = -\frac{53}{4}(\alpha_1, \alpha_1) < X, Y > /c \quad \text{for } X \in \mathfrak{m}_2, Y \in \mathfrak{m}.$$

Hence $(G/K, <\ ,\ >)$ is not Einstein.

Case (15) $G/K = E_8/\{E_7 \times T^1\}/\mathbb{Z}_2$

The roots of the canonical almost complex structure J are given by

$$-(x_1 + x_2),$$
$$x_i \pm x_j, \quad 1 \leqq i \leqq 2 < j \leqq 8,$$
$$(x_1 + x_2 + \sum_{i=3}^{8}(-1)^{\nu(i)}x_i)/2 \quad (\sum_{i=3}^{8}\nu(i) \text{ is odd}).$$

So the first Chern class is given by

$$c_1(G/K, J) = 27(x_1 + x_2) \qquad \text{mod } I_G^+,$$

which does not vanish. If we put

$$\begin{aligned}
\Phi_1 &= \{\pm(x_1 + x_2)\}, \\
\Phi_2 &= \{\pm(x_i \pm x_j) \quad (1 \leqq i \leqq 2 < j \leqq 8), \\
&\qquad \pm (x_1 + x_2 + \sum_{i=3}^{8}(-1)^{\nu(i)}x_i)/2 \quad (\sum_{i=3}^{8}\nu(i) \text{ is odd})\}, \\
\mathfrak{m}_i^{\mathbb{C}} &= \sum_{\alpha \in \Phi_i} \mathfrak{g}_\alpha,
\end{aligned}$$

then the Ricci tensor ρ is given by

$$\begin{aligned}
\rho(X, Y) &= -8(\alpha_1, \alpha_1) < X, Y > /c \quad \text{for } X \in \mathfrak{m}_1, Y \in \mathfrak{m}, \\
\rho(X, Y) &= -\frac{59}{4}(\alpha_1, \alpha_1) < X, Y > /c \quad \text{for } X \in \mathfrak{m}_2, Y \in \mathfrak{m}.
\end{aligned}$$

Hence $(G/K, < \ , \ >)$ is not Einstein.

Case (16) $G/K = G_2/SU(3) = S^6$

The roots of the canonical almost complex structure J are given by

$$x_1, \quad x_2, \quad -(x_1 + x_2).$$

Hence $c_1(G/K, J) = 0$. And by Theorem 2.4.5, $(G/K, < \ , \ >)$ is Einstein.

Case (17) $G/K = F_4/\{SU(3) \times SU(3)\}/\mathbb{Z}_3$

The roots of the canonical almost complex structure J are given by

$$\begin{aligned}
&-(x_1 \pm x_4), \quad -(x_2 + x_3), \\
&x_i \pm x_4, \quad i = 2, 3, \\
&x_1 - x_i, \quad i = 2, 3, \\
&-x_1, \quad x_2, \quad x_3, \quad -(x_1 + x_2 + x_3 \pm x_4)/2, \\
&(x_1 + x_2 - x_3 \pm x_4)/2, \quad (x_1 - x_2 + x_3 \pm x_4)/2.
\end{aligned}$$

Hence $c_1(G/K, J) = 0$. And by Theorem 2.4.5, $(G/K, <\ ,\ >)$ is Einstein.

Case (18) $G/K = \{E_6/\mathbb{Z}_3\}/\{SU(3) \times SU(3) \times SU(3)\}/\{\mathbb{Z}_3 \times \mathbb{Z}_3\}$

The roots of the canonical almost complex structure J are given by

$$\begin{aligned}
&(-x_1 - x_2 + \sum_{i=3}^{5}(-1)^{\nu(i)}x_i + \sqrt{3}x_6)/2 \quad (\sum_{i=3}^{5}\nu(i) = 2),\\
&-x_i + x_j, \quad 1 \leqq i \leqq 2 < j \leqq 5,\\
&-(-x_1 + x_2 + \sum_{i=3}^{5}(-1)^{\nu(i)}x_i + \sqrt{3}x_6)/2 \quad (\sum_{i=3}^{5}\nu(i) = 1),\\
&-(x_1 - x_2 + \sum_{i=3}^{5}(-1)^{\nu(i)}x_i + \sqrt{3}x_6)/2 \quad (\sum_{i=3}^{5}\nu(i) = 1),\\
&-(x_i + x_j), \quad 3 \leqq i < j \leqq 5,\\
&(x_1 + x_2 + \sum_{i=3}^{5}(-1)^{\nu(i)}x_i + \sqrt{3}x_6)/2 \quad (\sum_{i=3}^{5}\nu(i) = 2),\\
&x_i + x_j, \quad 1 \leqq i \leqq 2 < j \leqq 5.
\end{aligned}$$

Hence $c_1(G/K, J) = 0$. And by Theorem 2.4.5, $(G/K, <\ ,\ >)$ is Einstein.

Case (19) $G/K = \{E_7/\mathbb{Z}_2\}/\{SU(3) \times [SU(6)/\mathbb{Z}_2]\}/\mathbb{Z}_3$

The roots of the canonical almost complex structure J are given by

$$\begin{aligned}
&x_j \pm x_i, \quad 1 \leqq i \leqq 3 < j \leqq 6,\\
&-(x_j + x_i), \quad 4 \leqq i < j \leqq 6,\\
&-x_7/\sqrt{2} - (\sum_{j=4}^{6}(-1)^{\mu(j)}x_j + \sum_{i=1}^{3}(-1)^{\nu(i)}x_i)/2\\
&\qquad (\sum_{i=1}^{3}\nu(i) \text{ is even}, \sum_{j=4}^{6}\mu(j) = 1),\\
&x_7/\sqrt{2} + (\sum_{j=4}^{6}(-1)^{\mu(j)}x_j + \sum_{i=1}^{3}(-1)^{\nu(i)}x_i)/2\\
&\qquad (\sum_{i=1}^{3}\nu(i) \text{ is odd}, \sum_{j=4}^{6}\mu(j) = 2).
\end{aligned}$$

Hence $c_1(G/K, J) = 0$. And by Theorem 2.4.5, $(G/K, <\ ,\ >)$ is Einstein.

Case (20) $G/K = E_8/\{SU(3) \times E_6\}/\mathbb{Z}_3$

The roots of the canonical almost complex structure J are given by

$$
\begin{aligned}
&-(x_1 \pm x_i), \quad 4 \leqq i \leqq 8,\\
&-(x_2 + x_3), \quad x_1 - x_2,\\
&x_1 - x_3, \quad x_i \pm x_j, \quad 2 \leqq i \leqq 3 < j \leqq 8,\\
&-(x_1 + x_2 + x_3 + \sum_{i=4}^{8}(-1)^{\nu(i)}x_i)/2 \quad (\sum_{i=4}^{8}\nu(i) \text{ is odd}),\\
&(x_1 + \sum_{i=2}^{3}(-1)^{\nu(i)}x_i + \sum_{j=4}^{8}(-1)^{\mu(j)}x_j)/2\\
&\quad(\sum_{i=2}^{3}\nu(i) = 1, \sum_{j=4}^{8}\mu(j) \text{ is even})\}.
\end{aligned}
$$

Hence $c_1(G/K, J) = 0$. And by Theorem 2.4.5, $(G/K, <\ ,\ >)$ is Einstein.

Case (21) $G/K = E_8/\{SU(9)/\mathbb{Z}_3\}$

The roots of the canonical almost complex structure J are given by

$$
\begin{aligned}
&-(x_1 + x_8), \quad -(x_1 \pm x_i), \quad 2 \leqq i \leqq 7,\\
&x_i + x_j, \quad 2 \leqq i < j \leqq 7,\\
&x_i - x_8, \quad 2 \leqq i \leqq 7,\\
&-(x_1 + \sum_{i=2}^{7}(-1)^{\nu(i)}x_i + x_8)/2 \quad (\sum_{i=2}^{7}\nu(i) = 1),\\
&-(x_1 + \sum_{i=2}^{7}(-1)^{\nu(i)}x_i - x_8)/2 \quad (\sum_{i=2}^{7}\nu(i) = 2),\\
&(x_1 + \sum_{i=2}^{7}(-1)^{\nu(i)}x_i + x_8)/2 \quad (\sum_{i=2}^{7}\nu(i) = 3),\\
&(x_1 + \sum_{i=2}^{7}(-1)^{\nu(i)}x_i - x_8)/2 \quad (\sum_{i=2}^{7}\nu(i) = 4).
\end{aligned}
$$

Hence $c_1(G/K, J) = 0$. And by Theorem 2.4.5, $(G/K, <\ ,\ >)$ is Einstein.

Finally we shall calculate the second Stiefel-Whitney class of two kinds of Riemannian 3-symmetric spaces.

Case (22) $G/K = \{SU(n+1)/\mathbb{Z}_{n+1}\}/S(U(h) \times U(m-h) \times U(n-m+1))/\mathbb{Z}_{n+1}$

From Proposition 2.4.3, we may see that

$$\begin{aligned} &H^*(G/K, \mathbb{Z}) \\ &\quad \cong S\{x_1, \cdots, x_h\} \otimes S\{x_{h+1}, \cdots, x_m\} \otimes S\{x_{m+1}, \cdots, x_{n+1}\}/S\{x_1, \cdots, x_{n+1}\}. \end{aligned}$$

Since $\sum_{\lambda=1}^{h} x_\lambda$ and $\sum_{\mu=h+1}^{m} x_\mu$ generate $H^2(G/K, \mathbb{Z})$, we may see that G/K is a spin manifold $w_2(G/K) \equiv c_1(G/K, J) \mod 2$ vanishes if and only if

$$G/K = \{SU(2n+1)/\mathbb{Z}_{2n+1}\}/S(U(2r_1+1) \times U(2r_2-1) \times U(2n-2r_1-2r_2+1))/\mathbb{Z}_{2n+1},$$

or

$$G/K = \{SU(2n+2)/\mathbb{Z}_{2n+2}\}/S(U(2r_1) \times U(2r_2) \times U(2n-2r_1-2r_2+2))/\mathbb{Z}_{2n+2}.$$

Case (23) $G/K = \{Sp(n)/\mathbb{Z}_2\}/\{U(r) \times Sp(n-r)\}/\mathbb{Z}_2$

Since σ_1 is a generator of $H^2(G/K, \mathbb{Z})$, we may see that G/K is a spin manifold if and only if

$$G/K = Sp(n)/\{U(2r+1) \times Sp(n-2r-1)\}, \quad 0 \leqq r < \frac{n}{2}.$$

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References

[1] M. F. Atiyah, N. J. Hitchin and I. M. Singer, *Self-duality in four-dimensional Riemannian geometry*, Proc. R. Math. Soc. **362** (1978), 425–461.

[2] A. Besse, *Einstein Manifolds*, Springer-Verlag, Berlin Heidelberg, 1987.

[3] E. Bombieri and D. Husemoller, *Classification and embeddings of surfaces*, Proc. Sympos. Pure Math. **29** (1975), 329–420.

[4] A. Borel, *Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts*, Ann. of Math. **57** (1953), 115–207.

[5] ———, *La cohomologie mod 2 de certains espaces homogènes*, Commentrarii Mathematici Helvetici **27** (1953), 165–191.

[6] A. Borel and F. Hirzebruch, *Characteristic classes and homogeneous spaces*, I, Amer. J. Math. **80** (1958), 458–538; II, ibid. **81** (1959), 315–382; III, ibid. **82** (1960), 491–504.

[7] J. P. Bourguignon, *Les variétés de dimension 4 à signature non nulle dont la courbure est harmonique sont d'Einstein*, Invent. Math. **63** (1981), 263–286.

[8] C. P. Boyer, *Conformal duality and compact complex surfaces*, Math. Ann. **274** (1986), 517–526.

[9] ———, *Self-dual and anti-self-dual Hermitian metrics on compact complex surfaces*, Contemporary Math. **71** (1988), 105–114.

[10] R.L. Bryant, *Lie groups and twistor spaces*, Duke Math. J. **52** (1985), 223–261.

[11] B. Y. Chen, *Some topological obstructions to Bochner-Kaehler metrics and their applications*, Jour. Differential Geom. **13** (1978), 547–558.

[12] A. Derdzinski, *Self-dual Kähler manifolds and Einstein manifolds of dimension four*, Compositio Math. **49** (1983), 405–433.

[13] S.K. Donaldson and R. Friedman, *Connected sums of self-dual manifolds and deformations of singular spaces*, Nonlinearlity **2** (1989), 197–239.

[14] T. Friedrich and H. Kurke, *Compact four-dimensional self-dual Einstein manifolds*

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

with positive scalar curvature, Math. Nachr. **106** (1982), 271–299.

[15] A. Gray, *Riemannian manifolds with geodesic symmetries of order* 3, J. Differential Geom. **7** (1972), 343–369.

[16] ———, *Curvature identities for Hermitian and almost Hermitian manifolds*, Tôhoku Math. J. **28** (1976), 601–612.

[17] A. Gray and L. M. Hervella, *The sixteen classes of almost Hermitian manifolds*, Ann. Mat. Pura Appl. **123** (1980), 35–58.

[18] T. Heaps, *Almost complex structures on eight- and ten-dimensional manifolds*, Topology **9** (1970), 111–119.

[19] S. Helgason, *Differential Geometry, Lie Groups, and Symmetric Spaces*, Academic Press, New York, 1978.

[20] F. Hirzebruch, *Topological Methods in Algebraic Geometry*, Springer-Verlag, Berlin, 1966.

[21] N. J. Hitchin, *Compact four-dimensional Einstein manifolds*, J. Differential Geom. **9** (1974), 435–441.

[22] ———, *Kählerian twistor spaces*, Proc. London Math. Soc. **43** (1981), 133–150.

[23] M. Itoh, *Self-duality of Kähler surfaces*, Compositio Math. **51** (1984), 265–273.

[24] U. K. Kim, I.-B. Kim and J.-B. Jun, *On self-dual almost Hermitian 4-manifolds*, Nihonkai Math. J. **3** (1992), 163–176.

[25] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, I, II, Interscience Publishers, New York, 1963, 1969.

[26] T. Koda, *Self-dual and anti-self-dual Hermitian surfaces*, Kodai Math. J. **10** (1987), 335–342.

[27] ———, *The first Chern classes of compact Riemannian 3-symmetric spaces*, Math. Rep. Toyama Univ. **12** (1989), 113–138.

[28] ———, *The first Chern class of Riemmanian 3-symmetric spaces : the classical case*, Note di Matematica **10** (1990).

[29] ———, *A remark on the manifold* $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$ *with Bérard-Bergery's metric* **11** (1993) (to appearAnn. Global Analysis and Geometry).

[30] ———, *A remark on the second homotopy groups of compact Reimannian 3-symmetric spaces* **18** (1993) (to appearTsukuba J. Math.).

[31] T. Koda and K. Sekigawa, Self-dual Einstein Hermitian surfaces, "Recent Developments in Differential Geometry", Advanced Studies in Pure Mathematics, vol. 20, 1993, pp. 1–9 (to appear).

[32] T. Koda and Y. Watanabe, *Geometry of* $SU(2) \times SU(2) \times U(1)/U(1) \times U(1)$, Math. J. Toyama University **14** (1991), 193–207.

[33] K. Kodaira, *On the structure of compact analytic surfaces I, II, III, IV*, Amer. J. Math. **86, 88, 90** (1964; 1966; 1968), 751–798, 682–721, 55–83, 1048–1066.

[34] O. Kowalski, *Generalized Symmetric spaces*, Lecture Notes in Math. 805, Springer-Verlag, Berlin Heidelberg, 1980.

[35] C. LeBrun, *On the topology of self-dual* 4*-manifolds*, Proc. Amer. Math. Soc. **98** (1986), 637–640.

[36] ———, *Poon's self-dual metrics and Kähler geometry*, J. Differential Geometry **28** (1988), 341–343.

[37] ———, *Explicit self-dual metrics on* $\mathbb{CP}_2\#\ldots\#\mathbb{CP}_2$, J. Differential Geometry **34** (1991), 223–253.

[38] P. Libermann, *Classification and conformal properties of almost Hermitian structures*, Janos Bolyai 31., Differential Geometry, Budapest (Hungary), 1979, 371–391.

[39] J. Milnor and J. Stasheff, *Characteristic Classes*, Princeton Univ. Press, 1974.

[40] Y. Miyaoka, *On the Chern numbers of surfaces of general type*, Invent. Math. **42** (1977), 225–237.

[41] A. Newlander and L. Nirenberg, *Complex analytic coordinates in almost complex manifolds*, Ann. of Math. **65** (1957), 391–404.

[42] H. Osborn, *Vector Bundles Volume 1*, Academic Press, 1982.

[43] S. Salamon, Harmonic and holomorphic maps, Geometry Seminar "Luigi Bianchi II" –1984, Lecture Notes in Mathematics, 1164, Springer-Verlag, 1984, pp. 161–224.

[44] T. Sato and K. Sekigawa, *Hermitian surfaces of constant holomorphic sectional curvature*, Math. Z. **205** (1990), 659–668.

[45] ———, *Hermitian surfaces of constant holomorphic sectional curvature II*, (preprint).

[46] K. Sekigawa, *On some 4-dimensional compact Einstein almost Kähler manifolds*, Math. Ann. **271** (1985), 333–337.

[47] ———, *On some 4-dimensional compact almost Hermitian manifolds*, J. Ramanujan Math. Soc. **2** (1987), 101–116.

[48] K. Sekigawa and J. Watanabe, *On some compact Riemannian 3-symmetric spaces*, Science Reports of Niigata Univ. **19** (1983), 1–17.

[49] K. Sekigawa and H. Yoshida, *Riemannian 3-symmetric spaces defined by some outer automorphisms of compact Lie groups*, Tensor, N.S. **40** (1983), 261–268.

[50] R. Schoen, *Conformal deformation of a Riemannian metric to constant scalar curvature*, J. Differential Geom. **20** (1985), 479–495.

[51] Y. T. Siu, *Every K3 surface is Kähler*, Inventiones Math. **73** (1983), 139–150.

[52] N. Steenrod, *The Topology of Fibre Bundles*, Princeton Univ. Press, 1972.

[53] M. Takeuchi, *On Pontrjagin classes of compact symmetric spaces*, J. Fac. Sci. Univ. Tokyo, Sect.IA Math. **9** (1961–63), 313–328.

[54] S. Tanno, *Constancy of holomorphic sectional curvature in almost Hermitian manifolds*, Kôdai Math. Sem. Rep. **25** (1973), 190–201.

[55] F. Tricerri and I. Vaisman, *On some 2-dimensional Hermitian manifolds*, Math. Z. **192** (1986), 205–216.

[56] F. Tricerri and L. Vanhecke, *Curvature tensors on almost Hermitian manifolds*, Trans. Amer. Math. Soc. **267** (1981), 365–398.

[57] I. Vaisman, *On locally conformal almost Kähler manifolds*, Israel J. Math. **24** (1976), 338–351.

[58] ———, *Some survature properties of complex surfaces*, Ann. Mat. Pura Appl. **32** (1982), 1–18.

[59] R. O. Wells, Jr., *Differential Analysis on Complex Manifolds*, Springer-Verlag, New York, 1980.

[60] G. W. Whitehead, *Elements of Homotopy Theory*, Springer-Verlag, New York, 1978.

[61] J. A. Wolf, *The geometry and structure of isotropy irreducible homogeneous spaces*, Acta Math. **120** (1968), 59–148.

[62] J. A. Wolf and A. Gray, *Homogeneous spaces defined by Lie group automorphisms*, I, J. Differential Geom. **2** (1968), 77–114; II, ibid. **2** (1968), 115–159.

[63] W.T. Wu, *Sur les classes caractéristiques des structures fibrées sphériques*, Actual. Sci. Ind. **1183** (1952), 1–89.

[64] I. Yokota, *Realizations of involutive automorphisms σ and G^σ of exceptional linear Lie groups G, Part I, $G = G_2, F_4$, and E_6*, Tsukuba J. Math. **14** (1990), 185–223.

[65] ———, *Realizations of involutive automorphisms σ and G^σ of exceptional linear Lie groups G, Part II, $G = E_7$*, Tsukuba J. Math. **14** (1990), 379–404.

[66] ———, *Realizations of involutive automorphisms σ and G^σ of exceptional linear Lie groups G, Part III, $G = E_8$*, Tsukuba J. Math. **15** (1991), 301–314.