Homomorphisms and isometries on Banach algebras of continuous functions
（連続関数からなるバナッハ環の上の準同形写像と等距離写像）

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Contents

CHAPTER 1

Introduction

1. Organization

The aim of this dissertation is to study homomorphisms and isometries between spaces or algebras of continuous functions and vector-valued continuous maps. A long tradition of inquiry seeks sufficient sets of the properties of Banach algebras in terms of surjective linear isometries. A linear isometry on a Banach algebra encodes not only the geometric structure as a Banach space but also the algebraic structures of the underlying Banach algebra. The most prominent result along these lines is the Banach-Stone theorem on a linear map between the commutative C^*-algebra of all complex-valued continuous functions on a compact Hausdorff space. This theorem states that two compact Hausdorff spaces Y_1 and Y_2 are homeomorphic if and only if their corresponding algebras $C(Y_1)$ and $C(Y_2)$ are isomorphic if and only if they are isometrically isomorphic as Banach spaces. Thus the basic problem of interest is to derive extensions of the Banach-Stone theorem for several different settings. We consider the problem whether underlying commutative Banach algebras are isomorphic or not when there exists a linear map U between the algebras which preserves the distance of elements in the algebras.

The algebra of Lipschitz functions and the algebra of continuously differentiable functions are typical commutative Banach algebras. Let (X, d) be a compact metric space. A complex-valued continuous function $f : X \to \mathbb{C}$ is called a Lipschitz function if there exists a positive number L such that

$$|f(x) - f(y)| \leq Ld(x, y)$$

for every $x,\ y \in X$. For any Lipschitz function f, we define Lipschitz constant $L(f)$ by

$$L(f) = \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x, y)}. \tag{1.1}$$

We denote by $\mathrm{Lip}(X)$ the space of all Lipschitz functions on X. The space $\mathrm{Lip}(X)$ is a Banach space under the following two norms respectively:

(1) max norm $\|\cdot\|_{\max}$
For any $f \in \mathrm{Lip}(X)$, we define

$$\|f\|_{\max} = \max\{\sup_{x \in X} |f(x)|, L(f)\};$$

(2) sum norm $\|\cdot\|_{\Sigma}$
For any $f \in \mathrm{Lip}(X)$, we define

$$\|f\|_{\Sigma} = \sup_{x \in X} |f(x)| + L(f).$$

We point out that $\mathrm{Lip}(X)$ is an algebra and the sum norm $\|\cdot\|_{\Sigma}$ is submultiplicative in the sense of the inequality;

$$\|fg\|_{\Sigma} \leq \|f\|_{\Sigma}\|g\|_{\Sigma}, \quad f, g \in \mathrm{Lip}(X).$$

Thus, $\mathrm{Lip}(X)$ is a unital commutative Banach algebra with $\|\cdot\|_{\Sigma}$. Note that the algebra $\mathrm{Lip}(X)$ needs not be a Banach algebra with respect the max norm since $\|\cdot\|_{\max}$ needs not satisfy the submultiplicativity. We prefer to study $\mathrm{Lip}(X)$ with the sum norm in this dissertation. Although the algebra $\mathrm{Lip}(X)$ needs not be a Banach algebra in the strict sense, it does satisfy the following inequality

$$\|fg\|_{\max} \leq 2\|f\|_{\max}\|g\|_{\max}, \quad f, g \in \mathrm{Lip}(X).$$

Therefore, $\mathrm{Lip}(X)$ has a weak-$*$ topology with each norms and the algebraic structure of $\mathrm{Lip}(X)$ has been studied for many years. The seminal works of the study of Banach space and Banach algebra of $\mathrm{Lip}(X)$ are due to Mirkil [**81**] and de Leeuw [**28**]. They studied the space of periodic Lipschitz functions on the real line. Among them de Leeuw showed the existence of the predual of the space of Lipschitz functions by applying the so called de Leeuw's map, which is now very familiar to us. His result first turned the attention of mathematicians to the new theory. Sherbert has developed the structure of the algebra of Lipschitz functions [**111, 112**]. In [**111**], he exhibited the following foundamental result.

PROPOSITION 1.1. *Let A be a semi-simple commutative Banach algebra with identity. Then the Gelfand mapping is a continuous isomorphism of A onto a subalgebra of* $\mathrm{Lip}(\mathcal{M})$*, where $\mathcal{M}$ is the space of the maximal ideals with the topology induced by the operator norm.*

This implies that any unital semi-simple commutative Banach algebra can be seen as a subalgebra of the algebra of Lipschitz functions. This is the basic motivation for the author of the study on the algebras of Lipschitz functions and mappings. It is well known by the Gelfand theory that $\mathcal{M}$ is compact in the Gelfand topology, which is the relative weak-$*$ topology. On the other hand $\mathcal{M}$ with the topology induced by the operator norm is complete metric space which needs not be compact.

Sherbert [**111**, Theorem 5.1] characterized the unital algebra homomorphism on $\mathrm{Lip}(X)$.

THEOREM 1.2. *Let X_j be a compact metric space for $j = 1, 2$. Then every unital algebra homomorphism $T : \mathrm{Lip}(X_1) \to \mathrm{Lip}(X_2)$ is of the form*

$$(Tf)(x) = f(\varphi(x)), \quad f \in \mathrm{Lip}(X_1), x \in X_2, \tag{1.2}$$

where $\varphi : X_2 \to X_1$ satisfies

$$d(\varphi(x_1), \varphi(x_2)) \leq K d(x_1, x_2) \quad x_1, x_2 \in X_2 \tag{1.3}$$

for some positive constant K. Conversely, if T is defined on $\mathrm{Lip}(X_1)$ *by (1.2) where $\varphi : X_2 \to X_1$ satisfies (1.3), then T is a unital algebra homomorphism of* $\mathrm{Lip}(X_1)$ *into* $\mathrm{Lip}(X_2)$*. T is one-to-one if and only if $\varphi(X_2) = X_1$, T takes* $\mathrm{Lip}(X_1)$ *onto* $\mathrm{Lip}(X_2)$ *if and only if φ satisfies the additional condition*

$$K' d(x_1, x_2) \leq d(\varphi(x_1), \varphi(x_2)) \quad x_1, x_2 \in X_2$$

for some positive constant K'.

Recently the studies on the algebra of vector-valued Lipschitz maps are popular. Let E be a Banach space with the norm $\|\cdot\|_E$. For a metric space X, a continuous map $F : X \to E$ is a Lipschitz map if there exists a positive number L such that

$$\|F(x) - F(y)\|_E \leq Ld(x, y)$$

for every $x,\ y \in X$. The definition of Lipschitz constant $L(F)$ is the same with (1.1) by substituting $\|\cdot\|_E$ for $|\cdot|$. We define the algebra of E-valued Lipschitz maps by

$$\mathrm{Lip}(X, E) = \{F : X \to E\ ; F \text{ is a Lipschitz map}\}.$$

If E is a Banach algebra with $\|\cdot\|_E$, $\mathrm{Lip}(X, E)$ is also Banach algebra with $\|\cdot\|_\Sigma$. In addition, if E is a semi-simple Banach algebra then so is $\mathrm{Lip}(X, E)$ with $\|\cdot\|_\Sigma$. On the other hand, in the case of $\|\cdot\|_{\max}$, we have by the similar way as the case of complex-valued functions that

$$\|FG\|_{\max} \leq 2\|F\|_{\max}\|G\|_{\max}, \quad F, G \in \mathrm{Lip}(X, E).$$

The constant 2 in the above inequality is best and it implies that even if E is a Banach algebra $\|\cdot\|_{\max}$ needs not be a Banach algebra norm on $\mathrm{Lip}(X, E)$. This is one of the major differences between $\|\cdot\|_{\max}$ and $\|\cdot\|_\Sigma$. There are various studies on algebras of vector-valued Lipschitz maps with each of the norms.

On the other hand, what is a motivation of the study on $\mathrm{Lip}(X, E)$? One of the motivation of the author is that the comparison of a given Banach space E and the algebra of Lipschitz functions $\mathrm{Lip}(X)$, and to clarify how much different between E and $\mathrm{Lip}(X)$.

In Chapter 2 we study homomorphisms on the algebras of vector-valued maps. We introduce a notion of admissible quadruple (cf. Definition 2.16) which is equivalent to the one defined by Nikou and O'Farrell [**95**]. The Banach algebra of the all Lipschitz maps on a compact metric space with the value in a unital commutative C^*-algebra is an admissible quadruple. We prove that a unital homomorphism on certain admissible quadruple is of a peculiar form which is called of type BJ. In particular, we have under some additional assumption that a unital homomorphism between the Banach algebra of the all Lipschitz maps with the value in a unital commutative C^*-algebra is of type BJ. This result means that the Banach algebra of all Lipschitz functions and a unital commutative C^*-algebra are completely different in its manner.

In Chapter 3 isomorphisms on admissible quadruple which is a generalization of the Banach algebra of Lipschitz maps with the values in a uniform algebra. We prove that algebra isomorphisms on certain admissible quadruples with the value in a uniform algebra is of type BJ (Theorem 3.3), while unital homomorphisms need not be of type BJ (Example 2.32).

Not only algebra homomorphisms but also surjective linear isometries on algebras of Lipschitz functions have been studied for decades of years. Let B_j be a Banach space with the norm $\|\cdot\|_j$ for $j = 1, 2$. Recall that a map $T : B_1 \to B_2$ is an isometry if $\|T(b_1) - T(b_2)\|_{B_2} = \|b_1 - b_2\|_{B_1}$ for every pair $b_1, b_2 \in B_1$. For instance, a classical problem on the spaces of the Lipschitz maps is as follows. Let X_i be a compact metric space for $i = 1, 2$. Suppose that $\mathrm{Lip}(X_1, E)$ and $\mathrm{Lip}(X_2, E)$ are isometric. Does it follow that X_1 and X_2 are isometric? The answer is not. They are not even homeomorphic. For example, let $E_i = \mathrm{Lip}(X_i)$ for $i = 3, 4$, where X_i is a compact metric space. Let $T : \mathrm{Lip}(X_1, \mathrm{Lip}(X_3)) \to \mathrm{Lip}(X_2, \mathrm{Lip}(X_4))$ be an algebra homomorphism. Since we have $\mathrm{Lip}(X_1, \mathrm{Lip}(X_3)) = \mathrm{Lip}(X_1 \times X_3)$ and $\mathrm{Lip}(X_2, \mathrm{Lip}(X_4)) = \mathrm{Lip}(X_2 \times X_4)$, by Theorem 1.2 we get there exists Lipschitz map $\varphi : X_2 \times X_4 \to X_1 \times X_3$ such that

$$TF(x, y) = F(\varphi(x, y)), \quad F \in \mathrm{Lip}(X_1, \mathrm{Lip}(X_3)), \quad (x, y) \in X_2 \times X_4.$$

Even if φ is a homeomorphism and X_3 and X_4 are homeomorphic, X_1 needs not be homeomorphic to X_2. In this dissertation, we focus on this problem. In the other words, one of the purpose of this dissertation is to give a view point on the matter whether each operator from $\mathrm{Lip}(X_1, E)$ into $\mathrm{Lip}(X_2, E)$ can be induced by a homeomorphism between X_1 and X_2. We study surjective linear isometries on admissible quadruples of type L in Chapter 4.

Roy proved the following theorem by applying the de Leeuw's map in [**106**, Theorem 1.7].

THEOREM 1.3. *Let X be a compact, connected metric space with diameter at most 1. Then $T : \mathrm{Lip}(X) \to \mathrm{Lip}(X)$ is a surjective linear isometry with $\|\cdot\|_{\max}$ if and only if*

$$Tf(x) = e^{i\theta} f(\varphi(x)),$$

where $\varphi : X \to X$ is an isometry of X onto itself and θ is a constant in $[0, 2\pi)$.

As for the problem on isometries with respect to the sum norm, Rao and Roy [**104**] proved that every surjective linear isometry on $\mathrm{Lip}([0,1])$ is represented as a weighted composition operator, which is sometimes called the canonical form. Rao and Roy [**104**, p.189] posed a problem if a similar result is valid for $\|\cdot\|_\Sigma$ on a compact metric space X instead of $[0,1]$. This problem on surjective linear isometries $T : \mathrm{Lip}(X) \to \mathrm{Lip}(X)$ had not been solved until quite recently. We point out that the problem for $\|\cdot\|_\Sigma$ is substantially harder than that one for $\|\cdot\|_{\max}$ by the fact that the structure of the extreme points of the closed unit ball of the dual space are complicated in the former case. Jarosz and Pathak exhibited in [**48**, Example 8] that a surjective isometry on $\mathrm{Lip}(X)$ and $\mathrm{lip}(X)$ of a compact metric space X with respect to the norm $\|\cdot\|_\Sigma$ is canonical. There seems to be a confusion of the status of the result and we clarify the current situation. After the publication of [**48**] some authors expressed their suspicion about the argument there and the validity of the statement there had not been confirmed when the authors of [**75**] pointed out a gap by referring the comment of Weaver [**116**, p. 243]. While Weaver in [**116**] pointed out that the argument of [**48**] failed on p.200 in which the norm $\|\cdot\|_{\max}$ was studied, he did not seem to have stated explicitly that the argument in the Example 8 contained a flaw.

In Chapter 4, we prove that a form of surjective isometries on admissible quadruples of type L (Theorem 4.5) is of type BJ. As a corollary we solve the problem of Rao and Roy affirmatively (Corollary 4.15).

In Chapters 5 and 6 we study Hermitian operators. Recall that $C^1([0,1])$ is the algebra of complex-valued continuously differentiable functions on $[0,1]$. On $C^1([0,1])$, there are various types of norms on $C^1([0,1])$. We introduce the well known one among them.

(1) max norm $\|\cdot\|_{\max}$
For any $f \in C^1([0,1])$, we define

$$\|f\|_{\max} = \max\{\sup_{x\in[0,1]} |f(x)|, \sup_{x\in[0,1]} |f'(x)|\}.$$

(2) sum norm $\|\cdot\|_\Sigma$
For any $f \in C^1([0,1])$, we define

$$\|f\|_\Sigma = \sup_{x\in[0,1]} |f(x)| + \sup_{x\in[0,1]} |f'(x)|.$$

We also define the algebra of vector-valued continuously differentiable maps on $[0,1]$ with $\|\cdot\|_\Sigma$. We denote the set of all E-valued continuously differentiable maps on $[0,1]$ by $C^1([0,1],E)$. This is a Banach algebra with $\|\cdot\|_\Sigma$ provided that E is a Banach algebra. For $C^1([0,1])$ and $C^1([0,1],E)$, there are also wide range of studies for algebra homomorphisms, Hermitian operators and surjective linear isometries with respect to various norms. By previous results, we can see a lot of results for $C^1([0,1])$, which resemble the statement for $\mathrm{Lip}(X)$. This means that both Banach spaces must have similar properties. It is strange that the studies on the algebra of Lipschitz maps, and on the algebra of continuously differentiable maps on $[0,1]$ are independent. At this point as second purpose of this dissertation, we propose a unified approach for both of the Banach spaces. In Chapters 2 and 4, we define the admissible quadruples. The admissible quadruple enable us to study $\mathrm{Lip}(X)$, $C^1([0,1])$, $\mathrm{Lip}(X,E)$ and $C^1([0,1],E)$ simultaneously, where E is certain Banach spaces.

In Chapter 7 we study a Hermitian operators on the tensor product of a uniform algebra and a unital C^*-algebra and the Banach-Stone properties.

We study a local map in Chapter 8. A Local map has a long history. To among other important subjects, it is strongly related to the Kaplansky's problem which is posed in 1970; whether invertibility preserving linear operators on algebras is Jordan homomorphisms or not? Recall it as follows.

Problem 1.4. *For* $i = 1, 2$*, let* $\mathfrak{B}_i$ *be a semi-simple Banach algebras and let* $\phi : \mathfrak{B}_1 \to \mathfrak{B}_2$ *be a surjective linear operator such that*

$$\phi(\mathbf{1}) = 1$$

and

$$a \in \mathfrak{B}_1 \text{ is invertible} \Longrightarrow \phi(a) \in \mathfrak{B}_2 \text{ is invertible.}$$

Then is ϕ *a Jordan homomorphism?*

This concerns with the problem how the local properties of an operator determine the global properties. In 1967 and 1968, Gleason and Kahane and Żelazko proved a classical result in the theory of Banach algebras.

Theorem 1.5. *Let* $\mathfrak{B}$ *be a complex Banach algebra (we do not assume that* $\mathfrak{B}$ *is unital nor commutative). Let* $\Delta : \mathfrak{B} \to \mathbb{C}$ *be a linear functional such that*

$$\Delta(a) \in \sigma(a), \quad a \in \mathfrak{B},$$

where $\sigma(a)$ *is the spectrum of* a*. Then* Δ *is multiplicative, i.e.*

$$\Delta(ab) = \Delta(a)\Delta(b), \quad a, b \in \mathfrak{B}.$$

It is greatly attracted in what local behavior of the operator decide the operator globally. The study of a local map was initiated by Kadison [**63**], Larson [**71**], and Larson and Sourour [**72**]. Kadison in [**63**] proved that a local derivation on a von Neumann algebra R into a dual R-bimodule is a derivation. Larson and Sourour in [**72**] showed that every local derivation of the algebra of all bounded linear operators on a Banach space is a derivation and if a Banach space is infinite-dimensional, every invertible local automorphism is an automorphism. This results have been improved by Brešar and Šemrl [**19, 20, 21**]. Let A_i be a complex Banach space for $i = 1, 2$. Denote by $\mathcal{B}(A_1, A_2)$ the set of all bounded linear operators from A_1 into A_2. We call the subset $\mathcal{S} \subset \mathcal{B}(A_1, A_2)$ algebraically reflexive if every bounded linear operator T belongs to $\mathcal{S}$ whenever $T \in \mathcal{B}(A_1, A_2), Tf \in \mathcal{S}f$ for every $f \in A_1$. Recently, the study of algebraic reflexivity of the subspace of bounded linear operators is attracted greatly. Many researcher study the algebraic reflexivity for various operators on Banach spaces. In this dissertation, we consider the set of surjective linear isometries on unital semi-simple commutative Banach algebras.

In Chapter 9 and Appendix A, we study 2-local isometry. A 2-local map is initially studied by Šemrl in [**110**]. His motivation to define and study a 2-local map comes from the Kowalski-Słodkowski theorem as follows;

THEOREM 1.6. *Let* $\mathfrak{B}$ *be a complex Banach algebra. We do not assume that* $\mathfrak{B}$ *is unital or commutative. Let* $\Delta : \mathfrak{B} \to \mathbb{C}$ *such that*

$$\Delta(0) = 0$$

and

$$\Delta(a) - \Delta(b) \in \sigma(a - b), \quad a, b \in \mathfrak{B}.$$

Then Δ *is linear and multiplicative.*

This theorem is a generalization of the Gleason-Kahane-Żelazko Theorem. Our interest is whether 2-local map T is in $\mathcal{S}$ or not. In this dissertation, we mainly study a 2-local isometry on a various Banach space. In Chapter 9, we study 2-local reflexivity of the set of all surjective isometries between certain function spaces. We do not assume linearity for isometries. Without assuming the linearity of isometries, the problem is much harder. Whether every 2-local isometry (do not assume the linearity) on $C(Y)$, where Y is a first countable compact Hausdorff space, is a surjective isometry or not had been unsolved. This problem is posed by Molnár. We prove that a 2-local map in the group of all surjective isometries on the algebra of all continuously differentiable functions is a surjective isometry. In Appendix A, we generalize the Kowalski-Słodkowski theorem. By applying the generalization of the Kowalski-Słodkowski theorem, we prove that a 2-local isometry on a certain semi-simple commutative Banach algebra is a surjective isometry. In Section 3, we give a positive answer to the Molnár's problem.

2. Definitions

Throughout this section E is a Banach space and $\mathfrak{B}c$ is a unital commutative Banach algebra. We denote the unity for $\mathfrak{B}c$ by $\mathbf{1}_{\mathfrak{B}c}$. If no confusion can arise, we write just $\mathbf{1}$. A non-zero multiplicative linear functional on $\mathfrak{B}c$ is automatically continuous with norm 1. Hence the set $\mathcal{M}(\mathfrak{B}c)$ of all non-zero multiplicative linear functionals on $\mathfrak{B}c$ is a subset of the unit sphere of the dual space $\mathfrak{B}c^*$ with the usual functional norm $\|\cdot\|_*$. We call $\mathcal{M}(\mathfrak{B}c)$ maximal ideal space or Gelfand space. We sometime denote the maximal ideal space for $\mathfrak{B}c$ by $\mathcal{M}$ or $\mathcal{M}_j$ for $\mathfrak{B}c_j$, just for simplicity. The Gelfand topology on $\mathcal{M}(\mathfrak{B}c)$ is the relative topology of the weak-$*$ topology of $\mathfrak{B}c^*$; thus $\mathcal{M}(\mathfrak{B}c)$ is a compact Hausdorff space. The kernel $\phi^{-1}(0)$ for $\phi \in \mathcal{M}(\mathfrak{B}c)$ is a maximal ideal of $\mathfrak{B}c$. Conversely for any maximal ideal $\mathcal{M}$ of $\mathfrak{B}c$, there exists a unique $\phi \in \mathcal{M}(\mathfrak{B}c)$ with $\mathcal{M} = \phi^{-1}(0)$. The maximal ideal space of $\mathfrak{B}c$ is $\mathcal{M}(\mathfrak{B}c)$ with the Gelfand topology. The Gelfand transform of $a \in \mathfrak{B}c$ is denoted by $\Gamma_{\mathfrak{B}c}(a)$; $\Gamma_{\mathfrak{B}c}(a) : \mathcal{M}(\mathfrak{B}c) \to \mathbb{C}$, $\Gamma_{\mathfrak{B}c}(a)(\phi) = \phi(a)$ for $\phi \in \mathcal{M}(\mathfrak{B}c)$. For simplicity of notation, we sometimes denote the Gelfand transform of a by $\hat{a}$. The Gelfand topology on $\mathcal{M}(\mathfrak{B}c)$ is the weakest topology that $\hat{a}$ is continuous for every $a \in \mathfrak{B}c$. We denote the set $\{\Gamma_{\mathfrak{B}c}(a) : a \in S\}$ by $\Gamma_{\mathfrak{B}c}(S)$ for a subset S of $\mathfrak{B}c$. The set $\Gamma_{\mathfrak{B}c}(\mathfrak{B}c)$ is called the Gelfand transform of $\mathfrak{B}c$ and is also denoted by $\widehat{\mathfrak{B}c}$. We denote the spectrum of a by $\sigma(a)$, the spectral radius by $r(a)$, the group of all invertible elements by $\mathfrak{B}c^{-1}$. The Jacobson radical, the intersection of all maximal ideals, of $\mathfrak{B}c$ is denoted by $\mathrm{rad}(\mathfrak{B}c)$. We have $a \in \mathrm{rad}(\mathfrak{B}c)$ if and only if $r(a) = 0$ if and only if $\sigma(a) = \{0\}$ (see [**70**, Proposition 3.5.1, Theorem 3.5.1]). We say that $\mathfrak{B}c$ is semi-simple if $\mathrm{rad}(\mathfrak{B}c) = \{0\}$. Hence $\mathfrak{B}c$ is semi-simple if and only if the Gelfand map $\Gamma_{\mathfrak{B}c} : \mathfrak{B}c \to \widehat{\mathfrak{B}c}$ is an isomorphism. For the theory of commutative Banach algebras, see for instance [**22, 64, 70, 101**].

Let Y be a compact Hausdorff space. The space of all continuous maps from a compact Hausdorff space Y into E is denoted by $C(Y, E)$. For $S \subset Y$ we denote

$$\|f\|_{\infty(S)} = \sup_{x \in S} \|f(x)\|_E, \quad f \in C(Y, E).$$

When no confusion will result we omit the subscript S and write just $\|\cdot\|_\infty$. The supremum norm $\|\cdot\|_{\infty(Y)}$ makes $C(Y,E)$ a complex Banach space. Note that $C(Y,\mathfrak{B}c)$ is a unital commutative Banach algebra, and it is semisimple provided that so is $\mathfrak{B}c$. The algebra $C(Y,\mathbb{C})$ is abbreviated by $C(Y)$. The real-algebra of all real-valued continuous functions on Y is denoted by $C_{\mathbb{R}}(Y)$. A uniform algebra A on a compact Hausdorff space Y is a closed subalgebra of $C(Y)$ which contains the constants and separates the points of Y. A uniform algebra A on Y is a semi-simple commutative Banach algebra with the supremum norm on the set Y.

2.1. The space of Lipschitz maps. Let X be a compact metric space and $0<\alpha\leq 1$. For $F\in C(X,E)$, put

$$L_\alpha(F)=\sup_{x\neq y}\frac{\|F(x)-F(y)\|_E}{d(x,y)^\alpha},$$

which is called an α-Lipschitz constant of F, or just a Lipschitz constant of F. When $\alpha=1$ we usually omit the subscript α and write only $L(F)$. The space of all $F\in C(X,E)$ such that $L_\alpha(F)<\infty$ is denoted by $\mathrm{Lip}_\alpha(X,E)$. When $\alpha=1$ we usually omit the subscript α and write $\mathrm{Lip}(X,E)$. When $0<\alpha<1$ the closed subspace

$$\begin{aligned}&\mathrm{lip}(X,E)\\&=\{F\in \mathrm{Lip}_\alpha(X,E):\lim_{x\to x_0}\frac{\|f(x_0)-f(x)\|_E}{d(x_0,x)^\alpha}=0 \text{ for every } x_0\in X\}\end{aligned}$$

of $\mathrm{Lip}_\alpha(X,E)$ is called a little Lipschitz space. In this dissertation the norm $\|\cdot\|_\Sigma$ of $\mathrm{Lip}_\alpha(X,E)$ (resp. $\mathrm{lip}(X,E)$) is defined by

$$\|F\|_\Sigma=\|F\|_{\infty(X)}+L_\alpha(F),\quad F\in \mathrm{Lip}_\alpha(X,E)\ (\text{resp. } \mathrm{lip}(X,E)),$$

unless otherwise stated. Note that if $d(\cdot,\cdot)$ is a metric, then so is $d(\cdot,\cdot)^\alpha$, and is denoted by d^α which is called a Hölder metric. For a compact metric space (X,d), The space $\mathrm{Lip}_\alpha((X,d),E)$ (resp. $\mathrm{Lip}_\alpha((X,d),\mathfrak{B}c)$) is a Banach space (resp. unital commutative Banach algebra, and it is semi-simple provided that so is $\mathfrak{B}c$). $\mathrm{Lip}_\alpha((X,d),E)$ is isometrically isomorphic to $\mathrm{Lip}((X,d^\alpha),E)$ as a Banach space and $\mathrm{Lip}_\alpha((X,d),\mathfrak{B}c)$ is isometrically isomorphic to $\mathrm{Lip}((X,d^\alpha),\mathfrak{B}c))$ as a Banach algebra.

We introduce the algebraic tensor product space with a crossnorm. For any $g \in \mathrm{Lip}(X)$ and $e \in E$, we define $g \otimes e : X \to E$ by

$$(g \otimes e)(x) = g(x)e.$$

Then $g \otimes e \in \mathrm{Lip}(X, E)$ and we have two equalities

$$\|g \otimes e\|_\infty = \|g\|_\infty \|e\|_E,$$

$$L(g \otimes e) = L(g)\|e\|_E.$$

These imply that $\|g \otimes e\|_\Sigma = \|g\|_\Sigma \|e\|_E$. We define the algebraic tensor product space with a crossnorm.

Definition 1.7. Let

$$\mathrm{Lip}(X) \otimes E = \{\Sigma_{i=1}^n g_i \otimes e_i; g_i \in \mathrm{Lip}(X),\ e_i \in E\ n \in \mathbb{N}\}.$$

By a partition of unity we have that $C(Y) \otimes E$ is uniformly dense in $C(Y, E)$. For any compact metric space X, the Stone-Weierstrass theorem asserts that $\mathrm{Lip}(X)$ is uniformly dense in $C(X)$. By a partition of unity, the algebraic tensor product $\mathrm{Lip}(X) \otimes E$ is uniformly dense in $\mathrm{Lip}(X, E)$. On the other hand, for an infinite dimensional Banach space E, it is not an easy question of whether the closure of $\mathrm{Lip}(X) \otimes E$ with Banach space norm $\| \cdot \|_\Sigma$ is the space $\mathrm{Lip}(X, E)$ or not.

2.2. The space of continuously differentiable maps. Let $C^1([0, 1], E)$ be the space of all E-valued continuously differentiable maps on the unit interval $[0, 1]$. Note that $C^1([0, 1], E)$ is a Banach space with respect to the sum norm

$$\|F\|_\Sigma = \|F'\|_{\infty([0,1])} + \|F\|_{\infty([0,1])}$$

for $F \in C^1([0, 1], E)$. We mainly consider this norm on $C^1([0, 1])$ in this dissertation. Note also that $C^1([0, 1], \mathfrak{B}c)$ is a unital commutative Banach algebra with respect to the sum norm and it is semi-simple provided that so is $\mathfrak{B}c$. The algebra $C^1([0, 1], \mathbb{C})$ is abbreviated by $C^1([0, 1])$. In the same way as in the case of $\mathrm{Lip}(X, E)$ we define the algebraic tensor product space of $C^1([0, 1])$ and a Banach space E by

$$C^1([0, 1]) \otimes E = \{\Sigma_{i=1}^n g_i \otimes e_i; g_i \in C^1([0, 1]),\ e_i \in E\ n \in \mathbb{N}\}.$$

By a partition of unity $C^1([0,1])\otimes E$ is uniformly dense in $F \in C^1([0,1],E)$. Since $\mathrm{Lip}_\alpha(X,E)$ and $C^1([0,1],E)$ have common feature in some sense, the author expect that unified arguments for $C^1([0,1],E)$ and $\mathrm{Lip}(X,E)$ are possible.

2.3. Algebra homomorphisms on Banach algebras. Let A_j be an algebra for $j=1,2$. Then an algebra homomorphism $\psi : A_1 \to A_2$ is a complex-linear map such that

$$\psi(ab) = \psi(a)\psi(b)$$

for every $a,b \in A_1$. An algebra homomorphism is essential operator to study the structure of algebra. Moreover, if an algebra homomorphism is a bijection, we call it an isomorphism. The Banach-Stone theorem asserts that two compact Hausdorff spaces Y_1 and Y_2 are homeomorphic if and only if their corresponding algebras of all complex valued continuous functions on Y_1 and Y_2 respectively are isomorphic.

2.4. Surjective linear isometries. Let $(\mathfrak{M}_j, d_j)$ be a metric space for $j=1,2$. An isometry $U : \mathfrak{M}_1 \to \mathfrak{M}_2$ is a distance preserving map, i.e, $d_2(Ua,Ub) = d_1(a,b)$ for $a,b \in \mathfrak{M}_1$. Let N_j be a normed space for $j=1,2$ we call U an isometry from N_1 into N_2 if $\|Ua-Ub\| = \|a-b\|$ for any $a,b \in N_1$. Suppose that U is a real-linear. Then U is an isometry if and only if $\|Ua\| = \|a\|$ for $a \in N_1$. A celebrated Mazur-Ulam Theorem states that every surjective isometry (need not to be linear) between real normed spaces is affine, so if surjective isometry preserve 0, then it is a real-linear isometry. We denote the set of all surjective complex-linear isometry from $\mathfrak{M}_1$ onto $\mathfrak{M}_2$ by $\mathrm{Iso}_{\mathbb{C}}(\mathfrak{M}_1,\mathfrak{M}_2)$ and the set of all surjective real-linear isometry by $\mathrm{Iso}_{\mathbb{R}}(\mathfrak{M}_1,\mathfrak{M}_2)$.

2.5. Hermitian operators. The notion of a Hermitian operator on a Banach space dates back to the seminal papers by Vidav [**115**] and Lumer [**76**]. Lumer considered a definition in terms of semi-inner product. We introduce the definition of semi-inner product on a Banach space as presented in [**76**].

DEFINITION 1.8. Let E be a Banach space. A map $[\cdot,\cdot] : E\times E \to \mathbb{C}$ is called a *semi-inner product* if the following conditions hold for every $x, y, z \in E$ and $\lambda \in \mathbb{C}$:

(1) $[x + y, z] = [x, z] + [y, z]$,
(2) $[\lambda x, y] = \lambda[x, y]$,
(3) $[x, x] > 0$ if $x \neq 0$,
(4) $|[x, y]|^2 \leq [x, x][y, y]$.

A semi-inner product $[\cdot,\cdot]$ is said to be compatible with the norm if $[x, x] = \|x\|^2$ for every $x \in E$.

The following proposition is well known.

PROPOSITION 1.9. *Any Banach space E has a semi-inner product.*

PROOF. Let E^* be the dual space of E. By Hahn-Banach theorem, for any $x \in E$, there exists linear functional $\phi_x \in E^*$ such that $\phi_x(x) = \|x\|_E^2$. We define a semi-inner product on E by $[x, y] = \phi_y(x)$ for $x, y \in E$. We get a semi-inner product immediately. □

This proposition shows that a semi-inner product which is compatible with the norm is not unique in general.

A bounded linear operator T on E is called a Hermitian operator if there exists a semi-inner product on E compatible with the norm of E such that

$$[T(a), a] \in \mathbb{R}, \quad a \in E,$$

where $\mathbb{R}$ is the set of all real numbers. The operator norm is usually denoted by $\|\cdot\|$. Note that if T is a Hermitian operator on E, then for every semi-inner product $[\cdot,\cdot]$ on E compatible with the norm, $[Ta, a] \in \mathbb{R}$ for every $a \in E$ (see [**76, 7, 30**]). Let $\mathfrak{B}$ be a Banach algebra and $\mathfrak{B}^*$ its dual space as a Banach space. We define the algebraic numerical range $V(a)$ for $a \in \mathfrak{B}$. The algebraic numerical range for $a \in \mathfrak{B}$ is given by

$$V(a) = \{\varphi(a) : \varphi \in \mathfrak{B}^*,\ \|\varphi\| = \varphi(\mathbf{1}) = 1\}.$$

We call an element $a \in \mathfrak{B}$ Hermitian if $V(a) \subset \mathbb{R}$. It is known that $a \in \mathfrak{B}$ is Hermitian if and only if $\|\exp(ita)\| = 1$ for all $t \in \mathbb{R}$ (see

[**7, 30**]). The set of all Hermitian elements in $\mathfrak{B}$ is a real linear subspace of $\mathfrak{B}$ and is denoted by $\mathrm{Her}(\mathfrak{B})$.

Note that we denote the usual Banach algebra of all bounded linear operators on a Banach space E equipped with the operator norm by $\mathbf{B}(E)$. We introduce the famous and important theorem for Hermitian operator [**30**, Theorem 5.2.6].

THEOREM 1.10. [**30**, Theorem 5.2.6] *Let T be a bounded linear operator on a Banach space E. The following are equivalent:*

(1) *T is a Hermitian operator;*
(2) $\|I + itT\| = 1 + o(t), \qquad t \in \mathbb{R}$*;*
(3) $\|\exp(itT)\| = 1$ *for all* $t \in \mathbb{R}$*;*
(4) $\exp(itT)$ *is an isometry for each* $t \in \mathbb{R}$*.*

This characterization for Hermitian operators is useful.

2.6. Local maps. Next, we define a local map from a Banach space E_1 to E_2.

DEFINITION 1.11. Let E_i be a Banach space for $i = 1, 2$. Let T be a bounded linear operator from E_1 into E_2, i.e., $T \in \boldsymbol{B}(E_1, E_2)$. Let $\mathcal{S}$ be a non-empty subset of $\boldsymbol{B}(E_1, E_2)$. We call T local in $\mathcal{S}$ if for any $x \in E_1$, there exists a bounded linear operator $T_x \in \mathcal{S}$ such that

$$Tx = T_x(x).$$

2.7. 2-local maps. Motivated by the Kowalski-Słodkowski theorem, the concept of a 2-local map was introduced by Šemrl [**110**], who proved the first results on 2-local automorphisms and derivations on algebras of operators. We define a 2-local map. Let N_i be a normed space for $i = 1, 2$. The set of all maps from N_1 into N_2 is denoted by $M(N_1, N_2)$.

DEFINITION 1.12. Let N_i be a normed space for $i = 1.2$. Let T be a map from N_1 into N_2. Note that we do not assume that linearity and continuity for T. Let $\mathcal{S}$ be a non-empty subset of the set of all maps from N_1 into N_2; $\emptyset \neq \mathcal{S} \subset M(N_1, N_2)$. We call a $T \in M(N_1, N_2)$ is a

2-local map in $\mathcal{S}$ if for any $x, y \in N_1$, there exists a map $T_{x,y} \in \mathcal{S}$ such that

$$Tx = T_{x,y}(x), \quad Ty = T_{x,y}(y).$$

We call a 2-local map in the set of all surjective isometries 2-local isometry.

CHAPTER 2

Peculiar homomorphisms on algebras of vector-valued maps

1. Introduction to peculiar homomorphisms

Gelfand theory asserts that a unital homomorphism between unital semi-simple commutative Banach algebras is represented by a composition operator.

PROPOSITION 2.1. *Let $\mathfrak{B}_j$ be a unital semi-simple commutative Banach algebra for $j = 1, 2$. Let $\mathcal{M}_j$ be a maximal ideal space for $\mathfrak{B}_j$ for $j = 1, 2$. If $\psi : \mathfrak{B}_1 \to \mathfrak{B}_2$ is a unital algebra homomorphism, then there exists a map $h : \mathcal{M}_2 \to \mathcal{M}_1$ such that*

$$\widehat{\psi(b)} = \widehat{b} \circ h$$

for every $b \in \mathfrak{B}_1$, where the map $b \to \widehat{b}$ denotes Gelfand transformation.

EXAMPLE 2.2. Let X_j and Y_j be a compact Hausdorff space for $j = 1, 2$. Suppose that $\psi : C(X_1, C(Y_1)) \to C(X_2, C(Y_2))$ is an algebra homomorphism such that $\psi(\mathbf{1}) = \mathbf{1}$. Then there exists a continuous map $h : X_2 \times Y_2 \to X_1 \times Y_1$ such that

$$\psi(F)(x, y) = F \circ h(x, y), \quad F \in C(X_1, C(Y_1)).$$

The converse also holds.

As we see in the above simple example, there exists a lot of algebra homomorphisms which are composition operators defined by continuous maps from $X_2 \times Y_2$ into $X_1 \times Y_1$ which are homeomorphic to the maximal ideal spaces. Moreover, suppose that $\psi : \mathrm{Lip}(X_1, E_1) \to \mathrm{Lip}(X_2, E_2)$ is a unital homomorphism between the algebras of all Lipschitz maps on a compact metric space X_j into a unital semi-simple commutative Banach algebra E_j with the maximal ideal space $\mathcal{M}(E_j)$. The maximal ideal space of $\mathrm{Lip}(X_j, E_j)$ is homeomorphic to $X_j \times \mathcal{M}(E_j)$ and we may suppose that $\mathrm{Lip}(X_j, E_j)$ is a subalgebra of $C(X_j \times \mathcal{M}(E_j))$ of the algebra of all complex-valued continuous functions on $X_j \times \mathcal{M}(E_j)$ through the Gelfand transform (see (2.3)). Then there exists a continuous map $\Phi : X_2 \times \mathcal{M}(E_2) \to X_1 \times M(E_1)$ denoted by $\Phi(x, \phi) = (\varphi_1(x, \phi), \varphi_2(x, \phi))$ such that $\psi(F)(x, \phi) = F(\varphi_1(x, \phi), \varphi_2(x, \phi))$ for every $(x, \phi) \in X_2 \times \mathcal{M}(E_2)$ and $F \in \mathrm{Lip}(X_1, E_1)$. Does the converse hold in general? There exist variety of examples of Lipschitz algebras for which the converse statement does not hold.

On the other hand, Botelho and Jamison [**13**] proved the following theorem. If X_2 is connected, and both of E_1 and E_2 are the algebra of convergent sequences or the algebra of bounded sequences, then φ_2 depends only on $\mathcal{M}(E_2)$, not on X_2. The author [**98**] generalized this result by showing that it is the case where E_j is a unital commutative C^*-algebra as follows.

THEOREM 2.3. [**98**, Theorem 1] *Let X_j be a compact metric space for $j = 1, 2$. Suppose that X_2 is connected. Let Y_j be a compact Hausdorff spaces for $j = 1, 2$. If $\tau : Y_2 \to Y_1$ is a continuous map, $\{\varphi(\cdot, y)\}_{y \in Y_2}$ is a set of Lipschitz maps from X_2 into X_1 with the bounded Lipschitz constants such that $y \mapsto \varphi(x, y)$ is a continuous map from Y_2 into X_1 for every $x \in X_2$, then*

$$\psi(F)(x, y) = F(\varphi(x, y), \tau(y)), \tag{1.1}$$

$F \in \mathrm{Lip}(X_1, C(Y_1))$, $y \in Y_2$ and $x \in X_2$ gives an algebra homomorphism from $\mathrm{Lip}(X_1, C(Y_1))$ into $\mathrm{Lip}(X_2, C(Y_2))$ with $\psi(\mathbf{1}_{X_1}) = \mathbf{1}_{X_2}$.

Conversely suppose that $\psi : \mathrm{Lip}(X_1, C(Y_1)) \to \mathrm{Lip}(X_2, C(Y_2))$ is an algebra homomorphism such that $\psi(\mathbf{1}_{X_1}) = \mathbf{1}_{X_2}$. Then there exist a continuous map $\tau : Y_2 \to Y_1$, a Lipschitz map $\varphi(\cdot, y) : X_2 \to X_1$ for each $y \in Y_2$, where the set of Lipschitz constants $\{L(\varphi(\cdot, y))\}_{y \in Y_2}$ is bounded and $y \mapsto \varphi(x, y)$ is a continuous map from Y_2 into X_1 for every $x \in X_2$ such that the equation (1.1) holds for every $F \in \mathrm{Lip}(X_1, C(Y_1))$, $y \in Y_2$ and $x \in X_2$.

1.1. Proof of Theorem 2.3. We give a proof.

DEFINITION 2.4. Given $f \in C(Y)$, we denote the constant map $\Phi_X(f) \in \mathrm{Lip}(X, C(Y))$ by

$$\Phi_X(f)(x) = f(x)$$

for all $x \in X$.

The subalgebra of all constant maps in $\mathrm{Lip}(X, C(Y))$ is denoted by $\mathrm{Const}(X, C(Y))$. For any constant map f, as the Lipschitz constant $L(\Phi_X(f))$ vanishes, $f \mapsto \Phi_X(f)$ ($f \in C(Y)$) gives the natural isometric isomorphism from $C(Y)$ onto $\mathrm{Const}(X, C(Y))$.

DEFINITION 2.5. For any $y \in Y$, we define $P_y^X : \mathrm{Lip}(X, C(Y)) \to \mathrm{Lip}(X)$ by

$$(P_y^X F)(x) = F(x, y), \quad F \in \mathrm{Lip}(X, C(Y))$$

for all $x \in X$.

To prove the necessity part of Theorem 2.3 we need several lemmas. In the following Lemma 2.6 through Lemma 2.12, we assume the hypotheses in Theorem 2.3. We define $T : C(Y_1) \to \mathrm{Lip}(X_2, C(Y_2))$ by $T = \psi \circ \Phi_{X_1}$. It is straightforward that the map T is an algebra homomorphism. Fix $y \in Y_2$, and we consider the map $P_y^{X_2} \circ T : C(Y_1) \to \mathrm{Lip}(X_2)$. As $C(Y_1)$ and $\mathrm{Lip}(X_2)$ are semi-simple commutative Banach algebras, we can prove following lemma immediately.

LEMMA 2.6. *The map $P_y^{X_2} \circ T$ is a continuous algebra homomorphism from $C(Y_1)$ into* $\mathrm{Lip}(X_2)$.

Note that $(P_y^{X_2} \circ T)(\mathbf{1}) = \mathbf{1}$ and $\|P_y^{X_2} \circ T\| \geq 1$ are simple observations.

Let us now consider the adjoint $(P_y^{X_2} \circ T)^* : (\mathrm{Lip}(X_2))^* \to (C(Y_1))^*$ of $P_y^{X_2} \circ T$, where $(\mathrm{Lip}(X_2))^*$ and $(C(Y_1))^*$ are the dual spaces of $\mathrm{Lip}(X_2)$ and $C(Y_1)$ as Banach spaces respectively.

For any $x \in X_2$ (resp. $y' \in Y_1$), we denote by δ_x (resp. δ'_y) the evaluational functional at x on $\mathrm{Lip}(X_2)$ (resp. at y' on $C(Y_1)$). It is well known that the maximal ideal space of $\mathrm{Lip}(X_2)$ (resp. $C(Y_1)$) is $\{\delta_x; x \in X_2\}$ (resp. $\{\delta_{y'}; y' \in Y_1\}$) with the Gelfand topology.

LEMMA 2.7. *For any $x \in X_2$, there exists a unique $y' \in Y_1$ such that*

$$(P_y^{X_2} \circ T)^*(\delta_x) = \delta_{y'}.$$

PROOF. We obtain $(P_y^{X_2} \circ T)^* \delta_x$ is a multiplicative linear functional on $C(Y_1)$.

$$((P_y^{X_2} \circ T)^* \delta_x)(1) = \delta_x(P_y^{X_2} \circ T)(\mathbf{1}) = 1,$$

which asserts that $(P_y^{X_2} \circ T)^* \delta_x \neq 0$. Hence there corresponds a unique evaluational functional at a point $y' \in Y_1$ with $(P_y^{X_2} \circ T)^*(\delta_x) = \delta_{y'}$. □

Lemma 2.7 asserts that $(P_y^{X_2} \circ T)^*$ gives a map from the maximal ideal space of $\mathrm{Lip}(X_2)$ into that of $C(Y_1)$.

LEMMA 2.8. *If $y_1' \neq y_2' \in Y_1$, then $\|\delta_{y_1'} - \delta_{y_2'}\|^* = 2$, where $\| \cdot \|^*$ denotes the usual functional norm for the dual space of $C(Y_1)$.*

PROOF. Applying Urysohn's lemma, there exists $g \in C(Y_1)$ such that $0 \leq g \leq 1$, $g(y_1') = 1$, $g(y_2') = 0$. By Lemma 2.6.1 in [**22**], we infer that $\|\delta_{y_1'} - \delta_{y_2'}\|^* \geq 2$. Since it is also clear that $\|\delta_{y_1'} - \delta_{y_2'}\|^* \leq \|\delta_{y_1'}\|^* + \|\delta_{y_2'}\|^* = 2$, we conclude that $\|\delta_{y_1'} - \delta_{y_2'}\|^* = 2$. □

The following lemma shows that $(P_y^{X_2} \circ T)^*(\{\delta_x; x \in X_2\})$ is a singleton.

LEMMA 2.9. *Let $y \in Y_2$. Then there exists a unique $\tilde{y} \in Y_1$ such that*

$$(P_y^{X_2} \circ T)^*(\{\delta_x; x \in X_2\}) = \{\delta_{\tilde{y}}\}.$$

PROOF. By Lemma 2.7, the set $(P_y^{X_2} \circ T)^*(\{\delta_x; x \in X_2\})$ is a subset of $\{\delta_{y'}; y' \in Y_1\}$. Suppose that $(P_y^{X_2} \circ T)^*(\{\delta_x; x \in X_2\})$ contains at least two elements y_1' and y_2' of Y_1 with $\delta_{y_1'} \neq \delta_{y_2'}$. There exist $x_1, x_2 \in X_2$ such that

$$(P_y^{X_2} \circ T)^*(\delta_{x_1}) = \delta_{y_1'}, (P_y^{X_2} \circ T)^*(\delta_{x_2}) = \delta_{y_2'}.$$

Let $A = \{x \in X_2; (P_y^{X_2} \circ T)^*(\delta_x) = \delta_{y_1'}\}$ and $B = \{x \in X_2; (P_y^{X_2} \circ T)^*(\delta_x) \neq \delta_{y_1'}\}$. By Lemma 2.6, we have already known that $(P_y^{X_2} \circ T)^*$ is a continuous map from $(\mathrm{Lip}(X_2))^*$ into $(C(Y_1))^*$. Hence A is a closed set. We now prove that B is also closed. In order to verify this, let

$\{z_n\} \subseteq B$ be a sequence such that $z_n \to z_0 \in X_2$ $(n \to \infty)$. For any $n \in \mathbb{N} \cup \{0\}$, there corresponds $\xi_n \in Y_1$ with

$$(P_y^{X_2} \circ T)^*(\delta_{z_n}) = \delta_{\xi_n}$$

by Lemma 2.7. Since the sequence $\{z_n\}$ is a converge sequence, there exists $n_0 \in \mathbb{N}$ such that $d(z_n, z_0) < \frac{1}{3\|P_y^{X_2} \circ T\|}$ holds for every $n \geq n_0$. For any $x, z \in X_2$, we have

$$\|\delta_x - \delta_z\|^* = \sup_{\|f\|_\Sigma \leq 1} |f(x) - f(z)| \leq \sup_{\|f\|_\Sigma \leq 1} L(f)d(x,z) \leq d(x,z).$$

By [**107**, Theorem 4.10], we get $\|P_y^{X_2} \circ T\| = \|(P_y^{X_2} \circ T)^*\|^*$. Thus, if $n \geq n_0$, then

$$\begin{aligned} \|\delta_{\xi_n} - \delta_{\xi_0}\|^* &= \|(P_y^{X_2} \circ T)^*(\delta_{z_n}) - (P_y^{X_2} \circ T)^*(\delta_{z_0})\|^* \\ &\leq \|(P_y^{X_2} \circ T)^*\|^* \|\delta_{z_n} - \delta_{z_0}\|^* \\ &\leq \|P_y^{X_2} \circ T\| d(z_n, z_0) \\ &< \frac{1}{3}. \end{aligned} \tag{1.2}$$

By Lemma 2.8 and the inequality (1.2), it must be

$$\delta_{\xi_0} = \delta_{\xi_n} \neq \delta_{y_1'}.$$

By the definition of the set B, we have $z_0 \in B$. This implies that the set B is closed. However, since $x_1 \in A$, $x_2 \in B$ and $X_2 = A \cup B$, where A and B are disjoint and closed, it contradicts to the connectedness of X_2. Therefore $(P_y^{X_2} \circ T)^*(\{\delta_x; x \in X_2\})$ is a singleton. □

Applying Lemma 2.9, we define $\tau : Y_2 \to Y_1$ given by

$$(P_y^{X_2} \circ T)^*(\{\delta_x; x \in X_2\}) = \{\delta_{\tau(y)}\}$$

for all $y \in Y_2$. For any $f \in C(Y_1)$, due to the definition of τ, we have

$$((Tf)(x))(y) = f(\tau(y)) \tag{1.3}$$

for all $x \in X_2$. This implies that $T(C(Y_1)) \subseteq \mathrm{Const}(X_2, C(Y_2))$. In view of this, we can consider $\Phi_{X_2}^{-1} \circ T : C(Y_1) \to C(Y_2)$ such that

$$(\Phi_{X_2}^{-1} \circ T)(f)(y) = f(\tau(y)) \tag{1.4}$$

for all $y \in Y_2$.

Lemma 2.10. *The map $\tau : Y_2 \to Y_1$ is continuous.*

Due to the definition of τ and the equality (1.3), we claim that

$$\psi(\mathrm{Const}(X_1, C(Y_1))) \subset \mathrm{Const}(X_2, C(Y_2)).$$

Let $y \in Y_2$. We define an algebra homomorphism $J_y : \mathrm{Lip}(X_1) \to \mathrm{Lip}(X_2)$ by

$$(J_y u)(x) = \psi(u_{X_1})(x, y), \quad u \in \mathrm{Lip}(X_1)$$

for all $x \in X_2$, where u_{X_1} denote the function from X_1 into $C(Y_1)$ with $u_{X_1}(x) = u(x)$ on Y_1 for every $x \in X_1$. Since X_1 and X_2 are compact metric spaces, Theorem 5.1 in [**111**] asserts that there exists a Lipschitz map $\varphi(\cdot, y) : X_2 \to X_1$, which satisfies

$$(J_y u)(x) = u(\varphi(x, y)) \tag{1.5}$$

for all $u \in \mathrm{Lip}(X_1)$ and $x \in X_2$. Thus for any $u \in \mathrm{Lip}(X_1)$, we have

$$\psi(u_{X_1})(x, y) = u(\varphi(x, y)). \tag{1.6}$$

By (1.3) and the definition of T, for any $f \in C(Y_1)$, we also have

$$\psi(\Phi_{X_1}(f))(x, y) = f(\tau(y)). \tag{1.7}$$

Multiplying (1.6) by (1.7), we have

$$(\psi(u_{X_1})(x, y) \cdot (\psi(\Phi_{X_1}(f))(x))(y) = u(\varphi(x, y)) \cdot f(\tau(y)). \tag{1.8}$$

Since ψ is an algebra homomorphism, Lemma 2.11 follows immediately from (1.8) and Lemma 2.10.

LEMMA 2.11. *There exist a continuous map $\tau : Y_2 \to Y_1$ and a set of Lipschitz maps $\varphi(\cdot, y) : X_2 \to X_1$ such that, for every $u \in \mathrm{Lip}(X_1)$, $f \in C(Y_1)$, $y \in Y_2$ and $x \in X_2$,*

$$\begin{aligned}(\psi(&u_{X_1}\Phi_{X_1}(f))(x))(y) \\ &= ((u_{X_1}\Phi_{X_1}(f))(\varphi(x, y))(\tau(y)) = u(\varphi(x, y)) \cdot f(\tau(y)).\end{aligned}$$

LEMMA 2.12. *For any $x \in X_2$, a map $y \mapsto \varphi(x, y)$ from Y_2 into X_1 is continuous.*

We get by simple calculations, we omit the proof. We obtain the maximal ideal space of $\mathrm{Lip}(X, C(Y))$. We omit a proof since we prove the general case after (Proposition 2.18).

PROPOSITION 2.13. *Let X be a compact metric space and Y be a compact Hausdorff space. Then the maximal ideal space of* $\mathrm{Lip}(X, C(Y))$ *is homeomorphic to* $X \times Y$.

Proposition 2.13 and Proposition 2.1 imply that there exists a map $h : X_2 \times Y_2 \to X_1 \times Y_1$ such that

$$\widehat{\psi(F)} = \widehat{F} \circ h \tag{1.9}$$

for all $F \in \mathrm{Lip}(X_1, C(Y_1))$. We are now ready to prove Theorem 2.3.

Proof of the necessity part of Theorem 2.3. Let F $\in \mathrm{Lip}(X_1, C(Y_1))$. In the same way as in [**13**], we have that the algebraic tensor product space $C(X_1) \otimes C(Y_1)$, with the least crossnorm, is dense in $C(X_1, C(Y_1))$ equipped with $|| \cdot ||_\infty$ since X_1 is a compact metric space and $C(Y_1)$ is a Banach space. In addition, by the Stone-Weierstrass theorem, $\mathrm{Lip}(X_1)$ is dense in $C(X_1)$, therefore there exists a sequence $\{F_n\}$ in $\mathrm{Lip}(X_1) \otimes C(Y_1)$ that converges uniformly to F. Thus given $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $\|F - F_n\|_\infty < \epsilon$ for any $n \geq n_0$. We identify the space $\mathrm{Lip}(X_1) \otimes C(Y_1)$ with all the functions of the form $\sum_{i=1}^{k} (u_i)_{X_1} \Phi_{X_1}(f_i)$ with $u_i \in \mathrm{Lip}(X_1)$ and $f_i \in C(Y_1)$. Each function F_n is represented as follows :

$$F_n = \sum_{i=1}^{k_n} (u_i^{(n)})_{X_1} \Phi_{X_1}(f_i^{(n)})$$

with some $u_i^{(n)} \in \mathrm{Lip}(X_1)$ and $f_i^{(n)} \in C(Y_1)$. By Lemma 2.11, we have

$$\begin{aligned} \psi(F_n)(x, y) &= \psi(\sum_{i=1}^{k_n} (u_i^{(n)})_{X_1} \Phi_{X_1}(f_i^{(n)}))(x, y) \\ &= \sum_{i=1}^{k_n} \psi((u_i^{(n)})_{X_1} \Phi_{X_1}(f_i^{(n)}))(x, y) \\ &= \sum_{i=1}^{k_n} u_i^{(n)}(\varphi(x, y)) \cdot f_i^{(n)}(\tau(y)) = F_n(\varphi(x, y))(\tau(y)). \end{aligned}$$

For each F$\in \mathrm{Lip}(X_1, C(Y_1))$, $\widehat{F}$ is continuous on $X_1 \times Y_1$ in Gelfand topology. Hence we have $\|F - F_n\|_\infty = \|\widehat{F} - \widehat{F_n}\|_\infty$ for all $n \in \mathbb{N}$. Thus for any $n \geq n_0$, by the existence of a map $h : X_2 \times Y_2 \to X_1 \times Y_1$ as described in (1.9), we have

$$\begin{aligned}
&|\psi(F)(x,y) - F(\varphi(x,y))(\tau(y))| \\
&\le |\psi(F)(x,y) - \psi(F_n)(x,y)| + |F_n(\varphi(x,y))(\tau(y)) - F(\varphi(x,y))(\tau(y))| \\
&\le |\delta_{(x,y)}(\psi(F)) - \delta_{(x,y)}(\psi(F_n))| + \|F_n - F\|_\infty \\
&= |\widehat{\psi(F)}(\delta_{(x,y)}) - \widehat{\psi(F_n)}(\delta_{(x,y)})| + \|F_n - F\|_\infty \\
&= |(\widehat{F} \circ h)(\delta_{(x,y)}) - (\widehat{F_n} \circ h)(\delta_{(x,y)})| + \|F_n - F\|_\infty \\
&\le \|\widehat{F} \circ h - \widehat{F_n} \circ h\|_\infty + \|F_n - F\|_\infty \\
&\le \|\widehat{F} - \widehat{F_n}\|_\infty + \|F_n - F\|_\infty \le 2\epsilon.
\end{aligned}$$

As $\epsilon > 0$ was chosen arbitrarily, we obtain

$$\psi(F)(x,y) = F(\varphi(x,y))(\tau(y))$$

for all $x \in X_2$ and $y \in Y_2$. Finally we prove that the set of Lipschitz constants $\{L(\varphi(\cdot,y))\}_{y\in Y_2}$ is bounded. Let $y \in Y_2$. For $J_y : \mathrm{Lip}(X_1) \to \mathrm{Lip}(X_2)$, we recall the definition and the equation (1.5) as follows:

$$(J_y u)(x) = \psi(u_{X_1})(x,y) = u(\varphi(x,y)), \quad x \in X_2$$

for every $u \in \mathrm{Lip}(X_1)$. For any $x_1, x_2 \in X_2$, we have

$$\begin{aligned}
&|(J_y u)(x_1) - (J_y u)(x_2)| = |\psi(u_{X_1})(x_1,y) - \psi(u_{X_1})(x_2,y)| \\
&\le \|\psi(u_{X_1})(x_1) - \psi(u_{X_1})(x_2)\|_\infty \le L(\psi(u_{X_1}))d(x_1,x_2).
\end{aligned}$$

This implies that

$$L(J_y u) \le L(\psi(u_{X_1})) \le \|\psi(u_{X_1})\|_L.$$

As ψ is an algebra homomorphism between semi-simple commutative Banach algebras, ψ is continuous, so that

$$L(J_y u) \le \|\psi\|\|u_{X_1}\|_L = \|\psi\|\|u\|_L.$$

We also obtain

$$\|J_y u\|_\infty = \|u \circ \varphi(\cdot,y)\|_\infty \le \|u\|_\infty \le \|u\|_L.$$

Thus

$$\|J_y u\|_L = \|J_y u\|_\infty + L(J_y u) \le (\|\psi\| + 1)\|u\|_L.$$

for any $u \in \mathrm{Lip}(X_1)$. Putting $K = \|\psi\| + 1$ we have

$$\|J_y\| \le K$$

for any $y \in Y_2$. Let $y \in Y_2$. For any $x_1, x_2 \in X_2$, we define $u_y \in \mathrm{Lip}(X_1)$ by $u_y(x) = d(x, \varphi(x_2, y))$. We have

$$\frac{d(\varphi(x_1, y), \varphi(x_2, y))}{d(x_1, x_2)} = \frac{|u_y(\varphi(x_1, y)) - u_y(\varphi(x_2, y))|}{d(x_1, x_2)}$$
$$= \frac{|(J_y u_y)(x_1) - (J_y u_y)(x_2)|}{d(x_1, x_2)} \leq L(J_y u_y) \leq \|J_y u_y\|_L \leq \|J_y\| \|u_y\|_\Sigma$$
$$= \|J_y\|(\|u_y\|_\infty + L(u_y)) \leq K(\mathrm{diam}(X_1) + 1).$$

Therefore,

$$L(\varphi(\cdot, y)) \leq K(\mathrm{diam}(X_1) + 1).$$

We conclude that the set of Lipschitz constants $\{L(\varphi(\cdot, y))\}_{y \in Y_2}$ is bounded. $\square$

1.2. Remarks on a generalization of Theorem2.3. Since ψ in Theorem 2.3 is a unital algebra homomorphism on a semi-simple Banach algebra, the Gelfand theory asserts that there exists a continuous map between the maximal ideal spaces by which the given homomorphism is represented as a composition operator. The form of the homomorphism given in Theorem 2.3 is called of type BJ, in the sense that the second variable depends only on the second variable. This means that the Lipschitz algebra $\mathrm{Lip}(X)$ and the algebra $C(Y)$ are completely different; they are not miscible with each other.

We call a unital homomorphism represented by the composition operator induced by a continuous map $\Phi(x, \phi) = (\varphi_1(x, \phi), \varphi_2(\phi))$ is of type BJ. A precise definition of a unital homomorphism of type BJ is given at the end of the next section (see Definition 2.20). In this chapter, we show that several unital homomorphisms between certain Banach algebra are of type BJ. On the other hand, it is not the case in general; if a unital commutative C^*-algebra is replaced by a certain uniform algebra, a homomorphism needs not be of type BJ (see Example 2.32). It is interesting to note that isomorphisms between algebras of the all Lipschitz maps on connected compact metric spaces into uniform algebras are of type BJ (see Section 2 in Chapter 3). We give several sufficient conditions on Banach algebras which ensure that every unital homomorphism on them is of type BJ.

2. Preliminary with Definitions

Let E be a unital commutative Banach algebra. An E-valued function algebra in the strong sense is as follows.

DEFINITION 2.14. We say that A is an E-valued function algebra on a compact Hausdorff space X in the strong sense if A is a subalgebra of $C(X, E)$ for a unital commutative Banach algebra E such that the following conditions are satisfied.

(1.1) A is a Banach algebra under some norm $\|\cdot\|_A$,
(1.2) A contains the constant maps,
(1.3) A separates the points of X,
(1.4) for every $x \in X$ the evaluation map $e_x : A \to E$ defined by $f \mapsto f(x)$ is continuous.

The algebra $C(X, E)$ is an E-valued function algebra on X in the strong sense with the norm $\|\cdot\|_{\infty(X)}$. If E is semi-simple, and a subalgebra A of $C(X, E)$ is a Banach algebra under some norm, then $e_x : A \to E$ is automatically continuous for every $x \in X$ by a theorem of Šilov (see [**101**, Theorem 3.1.11]). Nikou and O'Farrell defined an E-valued function algebra [**95**, Definition 1.1]. But we have pointed out a minor error in [**95**, Definition 1.1] of an E-valued function algebra. Recently, Nikou and O'Farrell have published the corrigendum [**96**].

Let K be a compact metric space and E a unital commutative Banach algebra. We have $\|e_x(F)\|_E \leq \|F\|_L$ for every $x \in K$ and $F \in \mathrm{Lip}(K, E)$. Therefore $\mathrm{Lip}(K, E)$ is an E-valued function algebra on K.

If B is a $\mathbb{C}$-valued function algebra on X (in the strong sense), then $\{e_x : x \in X\} \subset \mathcal{M}(B)$ and the map $x \mapsto e_x$ from X into $\mathcal{M}(B)$ is a continuous injection. Hence X is embedded in $\mathcal{M}(B)$ as a compact subset. We introduce the definition of natural.

Definition 2.15. We call B is natural if the map $x \mapsto e_x$ is a surjection, that is, if X is homeomorphic to $\{e_x : x \in X\} = \mathcal{M}(B)$ through the map $x \mapsto e_x$.

The Gelfand transform of a unital commutative semi-simple Banach algebra is natural. As $\mathrm{Lip}(K, \mathbb{C})$ is dense in $C(K)$ and $\mathrm{Lip}(K, \mathbb{C})$ is inverse-closed, we can prove that $\mathrm{Lip}(K, \mathbb{C})$ is natural; K is homeomorphic to $\{e_x : x \in K\} = \mathrm{Lip}(K, \mathbb{C})$. We discuss natural in more detail (Proposition 4.2).

Definition 2.16 (see [**95**]). Let X be a compact Hausdorff space and E a commutative Banach algebra with unit. By an admissible quadruple we mean a quadruple $(X, E, B, \widetilde{B})$, where

(2.1) $B \subset C(X)$ is a natural $\mathbb{C}$-valued function algebra on X,
(2.2) $\widetilde{B} \subset C(X, E)$ is an E-valued function algebra on X in the strong sense,
(2.3) $B \otimes E \subset \widetilde{B}$ and
(2.4) $\{\lambda \circ f : f \in \widetilde{B}, \lambda \in M(E)\} \subset B$.

In fact, two definitions of an admissible quadruple by Definition 2.1 in [**95**] and Definition 2.16 are formally different. However An admissible quadruple defined by Definition 2.1 in [**95**] and one defined by Definition 2.16 in this dissertation are equivalent.

Let X be a compact Hausdorff space and E a unital commutative Banach algebra. Then $(X, E, C(X), C(X, E))$ is an admissible quadruple. Let K be a compact metric space. Then $(K, E, \mathrm{Lip}(K, \mathbb{C}), \mathrm{Lip}(K, E))$ is an admissible quadruple.

Definition 2.17. Let $(X, E, B, \widetilde{B})$ be an admissible quadruple. Let $\pi : X \times \mathcal{M}(E) \to \mathcal{M}(\widetilde{B})$ be given by $\pi(x, \phi) = \phi \circ e_x$, where $\phi \circ e_x(F) = \phi(F(x))$ for every $F \in \widetilde{B}$. Then by a routine argument π is a continuous injection. We say that an admissible quadruple $(X, E, B, \widetilde{B})$ is natural if the associated map π is bijective.

As $X \times \mathcal{M}(E)$ is compact and $\mathcal{M}(\widetilde{B})$ is Hausdorff, π is a homeomorphism if $(X, E, B, \widetilde{B})$ is natural. In this case the maximal ideal space of $\widetilde{B}$ coincides with $\{\phi \circ e_x : x \in X,\ \phi \in \mathcal{M}(E)\}$, which is homeomorphic to $X \times \mathcal{M}(E)$. Hence we may suppose that

$$\widehat{\widetilde{B}} \subset C(X \times \mathcal{M}(E)) \tag{2.1}$$

through the homeomorphism $\pi : X \times \mathcal{M}(E) \to \mathcal{M}(\widetilde{B})$; identifying (x, ϕ) and $\phi \circ e_x$ through π.

Proposition 2.18. *Let $(X, E, B, \widetilde{B})$ be an admissible quadruple. Suppose that B is dense in $C(X)$. Suppose also that $\widetilde{B}$ is inverse-closed; $F \in \widetilde{B}$ with $\Gamma_{\widetilde{B}}(F)(\phi \circ e_x) \neq 0$ for every pair $x \in X$ and $\phi \in \mathcal{M}(E)$ implies $F^{-1} \in \widetilde{B}$. Then $(X, E, B, \widetilde{B})$ is natural.*

Proof. Suppose that $\pi : X \times \mathcal{M}(E) \to \mathcal{M}(\widetilde{B})$ is defined by $(x, \phi) \mapsto \phi \circ e_x$. We prove that π is surjective. Suppose that π is not surjective. Choose any $\Phi_0 \in \mathcal{M}(\widetilde{B}) \setminus \pi(X \times \mathcal{M}(\widetilde{B}))$ and fix it. Since $\pi(X \times \mathcal{M}(E))$ is compact there exist a finite number of $F_1, F_2, \ldots, F_n \in \operatorname{Ker} \Phi_0 = \{F \in \widetilde{B} : \Phi_0(F) = 0\}$ such that

$$\sum_{j=1}^{n} |\phi(F_j(x))| = \sum_{j=1}^{n} |\phi \circ e_x(F_j)| > 1/2$$

for every $(x, \phi) \in X \times \mathcal{M}(E)$. As $\mathcal{M}(E)$ is the maximal ideal space of E, for each $x \in X$ there exist $b_1^x, b_2^x, \ldots, b_n^x \in E$ with

$$\sum_{j=1}^{n} \phi(F_j(x) b_j^x) = 1.$$

for every $\phi \in \mathcal{M}(E)$. For every $x \in X$ there exists an open neighborhood G_x with

$$\left| \phi \left(\sum_{j=1}^{n} F_j(y) b_j^x - 1 \right) \right| = \left| \sum_{j=1}^{n} \phi(F_j(y) b_j^x) - 1 \right| < 1/2, \quad y \in G_x.$$

for every ϕ. Note that $\cup_{x \in X} G_x = X$. Since X is compact, there exist a finite number of $x_1, x_2, \ldots, x_m \in X$such that $\cup_{i=1}^{m} G_{x_i} = X$. We have

$$\left| \phi \left(\sum_{j=1}^{n} F_j(y) b_j^{x_i} - 1 \right) \right| < 1/2, \quad y \in G_{x_i}$$

for every $\phi \in \mathcal{M}(E)$ and $i = 1, 2, \ldots, m$. Put $H_i = \sum_{j=1}^{n} F_j (1_B \otimes b_j^{x_i})$ for $i = 1, 2, \ldots, m$. As $1_B \otimes b_j^{x_i} \in B \otimes E \subset \widetilde{B}$, we have $H_i \in \widetilde{B}$. We also have $H_i \in \operatorname{Ker} \Phi_0$ since $F_j \in \operatorname{Ker} \Phi_0$ for $j = 1, 2, \ldots, n$. Let

$\{\lambda_i\}_{i=1}^m \subset C(X)$ be the decomposition of unity related to $\{G_i\}_{i=1}^m$; $0 \le \lambda_i \le 1$, $\lambda_i = 0$ on the complement of G_i $(i = 1, 2, \dots, m)$, and $\sum_{i=1}^m \lambda_i = 1$. Put

$$H^{\phi}(y) = \sum_{i=1}^{m} \lambda_i(y)\phi(H_i(y)), \quad y \in X.$$

Then for every $\phi \in \mathcal{M}(E)$ we have H^{ϕ} is in $C(X)$ and

$$|H^{\phi}(y) - 1| \le \sum_{i=1}^{m} \lambda_i(y)|\phi(H_i(y)) - 1| \le 1/2$$

for every $\phi \in \mathcal{M}(E)$ and $y \in X$ since $\lambda_i(y)|\phi(H_i(y)) - 1| = 0$ for $y \notin G_i$ and $|\phi(H_i(y)) - 1| < 1/2$ for $y \in G_i$. Since B is dense in $C(X)$, there exist $\lambda_1', \lambda_2', \dots, \lambda_m' \in B$ such that

$$\left| \sum_{i=1}^{m} \lambda_i'(y)\phi(H_i(y)) - \sum_{i=1}^{m} \lambda_i(y)\phi(H_i(y)) \right| < 1/6, \quad y \in X$$

for every $\phi \in \mathcal{M}(E)$. We infer that

$$\left| \sum_{i=1}^{m} \lambda_i'(y)\phi(H_i(y)) - 1 \right| < 2/3, \quad y \in X$$

for every $\phi \in \mathcal{M}(E)$. Put $\widetilde{H} = \sum_{i=1}^m (\lambda_i' \otimes 1_E) H_i \in \widetilde{B}$. Then $\widetilde{H} \in \operatorname{Ker} \Phi_0$ as $H_i \in \operatorname{Ker} \Phi_0$, and we have

$$\phi \circ e_y(\widetilde{H}) = \phi(\widetilde{H}(y)) = \sum_{i=1}^{m} \lambda_i'(y)\phi(H_i(y)), \quad (y, \phi) \in X \times \mathcal{M}(E).$$

Hence $|\phi \circ e_y(\widetilde{H}) - 1| < 2/3$ for every $(y, \phi) \in X \times \mathcal{M}(E)$. As $\widetilde{B}$ is inverse closed, $\widetilde{H}^{-1} \in \widetilde{B}$. This contradicts to $\widetilde{H} \in \operatorname{Ker} \Phi_0$. □

Let K be a compact metric space and E a unital commutative Banach algebra. By the Stone-Weierstrass theorem $\operatorname{Lip}(K, \mathbb{C})$ is dense in $C(K)$, and $\operatorname{Lip}(K, E)$ is inverse-closed by the definition of a vector-valued Lipschitz maps. Hence by Proposition 2.18 the maximal ideal space of $\operatorname{Lip}(K, E)$ is homeomorphic to $K \times \mathcal{M}(E)$. Thus the quadruple $(K, E, \operatorname{Lip}(K, \mathbb{C}), \operatorname{Lip}(K, E))$ is a natural admissible quadruple and

$$\widehat{\operatorname{Lip}(K, E)} \subset C(K \times \mathcal{M}(E)). \tag{2.2}$$

We say that $(X, E, B, \widetilde{B})$ is semi-simple if so is $\widetilde{B}$.

PROPOSITION 2.19. *An admissible quadruple* $(X, E, B, \widetilde{B})$ *is semi-simple if and only if* E *is semi-simple.*

PROOF. Suppose that E is semi-simple. Let $F \in \widetilde{B}$ satisfy $\Gamma_{\widetilde{B}}(F) = 0$ on $\mathcal{M}(\widetilde{B})$. Then $(\Gamma_{\widetilde{B}}(F))(\phi \circ e_x) = 0$ for every $(x, \phi) \in X \times \mathcal{M}(E)$ since $\pi(X \times \mathcal{M}(E)) \subset \mathcal{M}(\widetilde{B})$. Hence for every $x \in X$, we have $\phi(F(x)) = 0$ for $\phi \in \mathcal{M}(E)$. As E is semi-simple we get $F(x) = 0$ for every $x \in X$, hence $F = 0$. We have that $\widetilde{B}$ is semi-simple.

Conversely suppose that $\widetilde{B}$ is semi-simple. Suppose that $a \in E$ satisfies $\sigma(a) = \{0\}$. We show that $a = 0$. By a simple calculation we have $\sigma(a) = \sigma(1 \otimes a)$. Since $\widetilde{B}$ is semi-simple, we have that $1 \otimes a = 0$ and thus $a = 0$. □

We have by (2.2) and Proposition 2.19 that we may suppose that

$$\mathrm{Lip}(K, E) \subset C(K \times \mathcal{M}(E)) \tag{2.3}$$

if E is semi-simple. In general, we have the following. Suppose that E is semi-simple and $(X, E, B, \widetilde{B})$ is natural. Then $\widetilde{B}$ is semi-simple by Proposition 2.19; we may identify $\widetilde{B}$ and $\widehat{\widetilde{B}}$. Hence we may suppose that

$$\widetilde{B} \subset C(X \times \mathcal{M}(E)) \tag{2.4}$$

by (2.1).

Suppose that $\psi : (X_1, E_1, B_1, \widetilde{B_1}) \to (X_2, E_2, B_2, \widetilde{B_2})$ is a unital homomorphism between semi-simple and natural quadruples. Gelfand theory asserts that there exists a continuous map $\Phi : X_2 \times \mathcal{M}(E_2) \to X_1 \times \mathcal{M}(E_1)$ denoted by $\Phi(x, \phi) = (\varphi_1(x, \phi), \varphi_2(x, \phi))$ such that $\psi(F)(x, \phi) = F(\varphi_1(x, \phi), \varphi_2(x, \phi))$ for every $(x, \phi) \in X_2 \times \mathcal{M}(E_2)$ and $F \in (X_1, E_1, B_1, \widetilde{B_1})$.

DEFINITION 2.20. Suppose that E_j is semi-simple and $(X_j, E_j, B_j, \widetilde{B_j})$ is natural for $j = 1, 2$. Suppose that $\psi : (X_1, E_1, B_1, \widetilde{B_1}) \to (X_2, E_2, B_2, \widetilde{B_2})$ is a unital homomorphism. We say that ψ is of type BJ if φ_2 depends only on the second variable;

$$\psi(F)(x, \phi) = F(\varphi_1(x, \phi), \varphi_2(\phi)), \quad (x, \phi) \in X_2 \times \mathcal{M}(E_2)$$

for every $F \in (X_1, E_1, B_1, \widetilde{B_1})$.

3. Results and Proofs

Let E_j be a unital commutative Banach algebra. We say that a homomorphism $\psi : E_1 \to E_2$ is unital if $\psi(\mathbf{1}_{E_1}) = \mathbf{1}_{E_2}$. In this section we study sufficient conditions on an admissible quadruple which ensure that every unital homomorphism on it is of type BJ.

LEMMA 2.21. *Let E_j be a unital commutative Banach algebra. Suppose that $\psi : E_1 \to E_2$ is a unital homomorphism. Then we have $\psi(\mathrm{rad}(E_1)) \subset \mathrm{rad}(E_2)$. Hence the map $\widehat{\psi} : \widehat{E_1} \to \widehat{E_2}$ defined by $\widehat{\psi}(\Gamma_{E_1}(a)) = \Gamma_{E_2}(\psi(a))$, $\Gamma_{E_1}(a) \in \widehat{E_1}$, is well defined. $\widehat{\psi}$ is a unital homomorphism from $\widehat{E_1}$ into $\widehat{E_2}$. There exists a continuous map $h : \mathcal{M}(E_2) \to \mathcal{M}(E_1)$ such that*

$$\widehat{\psi}(\Gamma_{E_1}(a)) = \Gamma_{E_1}(a) \circ h, \quad \Gamma_{E_1}(a) \in \widehat{E_1}.$$

In particular, if ψ is an isomorphism, then $\widehat{\psi}$ is an isomorphism and h is a homeomorphism.

PROOF. Suppose that $a \in \mathrm{rad}(E_1)$. Then, by Proposition 3.5.1 and Theorem 3.5.1 in [**70**], $\lambda \mathbf{1}_{E_1} - a \in E_1^{-1}$ for every non-zero complex number λ. As we assume $\psi(\mathbf{1}_{E_1}) = \mathbf{1}_{E_2}$, $\lambda \mathbf{1}_{E_2} - \psi(a) \in E_2^{-1}$. As $\lambda \neq 0$ can be chosen arbitrary, we have that $\sigma(\psi(a)) = \{0\}$. We conclude that $\psi(a) \in \mathrm{rad}(E_2)$ and thus $\psi(\mathrm{rad}(E_1)) \subset \mathrm{rad}(E_2)$.

The rest of a proof is a routine argument and we omit it.

□

Let $A_{\mathbb{C}}$ be a $\mathbb{C}$-valued function algebra on a compact Hausdorff space X in the strong sense and E a unital commutative Banach algebra. For $f \in A_{\mathbb{C}}$ and $b \in E$, $f \otimes b$ denotes the map in $C(X, E)$ such that $(f \otimes b)(x) = f(x)b$ for $x \in X$. We denote

$$A_{\mathbb{C}} \otimes E = \left\{ \sum_{j=1}^{n} f_j \otimes b_j : n \in \mathbb{N},\ f_j \in A_{\mathbb{C}},\ b_j \in E\ (j = 1, 2, \ldots, n) \right\},$$

where $\mathbb{N}$ is the set of all positive integers.

Suppose that $(X, E, B, \widetilde{B})$ is an admissible quadruple. The subalgebra of all constant maps in $\widetilde{B}$ is denoted by $\mathrm{Const}(\widetilde{B})$; $\mathrm{Const}(\widetilde{B}) = \{\mathbf{1}_B \otimes a : a \in E\}$.

PROPOSITION 2.22. *Let E_j be a unital commutative Banach algebra and $(X_j, E_j, B_j, \widetilde{B_j})$ an admissible quadruple for $j = 1, 2$. Suppose that $\widehat{\widetilde{B_1}} \subset \overline{\Gamma_{\widetilde{B_1}}(B_1 \otimes E_1)}$, where $\bar{\cdot}$ denotes the uniform closure on $\mathcal{M}(\widetilde{B_1})$. Let $\psi : \widetilde{B_1} \to \widetilde{B_2}$ be a unital homomorphism. Suppose that for every $F \in \mathrm{Const}(\widetilde{B_1})$ there exists $b_F \in E_2$ such that $\Gamma_{E_2}(\psi(F)(x)) = \Gamma_{E_2}(b_F)$ for every $x \in X_2$. Then there exists continuous maps $\tau : \mathcal{M}(E_2) \to \mathcal{M}(E_1)$ and $\varphi : X_2 \times \mathcal{M}(E_2) \to X_1$ which satisfy that*

$$\Gamma_{\widetilde{B_2}}(\psi(F))(\phi \circ e_x) = \Gamma_{\widetilde{B_1}}(F)(\tau(\phi) \circ e_{\varphi(x,\phi)}), \quad (x, \phi) \in X_2 \times \mathcal{M}(E_2)$$

for every $F \in \widetilde{B_1}$.

Note that if a homomorphism $\psi : \widetilde{B_1} \to \widetilde{B_2}$ satisfies that $\psi(\mathrm{Const}(\widetilde{B_1})) \subset \mathrm{Const}(\widetilde{B_2})$, then for every $F \in \mathrm{Const}(\widetilde{B_1})$ we have $\Gamma_{E_2}(\psi(F)(x)) = \Gamma_{E_2}(b_F)$ for every $x \in X_2$, where $b_F = \psi(F)(x_0)$ for some $x_0 \in X_2$. The converse can be false unless E_2 is semi-simple.

PROOF. Recall that $X_2 = \mathcal{M}(B_2)$ by (2.1) of Definition 2.16. Let $x \in X_2$. For $b \in E_1$, we denote $\widetilde{b} = \mathbf{1}_{B_1} \otimes b$. Define $\widetilde{\psi_x} : E_1 \to E_2$ by $E_1 \ni b \mapsto \widetilde{\psi_x}(b) = \psi(\widetilde{b})(x)$. The map $\widetilde{\psi_x}$ is well defined since $B_j \otimes E_j \subset \widetilde{B_j}$ by (2.3) of Definition 2.16. By a simple calculation we have that $\widetilde{\psi_x}$ is a homomorphism. As $\mathbf{1}_{B_j} \otimes \mathbf{1}_{E_j} = \mathbf{1}_{\widetilde{B_j}}$ for $j = 1, 2$, we infer that

$$\begin{aligned}\widetilde{\psi_x}(\mathbf{1}_{E_1}) = \psi(\mathbf{1}_{B_1} \otimes \mathbf{1}_{E_1})(x) = \psi(\mathbf{1}_{\widetilde{B_1}})(x)& \\ = \mathbf{1}_{\widetilde{B_2}}(x) = (\mathbf{1}_{B_2} \otimes \mathbf{1}_{E_2})(x) = \mathbf{1}_{E_2}.&\end{aligned}$$

Thus the map $\widetilde{\psi_x}$ is a unital homomorphism. By Lemma 2.21 the induced map $\widehat{\widetilde{\psi_x}} : \widehat{E_1} \to \widehat{E_2}$ defined by $\widehat{\widetilde{\psi_x}}(\Gamma_{E_1}(b)) = \Gamma_{E_2}(\widetilde{\psi_x}(b))$, $\Gamma_{E_1}(b) \in \widehat{E_1}$, is a unital homomorphism. As $\mathbf{1}_{B_1} \otimes b \in \mathrm{Const}(\widetilde{B_1})$, there exists $b_F \in E_2$ such that $\Gamma_{E_2}(\psi(\mathbf{1}_{B_1} \otimes b)(x)) = \Gamma_{E_2}(b_F)$ for every $x \in X_2$. Thus

$$\widehat{\widetilde{\psi_x}}(\Gamma_{E_1}(b)) = \Gamma_{E_2}(\widetilde{\psi_x}(b)) = \Gamma_{E_2}(\psi(\mathbf{1}_{B_1} \otimes b)(x)) = \Gamma_{E_2}(b_F).$$

Hence we have that $\widehat{\widetilde{\psi_x}}$ does not depend on $x \in X_2$. By the Gelfand theory there exists a continuous map $\tau : \mathcal{M}(E_2) \to \mathcal{M}(E_1)$ such that

$$\widehat{\widetilde{\psi_x}}(\Gamma_{E_1}(b)) = \Gamma_{E_1}(b) \circ \tau, \quad \Gamma_{E_1}(b) \in \widehat{E_1}.$$

Thus we have that

$$\Gamma_{E_2}(\widetilde{\psi_x}(b)) = \Gamma_{E_1}(b) \circ \tau$$

for every $x \in X_2$ and $b \in E_1$.

Choose an arbitrary $\phi \in \mathcal{M}(E_2)$. Let $f \in B_1$. Since $f \otimes \mathbf{1}_{E_1} \in B_1 \otimes E_1 \subset \widetilde{B_1}$, $\psi(f \otimes \mathbf{1}_{E_1}) \in \widetilde{B_2}$. Then by (2.4) of Definition 2.16, we have that $\phi \circ \psi(f \otimes 1_{E_1}) \in B_2$. Define $\widetilde{\psi^\phi} : B_1 \to B_2$ by $\widetilde{\psi^\phi}(f) = \phi \circ \psi(f \otimes \mathbf{1}_{E_1})$, $f \in B_1$. By a simple calculation we infer that $\widetilde{\psi^\phi}$ is a unital homomorphism. By (2.3) of Definition 2.16, $X_j \ni x \mapsto e_x \in \mathcal{M}(B_j)$ is a surjection (homeomorphism) for $j = 1, 2$. Then by the Gelfand theory there exists a continuous map $\varphi(\cdot, \phi) : X_2 \to X_1$ such that

$$\widetilde{\psi^\phi}(f) = f \circ \varphi(\cdot, \phi)$$

holds for every $f \in B_1$.

Given $\sum_{j=1}^n f_j \otimes b_j \in B_1 \otimes E_1$, as ψ is a homomorphism, we have

$$\psi(\sum_{j=1}^n f_j \otimes b_j) = \sum_{j=1}^n \psi(f_j \otimes b_j) = \sum_{j=1}^n \psi(f_j \otimes \mathbf{1}_{E_1})\psi(1_{B_1} \otimes b_j).$$

For every $x \in X_2$ and $\phi \in \mathcal{M}(E_2)$ we have

$$\begin{aligned}\phi((\psi(\mathbf{1}_{B_1} \otimes b_j))(x)) &= \phi(\psi(\widetilde{b_j})(x)) = \phi(\widetilde{\psi_x}(b_j)) \\ &= \Gamma_{E_2}(\widetilde{\psi_x}(b_j))(\phi) = \Gamma_{E_1}(b_j) \circ \tau(\phi) = (\tau(\phi))(b_j),\end{aligned}$$

and

$$\begin{aligned}\phi((\psi(f_j \otimes \mathbf{1}_{E_1}))(x)) &= (\phi \circ \psi(f_j \otimes \mathbf{1}_{E_1}))(x) \\ &= \widetilde{\psi^\phi}(f_j)(x) = f_j(\varphi(x, \phi)) = f_j \circ \varphi(x, \phi).\end{aligned}$$

Considering ϕ and $\tau(\phi)$ being multiplicative linear functionals, we have

$$\begin{aligned}\phi\left((\psi(\sum_{j=1}^n f_j \otimes b_j))(x)\right) &= \sum_{j=1}^n f_j \circ \varphi(x, \phi)\tau(\phi)(b_j) \\ &= \tau(\phi)\left(\sum_{j=1}^n f_j(\varphi(x, \phi))b_j\right) = \tau(\phi)\left((\sum_{j=1}^n f_j \otimes b_j)(\varphi(x, \phi))\right).\end{aligned}$$

We have proved that

$$\phi\left((\psi(F))(x)\right) = \tau(\phi)\left(F(\varphi(x, \phi))\right) \tag{3.1}$$

for every $F \in B_1 \otimes E_1$, $x \in X_2$ and $\phi \in \mathcal{M}(E_2)$.

As $\psi : \widetilde{B_1} \to \widetilde{B_2}$ is a unital homomorphism we have by Lemma 2.21 that $\Gamma_{\widetilde{B_1}}(F) \mapsto \Gamma_{\widetilde{B_2}}(\psi(F))$ gives a unital homomorphism from $\widehat{\widetilde{B_1}}$ into $\widehat{\widetilde{B_2}}$. By the Gelfand theory there exists a continuous map $h : \mathcal{M}(\widetilde{B_2}) \to \mathcal{M}(\widetilde{B_1})$ such that $\Gamma_{\widetilde{B_2}}(\psi(F)) = \Gamma_{\widetilde{B_1}}(F) \circ h$. For $x \in X_2$ and $\phi \in \mathcal{M}(E_2)$, we have $\phi \circ e_x \in \mathcal{M}(\widetilde{B_2})$ and

$$\Gamma_{\widetilde{B_2}}(\psi(F))(\phi \circ e_x) = \phi((\psi(F))(x)$$

for every $F \in \widetilde{B_1}$. Suppose that $F_n, F \in \widetilde{B_1}$ and

$$\|\Gamma_{\widetilde{B_1}}(F_n) - \Gamma_{\widetilde{B_1}}(F)\|_{\infty(\mathcal{M}(\widetilde{B_1}))} \to 0$$

as $n \to \infty$. Then

(3.2)
$$\begin{aligned}\|\Gamma_{\widetilde{B_1}}(F_n) - \Gamma_{\widetilde{B_1}}(F)\|_{\infty(\mathcal{M}(\widetilde{B_1}))} &\geq \|\Gamma_{\widetilde{B_2}}(F_n) \circ h - \Gamma_{\widetilde{B_2}}(F) \circ h\|_{\infty(\mathcal{M}(\widetilde{B_2}))} \\ &\geq |\phi\left((\psi(F_n))(x)\right) - \phi\left((\psi(F))(x)\right)|\end{aligned}$$

for any $x \in X_2$ and $\phi \in \mathcal{M}(E_2)$.

Suppose that $F \in \widetilde{B_1}$. Then by the assumption $(\widehat{\widetilde{B_1}}) \subset \overline{\Gamma_{\widetilde{B_1}}(B_1 \otimes E_1)}$ there exists a sequence $\{F_n\} \subset B_1 \otimes E_1$ with

$$\|\Gamma_{\widetilde{B_1}}(F_n) - \Gamma_{\widetilde{B_1}}(F)\|_{\infty(\mathcal{M}(\widetilde{B_1}))} \to 0$$

as $n \to \infty$. By (3.1) we have

$$\phi\left((\psi(F_n))(x)\right) = \tau(\phi)\left(F_n(\varphi(x, \phi))\right)$$

for every positive integer n, $x \in X_2$ and $\phi \in \mathcal{M}(E_2)$. By (3.2) we have

$$\phi\left((\psi(F_n))(x)\right) \to \phi\left((\psi(F))(x)\right)$$

as $n \to \infty$. On the other hand as $\tau(\phi) \circ e_{\varphi(x,\phi)} \in \mathcal{M}(\widetilde{B_1})$ we have

$$\|\Gamma_{\widetilde{B_1}}(F_n) - \Gamma_{\widetilde{B_1}}(F)\|_{\infty(\widetilde{B_1})} \geq |\tau(\phi)\left(F_n(\varphi(x, \phi))\right) - \tau(\phi)\left(F(\varphi(x, \phi))\right)|.$$

Hence we have that

$$\tau(\phi)\left(F_n(\varphi(x, \phi))\right) \to \tau(\phi)\left(F(\varphi(x, \phi))\right)$$

as $n \to \infty$. As $\Gamma_{\widetilde{B_2}}(\psi(F))(\phi \circ e_x) = \phi\left(\psi(F))(x)\right)$ and $\Gamma_{\widetilde{B_1}}(F)(\tau(\phi) \circ e_{\varphi(x,\phi)}) = \tau(\phi)\left(F(\varphi(x, \phi))\right)$ we have

(3.3)
$$\begin{aligned}&\Gamma_{\widetilde{B_2}}(\psi(F))(\phi \circ e_x) \\ &\qquad = \Gamma_{\widetilde{B_1}}(F)(\tau(\phi) \circ e_{\varphi(x,\phi)}), \quad (x, \phi) \in X_2 \times \mathcal{M}(E_2).\end{aligned}$$

On the other hand we have

$$\Gamma_{\widetilde{B_2}}(\psi(F))(\lambda) = \widehat{\psi}(\Gamma_{\widetilde{B_1}}(F))(\lambda) = (\Gamma_{\widetilde{B_1}}(F))(h(\lambda)), \quad \lambda \in \mathcal{M}(\widetilde{B_2}),$$

hence

$$(3.4) \quad \Gamma_{\widetilde{B_2}}(\psi(F))(\phi \circ e_x) = \widehat{\psi}(\Gamma_{\widetilde{B_1}}(F))(\phi \circ e_x) \\ = (\Gamma_{\widetilde{B_1}}(F))(h(\phi \circ e_x)), \quad (x, \phi) \in X_2 \times \mathcal{M}(E_2).$$

Letting $\pi_j : X_j \times \mathcal{M}(E_j) \to \mathcal{M}(\widetilde{B_j})$ for $j = 1, 2$ by $\pi_j(x, \phi) = \phi \circ e_x$ we have by (3.3) and (3.4) that

$$\Gamma_{\widetilde{B_1}}(F)(\pi_1(\varphi(x, \phi), \tau(\phi)) = (\Gamma_{\widetilde{B_1}}(F))(h(\pi_2(x, \phi))), \quad (x, \phi) \in X_2 \times \mathcal{M}(E_2).$$

As $F \in \widetilde{B_1}$ is arbitrary, we have

$$h(\pi_2(x, \phi)) = \tau(\phi) \circ e_{\varphi(x,\phi)} = \pi_1(\varphi(x, \phi), \tau(\phi)), \quad (x, \phi) \in X_2 \times \mathcal{M}(E_2).$$

As π_1 can be seen a homeomorphism from $X_1 \times \mathcal{M}(E_1)$ onto $\{\phi \circ e_x : x \in X_1,\ \phi \in \mathcal{M}(E_1)\}$, we see that $\pi_1^{-1} \circ h \circ \pi_2$ is continuous, hence $\varphi : X_2 \times \mathcal{M}(E_2) \to X_1$ is continuous. □

We give a sufficient condition for admissible quadruples on which every unital homomorphism is of type BJ.

Theorem 2.23. *Let E_j be a unital commutative Banach algebra and $(X_j, E_j, B_j, \widetilde{B_j})$ an admissible quadruple for $j = 1, 2$. Suppose that $\widehat{\widetilde{B_1}} \subset \overline{\Gamma_{\widetilde{B_1}}(B_1 \otimes E_1)}$, where $\bar{\cdot}$ denotes the uniform closure on $\mathcal{M}(\widetilde{B_1})$. Suppose that X_2 is connected with respect to the relative topology induced by the metric inherited from the dual space of B_2 and that $\mathcal{M}(E_1)$ is totally disconnected with respect to the relative topology induced by the metric inherited from the dual space of E_1. Let $\psi : \widetilde{B_1} \to \widetilde{B_2}$ be a unital homomorphism. Then there exists a continuous map $\tau : \mathcal{M}(E_2) \to \mathcal{M}(E_1)$ and a continuous map $\varphi : X_2 \times \mathcal{M}(E_2) \to X_1$ which satisfies that*

$$\Gamma_{\widetilde{B_2}}(\psi(F))(\phi \circ e_x) = \Gamma_{\widetilde{B_1}}(F)(\tau(\phi) \circ e_{\varphi(x,\phi)}), \quad (x, \phi) \in X_2 \times \mathcal{M}(E_2)$$

for every $F \in \widetilde{B_1}$.

Proof. By Proposition 2.22 it is enough to prove that for every $F \in \mathrm{Const}(\widetilde{B_1})$ there exists $b_F \in E_2$ such that $\Gamma_{E_2}(\psi(F)(x)) = \Gamma_{E_2}(b_F)$ for every $x \in X_2$.

Let $\phi \in \mathcal{M}(E_2)$ be arbitrary. Define $S : E_1 \to B_2$ by $S(b) = \phi \circ \psi(\mathbf{1}_{B_1} \otimes b)$ for $b \in E_1$, where we consider ϕ as a multiplicative linear functional on E_2. Then S is a homomorphism. It is unital since $\mathbf{1}_{B_j} \otimes \mathbf{1}_{E_j} = \mathbf{1}_{\widetilde{B_j}}$ and $\psi(\mathbf{1}_{\widetilde{B_1}}) = \mathbf{1}_{\widetilde{B_2}}$. As B_2 is $\mathbb{C}$-valued function algebra on X_2 in the strong sense by Definition 2.16, it is semi-simple. Thus S is continuous by a theorem of Šilov (see [**101**, Theorem 3.1.11]). Let $S^* : B_2^* \to E_1^*$ be the dual of S. Then S^* is continuous with respect to the norm topology on B_2^* and E_1^*. Since S is multiplicative, by a simple calculation we have $S^*(\mathcal{M}(B_2)) \subset \mathcal{M}(E_1)$. Since $X_2 = \mathcal{M}(B_2)$ is connected with respect to the relative topology induced by the metric inherited from the B_2^* and $\mathcal{M}(E_1)$ is totally disconnected with respect to the relative topology induced by the metric inherited from E_1^*, we infer that $S(\mathcal{M}(B_2))$ is a singleton; there exists a unique $\nu_\phi \in \mathcal{M}(E_1)$ with $S^*(\mathcal{M}(B_2)) = \{\nu_\phi\}$. Hence $S^*(e_x) = \nu_\phi$ for every $x \in X_2$. Let $F \in \mathrm{Const}(\widetilde{B_1})$. Then there exists a unique $b \in E_1$ with $F(x) = b$ for every $x \in X_1$. Hence $F = 1_{B_1} \otimes b$, where e_x is a point evaluation on B_2 at x. It follows that

$$\begin{aligned}\phi\left((\psi(F))(x)\right) &= e_x(\phi \circ \psi(1_{B_1} \otimes b)) \\ &= e_x(S(b)) = (S^*(e_x))(b) = \nu_\phi(b)\end{aligned}$$

for every $x \in X_1$. Let x_0 be any point in X_1. Put $b_F = \psi(F)(x_0)$. Then $b_F \in E_2$ and

$$\phi\left((\psi(F))(x)\right) = \nu_\phi(b) = \phi\left((\psi(F))(x_0)\right) = \phi(b_F)$$

for every $x \in X_1$. As ϕ is an arbitrary element in $\mathcal{M}(E_2)$, we obtain $\Gamma_{E_2}(\psi(F)(x)) = \Gamma_{E_2}(b_F)$ for every $x \in X_2$. By Theorem 2.22 there exists a continuous map $\tau : \mathcal{M}(E_2) \to \mathcal{M}(E_1)$ and a continuous map $\varphi : X_2 \times \mathcal{M}(E_2) \to X_1$ which satisfies that

$$\Gamma_{\widetilde{B_2}}(\psi(F))(\phi \circ e_x) = \Gamma_{\widetilde{B_1}}(F)(\tau(\phi) \circ e_{\varphi(x,\phi)}), \quad (x, \phi) \in X_2 \times \mathcal{M}(E_2)$$

for every $F \in \widetilde{B_1}$. □

If E_j is semi-simple, and $(X_j, E_j, B_j, \widetilde{B_j})$ is natural, then we may suppose by (2.4) that $\widetilde{B_j} \subset C(X_j \times \mathcal{M}(E_j))$. Thus we have the following.

COROLLARY 2.24. *Suppose that* E_j *is semi-simple and* $(X_j, E_j, B_j, \widetilde{B_j})$ *is natural. Suppose that* $\widetilde{B_1} \subset \overline{B_1 \otimes E_1}$, *where* $\bar{\cdot}$ *denotes the uniform closure on* $\mathcal{M}(\widetilde{B_1})$. *Suppose that* X_2 *is connected with respect to the relative topology induced by the metric inherited from the dual space of* B_2 *and that* $\mathcal{M}(E_1)$ *is totally disconnected with respect to the relative topology induced by the metric inherited from the dual space of* E_1. *Let* $\psi : \widetilde{B_1} \to \widetilde{B_2}$ *be a unital homomorphism. Then there exists a continuous map* $\tau : \mathcal{M}(E_2) \to \mathcal{M}(E_1)$ *and a continuous map* $\varphi : X_2 \times \mathcal{M}(E_2) \to X_1$ *which satisfies that*

$$\psi(x, \phi) = F(\varphi(x, \phi), \tau(\phi)), \quad (x, \phi) \in X_2 \times \mathcal{M}(E_2)$$

for every $F \in \widetilde{B_1}$.

THEOREM 2.25. *Let* E_j *be a unital commutative Banach algebra and* $(X_j, E_j, B_j, \widetilde{B_j})$ *an admissible quadruple for* $j = 1, 2$. *Suppose that* $\widehat{\widetilde{B_1}} \subset \overline{\Gamma_{\widetilde{B_1}}(B_1 \otimes E_1)}$, *where* $\bar{\cdot}$ *denotes the uniform closure on* $\mathcal{M}(\widetilde{B_1})$. *Suppose that* X_2 *is connected and* $\mathcal{M}(E_1)$ *is totally disconnected. Let* $\psi : \widetilde{B_1} \to \widetilde{B_2}$ *be a unital homomorphism. Then there exists a continuous map* $\tau : \mathcal{M}(E_2) \to \mathcal{M}(E_1)$ *and a continuous map* $\varphi : X_2 \times \mathcal{M}(E_2) \to X_1$ *which satisfies that*

$$\Gamma_{\widetilde{B_2}}(\psi(F))(\phi \circ e_x) = \Gamma_{\widetilde{B_1}}(F)(\tau(\phi) \circ e_{\varphi(x,\phi)}), \quad (x, \phi) \in X_2 \times \mathcal{M}(E_2)$$

for every $F \in \widetilde{B_1}$.

PROOF. Defining S in the same way as in the proof of Theorem 2.23, S^* is continuous with respect to the weak-$*$ topology on B_2^* and E_1^*. The rest of the proof is similar to that of Theorem 2.23. □

As in the same way as Corollary 2.24 we have the following.

COROLLARY 2.26. *Suppose that* E_j *is semi-simple and* $(X_j, E_j, B_j, \widetilde{B_j})$ *is natural. Suppose that* $\widetilde{B_1} \subset \overline{B_1 \otimes E_1}$, *where* $\bar{\cdot}$ *denotes the uniform closure on* $\mathcal{M}(\widetilde{B_1})$. *Suppose that* X_2 *is connected and* $\mathcal{M}(E_1)$ *is totally disconnected. Let* $\psi : \widetilde{B_1} \to \widetilde{B_2}$ *be a unital homomorphism. Then there exists a continuous map* $\tau : \mathcal{M}(E_2) \to \mathcal{M}(E_1)$ *and a continuous map* $\varphi : X_2 \times \mathcal{M}(E_2) \to X_1$ *which satisfies that*

$$\psi(F)(x, \phi) = F(\varphi(x, \phi), \tau(\phi)), \quad (x, \phi) \in X_2 \times \mathcal{M}(E_2)$$

for every $F \in \widetilde{B_1}$.

4. The case of algebras of vector-valued Lipschitz maps

LEMMA 2.27. *Let* K *be a compact metric space. Suppose that* $G_1, \ldots, G_n$ *are open sets with* $\cup_{j=1}^n G_j = K$*. Then there exist* $f_1, \ldots, f_n \in \mathrm{Lip}(K, \mathbb{C})$ *such that* $0 \le f_j \le 1$ *on* K *and* $f_j = 0$ *on* $K \setminus G_j$ *for* $j = 1, 2, \ldots, n$ *and* $\sum_{j=1}^n f_j = 1$ *on* K.

It is well known and we omit a proof.

LEMMA 2.28. *Let* K *be a compact metric space and* E *a unital commutative Banach algebra. Then we have*

$$\widehat{\mathrm{Lip}(K, E)} \subset \overline{\Gamma_{\mathrm{Lip}(K,E)}(\mathrm{Lip}(K, \mathbb{C}) \otimes E)},$$

where $\bar{\cdot}$ *denotes the uniform closure in* $C(K \times \mathcal{M}(E))$.

PROOF. Let $F \in \mathrm{Lip}(K, E)$. Let $\varepsilon > 0$ be arbitrary. Then there exists a finite number of points $x_1, \ldots, x_n \in K$ and open neighborhoods $x_1 \in G_1, \ldots, x_n \in G_n$ such that $\cup_{j=1}^n G_j = K$ and

$$\|F(x) - F(x_j)\|_E \le \varepsilon, x \in G_j$$

for every $j = 1, 2, \ldots, n$. Then we have by Lemma 2.33 that there exist $\Lambda_1, \Lambda_2, \ldots, \Lambda_n \in \mathrm{Lip}(K, \mathbb{C})$ such that $0 \le \Lambda_j \le 1$ on K, $\lambda_j = 0$ on $K \setminus G_j$ for $j = 1, 2, \ldots, n$, and $\sum_{j=1}^n \Lambda_j = 1$ on K. Put $F_\varepsilon = \sum_{j=1}^n \Lambda_j F(x_j) \in \mathrm{Lip}(K, \mathbb{C}) \otimes E$. By some calculation we obtain that

$$\|\Gamma_{\mathrm{Lip}(K,E)}(F) - \Gamma_{\mathrm{Lip}(K,E)}(F_\varepsilon)\|_{\infty(K \times \mathcal{M}(E))} \le \varepsilon.$$

As $F \in \mathrm{Lip}(K, E)$ and ε are arbitrary, we have the conclusion. □

LEMMA 2.29. *Suppose that* K *is a compact metric space. Then, the original topology on* K*, the Gelfand topology induced by* $\mathrm{Lip}(K, \mathbb{C})$*, and the relative topology induced by the metric induced by operator norm topology on the dual space of* $\mathrm{Lip}(K, \mathbb{C})$ *all coincide with each other.*

PROOF. It is well known that the maximal ideal space $\mathcal{M}(\mathrm{Lip}(K, \mathbb{C}))$ with the Gelfand topology is homeomorphic to K. In fact, $x \mapsto e_x$ defines a homeomorphism from K onto $\mathcal{M}(\mathrm{Lip}(K, \mathbb{C}))$. We prove that the Gelfand topology of $\mathcal{M}(\mathrm{Lip}(K, \mathbb{C}))$, which is the topology induced by the weak-$*$ topology inherited from the dual space of $\mathrm{Lip}(K,\mathbb{C})$, is homeomorphic to the topology induced by the metric inherited from the dual space of $\mathrm{Lip}(K,\mathbb{C})$. Just for the simplicity we denote $\mathcal{M}(\mathrm{Lip}(K,\mathbb{C}))$ with the Gelfand topology by $\mathcal{M}_w$ and $\mathcal{M}(\mathrm{Lip}(K,\mathbb{C}))$ with the topology induced by the metric inherited from the dual space of $\mathrm{Lip}(K,\mathbb{C})$ by $\mathcal{M}_s$. Let $\mathrm{Id} : \mathcal{M}_s \to \mathcal{M}_w$ be the identity map. Since the topology induced by the metric inherited from the dual space of $\mathrm{Lip}(K,\mathbb{C})$ is stronger than the Gelfand topology, the map Id is continuous. For every pair $x, y \in K$ we have

$$\|e_x - e_y\|_* = \sup_{\|f\|_L \leq 1} |f(x) - f(y)| \leq \sup_{L(f)\leq 1} |f(x) - f(y)| \leq d(x, y).$$

Since the original topology and the Gelfand topology coincide we infer that Id^{-1} is continuous. We conclude that Id is a homeomorphism. $\square$

Corollary 2.30. *Let K_j be a compact metric space and E_j a unital commutative Banach algebra for $j = 1, 2$. Suppose that K_2 is connected. Suppose that $\mathcal{M}(E_1)$ is totally disconnected with respect to either the Gelfand topology (the original topology as the maximal ideal space) or the relative topology induced by the metric inherited from the dual space of E_1. Let $\psi : \mathrm{Lip}(K_1, E_1) \to \mathrm{Lip}(K_2, E_2)$ be a unital homomorphism. Then there exists a continuous map $\tau : \mathcal{M}(E_2) \to \mathcal{M}(E_1)$ and a continuous map $\varphi : K_2 \times \mathcal{M}(E_2) \to K_1$ such that the map $\varphi(\cdot, \phi) : K_2 \to K_1$ is a Lipschitz map for each $\phi \in \mathcal{M}(E_2)$, which satisfies that*

$$\Gamma_{\mathrm{Lip}(K_2,E_2)}(\psi(F))(\phi \circ e_x) = \Gamma_{\mathrm{Lip}(K_1,E_1)}F(\tau(\phi) \circ e_{\varphi(x,\phi)}), \quad (x, \phi) \in K_2 \times \mathcal{M}(E_2)$$

for every $F \in \mathrm{Lip}(K_1, E_1)$. Furthermore if E_j is semi-simple for $j = 1, 2$, then we may write

$$(\psi(F))(x, \phi) = F(\varphi(x, \phi), \tau(\phi)), \quad (x, \phi) \in K_2 \times \mathcal{M}(E_2)$$

for every $F \in \mathrm{Lip}(K_1, E_1)$; ψ is of type BJ.

Proof. We apply Theorems 2.23 or 2.25 for the admissible quadruple $(K_j, E_j, \mathrm{Lip}(K_j,\mathbb{C}), \mathrm{Lip}(K_j, E_j))$. The maximal ideal space of $\mathrm{Lip}(K_j, E_j)$ is homeomorphic to $K_j \times \mathcal{M}(E_j)$ by Lemma 2.18. We have the inclusion

$$\widehat{\mathrm{Lip}(K_1, E_1)} \subset \overline{\Gamma_{\mathrm{Lip}(K_1,E_1)}(\mathrm{Lip}(K_1,\mathbb{C}) \otimes E_1)}$$

by Lemma 2.34. Since K_2 is connected we have that K_2 is connected also with respect to the relative topology induced by the metric inherited from the dual space of $\mathrm{Lip}(K_2, \mathbb{C})$ by Lemma 2.35. By Theorems 2.23 or 2.25 there exists a continuous map $\tau : \mathcal{M}(E_2) \to \mathcal{M}(E_1)$ and a continuous map $\varphi : K_2 \times \mathcal{M}(E_2) \to K_1$ which satisfies that

$$(4.1)\quad \Gamma_{\mathrm{Lip}(K_2,E_2)}(\psi(F))(\phi \circ e_x) = \Gamma_{\mathrm{Lip}(K_1,E_1)}F(\tau(\phi) \circ e_{\varphi(x,\phi)}), \quad (x, \phi) \in K_2 \times \mathcal{M}(E_2)$$

for every $F \in \mathrm{Lip}(K_1, E_1)$. The rest is to prove that the map $\varphi(\cdot, \phi) : K_2 \to K_1$ is a Lipschitz map for each $\phi \in \mathcal{M}(E_2)$. Let $\phi \in \mathcal{M}(E_2)$. Define $\widetilde{\psi^\phi} : \mathrm{Lip}(K_1) \to \mathrm{Lip}(K_2)$ by $\widetilde{\psi^\phi}(f) = \phi(\psi(f \otimes \mathbf{1}_{E_1}))$, $f \in \mathrm{Lip}(K_1)$. Then $\widetilde{\psi^\phi}$ is a unital homomorphism from $\mathrm{Lip}(K_1)$ into $\mathrm{Lip}(K_2)$. Then $\widetilde{\psi^\phi}$ is continuous by a theorem of Šilov (see [**101**, Theorem 3.1.11]); there exists $K > 0$ such that $\|\widetilde{\psi^\phi}(f)\|_L \leq K\|f\|_L$ for every $f \in \mathrm{Lip}(K_1)$. On the other hand we have by (4.1) that $\widetilde{\psi^\phi}(f) = f(\varphi(\cdot, \phi))$, $f \in \mathrm{Lip}(K_1)$. Suppose that $\varphi(\cdot, \phi) : K_2 \to K_1$ is not a Lipschitz map. Then there exist sequences $\{x_n\}$ and $\{y_n\}$ in K_2 such that

$$0 < nd_2(x_n, y_n) \leq d_1(\varphi(x_n, \phi), \varphi(y_n, \phi))$$

for every positive integer n. Put $f_n : K_1 \to \mathbb{C}$ by $f_n(x) = d_1(x, \varphi(y_n, \phi))$, $x \in K_1$. Then $\|f_n\|_L \leq 1+\mathrm{diam}(K_1)$, where diam denotes the diameter. On the other hand

$$0 < nd_2(x_n, y_n) \leq d_1(\varphi(x_n, \phi), \varphi(y_n, \phi)) = |(\widetilde{\psi^\phi}(f_n))(x_n) - (\widetilde{\psi^\phi}(f_n))(y_n)|$$

for every positive integer n. Hence the Lipschitz constant $L_{\widetilde{\psi^\phi}(f_n)} \geq n$ for every positive integer n, so $\|\widetilde{\psi^\phi}(f_n)\|_L \geq L_{\widetilde{\psi^\phi}(f_n)} \to \infty$, which contradicts to the continuity of $\widetilde{\psi^\phi}$. We conclude that the map $\varphi(\cdot, \phi) : K_2 \to K_1$ is a Lipschitz map for each $\phi \in \mathcal{M}(E_2)$. □

We exhibit several examples of unital semi-simple commutative Banach algebras E such that the maximal ideal space are discrete with respect to the relative topology induced by the metric inherited from the dual space of E. We also exhibit an example of a unital semi-simple commutative Banach algebra whose maximal ideal space is totally disconnected. We say that a $\mathbb{C}$-valued function algebra on X in the strong sense is a uniform algebra on X if it is uniformly closed (see [**22**] for general theory of uniform algebras). Note that a uniform algebra is semi-simple.

EXAMPLE 2.31. (1) Let Y be a compact Hausdorff space. The Banach algebra $C(Y)$ of all complex-valued continuous functions on Y. Then Y is homeomorphic to the maximal ideal space of $C(Y)$. By the Urysohn's lemma we infer that Y is discrete with respect to the relative topology induced by the metric inherited from the dual space of $C(Y)$.

(2) Let $\mathbb{T}$ be the unit circle in the complex plane. Recall that the Wiener algebra is the algebra of all complex-valued continuous functions on $\mathbb{T}$ which have absolute converging Fourier series; $W(\mathbb{T}) = \{f \in C(\mathbb{T}) : \sum |\hat{f}(n)| < \infty\}$ with the norm $\|f\|_W = \sum_m |\hat{f}(m)|$ for $f \in W(\mathbb{T})$. The maximal ideal space of $W(\mathbb{T})$ is homeomorphic to $\mathbb{T}$. Let $w_1, w_2 \in \mathbb{T}$ be different. There exists a positive integer n such that $|w_1^n - w_2^n| > 1$. The function $g(z) = z^n$ is in $W(\mathbb{T})$ with $\|g\|_W = 1$. Hence the norm of $w_1 - w_2$ as the bounded linear functional on $W(\mathbb{T})$ is greater than 1. Hence $\mathbb{T}$ is discrete with respect to the relative topology induced by the metric inherited from the dual space of $W(\mathbb{T})$.

(3) Let A be a uniform algebra such that the maximal ideal space coincides with the Choquet boundary. The Choquet boundary for a uniform algebra A is discrete with respect to the relative topology induced by the metric inherited from the dual space of A. It is known as the Cole's counterexample to the peak point conjecture [**22**] that such a uniform algebra which is not a C^*-algebra exists.

(4) Let G be a compact Abelian group and Γ its dual group. The group algebra $A(G)$ of all Fourier transforms of functions in $L^1(\Gamma)$ is a unital semi-simple commutative Banach algebra with the maximal ideal space G. If Γ is a discrete group of bounded order, then G is a totally disconnected compact Abelian group [**108**, Example 2.5.7. (iii)]. See the paper of

Katznelson and Rudin [**65**] and a book of Rudin [**108**] for further examples and informations.

By (1) of Example 2.31 we see that Corollary 2.30 generalizes a part of Theorem 2.3 concerning to the case where E_j is a commutative C^*-algebra. On the other hand we cannot replace a commutative C^*-algebra by a uniform algebra. The following example shows that a unital homomorphism between algebras of Lipschitz maps with values in uniform algebras need not be of type BJ.

EXAMPLE 2.32. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $\bar{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$. Put

$$A(\bar{\mathbb{D}}) = \{f \in C(\bar{\mathbb{D}}) : f \text{ is analytic on } \mathbb{D}\}.$$

Then $A(\bar{\mathbb{D}})$ is a uniform algebra on $\bar{\mathbb{D}}$, which is called the disk algebra on $\bar{\mathbb{D}}$. By the celebrated Schwartz lemma the map

$$\psi(F)(t, \phi_z) = F(t, \phi_{\frac{1}{2}tz}), \quad F \in \mathrm{Lip}([0,1], A(\bar{\mathbb{D}}))$$

is well defined from $\mathrm{Lip}([0,1], A(\bar{\mathbb{D}}))$ into itself, where ϕ_w is the point evaluation on $A(\bar{\mathbb{D}})$ at $w \in \bar{\mathbb{D}}$. Then ψ is a homomorphism which is not of type BJ.

5. The case of algebras of vector-valued continuously differentiable maps

Let $C^1([0,1])$ be the algebra of all continuously differentiable complex-valued functions on the unit interval $[0,1]$.Let E be a unital commutative Banach algebra. By the Stone-Weierstrass theorem $C^1([0,1])$ is dense in $C([0,1])$, and $C^1([0,1], E)$ is inverse-closed by the definition of a vector-valued continuously differentiable maps. Hence by Proposition 2.18 the maximal ideal space of $C^1([0,1], E)$ is homeomorphic to $[0,1] \times \mathcal{M}(E)$. Hence the admissible quadruple $([0,1], E, C^1([0,1]), C^1([0,1], E))$ is natural and

$$C^1\widehat{([0,1]}, E) \subset C([0,1] \times \mathcal{M}(E)). \tag{5.1}$$

LEMMA 2.33. *Suppose that $G_1, \ldots, G_n$ are open sets with $\cup_{j=1}^n G_j = [0,1]$. Then there exist $f_1, \ldots, f_n \in C^1([0,1])$ such that $0 \leq f_j \leq 1$ on*

$[0,1]$ *and* $f_j = 0$ *on* $[0,1] \setminus G_j$ *for* $j = 1, 2, \ldots, n$ *and* $\sum_{j=1}^n f_j = 1$ *on* $[0,1]$.

Lemma 2.33 is well known and we omit a proof.

LEMMA 2.34. *Let* E *be a unital semi-simple commutative Banach algebra. Then we have*

$$C^1([0,1], E) \subset \overline{\Gamma_{C^1([0,1],E)}(C^1([0,1]) \otimes E)}$$

where $\bar{\cdot}$ *denotes the uniform closure on* $[0,1] \times \mathcal{M}(E)$.

PROOF. Let $F \in C^1([0,1], E)$. Let $\varepsilon > 0$ be arbitrary. Then there exists a finite number of points $x_1, \ldots, x_n \in [0,1]$ and open neighborhoods $x_1 \in G_1, \ldots, x_n \in G_n$ such that $\cup_{j=1}^n G_j = [0,1]$ and

$$\|F(x) - F(x_j)\|_E \leq \varepsilon, x \in G_j$$

for every $j = 1, 2, \ldots, n$. Then we have by Lemma 2.33 that there exist $\Lambda_1, \Lambda_2, \ldots, \Lambda_n \in C^1([0,1])$ such that $0 \leq \Lambda_j \leq 1$ on $[0,1]$, $\Lambda_j = 0$ on $[0,1] \setminus G_j$ for $j = 1, 2, \ldots, n$, and $\sum_{j=1}^n \Lambda_j = 1$ on $[0,1]$. Put $F_\varepsilon = \sum_{j=1}^n \Lambda_j F(x_j) \in C^1([0,1]) \otimes E$. By some calculation we obtain that $\|F - F_\varepsilon\|_{\infty([0,1]\times\mathcal{M}(E))} \leq \varepsilon$. As $F \in C^1([0,1], E)$ and ε are arbitrary, we have the conclusion. □

LEMMA 2.35. *The usual topology on* $[0,1]$, *the Gelfand topology induced by* $C^1([0,1])$, *and the relative topology induced by the operator norm topology on the dual space of* $C^1([0,1])$ *all coincide with each other.*

PROOF. It is well known that the maximal ideal space $\mathcal{M}(C^1([0,1]))$ with the Gelfand topology is homeomorphic to $[0,1]$ with the usual topology. In fact, $x \mapsto e_x$ defines a homeomorphism from $[0,1]$ onto $\mathcal{M}(C^1([0,1]))$. We prove that the Gelfand topology of $\mathcal{M}(C^1([0,1]))$, which is the topology induced by the weak-$*$ topology inherited from the dual space of $C^1([0,1])$, is homeomorphic to the topology induced by the metric inherited from the dual space of $C^1([0,1])$. Just for the simplicity we denote $\mathcal{M}(C^1([0,1]))$ with the Gelfand topology by $\mathcal{M}_w$ and $\mathcal{M}(C^1([0,1]))$ with the topology induced by the metric inherited from the dual space of $C^1([0,1])$ by $\mathcal{M}_s$. Let $\mathrm{Id} : \mathcal{M}_s \to \mathcal{M}_w$ be the identity map. Since the topology induced by the metric inherited from the dual space of $C^1([0,1])$ is stronger than the Gelfand topology, the map Id is continuous. For $x \in [0,1]$, e_x denotes the point evaluation on $C^1([0,1])$ at x. We denote the norm of the dual space of $C^1([0,1])$ by $\|\cdot\|_*$. Let $f \in C^1([0,1])$. Recall that the Lipschitz constant of f is $L(f) = \sup_{t\neq s} \frac{|f(t)-f(s)|}{|t-s|}$. It is easy to see that $\|f'\|_{\infty([0,1])} \le L(f)$. By the mean value theorem we have

$$\begin{aligned}\frac{|f(s)-f(t)|}{|s-t|} &\le \frac{|\operatorname{Re} f(s) - \operatorname{Re} f(t)|}{|s-t|} + \frac{|\operatorname{Im} f(s) - \operatorname{Im} f(t)|}{|s-t|} \\ &\le \|\operatorname{Re} f'\|_{\infty([0,1])} + \|\operatorname{Im} f'\|_{\infty([0,1])} \le 2\|f'\|_{\infty([0,1])}\end{aligned}$$

for every $s,t \in [0,1]$ with $s \neq t$. Thus $L(f) \le 2\|f'\|_{\infty([0,1])}$. Hence we have

$$\|e_x - e_y\|_* = \sup_{\|f\|_{C^1}\le 1} |f(x)-f(y)| \le \sup_{\|f'\|_{\infty([0,1])}\le 1} |f(x)-f(y)| \le 2|x-y|.$$

Since the usual topology and the Gelfand topology on $[0,1]$ coincide we infer that Id^{-1} is continuous. We conclude that Id is a homeomorphism. □

Applying Theorems 2.23 or 2.25 for $([0,1], E_j, C^1([0,1]), C^1([0,1],E_j))$ we obtain the following.

Corollary 2.36. *Suppose that E_j is a unital semi-simple commutative Banach algebra for $j=1,2$. Let $\psi : C^1([0,1],E_1) \to C^1([0,1],E_2)$ be a unital homomorphism. Suppose that $\mathcal{M}(E_1)$ is totally disconnected with respect to either the Gelfand topology or the relative topology induced by the metric inherited from the dual space of E_1. Then there exist a continuous map $\tau : \mathcal{M}(E_2) \to \mathcal{M}(E_1)$ and a continuous map $\varphi : [0,1] \times \mathcal{M}(E_2) \to [0,1]$ such that for each $\phi \in \mathcal{M}(E_2)$ the map $\varphi(\cdot,\phi) : [0,1] \to [0,1]$ is continuously differentiable, which satisfy that*

$$(\psi(F))(x,\phi) = F(\varphi(x,\phi), \tau(\phi)), \quad (x,\phi) \in [0,1] \times \mathcal{M}(E_2)$$

for every $F \in C^1([0,1],E_1)$.

Proof. The maximal ideal space of $C^1([0,1],E_j)$ is homeomorphic to $[0,1] \times \mathcal{M}(E_j)$ by Proposition 2.18. We have the inclusion

$$C^1([0,1],E_j) \subset \overline{\Gamma_{C^1([0,1],E_j)}(C^1([0,1]) \otimes E_j)}$$

by Lemma 2.34. Since $[0,1]$ is connected we have that $[0,1]$ is also connected with respect to the relative topology induced by the metric inherited from the dual space of $C^1([0,1])$ by Lemma 2.35. By Theorems 2.23 or 2.25 there exist a continuous map $\tau : \mathcal{M}(E_2) \to \mathcal{M}(E_1)$ and a continuous map $\varphi : [0,1] \times \mathcal{M}(E_2) \to [0,1]$ which satisfy that

$$(\psi(F))(x,\phi) = F(\varphi(x,\phi), \tau(\phi)), \quad (x,\phi) \in [0,1] \times \mathcal{M}(E_2) \tag{5.2}$$

for every $F \in C^1([0,1], E_1)$. To prove that the map $\varphi(\cdot,\phi) : [0,1] \to [0,1]$ is continuously differentiable for each $\phi \in \mathcal{M}(E_2)$, define $\widetilde{\psi^\phi} : C^1([0,1]) \to C^1([0,1])$ by $\widetilde{\psi^\phi}(f)(x) = \phi(\psi(f \otimes 1_{E_1})(x))$, $f \in C^1([0,1])$. Then $\widetilde{\psi^\phi}$ is a unital homomorphism from $C^1([0,1])$ into $C^1([0,1])$. Then $\widetilde{\psi^\phi}$ is continuous by a theorem of Šilov (cf. [**101**, Theorem 3.1.11]). On the other hand we have by (5.2) that $\widetilde{\psi^\phi}(f) = f(\varphi(\cdot,\phi))$, $f \in C^1([0,1])$. Letting f the identity function we have that $\varphi(\cdot,\phi) : [0,1] \to [0,1]$ is continuously differentiable. $\square$

CHAPTER 3

Peculiar isomorphisms on algebras of vector-valued maps

1. Results and Proofs

In this chapter 3, we study on algebra isomorphisms on algebras of vector-valued maps. We exhibit an example of a unital endomorphism of an algebra of Lipschitz maps with values in a uniform algebra which is not of type BJ (Example 2.32). We will prove in this chapter that any isomorphism between algebras of Lipschitz maps with values in uniform algebras, under some hypothesis of connectedness on the Choquet boundary, is of type BJ .

Let X be a compact Hausdorff space. Let L be a linear subspace of $C(X)$ such that L separates the points of X and contains constants. We set

$$K = \{\phi \in L^* : \phi(1) = 1 = \|\phi\|\},$$

where we consider L as a normed space with the supremum norm on X. The Choquet boundary of L, denoted by $\mathrm{Ch}(L)$, is the set of all $x \in X$ such that the point evaluation e_x on L is an extreme point of K. It is known [**102**, Proposition 6.2] that $x \in X$ is in $\mathrm{Ch}(L)$ if and only if the Dirac measure at x is a unique probability measure μ on X such that $u(x) = \int u d\mu$ for every $u \in L$. Let A be a uniform algebra on X. We say that $S \subset X$ is a peak set if there exists $f \in A$ such that

$$S = \{x \in X : f(x) = 1\} = \{x \in X : |f(x)| = \|f\|_{\infty(X)}\}.$$

A point $x \in X$ is called a peak point in the weak sense if $\{x\}$ is the intersection of some collection of peak sets. It is known that $x \in X$ is a peak point in the weak sense for A if and only if $x \in \mathrm{Ch}(A)$ [**22**, Theorem 2.3.4]. Hence $x \in X$ is in $\mathrm{Ch}(A)$ if and only if for every open neighborhood U of x there exists a function $f \in A$ such that

$f(x) = 1 = \|f\|_{\infty(X)}$ and $|f| < 1$ on $X \setminus U$. By a routine argument we see that $\mathrm{Ch}(A)$ with the relative topology induced by the metric inherited from the dual space of A as a Banach space is a discrete space. For the definition and properties of the Choquet boundary for function spaces and uniform algebras, we refer to [**102**] and [**22**], respectively.

LEMMA 3.1. *Let A be a uniform algebra and $(X, A, B, \widetilde{B})$ an admissible quadruple. Suppose that $\widetilde{B}$ is natural. Then $\pi(\mathrm{Ch}(B) \times \mathrm{Ch}(A)) = \mathrm{Ch}(\widetilde{B})$.*

PROOF. Let $\overline{B}$ be the uniform closure of B in $C(X)$ and $\overline{\widetilde{B}}$ the uniform closure of $\widetilde{B}$ in $C(M(\widetilde{B}))$. Then $\overline{B}$ and $\overline{\widetilde{B}}$ are uniform algebras. By a routine argument the restriction $\phi \mapsto \phi|_B$ gives a homeomorphism from $\mathcal{M}(\overline{B})$ onto $\mathcal{M}(B) = X$. Similarly the restriction gives a homeomorphism from $\mathcal{M}(\overline{\widetilde{B}})$ onto $\mathcal{M}(\widetilde{B})$. Thus we may suppose that $B \subset \overline{B} \subset C(X)$ and $\widetilde{B} \subset \overline{\widetilde{B}} \subset C(X \times \mathcal{M}(A))$. We infer that $\mathrm{Ch}(B) = \mathrm{Ch}(\overline{B})$ and $\mathrm{Ch}(\widetilde{B}) = \mathrm{Ch}(\overline{\widetilde{B}})$ by the definition of the Choquet boundary.

We prove that $\pi(\mathrm{Ch}(\overline{B}) \times \mathrm{Ch}(A)) = \mathrm{Ch}(\overline{\widetilde{B}})$. Let $(x, \phi) \in \mathrm{Ch}(\overline{B}) \times \mathrm{Ch}(A)$. Suppose that G is an open neighborhood of $\pi(x, \phi)$ in $\mathcal{M}(\overline{\widetilde{B}})$. Then there exists an open neighborhood U_1 of ϕ in $\mathcal{M}(A)$ and an open neighborhood V_1 of x in X such that $V_1 \times U_1 \subset G$. As $\phi \in \mathrm{Ch}(A)$, there exists $u \in A$ such that $u(\phi) = 1 = \|u\|_{\infty(\mathcal{M}(A))}$ and $|u| < 1$ on $X \setminus U_1$. As $x \in \mathrm{Ch}(\overline{B})$, there exists $f \in \overline{B}$ such that $f(x) = 1 = \|f\|_{\infty(X)}$ and $|f| < 1$ on $\mathcal{M}(A) \setminus V_1$. By (5) of Definition 2.16, we see that $\overline{B} \otimes A \subset \overline{\widetilde{B}}$. It follows that $f \otimes u \in \overline{\widetilde{B}}$ such that $f \otimes u(\pi(x, \phi)) = 1 = \|f \otimes u\|_{\infty(X \times \mathcal{M}(A))}$ and $|f \otimes u| < 1$ on $X \times \mathcal{M}(\overline{B}) \setminus G$. Thus $\pi(x, \phi) \in \mathrm{Ch}(\overline{\widetilde{B}})$. We have that $\pi(\mathrm{Ch}(\overline{B}) \times \mathrm{Ch}(A)) \subset \mathrm{Ch}(\overline{\widetilde{B}})$.

Next we prove that $\mathrm{Ch}(\overline{\widetilde{B}}) \subset \mathrm{Ch}(\overline{B}) \times \mathrm{Ch}(A)$. Let $\Phi \in \mathrm{Ch}(\overline{\widetilde{B}})$. Since $\mathrm{Ch}(\overline{\widetilde{B}}) = \mathrm{Ch}(\widetilde{B}) \subset \mathcal{M}(\widetilde{B})$ and $\pi : X \times \mathcal{M}(A) \to \mathcal{M}(\widetilde{B})$ is surjective, there exists $(x, \phi) \in X \times \mathcal{M}(A)$ with $\pi(x, \phi) = \Phi$. Let U be an open neighborhood of ϕ in $\mathcal{M}(A)$ and V an open neighborhood of x in X. Then $\pi(V \times U)$ is a open neighborhood of $\pi(x, \phi)$. As $\pi(x, \phi) = \Phi \in \mathrm{Ch}(\overline{\widetilde{B}})$ there exists $F \in \overline{\widetilde{B}}$ such that $F(\pi(x, \phi)) = 1 = \|F\|_{\infty(\mathcal{M}(\overline{\widetilde{B}}))}$ and $|F| < 1$ on $\pi(X \times \mathcal{M}(A)) \setminus \pi(V \times U)$. Now we look at $F(x)$. As $F \in \overline{\widetilde{B}}$, there is a sequence $\{F_n\} \subset \widetilde{B}$ such that $\|F_n - F\|_{\infty(\mathcal{M}(\widetilde{B}))} \to 0$ as $n \to \infty$. As $\pi(X \times \mathcal{M}(A)) = \mathcal{M}(\widetilde{B})$, we see that $\|F_n(x) - F(x)\|_{\infty(\mathcal{M}(A))} \to 0$ as $n \to \infty$. As $F_n(x) \in A$ for every positive integer n and $x \in X$, we have that $F(x) \in A$ for every $x \in X$. Then $\phi(F(x)) = 1 = \|F(x)\|_{\infty(\mathcal{M}(A))}$ and $|F(x)| < 1$ on $\mathcal{M}(A) \setminus U$. As U is arbitrary, we see that $\phi \in \mathrm{Ch}(A)$. By the condition (2.4) of Definition 2.16, we have $\phi \circ F_n \in B$ for every positive integer n. Then

$$\|\phi \circ F_n - \phi \circ F\|_{\infty(X)} \leq \|F_n - F\|_{\infty(\mathcal{M}(\widetilde{B}))} \to 0$$

as $n \to \infty$. Hence $\phi \circ F \in \overline{B}$. We obtain that $\phi \circ F(x) = 1 = \|\phi \circ F\|_{\infty(X)}$ and $|\phi \circ F| < 1$ on $X \setminus V$. As V is arbitrary, we see that $x \in \mathrm{Ch}(\overline{B})$. It follows that $\mathrm{Ch}(\overline{\widetilde{B}}) \subset \pi(\mathrm{Ch}(\overline{B}) \times \mathrm{Ch}(A))$. We conclude that $\mathrm{Ch}(\overline{\widetilde{B}}) = \pi(\mathrm{Ch}(\overline{B}) \times \mathrm{Ch}(A))$, hence $\pi(\mathrm{Ch}(B) \times \mathrm{Ch}(A)) = \mathrm{Ch}(\widetilde{B})$. □

THEOREM 3.2. *Let A_j be a uniform algebra and $(X_j, A_j, B_j, \widetilde{B_j})$ an admissible quadruple for $j = 1, 2$. Suppose that $\widetilde{B_j}$ is natural for $j = 1, 2$. Suppose that $\mathrm{Ch}(B_2)$ is connected with respect to the relative topology induced by the metric inherited from the dual space of B_2. Let $\psi : \widetilde{B_1} \to \widetilde{B_2}$ be an isomorphism. Then there exists a homeomorphism $\tau : \mathcal{M}(A_2) \to \mathcal{M}(A_1)$ and a continuous map $\varphi : X_2 \times \mathcal{M}(A_2) \to X_1$ such that the map $\varphi(\cdot, \phi) : X_2 \to X_1$ is a homeomorphism for each $\phi \in \mathcal{M}(A_2)$ which satisfies that*

$$(\psi(F))(x, \phi) = F(\varphi(x, \phi), \tau(\phi)), \quad (x, \phi) \in X_2 \times \mathcal{M}(A_2)$$

for every $F \in \widetilde{B_1}$. In particular, A_1 is isomorphic to A_2 and B_1 is isomorphic to B_2.

PROOF. Let $\phi \in \mathrm{Ch}(A_2)$ be arbitrary. Define $S : A_1 \to B_2$ by $S(b) = \phi \circ \psi(1_{B_1} \otimes b)$, $b \in A_1$. Note that S is well defined by the condition (2.4) in Definition 2.16. Then S is a homomorphism. It is unital since $\mathbf{1}_{B_1} \otimes \mathbf{1}_{A_1} = \mathbf{1}_{\widetilde{B_1}}$. As B_2 is $\mathbb{C}$-valued function algebra on X_2 in the strong sense by Definition 2.16, it is semi-simple. Thus S is continuous by a theorem of Šilov (see [**101**, Theorem 3.1.11]). Let $S^* : B_2^* \to A_1^*$ be the dual of S. Since S is multiplicative, by a simple calculation $S^*(\mathcal{M}(B_2)) \subset \mathcal{M}(A_1)$. As B_2 is natural, the map $x \mapsto e_x$, where e_x is the point evaluation on B_2 at $x \in X$, gives a homeomorphism from X_2 onto $\mathcal{M}(B_2)$. Hence we have

$$(S^*(e_x))(b) = (S(b))(x) = \phi((\psi(1_{B_1} \otimes b))(x))$$

for every $x \in X_2$ and $b \in A_1$. We prove that $S^*(\mathrm{Ch}(B_2)) \subset \mathrm{Ch}(A_1)$. As ψ is an isomorphism there exists a homeomorphism $h : \mathcal{M}(\widetilde{B_2}) \to \mathcal{M}(\widetilde{B_1})$ such that $\widehat{\psi}(\Gamma_{\widetilde{B_1}}(F)) = \Gamma_{\widetilde{B_2}}(\psi(F)) = \Gamma_{\widetilde{B_1}}(F) \circ h$ for every $F \in \widetilde{B_1}$ by Lemma 2.21. As $\widetilde{B_j}$ is natural, we may suppose that $\mathcal{M}(\widetilde{B_j}) = X_j \times \mathcal{M}(A_j)$ for $j = 1, 2$. Hence we may write $h : X_2 \times \mathcal{M}(A_2) \to X_1 \times \mathcal{M}(A_1)$. As A_j is semi-simple, we have by Proposition 2.19 that $\widetilde{B_j}$ is also semi-simple. Thus we may consider that $\widetilde{B_j}$ is a $\mathbb{C}$-valued function algebra on $X_j \times \mathcal{M}(A_j)$ in the strong sense. Then we may suppose that

$$\psi(F)(x, \phi) = F \circ h(x, \phi), \quad (x, \phi) \in X_2 \times \mathcal{M}(A_2)$$

for every $F \in \widetilde{B_1}$. Defining

$$\psi(G)(x, \phi) = G \circ h(x, \phi), \quad (x, \phi) \in X_2 \times \mathcal{M}(A_2)$$

for G in the uniform closure of $\widetilde{B_1}$ on $X_1 \times \mathcal{M}(A_1)$, it follows that ψ can be extended to an isomorphism from the uniform closure of $\widetilde{B_1}$ in $C(X_1 \times \mathcal{M}(A_1))$ onto the uniform closure of $\widetilde{B_2}$ in $C(X_2 \times \mathcal{M}(A_2))$. As the uniform closure of $\widetilde{B_j}$ is a uniform algebra on $X_j \times \mathcal{M}(A_j)$, $\mathrm{Ch}(\widetilde{B_j})$ coincides with the Choquet boudary of the uniform closure of $\widetilde{B_j}$. Hence we have that $h(\mathrm{Ch}(\widetilde{B_2})) = \mathrm{Ch}(\widetilde{B_1})$. As $\mathrm{Ch}(\widetilde{B_j}) = \mathrm{Ch}(B_j) \times \mathrm{Ch}(A_j)$ by Lemma 3.1 we may consider $h|_{\mathrm{Ch}(\widetilde{B_2})} : \mathrm{Ch}(B_2) \times \mathrm{Ch}(A_2) \to \mathrm{Ch}(B_1) \times \mathrm{Ch}(A_1)$ denoted by $h(x, \gamma) = (h_1(x, \gamma), h_2(x, \gamma))$ with continuous maps $h_1 : \mathrm{Ch}(B_2) \times \mathrm{Ch}(A_2) \to \mathrm{Ch}(B_1)$ and $h_2 : \mathrm{Ch}(B_2) \times \mathrm{Ch}(A_2) \to \mathrm{Ch}(A_1)$. For $x \in \mathrm{Ch}(B_2)$ we have for every $b \in A_1$

$$\begin{aligned} b(S^*(e_x)) = \phi\left((\psi(1_{B_1} \otimes b))(x)\right) &= \phi \circ e_x\left(\psi(1_{B_1} \otimes b)\right) \\ &= \Gamma_{\widetilde{B_1}}(1_{B_1} \otimes b)(h(x, \phi)) = b(h_2(x, \phi)). \end{aligned}$$

As $b \in A_1$ is arbitrary we infer that $S^*(e_x) = h_2(x, \phi) \in \mathrm{Ch}(A_1)$. We have proved that $S^*(\mathrm{Ch}(B_2)) \subset \mathrm{Ch}(A_1)$.

As A_1 is uniform algebra, $\mathrm{Ch}(A_1)$ is discrete with respect to the relative topology induced by the metric inherited from the dual space of A_1. Since $\mathrm{Ch}(B_2)$ is connected with respect to the relative topology induced by the metric inherited from the dual space of B_2 and S^* is continuous with respect to the norm topology on the dual spaces, we conclude that $S^*(\mathrm{Ch}(B_2))$ is a singleton; there exists a unique $k(\phi) \in \mathrm{Ch}(A_1)$ that $S^*(e_x) = k(\phi)$ for every $x \in \mathrm{Ch}(B_2)$. Note that $k(\phi)$ depends on ϕ.

We now prove that $\psi(\mathrm{Const}(\widetilde{B_1})) \subset \mathrm{Const}(\widetilde{B_2})$. Let $F \in \mathrm{Const}(\widetilde{B_1})$. Then there exists $b \in A_1$ such that $F = \mathbf{1}_{B_1} \otimes b$. Suppose that there exist different points $y_1, y_2 \in X_2$ with $(\psi(F))(y_1) \neq (\psi(F))(y_2)$. As A_2 is a uniform algebra, there exists $\phi_0 \in \mathrm{Ch}(A_2)$ with

$$b(k(\phi_0)) = \phi_0\left((\psi(F))(y_1)\right) \neq \phi_0\left((\psi(F))(y_2)\right) = b(k(\phi_0)),$$

which is a contradiction proving that $\psi(F)$ is constant on X_2. Thus $\psi(\mathrm{Const}(\widetilde{B_1})) \subset \mathrm{Const}(\widetilde{B_2})$. Then by Proposition 2.22 there exists continuous maps $\tau : \mathcal{M}(A_2) \to \mathcal{M}(A_1)$ and $\varphi : X_2 \times \mathcal{M}(A_2) \to X_1$ which satisfy that

$$\Gamma_{\widetilde{B_2}}(\psi(F))(\phi \circ e_x) = \Gamma_{\widetilde{B_1}}(F)(\tau(\phi) \circ e_{\varphi(x,\phi)}), \quad (x, \phi) \in X_2 \times \mathcal{M}(A_2)$$

for every $F \in \widetilde{B_1}$. As A_j is semi-simple and $(X_j, A_j, B_j, \widetilde{B_j})$ is natural we have by (2.4) that

$$(\psi(F))(x, \phi) = F(\varphi(x, \phi), \tau(\phi)), \quad (x, \phi) \in X_2 \times \mathcal{M}(A_2)$$

for every $F \in \widetilde{B_1}$.

By considering ψ^{-1} instead of ψ we see that $\psi^{-1}(\mathrm{Const}(\widetilde{B_2})) \subset \mathrm{Const}(\widetilde{B_1})$ and there exists continuous maps $\tau' : \mathcal{M}(A_1) \to \mathcal{M}(A_2)$ and $\varphi' : X_1 \times \mathcal{M}(A_1) \to X_2$ which satisfy that

$$\begin{aligned}(1.1)\quad & \Gamma_{\widetilde{B_1}}(\psi^{-1}(F'))(\phi' \circ e_{x'}) \\ &\qquad = \Gamma_{\widetilde{B_2}}(F')(\tau'(\phi') \circ e_{\varphi'(x',\phi')}), \quad (x', \phi') \in X_1 \times \mathcal{M}(A_1)\end{aligned}$$

for every $F' \in \widetilde{B_2}$. Substituting $F' = \psi(F)$ for $F \in \widetilde{B_1}$, $\phi' = \tau(\phi)$ and $x' = \varphi(x, \phi)$ for $\phi \in \mathcal{M}(A_2)$ and $x \in X_2$ in (1.1) we get

$$\Gamma_{\widetilde{B_1}}(F)(\tau(\phi) \circ e_{\varphi(x,\phi)}) = \Gamma_{\widetilde{B_2}}(\psi(F))(\tau'(\tau(\phi)) \circ e_{\varphi'(\varphi(x,\phi),\tau(\phi))}).$$

As

$$\Gamma_{\widetilde{B_2}}(\psi(F))(\phi \circ e_x) = \Gamma_{\widetilde{B_1}}(F)(\tau(\phi) \circ e_{\varphi(x,\phi)})$$

we have

$$\Gamma_{\widetilde{B_2}}(\psi(F))(\phi \circ e_x) = \Gamma_{\widetilde{B_2}}(\psi(F))(\tau'(\tau(\phi)) \circ e_{\varphi'(\varphi(x,\phi),\tau(\phi))}). \tag{1.2}$$

On the other hand we have $\psi(\mathrm{Const}(\widetilde{B_1})) = \mathrm{Const}(\widetilde{B_2})$ since $\psi(\mathrm{Const}(\widetilde{B_1})) \subset \mathrm{Const}(\widetilde{B_2})$ and $\psi^{-1}(\mathrm{Const}(\widetilde{B_2})) \subset \mathrm{Const}(\widetilde{B_1})$. Hence for every $1_{B_2} \otimes b' \in \mathrm{Const}(\widetilde{B_2})$ there exists $F \in \mathrm{Const}(\widetilde{B_1})$ with $\psi(F) = 1_{B_2} \otimes b'$. Thus

$$\psi(F)(x) = b',\ \psi(F)(\varphi'(\varphi(x,\phi),\tau(\phi))) = b'$$

for every $b' \in A_2$ and $x \in X_2$. By (2.1) we have $\phi(b') = \tau'(\tau(\phi))(b')$ for every $b' \in A_2$. Thus $\phi = \tau'(\tau(\phi))$ for every $\phi \in \mathcal{M}(A_2)$. In the same way we see that $\phi' = \tau(\tau'(\phi'))$ for every $\phi' \in \mathcal{M}(A_1)$. We have that τ is a homeomorphisms from $\mathcal{M}(A_2)$ onto $\mathcal{M}(A_1)$ and τ' is a homeomorphism from $\mathcal{M}(A_1)$ onto $\mathcal{M}(A_2)$. Then by (2.1) we have

$$\Gamma_{\widetilde{B_2}}(\psi(F))(\phi \circ e_x) = \Gamma_{\widetilde{B_2}}(\psi(F))(\phi \circ e_{\varphi'(\varphi(x,\phi),\tau(\phi))})$$

for every $\phi \in \mathcal{M}(A_2)$. Thus we have that

$$\psi(F)(x) = \psi(F)(\varphi'(\varphi(x,\phi),\tau(\phi)))$$

for every $F \in \widetilde{B_1}$. As ψ is a surjection, we have by the condition (2.4) of Definition 2.16 that

$$x = \varphi'(\varphi(x,\phi),\tau(\phi)) \tag{1.3}$$

for every $x \in X_2$; $\varphi'(\varphi(\cdot,\phi),\tau(\phi))$ is the identity on X_2 for every $\phi \in \mathcal{M}(A_2)$. In the same way

$$y = \varphi(\varphi'(y,\phi'),\tau'(\phi'))$$

for every $y \in X_1$ and $\phi' \in \mathcal{M}(A_1)$. Letting $\phi' = \tau(\phi)$ we have $\varphi(\varphi'(\cdot,\tau(\phi)),\phi)$ is the identity on X_1 for every $\phi \in \mathcal{M}(A_2)$ since $\tau' \circ \tau$ is the identity on $\mathcal{M}(A_2)$. It follows that $\varphi(\cdot,\phi) : \mathcal{M}(A_2) \to \mathcal{M}(A_1)$ is a homeomorphism for every $\phi \in \mathcal{M}(A_2)$. In the same way $\varphi'(\cdot,\phi') : \mathcal{M}(A_1) \to \mathcal{M}(A_2)$ is a homeomorphism for every $\phi' \in \mathcal{M}(A_1)$.

Let $x \in X_2$ arbitrary. Define $T : A_1 \to A_2$ by $T(b) = \psi(1_{B_1} \otimes b)(x)$. Since $\psi(\mathrm{Const}(\widetilde{B_1})) \subset \mathrm{Const}(\widetilde{B_2})$ we have

$$\Gamma_{\widetilde{B_2}}(\psi(\mathbf{1}_{B_1} \otimes b))(\phi \circ e_x) = \Gamma_{\widetilde{B_1}}(\mathbf{1}_{B_1} \otimes b)(\tau(\phi) \circ e_{\varphi(x,\phi)}) = (\tau(\phi))(b),$$

hence $\phi(T(b)) = (\tau(\phi))(b)$ for every $b \in A_1$. Thus T does not depend on $x \in X_2$ and T is an isomorphism from A_1 onto A_2.

Next let $\phi \in \mathcal{M}(A_2)$ be arbitrary. Define $U : B_1 \to B_2$ by $U(f) = f \circ \varphi(\cdot, \phi)$. It is well defined. A proof is the following. Let $f \in B_1$. Then $f \otimes \mathbf{1}_{A_1} \in \widetilde{B_1}$ and

$$\begin{aligned}\phi\left(\psi(f \otimes \mathbf{1}_{A_1})(x)\right) \\ &= \Gamma_{\widetilde{B_2}}(\psi(f \otimes \mathbf{1}_{A_1}))(\phi \circ e_x) = \Gamma_{\widetilde{B_1}}(f \otimes \mathbf{1}_{A_1})(\tau(\phi) \circ e_{\varphi(x,\phi)}) \\ &= f(\varphi(x, \phi)).\end{aligned}$$

By the condition (2.4) of Definition 2.16 we have $\phi \circ \psi(f \otimes \mathbf{1}_{A_1}) \in B_2$. Thus U is well defined. It is a homomorphism. In the same way a homomorphism $U' : B_2 \to B_1$ defined by $g \mapsto g \circ \varphi'(\cdot, \tau(\phi))$ is well defined. As $\varphi'(\varphi(\cdot, \phi), \tau(\phi))$ is the identity on X_2 for every $\phi \in \mathcal{M}(A_2)$, and $\varphi(\varphi'(\cdot, \phi'), \tau'(\phi'))$ is identity on X_1 for every $\phi' \in \mathcal{M}(A_1)$ we see that U is an isomorphism from B_1 onto B_2. □

THEOREM 3.3. *Let A_j be a uniform algebra and $(X_j, A_j, B_j, \widetilde{B_j})$ an admissible quadruple for $j = 1, 2$. Suppose that $\widetilde{B_j}$ is natural for $j = 1, 2$. Suppose that* $\mathrm{Ch}(B_2)$ *is connected with respect to the relative topology induced by the Gelfand topology of $\mathcal{M}(B_2)$. Suppose that* $\mathrm{Ch}(A_1)$ *is totally disconnected with respect to the relative topology induced by the Gelfand topology of $\mathcal{M}(A_1)$. Let $\psi : \widetilde{B_1} \to \widetilde{B_2}$ be an isomorphism. Then there exists a homeomorphism $\tau : \mathcal{M}(A_2) \to \mathcal{M}(A_1)$ and a continuous map $\varphi : X_2 \times \mathcal{M}(A_2) \to X_1$ such that the map $\varphi(\cdot, \phi) : X_2 \to X_1$ is a homeomorphism for each $\phi \in \mathcal{M}(A_2)$, which satisfies that*

$$(\psi(F))(x, \phi) = F(\varphi(x, \phi), \tau(\phi)), \quad (x, \phi) \in X_2 \times \mathcal{M}(E_2)$$

for every $F \in \widetilde{B_1}$. In particular, A_1 is isomorphic to A_2 and B_1 is isomorphic to B_2.

PROOF. We can prove Theorem 3.3 in a similar way as in the proof of Theorem 3.2. Let $\phi \in \mathrm{Ch}(A_2)$ be arbitrary. Define $S : A_1 \to B_2$ by $S(b) = \phi \circ \psi(1_{B_1} \otimes b)$, $b \in A_1$. Note that S is well defined by the condition (2.4) in Definition 2.16. Then $S^* : B_2^* \to A_1^*$ is well defined and continuous with respect to the weak-$*$ topology. We can prove

$S^*(\mathrm{Ch}(B_2)) \subset \mathrm{Ch}(A_1)$ in a similar way as in the proof of Theorem 3.2. The Gelfand topology is the relative topology induced by the weak-$*$ topology. Hence $S^*|_{\mathrm{Ch}(B_2)} : \mathrm{Ch}(B_2) \to \mathrm{Ch}(A_1)$ is continuous with the Gelfand topology. Since $\mathrm{Ch}(B_2)$ is connected and $\mathrm{Ch}(A_1)$ is totally disconnected with the Gelfand topology, $S^*(\mathrm{Ch}(B_2))$ is a singleton. The rest of the proof is the same as in the proof of Theorem 3.2. □

Let $\mathbb{T}$ be the unit circle in the complex plane $\mathbb{C}$. A Douglas algebra is a closed subalgebra of L^∞ which properly contains $H^\infty(\mathbb{T})$. The Šilov boundary of a Douglas algebra and the Choquet boundary coincides with each other and it is homeomorphic to the maximal ideal space of $L^\infty(\mathbb{T})$, which is totally disconnected (see [**32**]).

2. The case of algebras of vector-valued Lipschitz maps

COROLLARY 3.4. *Let A_j be a uniform algebra for $j = 1, 2$. Let K_j be a compact metric space for $j = 1, 2$. Suppose that K_2 is connected. Let $\psi : \mathrm{Lip}(K_1, A_1) \to \mathrm{Lip}(K_2, A_2)$ be an isomorphism. Then there exists a homeomorphism $\tau : \mathcal{M}(A_2) \to \mathcal{M}(A_1)$ and a continuous map $\varphi : K_2 \times \mathcal{M}(A_1) \to K_1$ such that the map $\varphi(\cdot, \phi) : K_2 \to K_1$ is a lipeomorphism for each $\phi \in \mathcal{M}(A_2)$, which satisfies that*

$$(\psi(F))(x, \phi) = F(\varphi(x, \phi), \tau(\phi)), \quad (x, \phi) \in K_2 \times \mathcal{M}(A_2)$$

for every $F \in \widetilde{B_1}$. In particular, A_1 is isomorphic to A_2.

PROOF. Applying Theorem 3.2 we can prove Corollary 3.4 as in the proof of Corollary 2.30 except that the map $\varphi(\cdot, \phi) : K_2 \to K_1$ is a lipeomorphism for every $\phi \in \mathcal{M}(A_2)$. We prove that the map $\varphi(\cdot, \phi) : K_2 \to K_1$ is a lipeomorphism for every $\phi \in \mathcal{M}(A_2)$. Let $\phi \in \mathcal{M}(A_2)$ be fixed. By Theorem 3.2 $\varphi(\cdot, \phi)$ is already a homeomorphism. On the other hand, as in the same way as in the proof of Corollary 2.30 we see that $\varphi(\cdot, \phi)$ is a Lipschitz map for every $\phi \in \mathcal{M}(A_2)$. By the equation (1.3) of Theorem 3.2 we see that $\varphi(\cdot, \phi)^{-1} = \varphi'(\cdot, \tau(\phi))$, and it is a Lipschitz map. It follows that $\varphi(\cdot, \phi)$ is a lipeomorphisms for every $\phi \in \mathcal{M}(A_2)$. □

3. The case of algebras of vector-valued continuously differentiable maps

Applying Theorem 3.2, we also obtain the case of $C^1([0,1],A_j)$ for $j=1,2$.

COROLLARY 3.5. *Let A_j be a uniform algebra for $j=1,2$. Suppose that $\psi: C^1([0,1],A_1)\to C^1([0,1],A_2)$ is an algebra isomorphism. Then there exist a homeomorphism $\tau:\mathcal{M}(A_2)\to\mathcal{M}(A_1)$ and a continuous map $\varphi:[0,1]\times\mathcal{M}(A_2)\to[0,1]$ such that for each $\phi\in\mathcal{M}(A_2)$, the map $\varphi(\cdot,\phi):[0,1]\to[0,1]$ is a C^1-diffeomorphism which satisfy that*

$$(\psi(F))(x,\phi)=F(\varphi(x,\phi),\tau(\phi)),\quad (x,\phi)\in[0,1]\times\mathcal{M}(A_2)$$

for every $F\in C^1([0,1],A_1)$. In particular, A_1 is algebraically isomorphic to A_2.

PROOF. The Choquet boundary for $C^1([0,1])$ is $[0,1]$. By Lemma 2.35, $[0,1]$ is connected with respect to the relative topology induced by the metric inherited from the dual space of $C^1([0,1])$. Applying Theorem 3.2 we can prove Corollary 3.5 as in the proof of Corollary 2.36 except that the map $\varphi(\cdot,\phi)$ is a C^1-diffeomorphism for every $\phi\in\mathcal{M}(A_2)$. Let $\phi\in\mathcal{M}(A_2)$ be fixed. By Theorem 3.2 and Corollary 2.36 we see that $\varphi(\cdot,\phi)$ is continuouly differentiable, and is a homeomorphism. Applying the equation (12) in the proof of [**45**, Theorem 19] we see that $\varphi(\cdot,\phi)^{-1}$ is continuouly differentiable. It follows that $\varphi(\cdot,\phi)$ is C^1-diffeomorphism. □

CHAPTER 4

Surjective linear isometry

1. Preliminary

In this chapter an isometry means a complex-linear isometry. De Leeuw [**28**] probably initiated the study of isometries on the algebra of Lipschitz functions on the real line. Roy [**106**] studied isometries on the Banach space $\mathrm{Lip}(X)$ of Lipschitz functions on a compact metric space X, equipped with the norm $\|f\| = \max\{\|f\|_\infty, L(f)\}$, where $L(f)$ denotes the Lipschitz constant. Cambern [**26**] considered isometries on spaces of scalar-valued continuously differentiable functions $C^1([0,1])$ with norm given by $\|f\| = \max_{x\in[0,1]}\{|f(x)| + |f'(x)|\}$ for $f \in C^1([0,1])$ and proved that the forms of them are canonical. Jiménez-Vargas and Villegas-Vallecillos in [**54**] considered isometries of spaces of vector-valued Lipschitz maps on a compact metric space with values in a strictly convex Banach space, equipped with the norm $\|f\| = \max\{\|f\|_\infty, L(f)\}$, see also [**52**]. Botelho and Jamison [**9**] studied isometries on $C^1([0,1],E)$ with $\max_{x\in[0,1]}\{\|f(x)\|_E+\|f'(x)\|_E\}$. See also [**104, 80, 55, 2, 8, 67, 18, 103, 14, 84, 57, 58, 59, 74, 60, 75, 50**]

In what follows, unless otherwise mentioned, α denotes a real scalar in $(0,1)$. Jarosz and Pathak [**48**] studied a problem when an isometry on a space of continuous functions is a weighted composition operator. They provided a unified approach for certain function spaces including $C^1(X)$, $\mathrm{Lip}(X)$, $\mathrm{lip}(X)$ and $AC[0,1]$. On the other hand, in many cases, isometries on algebras of Lipschitz maps and continuously differentiable maps have been somehow studied independently.

We propose a unified approach to the study of isometries on algebras $\mathrm{Lip}(X, C(Y))$, $\mathrm{lip}(X, C(Y))$ and $C^1(K, C(Y))$, where X is a compact metric space, $K = [0,1]$ or $\mathbb{T}$ (in this dissertation $\mathbb{T}$ denotes the unit circle on the complex plane), and Y is a compact Hausdorff space. We define an admissible quadruple of type L (see Definition 4.4) as a common abstraction of Lipschitz algebras and algebras of continuously differentiable maps. We prove that a surjective isometry between admissible quadruple of type L is canonical (Theorem 4.5), in the sense that it is represented as a weighted composition operator. As corollaries we describe isometries on $\mathrm{Lip}(X, C(Y))$, $\mathrm{lip}(X, C(Y))$ and $C^1(K, C(Y))$ respectively (Corollaries 4.14, 4.18, 4.19). There is a variety of norms on $\mathrm{Lip}(X, C(Y))$, $\mathrm{lip}(X, C(Y))$ and $C^1(K, C(Y))$. In this dissertation we consider the norm of sum type; $\|F\|_{\infty(X\times Y)} + L(F)$ for $F \in \mathrm{Lip}(X, C(Y))$, $\|F\|_{\infty(X\times Y)} + L_\alpha(F)$ for $F \in \mathrm{lip}(X, C(Y))$ and $\|F\|_{\infty(K\times Y)} + \|F'\|_{\infty(K\times Y)}$ for $F \in C^1(K, C(Y))$. With these norms $\mathrm{Lip}(X, C(Y))$, $\mathrm{lip}(X, C(Y))$ and $C^1(K, C(Y))$ are commutative Banach algebras respectively.

As is pointed above, the description in [**48**, Example 8] has a gap. We find it difficult to follow the argument given in the Example 8. Besides non-substantial typos, the well-definedness of the map $\Psi_\vartheta : \mathrm{ext}\, B^* \to \mathrm{ext}\, B^*$ ([**48**, p. 205, line 8]), where $\mathrm{ext}\, B^*$ is the set of all extreme points in the closed unit ball of the dual space of $B = \mathrm{Lip}_{\alpha'}(Y)$ given by $\Psi_\vartheta(\gamma\delta_{(y,\omega,\beta)}) = \gamma\delta_{(y,\omega,e^{i\vartheta}\beta)}$ (note that the formula on the line 9 of [**48**, p. 205] reads as in this way) seems to require further explanation. On the other hand Corollary 4.15 of this dissertation confirms the statement of [**48**, Example 8]. Our proof uses a similar but slightly different vein than that of Jarosz-Pathak's argument.

The main result in this chapter is Theorem 4.5, which gives the form of a surjective isometry U between admissible quadruples of type L. The proof of the necessity of the isometry in Theorem 4.5 comprises several steps. We give an outline of the proof. The crucial part of the proof of Theorem 4.5 is to prove that $U(\mathbf{1}) = \mathbf{1} \otimes h$ for $h \in C(Y_2)$ with $|h| = 1$ on Y_2 (Proposition 4.9). To prove Proposition 4.9 we apply Choquet's theory with measure theoretic arguments (Lemmas 4.10,4.11). By Proposition 4.9 we have that $U_0 = (\mathbf{1} \otimes \bar{h})U$ is a surjective isometry fixing the unit. Then by applying a theorem of Jarosz [**47**] we see that U_0 is also an isometry with respect to the supremum norm. By the Banach-Stone theorem U_0 is an algebra isomorphism and applying [**45**] we see that U_0 is a composition operator of type BJ. Analyzing the maps which produce the composition operator, the final conclusion of Theorem 4.5 is derived.

In this chapter, we consider the surjective linear isometry on $\mathrm{Lip}(X,E)$, $\mathrm{Lip}_\alpha(X,E)$, $\mathrm{lip}(X,E)$,where X is a compact metric space and $C^1(K,E)$ for $K=[0,1]$ or $\mathbb{T}$. As we noted, the norm $\|\cdot\|$ of $\mathrm{Lip}_\alpha(X,E)$ (resp. $\mathrm{lip}(X,E)$) is defined by

$$\|F\| = \|F\|_{\infty(X)} + L_\alpha(F), \quad F \in \mathrm{Lip}_\alpha(X,E) \text{ (resp. } \mathrm{lip}(X,E)).$$

We mainly concern with $E=C(Y)$. In this case $\mathrm{Lip}_\alpha(X,C(Y))$ and $\mathrm{lip}(X,C(Y))$ are unital semi-simple commutative Banach algebras with $\|\cdot\|$. When $E=\mathbb{C}$ we abbreviate $\mathrm{Lip}(X,\mathbb{C})$ (resp. $\mathrm{lip}(X,\mathbb{C})$) by $\mathrm{Lip}(X)$ (resp. $\mathrm{lip}(X)$). There are a variety of complete norms other than $\|\cdot\|$. For example $\|\cdot\|_{\max} = \max\{\|\cdot\|_\infty, L_\alpha(\cdot)\}$ is one, but it needs not be submultiplicative. Hence $\mathrm{Lip}_\alpha(X,C(Y))$ and $\mathrm{lip}(X,C(Y))$ need not be Banach algebras with respect to the norm $\|\cdot\|_{\max}$.

Let $F \in C(K,C(Y))$ for $K=[0,1]$ or $\mathbb{T}$. We say that F is continuously differentiable if there exists $G \in C(K,C(Y))$ such that

$$\lim_{K \ni t \to t_0} \left\| \frac{F(t_0)-F(t)}{t_0-t} - G(t_0) \right\|_{\infty(Y)} = 0$$

for every $t_0 \in K$. We denote $F' = G$. Put $C^1(K,C(Y)) = \{F \in C(K,C(Y)) : F \text{ is continuously differentiable}\}$. Then $C^1(K,C(Y))$ with norm $\|F\| = \|F\|_\infty + \|F'\|_\infty$ is a unital semi-simple commutative Banach algebra. If Y is singleton we may suppose that $C(Y)$ is isometrically isomorphic to $\mathbb{C}$ and we abbreviate $C^1(K,C(Y))$ by $C^1(K)$.

By identifying $C(X,C(Y))$ with $C(X\times Y)$ we may assume that $\mathrm{Lip}(X,C(Y))$ is a subalgebra of $C(X\times Y)$ by the correspondence

$$F \in \mathrm{Lip}(X,C(Y)) \leftrightarrow ((x,y) \mapsto (F(x))(y)) \in C(X\times Y).$$

In the same way we may assume that $\mathrm{lip}(X,C(Y))$ is a subalgebra of $C(X\times Y)$. We may also assume that $C^1(K,C(Y))$ is a subalgebra of

$C(K \times Y)$. We may assume that

$$\begin{aligned}\mathrm{Lip}(X, C(Y)) &\subset C(X \times Y),\\ \mathrm{lip}(X, C(Y)) &\subset C(X \times Y),\\ C^1(K, C(Y)) &\subset C(K \times Y).\end{aligned}$$

We say that a subset S of $C(Y)$ is point separating if S separates the points of Y. Suppose that B is a unital point separating subalgebra of $C(Y)$ equipped with a Banach algebra norm. Then B is semi-simple because $\{f \in B : f(x) = 0\}$ is a maximal ideal of B for every $x \in X$ and the Jacobson radical of B vanishes. The maximal ideal space of B is denoted by $\mathcal{M}(B)$. We now introduce some definitions. Although the definition of natural is the same with Definition 2.15, we note it for the convenience of the reader.

Definition 4.1. We say that B is inverse-closed if $f \in B$ with $f(y) \neq 0$ for every $y \in Y$ implies $f^{-1} \in B$. We say that B is natural if the map $e : Y \to \mathcal{M}(B)$ defined by $y \mapsto \phi_y$, where $\phi_y(f) = f(y)$ for every $f \in B$, is bijective. We say that B is self-adjoint if B is natural and satisfies that $f \in B$ implies that $\bar{f} \in B$ for every $f \in B$, where $\bar{\cdot}$ denotes the complex conjugation on $Y = \mathcal{M}(B)$.

Note that conjugate closedness of B ($f \in B$ implies $\bar{f} \in B$) needs not satisfy the self-adjointness of B. If B is conjugate closed and natural, then B is self-adjoint.

Proposition 4.2. *Let Y be a compact Hausdorff space. Suppose that B is a unital point separating subalgebra of $C(Y)$ equipped with a Banach algebra norm. If B is uniformly dense in $C(Y)$ and inverse-closed, then B is natural.*

Proof. Suppose that $e : Y \to \mathcal{M}(B)$ is not surjective. Then there exists $\phi \in \mathcal{M}(B)$ such that for every $y \in Y$ there exists $f_y \in B$ with $\phi(f_y) = 0$ such that $f_y(y) = 1$. As Y is compact, there exists a finite number of $f_1, \ldots, f_n \in B$ with $\phi(f_j) = 0$ for $j = 1, \ldots, n$ such that $\sum_{j=1}^n |f_j|^2 > 0$ on Y. Since B is uniformly dense in $C(Y)$ there exist $g_1, \ldots, g_n \in B$ such that $\sum_{j=1}^n f_j g_j > 0$ on Y. As B is inverse-closed, there exists $h \in B$ such that $h \sum_{j=1}^n f_j g_j = \mathbf{1}$. As $\phi(f_j) = 0$

for $j = 1, \ldots, n$ we have $0 = \phi(h \sum_{j=1}^{n} f_j g_j) = \phi(\mathbf{1}) = 1$, which is a contradiction. □

Corollary 4.3. *The unital Banach algebras* $\mathrm{Lip}(X)$ *with* $\|\cdot\|_\infty + L(\cdot)$ *and* $\mathrm{Lip}(X, C(Y))$ *with* $\|\cdot\|_\infty + L(\cdot)$ *are point separating and self-adjoint. Let* $0 < \alpha < 1$. *The unital Banach algebras* $\mathrm{lip}(X)$ *with* $\|\cdot\|_\infty + L_\alpha(\cdot)$ *and* $\mathrm{lip}(X, C(Y))$ *with* $\|\cdot\|_\infty + L_\alpha(\cdot)$ *are point separating and self-adjoint. The unital Banach algebras* $C^1(K)$ *with* $\|\cdot\|_\infty + \|\cdot'\|_\infty$ *and* $C^1(K, C(Y))$ *with* $\|\cdot\|_\infty + \|\cdot'\|_\infty$ *are point separating and self-adjoint, where* $K = [0, 1]$ *or* $K = \mathbb{T}$.

Proof. The Lipschitz algebra $\mathrm{Lip}(X)$ is a unital point separating subalgebra of $C(X)$ equipped with a Banach algebra norm $\|\cdot\|_\infty + L(\cdot)$. As $\mathrm{Lip}(X)$ is conjugate closed, the Stone-Weierstrass theorem asserts that $\mathrm{Lip}(X)$ is uniformly dense in $C(X)$. Thus it is natural by Proposition 4.2, so that it is self-adjoint. In a similar way as $\mathrm{Lip}(X)$ we infer that $\mathrm{Lip}(X, C(Y))$ is self-adjoint.

Suppose that $0 < \alpha < 1$. Then we see that $\mathrm{lip}(X)$ separates the points of X. (Let x, y be different points in X. Put $f : X \to \mathbb{C}$ by $f(\cdot) = d(\cdot, y)$. By a simple calculation we infer that $f \in \mathrm{lip}(X)$ and $f(x) \neq f(y)$.) In the same way as above we see that $\mathrm{lip}(X)$ and $\mathrm{lip}(X, C(Y))$ are natural, hence self-adjoint.

Let $K = [0, 1]$ or $K = \mathbb{T}$. In the same way as above we see that $C^1(K)$ is self-adjoint. In the same way as above $C^1(K, C(Y))$ is self-adjoint. □

2. Admissible quadruples of type L

An admissible quadruple was defined by Nikou and O'Farrell in [**95**]. The definition is little complicated and we adopt a simpler definition that is sufficient for our purpose. For a detailed account of admissible quadruples see [**95**] and Definition 2.16. Let X and Y be compact Hausdorff spaces. For functions $f \in C(X)$ and $g \in C(Y)$, let $f \otimes g \in C(X \times Y)$ be the function defined by $f \otimes g(x, y) = f(x)g(y)$, and for a subspace E_X of $C(X)$ and a subspace E_Y of $C(Y)$, let

$$E_X \otimes E_Y = \left\{ \sum_{j=1}^{n} f_j \otimes g_j : n \in \mathbb{N},\ f_j \in E_X,\ g_j \in E_Y \right\}.$$

An admissible quadruple $(X, C(Y), B, \widetilde{B})$ in this chapter is as follows.

Definition 4.4. Let X and Y be compact Hausdorff spaces. Let B and $\widetilde{B}$ be unital point separating subalgebras of $C(X)$ and $C(X \times Y)$ equipped with Banach algebra norms respectively which satisfy

$$B \otimes C(Y) \subset \widetilde{B},\ \{F(\cdot, y) : F \in \widetilde{B},\ y \in Y\} \subset B.$$

We say that $(X, C(Y), B, \widetilde{B})$ is an admissible quadruple of type L if the following conditions are satisfied.

① The algebras B and $\widetilde{B}$ are self-adjoint.

② There exists a compact Hausdorff space $\mathfrak{M}$ and a complex-linear operator $D : \widetilde{B} \to C(\mathfrak{M})$ such that

$$D(\widetilde{B} \cap C_{\mathbb{R}}(X \times Y)) \subset C_{\mathbb{R}}(\mathfrak{M})$$

and also

(1) the norm $\|\cdot\|$ on $\widetilde{B}$ satisfies

$$\|F\| = \|F\|_{\infty(X\times Y)} + \|D(F)\|_{\infty(\mathfrak{M})}, \quad F \in \widetilde{B},$$

(2) $\operatorname{Ker} D = \mathbf{1}_B \otimes C(Y)$,

(3) $\|D((\mathbf{1}_B \otimes g)F)\|_{\infty(\mathfrak{M})} = \|D(F)\|_{\infty(\mathfrak{M})}$ for every $F \in \widetilde{B}$ and $g \in C(Y)$ such that $|g| = 1$ on Y

It will be appropriate to make a few comments on the above definition. First we do not assume that $D(\widetilde{B})$ is point separating. Next B and $\widetilde{B}$ are semi-simple since they are point separating. For a point $x \in X$ define $e_x : \widetilde{B} \to C(Y)$ by $e_x(F) = F(x, \cdot)$ for every $F \in \widetilde{B}$. A theorem of Šilov (see [**101**, Theorem 3.1.11] asserts that the evaluation map $e_x : \widetilde{B} \to C(Y)$ is automatically continuous for every $x \in X$ since $C(Y)$ is semi-simple. Hence it is straightforward to check that an admissible quadruple of type L is in fact an admissible quadruple defined by Nikou and O'Farrell in [**95**] (see also [**45**]). In particular if X is a compact metric space, then $(X, C(Y), \mathrm{Lip}(X), \mathrm{Lip}(X, C(Y)))$,

$(X, C(Y), \mathrm{lip}(X), \mathrm{lip}(X, C(Y)))$ and $(K, C(Y), C^1(K), C^1(K, C(Y)))$ for $K = [0,1], \mathbb{T}$ are admissible quadruples of type L. See Section 5.

We define a seminorm $|||\cdot|||$ on $\widetilde{B}$ by $|||F||| = \|D(F)\|_{\infty(\mathfrak{M})}$ for $F \in \widetilde{B}$. Note that $|||\cdot|||$ is one-invariant in the sense of Jarosz [**47**] ($|||F||| = |||F + \mathbf{1}_{\widetilde{B}}|||$ for every $F \in \widetilde{B}$) since $\mathbf{1}_{\widetilde{B}} = \mathbf{1}_B \otimes \mathbf{1}_{C(Y)}$ and $D(\mathbf{1}_{\widetilde{B}}) = 0$. The norm $\|\cdot\| = \|\cdot\|_\infty + |||\cdot|||$ is a p-norm (see [**47**, p.67]).

3. Main Results

The main result in this chapter is the following.

THEOREM 4.5. *Suppose that $(X_j, C(Y_j), B_j, \widetilde{B_j})$ is an admissible quadruple of type L for $j = 1, 2$. Suppose that $U : \widetilde{B_1} \to \widetilde{B_2}$ is a surjective isometry. Then there exists $h \in C(Y_2)$ such that $|h| = 1$ on Y_2, a continuous map $\varphi : X_2 \times Y_2 \to X_1$ such that $\varphi(\cdot, y) : X_2 \to X_1$ is a homeomorphism for each $y \in Y_2$, and a homeomorphism $\tau : Y_2 \to Y_1$ which satisfy*

$$U(F)(x, y) = h(y)F(\varphi(x, y), \tau(y)), \qquad (x, y) \in X_2 \times Y_2$$

for every $F \in \widetilde{B_1}$.

In short a surjective isometry between admissible quadruples of type L is canonical, that is, a weighted composition operator of a specific form: the homeomorphism $X_2 \times Y_2 \to X_1 \times Y_1$, $(x, y) \mapsto (\varphi(x, y), \tau(y))$ has the second coordinate that depends only on the second variable $y \in Y_2$. A composition operator induced by such a homeomorphism is of type BJ (see Definition 2.20, this is the definition of type BJ). That every composition operator on an admissible quadruple $(X, E, B, \widetilde{B})$ onto itself is of type BJ indicates that B and E are totally different Banach algebras.

4. Proofs

We recall some basic properties of regular Borel measures for the convenience of the readers. As the authors could not find appropriate references, we exhibit the properties in Lemmas 4.6, 4.7 and 4.8. In Lemmas 4.6 and 4.7, X is a compact Hausdorff space and μ is a Borel probability measure (a positive measure on the σ-algebra of Borel sets whose total measure is 1). For a non-empty Borel subset S of X, $\mu|S$ denotes the measure on S which is the restriction of μ; $\mu|S(E) = \mu(E)$ for a Borel set $E \subset S$. Recall that the support of μ is the set defined by

$$\operatorname{supp}\mu = \{x \in X : \mu(U) > 0 \text{ for every open neighborhood } U \text{ of } x\}.$$

LEMMA 4.6. *Let K be a non-empty compact subset of X and $f \in C(X)$. Assume that $f \leq c$ on K for a constant $c > 0$. If*

$$\int_K f d\mu = c\mu(K),$$

then $\operatorname{supp}(\mu|K) \subset f^{-1}(c) \cap K$.

PROOF. Let $x \in \operatorname{supp}(\mu|K)$. Then $x \in K$ by the definition of the support of $\mu|K$. Suppose that $f(x) \neq c$. As $f \leq c$ on K, we have $f(x) < c$. Since $f|K$ is continuous on K, there exists an open neighborhood U of x relative to K such that $f < (f(x) + c)/2$ on U. As $x \in \operatorname{supp}(\mu|K)$ we have that $\mu(U) > 0$. Then

$$\begin{aligned}\int_K f d\mu &= \int_U f d\mu + \int_{K\setminus U} f d\mu \\ &\leq \frac{f(x) + c}{2}\mu(U) + c\mu(K \setminus U) \\ &= c\mu(K) - \frac{c - f(x)}{2}\mu(U) < c\mu(K),\end{aligned}$$

which is a contradiction proving that $f(x) = c$. Thus we conclude that $\operatorname{supp}(\mu|K) \subset f^{-1}(c) \cap K$. □

LEMMA 4.7. *Suppose that K_1 and K_2 are non-empty compact subsets of X. Then*

$$\operatorname{supp}(\mu|K_1) \cup \operatorname{supp}(\mu|K_2) = \operatorname{supp}(\mu|(K_1 \cup K_2)).$$

PROOF. Suppose that $x \in \operatorname{supp}(\mu|K_1)$. Suppose that G is an arbitrary open neighborhood of x relative to $K_1 \cup K_2$. Then there is an open set $\tilde{G}$ in X with $\tilde{G} \cap (K_1 \cup K_2) = G$. Then $\tilde{G} \cap K_1$ is an open neighborhood of x relative to K_1 and $G = \tilde{G} \cap (K_1 \cup K_2) \supset \tilde{G} \cap K_1$. As $x \in \operatorname{supp}(\mu|K_1)$ we have $0 < \mu(\tilde{G} \cap K_1) \leq \mu(G)$. Since G is arbitrary we conclude that $x \in \operatorname{supp}(\mu|(K_1 \cup K_2))$; $\operatorname{supp}(\mu|K_1) \subset \operatorname{supp}(\mu|(K_1 \cup K_2))$.

In the same way we have $\operatorname{supp}(\mu|K_2) \subset \operatorname{supp}(\mu|(K_1 \cup K_2))$. Thus we have $\operatorname{supp}(\mu|K_1) \cup \operatorname{supp}(\mu|K_2) \subset \operatorname{supp}(\mu|(K_1 \cup K_2))$.

Suppose conversely that $x \in \operatorname{supp}(\mu|(K_1 \cup K_2))$. Then $x \in K_1 \cup K_2$. Suppose that $x \notin \operatorname{supp}(\mu|K_1) \cup \operatorname{supp}(\mu|K_2)$. First we consider the case that $x \in K_1$ and $x \in K_2$. Then there is an open neighborhood G_1 of x relative to K_1 and an open neighborhood G_2 of x relative to K_2 such that $\mu(G_1) = \mu(G_2) = 0$ since we have assumed that $x \notin \operatorname{supp}(\mu|K_1) \cup \operatorname{supp}(\mu|K_2)$. There exists open sets $\tilde{G}_1$ and $\tilde{G}_2$ in X such that $\tilde{G}_1 \cap K_1 = G_1$ and $\tilde{G}_2 \cap K_2 = G_2$. Put $\tilde{G} = \tilde{G}_1 \cap \tilde{G}_2$. Then $\tilde{G}$ is an open set in X and $x \in \tilde{G}$. Then $\tilde{G} \cap (K_1 \cup K_2)$ is an open neighborhood of x relative to $K_1 \cup K_2$ and

$$\tilde{G} \cap (K_1 \cup K_2) = (\tilde{G} \cap K_1) \cup (\tilde{G} \cap K_2) \subset (\tilde{G}_1 \cap K_1) \cup (\tilde{G}_2 \cap K_2) = G_1 \cap G_2.$$

Then

$$0 \leq \mu(\tilde{G} \cap (K_1 \cup K_2)) \leq \mu(G_1 \cup G_2) \leq \mu(G_1) + \mu(G_2) = 0,$$

so that $\mu(\tilde{G} \cap (K_1 \cup K_2)) = 0$, which is a contradiction since $x \in \operatorname{supp}(\mu|(K_1 \cup K_2))$. Next we consider the case where $x \in K_1$ and $x \notin K_2$. Then there exists an open neighborhood G_1 of x relative to K_1 with $\mu(G_1) = 0$ since we have assumed that $x \notin \operatorname{supp}(\mu|K_1)$. There exists an open set $\tilde{G}_1$ in X such that $\tilde{G}_1 \cap K_1 = G_1$. Since $x \notin K_2$ we infer that $\tilde{G}_1 \cap K_2^c$ is an open neighborhood of x in X. Then $(\tilde{G}_1 \cap K_2^c) \cap (K_1 \cup K_2)$ is an open neighborhood of x relative to $K_1 \cup K_2$ and

$$(\tilde{G}_1 \cap K_2^c) \cap (K_1 \cup K_2) = \tilde{G}_1 \cap K_2^c \cap K_1 \subset \tilde{G}_1 \cap K_1 = G_1.$$

As $(\tilde{G}_1 \cap K_2^c) \cap (K_1 \cup K_2)$ is an open neighborhood of x relative to $K_1 \cup K_2$, we infer that $0 < \mu((\tilde{G}_1 \cap K_2^c) \cap (K_1 \cup K_2))$ since $x \in \operatorname{supp}(\mu|(K_1 \cup K_2))$. On the other hand $(\tilde{G}_1 \cap K_2^c) \cap (K_1 \cup K_2) \subset G_1$ assures that

$$0 < \mu((G_1 \cap K_2^c) \cap (K_1 \cup K_2)) \leq \mu(G_1) = 0,$$

which is a contradiction. In the same way we will arrive at a contradiction also for the case where $x \notin K_1$ and $x \in K_2$. Therefore we have the conclusion that $x \in \operatorname{supp}(\mu|K_1) \cup \operatorname{supp}(\mu|K_2)$. □

We assume the regularity for the measure μ in Lemma 1.2. If μ is a regular Borel probability measure on a compact Hausdorff space Y, then for any Borel set S in $Y \setminus \operatorname{supp}(\mu)$ we have $\mu(S) = 0$. Indeed the regularity of μ assures that $\mu(S)$ is approximated arbitrarily closely by $\mu(E)$ for a compact subset $E \subset S$. Since $S \cap \operatorname{supp}(\mu) = \emptyset$, we use the compactness to cover E by a finitely many open sets with measure zero. This implies $\mu(E) = 0$ and thus $\mu(S) = 0$.

LEMMA 4.8. *Let Y be a compact Hausdorff space and let K be a non-empty compact subset of Y and let μ be a regular Borel probability measure on $Y \times \mathbb{T}$. Let $g \in C_{\mathbb{R}}(Y)$ such that $|g| \leq c$ on K for some $c > 0$. Suppose that there exists $\gamma_0 \in \mathbb{T}$ such that*

$$\int_{K\times\mathbb{T}} \gamma g \otimes 1_{C(\mathbb{T})}(m,\gamma) d\mu(m,\gamma) = \gamma_0 c \mu(K \times \mathbb{T}).$$

Then we have the inclusion

$$\begin{aligned}&\operatorname{supp}(\mu|K \times \mathbb{T})\\ &\qquad \subset \left\{(g^{-1}(c) \cap K) \times \{\gamma_0\}\right\} \cup \left\{(g^{-1}(-c) \cap K) \times \{-\gamma_0\}\right\}.\end{aligned}$$

Note that we write $g(m)$ instead of $g \otimes 1_{C(\mathbb{T})}(m,\gamma)$ for simplicity.

PROOF. As $|\gamma g| = |g| \leq c$ on $K \times \mathbb{T}$ we have

$$c\mu(K \times \mathbb{T}) = \left| \int_{K\times\mathbb{T}} \gamma g(m) d\mu \right| \leq \int_{K\times\mathbb{T}} |g(m)| d\mu \leq c\mu(K \times \mathbb{T}),$$

hence $\int_{K\times\mathbb{T}} |g(m)| d\mu = c\mu(K \times \mathbb{T})$. By Lemma 4.6 we have

$$\operatorname{supp}(\mu|K \times \mathbb{T}) \subset (|g|^{-1}(c) \cap K) \times \mathbb{T}.$$

As g is a real-valued function we infer by a simple calculation that

$$|g|^{-1}(c) = g^{-1}(c) \cup g^{-1}(-c).$$

Put $K_1 = g^{-1}(c)$ and $K_2 = g^{-1}(-c)$. As $c > 0$, we have $K_1 \cap K_2 = \emptyset$. Then

$$\begin{aligned}\operatorname{supp}(\mu|K \times \mathbb{T}) &\subset (K_1 \cap K) \times \mathbb{T} \cup (K_2 \cap K) \times \mathbb{T}\\ &= ((K_1 \cup K_2) \cap K) \times \mathbb{T}.\end{aligned}$$

As μ is regular, we have that

$$\mu(K \times \mathbb{T} \setminus [(K_1 \cap K) \times \mathbb{T} \cup (K_2 \cap K) \times \mathbb{T}]) = 0.$$

It follows that

$$\gamma_0 c\mu(K\times\mathbb{T}) = \int_{K\times\mathbb{T}} \gamma g(m)d\mu$$
$$= \int_{(K_1\cap K)\times\mathbb{T}} \gamma g(m)d\mu + \int_{(K_2\cap K)\times\mathbb{T}} \gamma g(m)d\mu$$
$$= c\int_{(K_1\cap K)\times\mathbb{T}} \gamma d\mu - c\int_{(K_2\cap K)\times\mathbb{T}} \gamma d\mu.$$

Thus we have

$$\mu(K\times\mathbb{T}) = \int_{(K_1\cap K)\times\mathbb{T}} \overline{\gamma_0}\gamma d\mu - \int_{(K_2\cap K)\times\mathbb{T}} \overline{\gamma_0}\gamma d\mu. \tag{4.1}$$

Put $M_1 = \int_{(K_1\cap K)\times\mathbb{T}} 1d\mu$ and $M_2 = \int_{(K_2\cap K)\times\mathbb{T}} 1d\mu$. As μ is regular and $K_1\cap K_2 = \emptyset$ we have

$$M_1 + M_2 = \int_{((K_1\cup K_2)\cap K)\times\mathbb{T}} 1d\mu = \int_{K\times\mathbb{T}} 1d\mu = \mu(K\times\mathbb{T}). \tag{4.2}$$

Put

$$\int_{(K_1\cap K)\times\mathbb{T}} \overline{\gamma_0}\gamma d\mu = e^{i\delta_1}N_1, \quad \int_{(K_2\cap K)\times\mathbb{T}} \overline{\gamma_0}\gamma d\mu = e^{i\delta_2}N_2,$$

where $N_1, N_2 \geq 0$. We may assume that $e^{i\delta_1} = 1$ if $N_1 = 0$ and $e^{i\delta_2} = -1$ if $N_2 = 0$. Note that $N_1 \leq M_1$ and $N_2 \leq M_2$. By (4.1) and (4.2) we obtain

$$M_1 + M_2 = e^{i\delta_1}N_1 - e^{i\delta_2}N_2.$$

Then by a simple calculation we have that $e^{i\delta_1} = -e^{i\delta_2} = 1$, $N_1 = M_1$, and $N_2 = M_2$, that is,

$$\int_{(K_1\cap K)\times\mathbb{T}} \overline{\gamma_0}\gamma d\mu = \mu((K_1\cap K)\times\mathbb{T}), \quad \int_{(K_2\cap K)\times\mathbb{T}} -\overline{\gamma_0}\gamma d\mu = \mu((K_2\cap K)\times\mathbb{T}).$$

Then

$$\mu((K_1\cap K)\times\mathbb{T}) = \operatorname{Re}\int_{(K_1\cap K)\times\mathbb{T}} \overline{\gamma_0}\gamma d\mu = \int_{(K_1\cap K)\times\mathbb{T}} \operatorname{Re}\overline{\gamma_0}\gamma d\mu, \tag{4.3}$$

$$\mu((K_2\cap K)\times\mathbb{T}) = \operatorname{Re}\int_{(K_2\cap K)\times\mathbb{T}} -\overline{\gamma_0}\gamma d\mu = \int_{(K_2\cap K)\times\mathbb{T}} \operatorname{Re}(-\overline{\gamma_0}\gamma) d\mu. \tag{4.4}$$

Applying Lemma 4.6 for (4.3) we infer that

$$\operatorname{supp}(\mu|((K_1\cap K)\times\mathbb{T})) \subset (K_1\cap K)\times\{\gamma_0\}.$$

In the same way we have by (4.4) that

$$\mathrm{supp}(\mu|((K_2 \cap K) \times \mathbb{T})) \subset (K_2 \cap K) \times \{-\gamma_0\}.$$

By Lemma 4.7 we have that

$$\begin{aligned}&\mathrm{supp}(\mu|(((K_1 \cup K_2) \cap K)) \times \mathbb{T})) \\ &\qquad\qquad \subset \{(K_1 \cap K) \times \{\gamma_0\}\} \cup \{(K_2 \cap K) \times \{-\gamma_0\}\}.\end{aligned}$$

Since μ is regular, so is $\mu|(K \times \mathbb{T})$. Thus $\mu|(K \times \mathbb{T})$ is a regular Borel measure on $K \times \mathbb{T}$ such that $\mathrm{supp}(\mu|(K \times \mathbb{T})) \subset ((K_1 \cup K_2) \cap K) \times \mathbb{T}$. Thus

$$\begin{aligned}\mathrm{supp}(\mu|(K \times \mathbb{T})) &= \mathrm{supp}((\mu|(K \times \mathbb{T}))|(((K_1 \cup K_2) \cap K) \times \mathbb{T})) \\ &= \mathrm{supp}(\mu|(((K_1 \cup K_2) \cap K) \times \mathbb{T})),\end{aligned}$$

hence the conclusion holds. □

Throughout this Section we assume all the hypotheses in Theorem 4.5 without further mention. For the simplicity of the proof of Theorem 4.5 we assume that X_2 is not a singleton in this Section. Now, we aim to prove Proposition 4.9, which is a crucial part of proof of Theorem 4.5.

PROPOSITION 4.9. *There exists $h \in C(Y_2)$ with $|h| = 1$ on Y_2 such that $U(\mathbf{1}_{\widetilde{B}_1}) = \mathbf{1}_{B_2} \otimes h$.*

Lemma 4.11 is crucial for the proof of Proposition 4.9. We prove Lemma 4.11 by applying Choquet's theory ([**102**]) which studies the extreme point of the dual unit ball of the space of continuous functions with the supremum norm. To apply the theory we first define an isometry from $\widetilde{B_j}$ into a uniformly closed space of complex-valued continuous functions. Let $j = 1, 2$. Define a map

$$I_j : \widetilde{B_j} \to C(X_j \times Y_j \times \mathfrak{M}_j \times \mathbb{T})$$

by $I_j(F)(x, y, m, \gamma) = F(x, y) + \gamma D_j(F)(m)$ for $F \in \widetilde{B_j}$ and $(x, y, m, \gamma) \in X_j \times Y_j \times \mathfrak{M}_j \times \mathbb{T}$. (Recall that $\mathbb{T}$ is the unit circle in the complex plane.) As D_j is a complex linear map, so is I_j. Let $S_j = X_j \times Y_j \times \mathfrak{M}_j \times \mathbb{T}$. For simplicity we just write I and D instead of I_j and D_j without any confusion. For every $F \in \widetilde{B_j}$ the supremum norm $\|I(F)\|_\infty$ on S_j of $I(F)$ is

$$\begin{aligned}\|I(F)\|_\infty &= \sup\{|F(x,y) + \gamma D(F)(m)| : (x,y,m,\gamma) \in S_j\} \\ &= \sup\{|F(x,y)| : (x,y) \in X_j \times Y_j\} \\ &\qquad + \sup\{|D(F)(m)| : m \in \mathfrak{M}_j\} \\ &= \|F\|_{\infty(X_j \times Y_j)} + \|D(F)\|_{\infty(\mathfrak{M})}.\end{aligned}$$

The second equality follows by an inspection that γ runs through the whole $\mathbb{T}$. It follows that

$$\|I(F)\|_\infty = \|F\|_\infty + \|D(F)\|_\infty = \|F\|$$

for every $F \in \widetilde{B_j}$. Since $0 = \|D(1)\|_\infty$, we have $D(\mathbf{1}) = 0$ and $I(\mathbf{1}) = \mathbf{1}$. Hence I is a complex-linear isometry with $I(\mathbf{1}) = \mathbf{1}$. In particular, $I(\widetilde{B_j})$ is a complex-linear closed subspace of $C(S_j)$ which contains 1. In general $I(\widetilde{B_j})$ needs not separate the points of S_j.

It follows from the definition in [**102**] of the Choquet boundary $\mathrm{Ch}\, I(\widetilde{B_2})$ of $I(\widetilde{B_2})$, we see that a point $p = (x,y,m,\gamma) \in X_2 \times Y_2 \times \mathfrak{M} \times \mathbb{T}$ is in $\mathrm{Ch}\, I(\widetilde{B_2})$ if the point evaluation ϕ_p at p is an extreme point of the state space, or equivalently ϕ_p is an extreme point of the closed unit ball $(I(\widetilde{B_2}))^*_1$ of the dual space $(I(\widetilde{B_2}))^*$ of $I(\widetilde{B_2})$.

LEMMA 4.10. *Suppose that* $(x_0, y_0) \in X_2 \times Y_2$ *and* $\mathfrak{U}$ *is an open neighborhood of* (x_0, y_0)*. Then there exists a function* $F_0 = b_0 \otimes f_0 \in \widetilde{B_2}$ *with* $0 \le b_0 \le 1$ *and* $0 \le f_0 \le 1$ *such that* $0 \le F_0 \le 1 = F_0(x_0, y_0)$ *on* $X_2 \times Y_2$ *and* $F_0 < 1/2$ *on* $X_2 \times Y_2 \setminus \mathfrak{U}$*. Furthermore there exists a point* $(x_c, y_c, m_c, \gamma_c)$ *in the Choquet boundary for* $I_2(\widetilde{B_2})$ *such that* $(x_c, y_c) \in \mathfrak{U} \cap (b_0^{-1}(1) \times f_0^{-1}(1))$ *and* $\gamma_c D(F_0)(m_c) = \|D(F_0)\|_\infty \neq 0$*.*

PROOF. Suppose that $\mathfrak{G}$ and $\mathfrak{H}$ are open neighborhoods of x_0 and y_0 respectively such that $\mathfrak{G} \times \mathfrak{H} \subset \mathfrak{U}$. Since B_2 is unital and self-adjoint, also separates the points of X_2, the Stone-Weierstrass theorem asserts that B_2 is uniformly dense in $C(X_2)$. By the Urysohn's lemma there exists $v \in C(X_2)$ such that $0 \le v \le 4/5$ on X_2, $v(x_0) = 0$, and $v = 4/5$ on $X_2 \setminus \mathfrak{G}$. As B_2 is uniformly dense in $C(X_2)$, there exists $u_1 \in B_2$ such that $\|v - u_1\|_\infty < 1/40$. Put $u = u_1 - u_1(x_0)$. By a simple calculation we infer that $u \in B_2$ with $u(x_0) = 0$ and $-1 \le u \le 1$ on X_2

and $u^2 > 1/2$ on $X_2 \setminus \mathfrak{G}$. Then $b_0 = 1 - u^2 \in B_2$, $0 \leq b_0 \leq 1 = b_0(x_0)$ on X_2, and $b_0 < 1/2$ on $X_2 \setminus \mathfrak{G}$. We may suppose that b_0 is not constant as we assume that X_2 is not a singleton in this Section. In a similar way, there exists $f_0 \in C(Y_2)$ with $0 \leq f_0 \leq 1 = f_0(y_0)$ and $f_0 < 1/2$ on $Y_2 \setminus \mathfrak{H}$. Put $F_0 = b_0 \otimes f_0$. Hence we have that $0 \leq F_0 \leq 1 = F_0(x_0, y_0)$ and $F_0 < 1/2$ on $X_2 \times Y_2 \setminus \mathfrak{U}$. Since $B_2 \otimes C(Y_2) \subset \widetilde{B_2}$ by Definition 4.4, we infer that $F_0 \in \widetilde{B_2}$.

By Proposition 6.3 in [**102**] there exists $c = (x_c, y_c, m_c, \gamma_c)$ in the Choquet boundary for $I(\widetilde{B_2})$ with

$$\|I(F_0)\|_\infty = |I(F_0)(c)|.$$

As in a similar way as we have mentioned before we see that

$$\begin{aligned} (4.5) \quad |I(F_0)(c)| &= |F_0(x_c, y_c) + \gamma_c D(F_0)(m_c)| \\ &= |F_0(x_c, y_c)| + |D(F_0)(m_c)| = \|F_0\|_\infty + \|D(F_0)\|_\infty. \end{aligned}$$

As $0 \leq F_0 \leq 1 = \|F_0\|_\infty$ we have by (4.5) that $F_0(x_c, y_c) = 1 = \|F_0\|_\infty$. Thus $(x_c, y_c) \in \mathfrak{U} \cap (b_0^{-1}(1) \times f_0^{-1}(1))$. Applying that $F_0(x_c, y_c) = 1$ and (4.5), we also have that $\gamma_c D(F_0)(m_c) = |D(F_0)(m_c)| = \|D(F_0)\|_\infty$. As b_0 is not a constant function, we have $F_0 = b_0 \otimes f_0 \notin 1 \otimes C(Y_2) = \operatorname{Ker} D$. Hence we have $\|D(F_0)\|_\infty \neq 0$, so that $D(F_0) \neq 0$. As F_0 is real-valued, so is $D(F_0)$ by the condition ② of Definition 4.4. Hence we see that $\gamma_c D(F_0)(m_c) = \|D(F_0)\|_\infty$ and $\gamma_c = 1$ or -1. □

Note that $\gamma_c = 1$ if $D(F_0)(m_c) > 0$ and $\gamma_c = -1$ if $D(F_0)(m_c) < 0$.

LEMMA 4.11. *Let b_0, f_0 and $F_0 = b_0 \otimes f_0$ be functions obtained in Lemma 4.10. Suppose that $(x_c, y_c, m_c, \gamma_c)$ is in the Choquet boundary for $I(\widetilde{B_2})$ such that $(x_c, y_c) \in b_0^{-1}(1) \times f_0^{-1}(1)$. Then for any $0 < \theta < \pi/2$, $c_\theta = (x_c, y_c, m_c, e^{i\theta}\gamma_c)$ is also in the Choquet boundary for $I(\widetilde{B_2})$.*

PROOF. Let θ be $0 < \theta < \pi/2$. The point evaluation $\phi_\theta(I(F)) = F(x_c, y_c) + e^{i\theta}\gamma_c D(F)(m_c)$ at c_θ is well defined for $I(F) \in I(\widetilde{B_2})$ since I is injective. We prove that the point evaluation ϕ_θ is an extreme point of the closed unit ball $I(\widetilde{B_2})_1^*$ of the dual space $I(\widetilde{B_2})^*$ of $I(\widetilde{B_2})$. Suppose that $\phi_\theta = \frac{1}{2}(\phi_1 + \phi_2)$ for $\phi_1, \phi_2 \in I(\widetilde{B_2})^*$ with $\|\phi_1\| = \|\phi_2\| = 1$, where $\|\cdot\|$ denotes the operator norm here. Let $\check{\phi}_j$ be a Hahn-Banach extension of ϕ_j for each $j = \theta, 1, 2$. By the Riesz-Markov-Kakutani representation theorem there exists a complex regular Borel measure μ_j on $X_2 \times Y_2 \times \mathfrak{M}_2 \times \mathbb{T}$ with $\|\mu_j\| = 1$ which represents $\check{\phi}_j$ for $j = \theta, 1, 2$ respectively. In particular, we have

$$\int I(F) d\mu_j = \phi_j(I(F)), \qquad I(F) \in I(\widetilde{B_2})$$

for $j = \theta, 1, 2$. As $\int \mathbf{1} d\mu_\theta = \phi_\theta(\mathbf{1}) = 1$ we see that μ_θ is a probability measure. By the equation

$$1 = \int \mathbf{1} d\mu_\theta = \frac{1}{2} \int \mathbf{1} d\mu_1 + \frac{1}{2} \int \mathbf{1} d\mu_2$$

we see that μ_1 and μ_2 are also probability measures.

We prove that the support $\mathrm{supp}(\mu_j)$ of the measure μ_j satisfies

$$\mathrm{supp}(\mu_j) \subset b_0^{-1}(1) \times f_0^{-1}(1) \times \left\{ (K_1 \times \{e^{-\theta}\gamma_c\}) \cup (K_2 \times \{-e^{i\theta}\gamma_c\}) \right\},$$

where $K_1 = D(F_0)^{-1}(D(F_0)(m_c))$ and $K_2 = D(F_0)^{-1}(-D(F_0)(m_c))$, for $j = \theta, 1, 2$. Note that $m_c \in K_1$ and K_2 can be empty. Note also that $K_1 \cap K_2 = \emptyset$ since $|D(F_0)(m_c)| = \|D(F_0)\|_\infty \neq 0$. We first consider the case for $j = \theta$. As $(x_c, y_c) \in b_0^{-1}(1) \times f_0^{-1}(1)$ we have

$$\phi_\theta(I(F_0)) = F_0(x_c, y_c) + e^{i\theta}\gamma_c D(F_0)(m_c) = 1 + e^{i\theta}\gamma_c D(F_0)(m_c).$$

As $\phi_\theta(I(F_0)) = \int I(F_0) d\mu_\theta$ we have

$$\begin{aligned} &1 + e^{i\theta}\gamma_c D(F_0)(m_c) \\ &\qquad = \int F_0(x, y) d\mu_\theta(x, y, m, \gamma) + \int \gamma D(F_0)(m) d\mu_\theta(x, y, m, \gamma). \end{aligned}$$

Note that $0 \leq \int F_0(x, y) d\mu_\theta \leq 1$ since $0 \leq F_0 \leq 1$ and μ_θ is a probability measure. As $\gamma_c D(F_0)(m_c) = \|D(F_0)\|_\infty$, we have

$$\left| \int \gamma D(F_0)(m) d\mu_\theta \right| \leq \gamma_c D(F_0)(m_c).$$

Taking into account that $0 < \theta < \pi/2$ we have by an elementary calculation that

$$1 = \int F_0(x, y) d\mu_\theta, \tag{4.6}$$

$$e^{i\theta}\gamma_c D(F_0)(m_c) = \int \gamma D(F_0)(m) d\mu_\theta. \tag{4.7}$$

Since μ_θ is a regular Borel measure, $\mu_\theta(L) = 0$ for any Borel set L with $L \cap \operatorname{supp}(\mu_\theta) = \emptyset$. Hence we have $\int G d\mu_\theta = \int_{\operatorname{supp}(\mu_\theta)} G d\mu_\theta$ for every $G \in C(X_2 \times Y_2 \times \mathfrak{M}_2 \times \mathbb{T})$. Then by the equality (4.6) we have

$$1 = \int_{\operatorname{supp}(\mu_\theta)} F_0(x,y) d\mu_\theta.$$

As $0 \leq F_0 \leq 1$ we have by Lemma 4.6 that

$$\operatorname{supp}(\mu_\theta) \subset F_0^{-1}(1) \times \mathfrak{M}_2 \times \mathbb{T} = b_0^{-1}(1) \times f_0^{-1}(1) \times \mathfrak{M}_2 \times \mathbb{T}. \tag{4.8}$$

Letting $K = X_2 \times Y_2 \times \mathfrak{M}_2$, $g = 1_{C(X_2 \times Y_2)} \otimes D(F_0)$, then applying Lemma 4.8 to the equation (4.7) we get

$$\operatorname{supp}(\mu_\theta) \subset X_2 \times Y_2 \times \left\{ (K_1 \times \{e^{i\theta}\gamma_c\}) \cup (K_2 \times \{-e^{i\theta}\gamma_c\}) \right\}.$$

Combining this inclusion with (4.8) we infer that

$$\operatorname{supp}(\mu_\theta) \subset b_0^{-1}(1) \times f_0^{-1}(1) \times \left\{ (K_1 \times \{e^{i\theta}\gamma_c\}) \cup (K_2 \times \{-e^{i\theta}\gamma_c\}) \right\},$$

the desired inclusion for μ_θ. In order to prove the corresponding inclusion for μ_j for $j = 1, 2$, we first have

$$\begin{aligned} 1 + e^{i\theta}\gamma_c D(F_0)(m_c) &= \phi_\theta(I(F_0)) \\ &= \int I(F_0) d\frac{\mu_1 + \mu_2}{2} + \int \gamma D(F_0) d\frac{\mu_1 + \mu_2}{2} \end{aligned}$$

by the equation $\phi_\theta(I(F_0)) = \frac{1}{2}(\phi_1(I(F_0)) + \phi_2(I(F_0)))$. Applying a similar argument as μ_θ for $\frac{\mu_1+\mu_2}{2}$ we get

$$\operatorname{supp}(\frac{\mu_1 + \mu_2}{2}) \subset b_0^{-1}(1) \times f_0^{-1}(1) \times \left\{ (K_1 \times \{e^{i\theta}\gamma_c\}) \cup (K_2 \times \{-e^{i\theta}\gamma_c\}) \right\}.$$

As μ_1 and μ_2 are positive measures we have

$$\operatorname{supp}(\mu_j) \subset b_0^{-1}(1) \times f_0^{-1}(1) \times \left\{ (K_1 \times \{e^{i\theta}\gamma_c\}) \cup (K_2 \times \{-e^{i\theta}\gamma_c\}) \right\}$$

for $j = 1, 2$.

Next we prove equations

$$F(x_c, y_c) = \int F(x,y) d\mu_\theta \tag{4.9}$$

and

$$\begin{aligned} D(F)(m_c) &= (e^{i\theta}\gamma_c)^{-1} \int \gamma D(F)(m) d\mu_\theta \\ &= \int_{L_1} D(F)(m) d\mu_\theta - \int_{L_2} D(F)(m) d\mu_\theta \end{aligned} \tag{4.10}$$

for every $F \in \widetilde{B_2}$, where $L_j = b_0^{-1}(1) \times f_0^{-1}(1) \times K_j \times \{(-1)^{j+1}e^{i\theta}\gamma_c\}$ for $j = 1, 2$. We first show (4.9) and (4.10) for a real-valued function $F \in \widetilde{B_2}$. Suppose that $F \in \widetilde{B_2} \cap C_{\mathbb{R}}(X_2 \times Y_2)$. Then we have

(4.11)
$$\begin{aligned} F(x_c, y_c) + e^{i\theta}\gamma_c D(F)(m_c) &= \phi_\theta(I(F)) \\ &= \int F(x,y)d\mu_\theta + \int \gamma D(F)(m)d\mu_\theta \\ &= \int F(x,y)d\mu_\theta + \int_{L_1} \gamma D(F)(m)d\mu_\theta + \int_{L_2} \gamma D(F)(m)d\mu_\theta \\ &= \int F(x,y)d\mu_\theta \\ &\quad + e^{i\theta}\gamma_c \left(\int_{L_1} D(F)(m)d\mu_\theta - \int_{L_2} D(F)(m)d\mu_\theta \right). \end{aligned}$$

Note that $F(x_c, y_c)$, $D(F)(m_c)$, $\int F(x,y)d\mu_\theta$, $\int_{L_j} D(F)(m)d\mu_\theta$ for $j = 1, 2$ are all real numbers since F and $D(F)$ are real-valued functions (see Definition 4.4). We also note that $e^{i\theta}\gamma_c \notin \mathbb{R}$ since $0 < \theta < \pi/2$ and $\gamma_c = 1$ or -1. Then comparing the real and the imaginary parts of the equation (4.11) we have (4.9) and (4.10) for every $F \in \widetilde{B_2} \cap C_{\mathbb{R}}(X_2 \times Y_2)$. Let $F \in \widetilde{B_2}$ in general. We have assumed that $\widetilde{B_2}$ is self-adjoint by the condition ① in Definition 4.4, therefore the real part $\operatorname{Re} F$ and the imaginary part $\operatorname{Im} F$ of F both are in $\widetilde{B_2} \cap C_{\mathbb{R}}(X_2 \times Y_2)$. Then by the case for real-valued map in $\widetilde{B_2}$ we have

$$\operatorname{Re} F(x_c, y_c) = \int \operatorname{Re} F(x,y)d\mu_\theta,$$

$$\operatorname{Im} F(x_c, y_c) = \int \operatorname{Im} F(x,y)d\mu_\theta.$$

Hence we have

$$F(x_c, y_c) = \int \operatorname{Re} F(x,y)d\mu_\theta + i \int \operatorname{Im} F(x,y)d\mu_\theta = \int F(x,y)d\mu_\theta,$$

(4.9) is proved for every $F \in \widetilde{B_2}$. As D is complex-linear we have by (4.10) for real-valued functions that

$$\begin{aligned} D(F)(m_c) &= D(\operatorname{Re} F)(m_c) + iD(\operatorname{Im} F)(m_c) \\ &= (e^{i\theta}\gamma_c)^{-1} \int \gamma D(\operatorname{Re} F)(m) d\mu_\theta + i(e^{i\theta}\gamma_c)^{-1} \int \gamma D(\operatorname{Im} F)(m) d\mu_\theta \\ &= (e^{i\theta}\gamma_c)^{-1} \int \gamma D(F) d\mu_\theta \\ &= \int_{L_1} D(F)(m) d\mu_\theta - \int_{L_2} D(F)(m) d\mu_\theta. \end{aligned}$$

Thus we have just proved (4.10) for every $F \in \widetilde{B_2}$.

For every $F \in \widetilde{B_2}$ we have

$$\begin{aligned} \phi_\theta(I(F)) &= \frac{1}{2}\left(\phi_1(I(F)) + \phi_2(I(F))\right) \\ &= \int F(x,y) d\frac{\mu_1+\mu_2}{2} + \int \gamma D(F)(m) d\frac{\mu_1+\mu_2}{2}. \end{aligned}$$

By the same way as the proof of (4.9) and (4.10) we have

$$F(x_c, y_c) = \int F(x,y) d\frac{\mu_1+\mu_2}{2} \tag{4.12}$$

and

$$\begin{aligned} D(F)(m_c) &= (e^{i\theta}\gamma_c)^{-1} \int \gamma D(F)(m) d\frac{\mu_1+\mu_2}{2} \\ &= \int_{L_1} D(F)(m) d\frac{\mu_1+\mu_2}{2} - \int_{L_2} D(F)(m) d\frac{\mu_1+\mu_2}{2} \end{aligned} \tag{4.13}$$

for every $F \in \widetilde{B_2}$.

Next define a regular Borel probability measure ν_j on $X_2 \times Y_2 \times \mathfrak{M}_2 \times \mathbb{T}$ for $j = \theta, 1, 2$ by

$$\nu_j(E) = \mu_j(\{(x, y, m, e^{i\theta}\gamma) : (x, y, m, \gamma) \in E\})$$

for a Borel set $E \subset X_2 \times Y_2 \times \mathfrak{M}_2 \times \mathbb{T}$. Then we have

$$\int F(x,y) d\nu_j = \int F(x,y) d\mu_j \tag{4.14}$$

for every $F \in \widetilde{B_2}$ and $j = \theta, 1, 2$. As

$$\operatorname{supp}(\mu_j) \subset b_0^{-1}(1) \times f_0^{-1}(1) \times \left[(K_1 \times \{e^{i\theta}\gamma_c\}) \cup (K_2 \times \{-e^{i\theta}\gamma_c\})\right]$$

for $j = \theta, 1, 2$, we have

$$\mathrm{supp}(\nu_j) \subset b_0^{-1}(1) \times f_0^{-1}(1) \times [(K_1 \times \{\gamma_c\}) \cup (K_2 \times \{-\gamma_c\})]$$

for $j = \theta, 1, 2$. Put $T_j = b_0^{-1}(1) \times f_0^{-1}(1) \times K_j \times \{(-1)^{j+1}\gamma_c\}$. As ν_j is regular and $K_1 \cap K_2 = \emptyset$, we have by (4.10) that

$$\begin{aligned} \int \gamma D(F)(m) d\nu_j &= \int_{T_1} \gamma D(F)(m) d\nu_j + \int_{T_2} \gamma D(F)(m) d\nu_j \\ &= \gamma_c \int_{T_1} D(F)(m) d\nu_j - \gamma_c \int_{T_2} D(F)(m) d\nu_j \\ &= \gamma_c \int_{L_1} D(F)(m) d\mu_j - \gamma_c \int_{L_2} D(F)(m) d\mu_j \\ &= e^{-i\theta} \int \gamma D(F)(m) d\mu_j \end{aligned} \tag{4.15}$$

for every $F \in \widetilde{B_2}$ and $j = \theta, 1, 2$. For $j = \theta, 1, 2$, put $\psi_j : I(\widetilde{B_2}) \to \mathbb{C}$ by

$$\psi_j(I(F)) = \int I(F) d\nu_j, \quad I(F) \in I(\widetilde{B_2}).$$

As ν_j is a probability measure we see that $\psi_j \in I(\widetilde{B_2})_1^*$. Let $I(F) \in I(\widetilde{B_2})$. Then by (4.14) and (4.15) we have

$$\begin{aligned} \psi_\theta(I(F)) &= \int I(F) d\nu_\theta \\ &= \int F(x, y) d\nu_\theta + \int \gamma D(F)(m) d\nu_\theta \\ &= \int F(x, y) d\mu_\theta + e^{-i\theta} \int \gamma D(F)(m) d\mu_\theta. \end{aligned}$$

Then by (4.9) and (4.10) we have

$$\psi_\theta(I(F)) = F(x_c, y_c) + \gamma_c D(F)(m_c) = I(F)(x_c, y_c, m_c, \gamma_c).$$

That is ψ_θ is the point evaluation for $I(\widetilde{B_2})$ at $(x_c, y_c, m_c, \gamma_c)$. By (4.14), (4.15), (4.12) and (4.13) we have

$$\begin{aligned}\frac{1}{2}&(\psi_1(I(F)) + \psi_2(I(F)))\\ &= \int F(x,y)d\frac{\nu_1+\nu_2}{2} + \int \gamma D(F)(m)d\frac{\nu_1+\nu_2}{2}\\ &= \int F(x,y)d\frac{\mu_1+\mu_2}{2} + e^{-i\theta}\int \gamma D(F)(m)d\frac{\mu_1+\mu_2}{2}\\ &= F(x_c,y_c) + \gamma_c D(F)(m_c)\end{aligned}$$

for every $F \in \widetilde{B_2}$. Hence we have

$$\psi_\theta(I(F)) = \frac{1}{2}\left(\psi_1(I(F)) + \psi_2(I(F))\right)$$

for every $I(F) \in I(\widetilde{B_2})$; $\psi_\theta = \frac{1}{2}(\psi_1 + \psi_2)$. Since $(x_c, y_c, m_c, \gamma_c)$ is in the Choquet boundary for $I(\widetilde{B_2})$, ψ_θ is an extreme point for $I(\widetilde{B_2})^*_1$. Thus we have that $\psi_\theta = \psi_1 = \psi_2$.

Applying the equations $\psi_\theta = \psi_1 = \psi_2$ we prove that $\phi_\theta = \phi_1 = \phi_2$. By (4.14) and (4.15) we have

$$\begin{aligned}\phi_j(I(F)) &= \int F(x,y)d\mu_j + \int \gamma D(F)(m)d\mu_j\\ &= \int F(x,y)d\nu_j + e^{i\theta}\int \gamma D(F)(m)d\nu_j, \quad F \in I(\widetilde{B_2})\end{aligned} \tag{4.16}$$

for every $j = \theta, 1, 2$. Put

$$P = \{G \in \widetilde{B_2} : 0 \le G \le 1 = G(x_c, y_c)\}.$$

Then the set P separates the points of $X_2 \times Y_2$. Suppose that (x_1, y_1) and (x_2, y_2) are different points in $X_2 \times Y_2$. We may assume that $(x_c, y_c) \neq (x_2, y_2)$. Let $\mathfrak{U}$ be an open neighborhood of (x_c, y_c) such that $(x_2, y_2) \notin \mathfrak{U}$. By Lemma 4.10 there is $F_c \in \widetilde{B_2}$ such that $0 \le F_c \le 1 = F_c(x_c, y_c)$ on $X_2 \times Y_2$ and $F_c < 1/2$ on $X_2 \times Y_2 \setminus \mathfrak{U}$. Hence $0 \le F_c(x_2, y_2) < 1/2$. In the same way there exists $F_1 \in \widetilde{B_2}$ such that $0 \le F_1 \le 1 = F_1(x_1, y_1)$ on $X_2 \times Y_2$ and $0 \le F_1(x_2, y_2) < 1/2$. Put $H = 1 - (1 - F_c)(1 - F_1) \in \widetilde{B_2}$. Then we infer that $1 \le H \le 1$ on $X_2 \times Y_2$, $H(x_c, y_c) = H(x_1, y_1) = 1$, and $H(x_2, y_2) \neq 1$. Hence we have that $H \in P$ and $H(x_1, y_1) \neq H(x_2, y_2)$. Let $G \in P$ be arbitrary. Since

$P \subset \widetilde{B_2}$, we have $G \in \widetilde{B_2}$. Hence by the equality (4.14) we have

$$\frac{1}{2}\left(\int G(x,y)d\nu_1 + \int G(x,y)d\nu_2\right)$$
$$= \frac{1}{2}\left(\int G(x,y)d\mu_1 + \int G(x,y)d\mu_2\right) = \int G(x,y)d\frac{\mu_1+\mu_2}{2}.$$

By (4.12)

$$\int G(x,y)d\frac{\mu_1+\mu_2}{2} = G(x_c,y_c) = 1.$$

Hence we have

$$\frac{1}{2}\left(\int G(x,y)d\nu_1 + \int G(x,y)d\nu_2\right) = 1.$$

Since $0 \le G \le 1$ we have $0 \le \int G(x,y)d\nu_j \le 1$ for $j = 1,2$. It follows that

$$\int G(x,y)d\nu_1 = \int G(x,y)d\nu_2 = 1.$$

As $G \in P$ is arbitrary we have

$$\int \sum a_n G_n(x,y)d\nu_1 = \sum a_n = \int \sum a_n G_n(x,y)d\nu_2$$

for any complex linear combination $\sum a_n G_n$ for $G_n \in P$. Since P is closed under multiplication and separates the points in $X_2 \times Y_2$, we have that

$$\left\{\sum a_n G_n : a_n \in \mathbb{C},\, G_n \in P\right\}$$

is a unital subalgebra of $\widetilde{B_j}$ which is conjugate-closed and separates the points of $X_2 \times Y_2$. The Stone-Weierstrass theorem asserts that it is uniformly dense in $C(X_2 \times Y_2)$, hence so is in $\widetilde{B_2}$. It follows that we have

$$\int F(x,y)d\nu_1 = \int F(x,y)d\nu_2 \tag{4.17}$$

for every $F \in \widetilde{B_2}$. On the other hand, since $\psi_1 = \psi_2$ we have

$$\begin{aligned}\int F(x,y)d\nu_1 + \int \gamma D(F)(m)d\nu_1 &= \psi_1(I(F)) \\ = \psi_2(I(F)) &= \int F(x,y)d\nu_2 + \int \gamma D(F)(m)d\nu_2\end{aligned} \tag{4.18}$$

for every $F \in \widetilde{B_2}$. By (4.17) and (4.18) we have

$$\int \gamma D(F)(m)d\nu_1 = \int \gamma D(F)(m)d\nu_2$$

for every $F \in \widetilde{B_2}$. It follows by (4.16) that $\phi_1(I(F)) = \phi_2(I(F))$ for every $F \in \widetilde{B_2}$. We infer that $\phi_\theta = \phi_1 = \phi_2$. We conclude that ϕ_θ is an extreme point for any $0 < \theta < \pi/2$, that is, $(x_c, y_c, m_c, e^{i\theta}\gamma_c)$ is in the Choquet boundary for $I(\widetilde{B_2})$ for any $0 < \theta < \pi/2$. $\square$

Proof of Proposition 4.9. Define a map $\tilde{U} : I_1(\widetilde{B_1}) \to I_2(\widetilde{B_2})$ by $\tilde{U}(I_1(H)) = I_2(U(H))$ for $I_1(H) \in I_1(\widetilde{B_1})$. The map $\tilde{U}$ is well defined since I_1 is injective. Due to the definition of I_j, we see that $\tilde{U}$ is a surjective isometry. Then the dual map $\tilde{U}^* : I_2(\widetilde{B_2})^* \to I_1(\widetilde{B_1})^*$ is an isometry and it preserves the extreme points of the closed unit ball $I_2(\widetilde{B_2})^*_1$ of $I_2(\widetilde{B_2})^*$. Let (x_0, y_0) be an arbitrary point in $X_2 \times Y_2$ and $\mathfrak{U}$ an arbitrary open neighborhood of (x_0, y_0). Then by Lemmas 4.10 and 4.11 there exists $(x_c, y_c, m_c, \gamma_c) \in \mathfrak{U} \times \mathfrak{M}_2 \times \mathbb{T}$ such that $(x_c, y_c, m_c, e^{i\theta}\gamma_c)$ is in the Choquet boundary of $I(\widetilde{B_2})$ for every $0 \le \theta < \pi/2$. Let ϕ_θ be the point evaluation on $I(\widetilde{B_2})$ at $(x_c, y_c, m_c, e^{i\theta}\gamma_c)$. Then ϕ_θ is an extreme point of the closed unit ball $I(\widetilde{B_2})^*_1$. As $\tilde{U}^*$ preserves the extreme point of the closed unit ball, $\tilde{U}^*(\phi_\theta)$ is an extreme points of the closed unit ball $I_1(\widetilde{B_1})^*_1$ of $I_1(\widetilde{B_1})^*$. By the Arens-Kelly theorem we see that there exists a complex number γ with absolute value 1 and a point d in the Choquet boundary for $I_1(\widetilde{B_1})$ such that $\tilde{U}^*(\phi_\theta) = \gamma\phi_d$, where ϕ_d denotes the point evaluation for $I_1(\widetilde{B_1})$ at d. Thus we have that

$$|\tilde{U}^*(\phi_\theta)(\mathbf{1})| = 1.$$

As $\tilde{U}(\phi_\theta)(\mathbf{1}) = \phi_\theta(I_2(U(\mathbf{1}))$ we have

$$1 = |U(\mathbf{1})(x_c, y_c) + e^{i\theta}\gamma_c D(U(\mathbf{1}))(m_c)|$$

for every $0 \le \theta < \pi/2$. Hence one of the following (i) or (ii) occurs:

(i) $U(\mathbf{1})(x_c, y_c) = 0$ and $|D(U(\mathbf{1}))(m_c)| = 1$,
(ii) $|U(\mathbf{1})(x_c, y_c)| = 1$ and $D(U(\mathbf{1}))(m_c) = 0$.

But (i) never occur. The reason is as follows. Since U is an isometry we have

$$1 = \|\mathbf{1}\| = \|U(\mathbf{1})\| = \|U(\mathbf{1})\|_\infty + \|D(U(\mathbf{1}))\|_\infty. \tag{4.19}$$

Suppose that (i) holds. By the second equation of (i) we have $\|D(U(\mathbf{1}))\|_\infty \geq 1$. Then by (4.19) we have $\|U(\mathbf{1})\|_\infty = 0$, and $U(\mathbf{1}) = 0$, which contradicts (4.19). Thus we conclude that only (ii) occurs.

By the first equation of (ii) we infer that $\|U(\mathbf{1})\|_\infty \geq 1$. Then by the equation (4.19), we have $0 = \|D(U(\mathbf{1}))\|_\infty$. By the condition ②(2) of Definition 4.4 we have $U\mathbf{1}1) \in \mathbf{1} \otimes C(Y_2)$; there exists $h \in C(Y_2)$ with $U(\mathbf{1}) = \mathbf{1} \otimes h$. As $|U(\mathbf{1})(x_c, y_c)| = 1$ we have $|h(y_c)| = 1$. Note that h does not depend on the point (x_0, y_0) nor a neighborhood $\mathfrak{U}$. As $\mathfrak{U}$ is an arbitrary neighborhood of (x_0, y_0) and $(x_c, y_c) \in \mathfrak{U}$, the continuity of h asserts that $|h(y_0)| = 1$. Since y_0 is an arbitrary point in Y_2, we infer that $|h| = 1$ on Y_2. □

Finally, we prove Theorem 4.5.

Proof of Theorem 4.5. Suppose first $X_1 = \{x_1\}$ and $X_2 = \{x_2\}$ are singletons. In this case B_j is isometrically isomorphic to $\mathbb{C}$ as a Banach algebra and $\widetilde{B_j} = 1 \otimes C(Y_j)$. Thus $\|D(F)\|_\infty = 0$ for every $F \in \widetilde{B_j}$. Therefore $\widetilde{B_j}$ is isometrically isomorphic to $C(Y_j)$ for $j = 1, 2$. Thus we may suppose that U is a surjective isometry from $C(Y_1)$ onto $C(Y_2)$. Then applying the Banach-Stone theorem, we see that $|U(1)| = 1$ on Y_2 and there exists a homeomorphism $\tau : Y_2 \to Y_1$ such that

$$U(F) = U(\mathbf{1}) F \circ \tau, \qquad F \in C(Y_1).$$

Letting $U(\mathbf{1}) = \mathbf{1} \otimes h$ and $\phi : X_2 \times Y_2 \to X_1$ by $\phi(x_1, y) = x_2$ for every $y \in Y_2$, we have

$$U(F)(x, y) = h(y) F(\varphi(x, y), \tau(y)), \qquad (x, y) \in X_2 \times Y_2$$

for every $F \in \widetilde{B_1}$.

Suppose that X_2 is not a singleton. We prove the conclusion applying Proposition 4.9. By Proposition 4.9 there exists $h \in C(Y_2)$ with $|h| = 1$ on Y_2 such that $U(\mathbf{1}) = \mathbf{1} \otimes h$. Define $U_0 : \widetilde{B_1} \to \widetilde{B_2}$ by

$U_0(F) = \mathbf{1} \otimes \bar{h}U(F)$ for $F \in \widetilde{B_1}$, where $\bar{h}$ denotes the complex conjugate of h. It is easy to see that U_0 is a bijection with $U_0(\mathbf{1}) = \mathbf{1}$. By the condition ②(3) of Definition 4.4 it is also easy to check that U_0 is an isometry. As $\widetilde{B_j}$ is a unital Banach algebra which is contained in $C(X_j \times Y_j)$ which separates the points of $X_j \times Y_j$. As $\widetilde{B_j}$ is natural, by [**47**, Proposition 2] it is a regular subspace of $C(X_j \times Y_j)$ in the sense of Jarosz [**47**, p. 67]. As the norm $\|\cdot\| = \|\cdot\|_\infty + |||\cdot|||$ is a p-norm (see [**47**, p. 67]) and $U_0(\mathbf{1}) = \mathbf{1}$, we infer by Theorem in [**47**] that U_0 is also an isometry with respect to the supremum norm $\|\cdot\|_\infty$ on $X_j \times Y_j$. As $\widetilde{B_j}$ is a self-adjoint unital subalgebra of $C(X_j \times Y_j)$ which separates the points of $X_j \times Y_j$, the Stone-Weierstrass theorem asserts that $\widetilde{B_j}$ is uniformly dense in $C(X_j \times Y_j)$. Then the Banach-Stone theorem asserts that U_0 is an algebra isomorphism. Since U_0 is an isometry with respect to the original norm $\|\cdot\|$ on $\widetilde{B_j}$ we have for every $\mathbf{1} \otimes g \in \mathbf{1} \otimes C(Y_1)$ that

$$\begin{aligned}\|\mathbf{1} \otimes g\|_\infty + \|D(\mathbf{1} \otimes g)\|_\infty = \|\mathbf{1} \otimes g\| = \|U_0(\mathbf{1} \otimes g)\| \\ = \|U_0(\mathbf{1} \otimes g)\|_\infty + \|D(U_0(\mathbf{1} \otimes g))\|_\infty.\end{aligned}$$

By the condition ②(2) of Definition 4.4 we have $\|D(1 \otimes g)\|_\infty = 0$. Since U_0 is also an isometry with respect to the supremum norm we have $\|\mathbf{1} \otimes g\|_\infty = \|U_0(\mathbf{1} \otimes g)\|_\infty$. Therefore we have that $\|D(U_0(\mathbf{1} \otimes g))\|_\infty = 0$. By the condition ②(2) of Definition 4.4 we have that $U_0(\mathbf{1} \otimes g) \in \mathbf{1} \otimes C(Y_2)$. Hence we see that $U_0(\mathbf{1} \otimes C(Y_1)) \subset \mathbf{1} \otimes C(Y_2)$. By the Stone-Weierstrass theorem $B_1 \otimes C(Y_1)$ is uniformly dense in $C(X_1 \times Y_1)$, hence $\widetilde{B_1} \subset \overline{B_1 \otimes C(Y_1)}$, where $\bar{\cdot}$ denotes the uniform closure on $X_1 \times Y_1$. Then by Proposition 3.2 and the following comments in [**45**] there exists continuous maps $\varphi : X_2 \times Y_2 \to X_1$ and $\tau : Y_2 \to Y_1$ such that

$$U_0(F)(x, y) = F(\varphi(x, y), \tau(y)), \qquad (x, y) \in X_2 \times Y_2$$

for every $F \in \widetilde{B_1}$. Applying a similar argument for U_0^{-1} instead of U_0 we observe that there exists continuous maps $\varphi_1 : X_1 \times Y_1 \to X_2$ and $\tau_1 : Y_1 \to Y_2$ such that

$$U_0^{-1}(G)(u, v) = G(\varphi_1(u, v), \tau_1(v)), \qquad (u, v) \in X_1 \times Y_1$$

for every $G \in \widetilde{B_2}$. Thus we have

$$(4.20)\quad \begin{aligned} G(x,y) &= U_0(U_0^{-1}(G))(x,y) = U_0^{-1}(G)(\varphi(x,y),\tau(y)) \\ &= G(\varphi_1(\varphi(x,y),\tau(y)),\tau_1(\tau(y))), \quad (x,y) \in X_2 \times Y_2 \end{aligned}$$

for every $G \in \widetilde{B_2}$ and

$$(4.21)\quad \begin{aligned} F(u,v) &= U_0^{-1}(U_0(F))(u,v) = U_0(F)(\varphi_1(u,v),\tau_1(v)) \\ &= F(\varphi(\varphi_1(u,v),\tau_1(v)),\tau(\tau_1(v))), \quad (u,v) \in X_1 \times Y_1 \end{aligned}$$

for every $F \in \widetilde{B_1}$. As $\widetilde{B_1}$ separates the points in $X_1 \times Y_1$ and $\widetilde{B_2}$ separates the points in $X_2 \times Y_2$, we infer that $y = \tau_1(\tau(y))$ for every $y \in Y_2$ and $v = \tau(\tau_1(v))$ for every $v \in Y_1$. Hence $\tau : Y_2 \to Y_1$ and $\tau_1 : Y_1 \to Y_2$ are homeomorphisms and $\tau_1^{-1} = \tau$. We have by (4.21) that $u = \varphi(\varphi_1(u,v),\tau_1(v))$ for every $(u,v) \in X_1 \times Y_1$. As τ_1 is a homeomorphism, we infer that $u = \varphi(\varphi_1(u,\tau_1^{-1}(y)),y)$ holds for every pair $u \in X_1$ and $y \in Y_2$. It means that for every $y \in Y_2$ the map $\varphi(\cdot,y) : X_2 \to X_1$ is a surjection.

We prove that $\varphi(\cdot,y)$ is an injection for every $y \in Y_2$. Let $y \in Y_2$. Suppose that $\varphi(a,y) = \varphi(b,y)$ for $a,b \in X_2$. Then $\varphi_1(\varphi(a,y),\tau(y)) = a$ and $\varphi_1(\varphi(b,y),\tau(y)) = b$ by the equation (4.20). Thus we have $a = b$. Hence we conclude that $\varphi(\cdot,y)$ is an injection. It follows that $\varphi(\cdot,y) : X_2 \to X_1$ is a bijective continuous map. As X_2 is compact and X_1 is Hausdorff, we at once see that $\varphi(\cdot,y)$ is a homeomorphism. As $U_0(F) = 1 \otimes \bar{h}U(F)$ for every $F \in \widetilde{B_1}$ we conclude that

$$U(F)(x,y) = h(y)F(\varphi(x,y),\tau(y)), \qquad (x,y) \in X_2 \times Y_2.$$

Suppose that X_1 is not a singleton. By a similar argument for U^{-1} instead of U we see that there exists a continuous map $\varphi_1 : X_1 \times Y_1 \to X_2$ such that $\varphi_1(\cdot,y) : X_1 \to X_2$ is a homeomorphism. As X_1 is not a singleton we infer that X_2 is not a singleton. Then the conclusion follows from the proof for the case where X_2 is not a singleton. □

5. Examples of admissible quadruples of type L with applications of Main Results

EXAMPLE 4.12. Let (X,d) be a compact metric space and Y a compact Hausdorff space. Let $0 < \alpha \leq 1$. Suppose that B is a closed subalgebra of $\mathrm{Lip}((X,d^\alpha))$ which contains the constants and separates the points of X, where d^α is the Hölder metric induced by d. Suppose that $\widetilde{B}$ is a closed subalgebra of $\mathrm{Lip}((X,d^\alpha),C(Y))$ which contains the constants and separates the points of $X\times Y$. Suppose that B and $\widetilde{B}$ are self-adjoint. Suppose that

$$B\otimes C(Y)\subset \widetilde{B}$$

and

$$\{F(\cdot,y):F\in\widetilde{B},y\in Y\}\subset B.$$

Let $\mathfrak{M}$ be the Stone-Čech compactification of $\{(x,x')\in X^2:x\neq x'\}\times Y$. For $F\in\widetilde{B}$, let $D(F)$ be the continuous extension to $\mathfrak{M}$ of the function $(F(x,y)-F(x',y))/d^\alpha(x,x')$ on $\{(x,x')\in X^2:x\neq x'\}\times Y$. Then $D:\widetilde{B}\to C(\mathfrak{M})$ is well defined. We have $\|D(F)\|_\infty=L_\alpha(F)$ for every $F\in\widetilde{B}$. It is easy to see that the condition ② of Definition 4.4 is satisfied. Hence we have that $(X,C(Y),B,\widetilde{B})$ is an admissible quadruple of type L.

There are two typical example of $(X,C(Y),B,\widetilde{B})$ above. One is

$$(X,C(Y),\mathrm{Lip}((X,d^\alpha)),\mathrm{Lip}((X,d^\alpha),C(Y)))$$

By Corollary 4.3 $\mathrm{Lip}((X,d^\alpha))$ and $\mathrm{Lip}((X,d^\alpha),C(Y))$ are self-adjoint. The inclusions

$$\mathrm{Lip}((X,d^\alpha))\otimes C(Y)\subset \mathrm{Lip}((X,d^\alpha),C(Y))$$

and

$$\{F(\cdot,y):F\in\mathrm{Lip}((X,d^\alpha),C(Y)),y\in Y\}\subset \mathrm{Lip}((X,d^\alpha))$$

is obvious. The other example of $(X,C(Y),B,\widetilde{B})$ above is

$$(X,C(Y),\mathrm{lip}(X),\mathrm{lip}(X,C(Y)))$$

for $0<\alpha<1$. In fact $\mathrm{lip}(X)$ (resp. $\mathrm{lip}(X,C(Y))$) is a closed subalgebra of $\mathrm{Lip}((X,d^\alpha))$ (resp. $\mathrm{Lip}((X,d^\alpha),C(Y))$ which contains the constants. In this case Corollary 4.3 asserts that $\mathrm{lip}(X)$ separates the points of X. As $\mathrm{lip}(X)\otimes C(Y)\subset\mathrm{lip}(X,C(Y))$ we see that $\widetilde{B}=\mathrm{lip}(X,C(Y))$ separates the points of $X\times Y$. By Corollary 4.3 $\mathrm{lip}(X)$ and $\mathrm{lip}(X,C(Y))$ are self-adjoint. The inclusions

$$\mathrm{lip}(X)\otimes C(Y)\subset\mathrm{lip}(X,C(Y))$$

and

$$\{F(\cdot, y) : F \in \mathrm{lip}(X, C(Y)), y \in Y\} \subset \mathrm{lip}(X)$$

is obvious.

COROLLARY 4.13. *Let $j = 1, 2$. Let (X_j, d_j) be a compact metric space and Y_j a compact Hausdorff space. Let α be $0 < \alpha \leq 1$. Suppose that B_j is a closed subalgebra of $\mathrm{Lip}((X_j, d_j^\alpha))$ which contains the constants and separates the points of X_j. Suppose that $\widetilde{B_j}$ is a closed subalgebra of $\mathrm{Lip}((X_j, d_j^\alpha), C(Y_j))$ which contains the constants and separates the points of $X_j \times Y_j$. Suppose that B_j and $\widetilde{B}_j$ are self-adjoint. Suppose that*

$$B_j \otimes C(Y_j) \subset \widetilde{B_j}$$

and

$$\{F(\cdot, y) : F \in \widetilde{B_j}, y \in Y_j\} \subset B_j.$$

Suppose that

$$U : \widetilde{B_1} \to \widetilde{B_2}$$

is a surjective isometry. Then there exists $h \in C(Y_2)$ such that $|h| = 1$ on Y_2, a continuous map $\varphi : X_2 \times Y_2 \to X_1$ such that $\varphi(\cdot, y) : X_2 \to X_1$ is a homeomorphism for each $y \in Y_2$, and a homeomorphism $\tau : Y_2 \to Y_1$ which satisfy

$$U(F)(x, y) = h(y)F(\varphi(x, y), \tau(y)), \qquad (x, y) \in X_2 \times Y_2$$

for every $F \in \widetilde{B_1}$.

PROOF. As in a similar way as in Example 5.6 we see that $(X_j, C(Y_j), B_j, \widetilde{B_j})$ is an admissible quadruple of type L. Then applying Theorem 4.5 the conclusion holds. □

Note that Corollary 4.13 holds for $\widetilde{B_j} = \mathrm{Lip}(X_j, C(Y_j))$ and $\widetilde{B_j} = \mathrm{lip}(X_j, C(Y_j))$ for $0 < \alpha < 1$. In this case we have a complete description of a surjective isometry for $\widetilde{B_j} = \mathrm{Lip}(X_j, C(Y_j))$ and $\widetilde{B_j} = \mathrm{lip}(X_j, C(Y_j))$ for $0 < \alpha < 1$. Note that $\mathrm{Lip}_\alpha((X_j, d_j), C(Y_j))$ for $0 < \alpha < 1$ is isometrically isomorphic to $\mathrm{Lip}((X_j, d_j^\alpha), C(Y_j))$ by considering the Hölder metric $d_j(\cdot, \cdot)^\alpha$ for the original metric $d_j(\cdot, \cdot)$ on X_j.

COROLLARY 4.14. *Let (X_j, d_j) be a compact metric space and Y_j a compact Hausdorff space for $j = 1, 2$. Suppose that $U : \mathrm{Lip}(X_1, C(Y_1)) \to \mathrm{Lip}(X_2, C(Y_2))$ (resp. $U : \mathrm{lip}(X_1, C(Y_1)) \to \mathrm{lip}(X_2, C(Y_2))$) is a map. Then U is a surjective isometry with respect to the sum norm $\|\cdot\| = \|\cdot\|_\infty + L(\cdot)$ (resp. $\|\cdot\| = \|\cdot\|_\infty + L_\alpha(\cdot)$) if and only if there exists $h \in C(Y_2)$ with $|h| = 1$ on Y_2, a continuous map $\varphi : X_2 \times Y_2 \to X_1$ such that $\varphi(\cdot, y) : X_2 \to X_1$ is a surjective isometry for every $y \in Y_2$, and a homeomorphism $\tau : Y_2 \to Y_1$ which satisfy that*

$$U(F)(x,y) = h(y)F(\varphi(x,y), \tau(y)), \qquad (x,y) \in X_2 \times Y_2$$

for every $F \in \mathrm{Lip}(X_1, C(Y_1))$ (resp. $F \in \mathrm{lip}(X_1, C(Y_1))$).

PROOF. Suppose that there exists $h \in C(Y_2)$ with $|h| = 1$ on Y_2, a continuous map $\varphi : X_2 \times Y_2 \to X_1$ such that $\varphi(\cdot, y) : X_2 \to X_1$ is a surjective isometry for every $y \in Y_2$, and a homeomorphism $\tau : Y_2 \to Y_1$ which satisfy that

$$U(F)(x,y) = h(y)F(\varphi(x,y), \tau(y)), \qquad (x,y) \in X_2 \times Y_2$$

for every $F \in \mathrm{Lip}(X_1, C(Y_1))$ (resp. $F \in \mathrm{lip}(X_1, C(Y_1))$). We prove that U is a surjective isometry on $\mathrm{Lip}(X_j, C(Y_j))$. A proof for the case of $\mathrm{lip}(X_j, C(Y_j))$ is the same and we omit it. Since $\varphi(\cdot, y)$ is an isometry for every $y \in Y_2$, we have

$$\begin{aligned} &\frac{|(U(F))(x,y) - (U(F))(x',y)|}{d_2(x,x')} \\ &= \frac{|h(y)F(\varphi(x,y),\tau(y)) - h(y)F(\varphi(x',y),\tau(y))|}{d_2(x,x')} \qquad (5.1) \\ &= \frac{|F(\varphi(x,y),\tau(y)) - F(\varphi(x',y),\tau(y))|}{d_2(\varphi(x,y),\varphi(x',y))}, \quad x, x' \in X_2, y \in Y_2 \end{aligned}$$

for $F \in \mathrm{Lip}(X_1, C(Y_1))$. Since $\varphi(\cdot, y)$ is bijective and the map $(x,y) \mapsto (\varphi(x,y), \tau(y))$ gives a bijection from $X_2 \times Y_2$ onto $X_1 \times Y_1$, we see by (5.1) that $L(F) = L(U(F))$ for every $F \in \mathrm{Lip}(X_1, C(Y_1))$. Since $\|F\|_\infty = \|U(F)\|_\infty$ is trivial, we conclude that

$$\|F\| = \|F\|_\infty + L(F) = \|U(F)\|_\infty + L(U(F)) = \|U(F)\|$$

for every $F \in \mathrm{Lip}(X_1, C(Y_1))$; U is an isometry. We prove that U is surjective. Let $G \in \mathrm{Lip}(X_2, C(Y_2))$ be arbitrary. Put F by $F(x,y) =$

$\bar{h}(y)G((\varphi(\cdot,\tau^{-1}(y)))^{-1}(x),\tau^{-1}(y))$ for $(x,y)\in X_1\times Y_1$, where $(\varphi(\cdot,\tau^{-1}(y)))^{-1}$ denotes the inverse of $\varphi(\cdot,\tau^{-1}(y)) : X_2 \to X_1$. Then we infer that $F\in \mathrm{Lip}(X_1,C(Y_1))$ and $U(F)=G$. As G is an arbitrary elements in $\mathrm{Lip}(X_2,C(Y_2))$, we conclude that U is surjective. It follows that U is a surjective isometry.

Next we prove the converse. First consider the case of $\mathrm{Lip}(X_j,C(Y_j))$. Suppose that $U:\mathrm{Lip}(X_1,C(Y_1))\to \mathrm{Lip}(X_2,C(Y_2))$ is a surjective isometry. Then by Corollary 4.13 there exists $h\in C(Y_2)$ with $|h|=1$ on Y_2, a continuous map $\varphi : X_2\times Y_2\to X_1$ such that $\varphi(\cdot,y):X_2\to X_1$ is a homeomorphism, and a homeomorphism $\tau : Y_2\to Y_1$ which satisfy that

$$U(F)(x,y)=h(y)F(\varphi(x,y),\tau(y)),\qquad (x,y)\in X_2\times Y_2 \tag{5.2}$$

for every $F\in \mathrm{Lip}(X_1,C(Y_1))$. We only need to prove that $\varphi(\cdot,y) : X_2\to X_1$ is a surjective isometry for every $y\in Y_2$. Let $x_1,x_2\in X_2$ and $y\in Y_2$ be arbitrary. Set $f:X_1\to\mathbb{C}$ by $f(x)=d_1(x,\varphi(x_2,y))$ for $x\in X_1$. Then $L(f\otimes 1)=1$ and $f\otimes 1\in \mathrm{Lip}(X_1,C(Y_1))$. Then we have

$$\begin{aligned} d_1(\varphi(x_1,y),\varphi(x_2,y)) &= f(\varphi(x_1,y)) = |f(\varphi(x_1,y)-f(\varphi(x_2,y))| \\ &= |f\otimes 1(\varphi(x_1,y),\tau(y))-f\otimes 1(\varphi(x_2,y),\tau(y))| \\ &= |(U(f\otimes 1))(x_1,y)-(U(f\otimes 1))(x_2,y)| \\ &\le L(U(f\otimes 1))d_2(x_1,x_2). \end{aligned} \tag{5.3}$$

By (5.2) the map U is an isometry with respect to $\|\cdot\|_\infty$, thus $1=L(f\otimes \mathbf{1})=L(U(f\otimes\mathbf{1}))$ since U is an isometry for $\|\cdot\|=\|\cdot\|_\infty+L(\cdot)$. It follows by (5.3) that $d_1(\varphi(x_1,y),\varphi(x_2,y))\le d_2(x_1,x_2)$. Since U^{-1} is a surjective isometry we have by Corollary 4.13 that there exists h_1, φ_1 and τ_1 such that

$$U^{-1}(G)(x,y)=h_1(y)G(\varphi_1(x,y),\tau_1(y)),\qquad (x,y)\in X_1\times Y_1$$

for $G\in \mathrm{Lip}(X_2,C(Y_2))$. Then by a similar way as above we infer that $d_2(\varphi_1(x_1',y'),\varphi_1(x_2',y'))\le d_1(x_1',x_2')$ for every pair $x_1',x_2'\in X_1$ and $y'\in Y_1$. By a simple calculation we obtain that $x=\varphi_1(\varphi(x,y),\tau(y))$ for every $x\in X_2$ and $y\in Y_2$ (see a similar calculation in the proof of

Theorem 4.5 or on p. 386 of [**40**]). Thus we have

$$\begin{aligned} d_2(x_1, x_2) &= d_2(\varphi_1(\varphi(x_1, y), \tau(y)), \varphi_1(\varphi(x_2, y), \tau(y)) \\ &\leq d_1(\varphi(x_1, y), \varphi(x_2, y)). \end{aligned}$$

Therefore $d_2(x_1, x_2) = d_1(\varphi(x_1, y), \varphi(x_2, y))$ holds for every pair $x_1, x_2 \in X_2$ and $y \in Y_2$, that is, $\varphi(\cdot, y)$ is an isometry for every $y \in Y_2$.

Next we consider the case of $\mathrm{lip}(X_j, C(Y_j))$. Suppose that $0 < \alpha < 1$ and $U : \mathrm{lip}(X_1, C(Y_1)) \to \mathrm{lip}(X_2, C(Y_2))$ is a surjective isometry. As in the same way as before there exists $h \in C(Y_2)$ with $|h| = 1$ on Y_2, a continuous map $\varphi : X_2 \times Y_2 \to X_1$ such that $\varphi(\cdot, y) : X_2 \to X_1$ is a homeomorphism for every $y \in Y_2$, and a homeomorphism $\tau : Y_2 \to Y_1$ which satisfy that

$$U(F)(x, y) = h(y)F(\varphi(x, y), \tau(y)), \qquad (x, y) \in X_2 \times Y_2$$

for every $F \in \mathrm{lip}(X_1, C(Y_1))$. We prove $\varphi(\cdot, y) : X_2 \to X_1$ is an isometry for every $y \in Y_2$. Let $x_1, x_2 \in X_2$ and $y \in Y_2$ be arbitrary. Let β with $\alpha < \beta < 1$ be arbitrary. Set $f^\beta : X_1 \to \mathbb{C}$ by $f^\beta(x) = d_1(x, \varphi(x_2, y))^\beta$. We have

$$\begin{aligned} \frac{|f^\beta(s) - f^\beta(t)|}{d_1(s, t)^\alpha} &= \frac{|d_1(s, \varphi(x_2, y))^\beta - d_1(t, \varphi(x_2, y))^\beta|}{d_1(s, t)^\alpha} \\ &\leq \frac{d_1(s, t)^\beta}{d_1(s, t)^\alpha} = d_1(s, t)^{\beta - \alpha}, \quad s, t \in X_1. \end{aligned} \tag{5.4}$$

Since X_1 is compact we have $\sup_{s,t \in X_1} d_1(s, t) < \infty$. Put $M = \sup_{s,t \in X_1} d_1(s, t)$. Then by (5.4) we infer that $L_\alpha(f^\beta \otimes 1) \leq M^{\beta - \alpha}$. We also infer by (5.4) that $\lim_{s \to t} \frac{|f^\beta(s) - f^\beta(t)|}{d_1(s,t)^\alpha} = 0$. Hence we have $f^\beta \otimes 1 \in \mathrm{lip}(X_1, C(Y_1))$. We have, as before,

(5.5)
$$\begin{aligned} d_1(\varphi(x_1, y), \varphi(x_2, y))^\beta &= |f^\beta \otimes 1(\varphi(x_1, y), \tau(y)) - f^\beta(\varphi(x_2, y), \tau(y))| \\ &= |(U(f^\beta \otimes 1)(x_1, y) - (U(f^\beta \otimes 1)(x_2, y)| \\ &\leq L_\alpha(U(f^\beta \otimes 1)) d_2(x_1, x_2)^\alpha \\ &= L_\alpha(f^\beta \otimes 1) d_2(x_1, x_2)^\alpha = M^{\beta - \alpha} d_2(x_1, x_2)^\alpha. \end{aligned}$$

Letting $\beta \to \alpha$ we have by (5.5) that $d_1(\varphi(x_1, y), \varphi(x_2, y))^\alpha \leq d_2(x_1, x_2)^\alpha$, hence $d_1(\varphi(x_1, y), \varphi(x_2, y)) \leq d_2(x_1, x_2)$. Applying the same argument for U^{-1} as in the case of $\mathrm{Lip}(X_j, C(Y_j))$ we get

$$d_2(x_1,x_2)^\beta \le M'^{\beta-\alpha} d_1(\varphi(x_1,y),\varphi(x_2,y))^\alpha$$

for every β with $\alpha < \beta < 1$, where $M' = \sup_{s,t\in X_2} d_2(s,t)$. Letting $\beta \to \alpha$ we get $d_2(x_1,x_2)^\alpha \le d_1(\varphi(x_1,y),\varphi(x_2,y))^\alpha$ and $d_2(x_1,x_2) \le d_1(\varphi(x_1,y),\varphi(x_2,y))$. It follows that $d_2(x_1,x_2) = d_1(\varphi(x_1,y),\varphi(x_2,y))$ for every pair $x_1,x_2 \in X_2$ and $y \in Y_2$, that is, $\varphi(\cdot,y)$ is an isometry for every $y \in Y_2$. □

Note that if Y_j is a singleton in Corollary 4.14, then we may suppose that $\mathrm{Lip}(X_j, C(Y_j))$ (resp. $\mathrm{lip}(X_j, C(Y_j))$) is $\mathrm{Lip}(X_j)$ (resp. $\mathrm{lip}(X_j)$). Then Corollary 4.14 states that Example 8 in [**48**] is indeed true.

COROLLARY 4.15. [**48**, Example 8] *The map $U : \mathrm{Lip}(X_1) \to \mathrm{Lip}(X_2)$ (resp. $U : \mathrm{lip}(X_1) \to \mathrm{lip}(X_2)$) is a surjective isometry with respect to the norm $\|\cdot\| = \|\cdot\|_\infty + L(\cdot)$ (resp. $\|\cdot\| = \|\cdot\|_\infty + L_\alpha(\cdot)$) if and only if there exists a complex number c with the unit modulus and a surjective isometry $\varphi : X_2 \to X_1$ such that*

$$U(F)(x) = cF(\varphi(x)), \qquad x \in X_2$$

for every $F \in \mathrm{Lip}(X_1)$ (resp. $F \in \mathrm{lip}(X_1)$).

PROOF. Suppose that U is a surjective isometry, then by Corollary 4.14 there exists a complex number c with the unit modulus and a surjective isometry $\varphi : X_2 \to X_1$ such that the desired equality holds.

Suppose that c is a complex number with the unit modulus and $\varphi : X_2 \to X_1$ is a surjective isometry. Then $U : \mathrm{Lip}(X_1) \to \mathrm{Lip}(X_2)$ (resp. $U : \mathrm{lip}(X_1) \to \mathrm{lip}(X_2)$) by $U(F)(x) = cF(\varphi(x))$, $x \in X_2$ for $F \in \mathrm{Lip}(X_1)$ (resp. $F \in \mathrm{lip}(X_1)$) is well defined. Then by Corollary 4.14 we have that U is a surjective isometry. □

EXAMPLE 4.16. Let Y be a compact Hausdorff space. Then

$$([0,1], C(Y), C^1([0,1]), C^1([0,1], C(Y)))$$

is an admissible quadruple of type L, where the norm of $f \in C^1([0,1])$ is defined by $\|f\| = \|f\|_\infty + \|f'\|_\infty$ and the norm of $F \in C^1([0,1], C(Y))$

is defined by $\|F\| = \|F\|_\infty + \|F'\|_\infty$. It is easy to see that $C^1([0,1]) \otimes C(Y) \subset C^1([0,1], C(Y))$ and

$$\{F(\cdot, y) : F \in C^1([0,1], C(Y)),\ y \in Y\} \subset C^1([0,1]).$$

Let $\mathfrak{M} = [0,1] \times Y$ and $D : C^1([0,1], C(Y)) \to C(\mathfrak{M})$ be defined by $D(F)(x,y) = F'(x,y)$ for $F \in C^1([0,1], C(Y))$. Then $\|F'\|_\infty = \|D(F)\|_\infty$ for $F \in C^1([0,1], C(Y))$. Then the conditions from ① through ②(3) of Definition 4.4 are satisfied.

Example 4.17. Let Y be a compact Hausdorff space. Then

$$(\mathbb{T}, C(Y), C^1(\mathbb{T}), C^1(\mathbb{T}, C(Y)))$$

is an admissible quadruple of type L, where the norm of $f \in C^1(\mathbb{T})$ is defined by $\|f\| = \|f\|_\infty + \|f'\|_\infty$ and the norm of $F \in C^1(\mathbb{T}, C(Y))$ is defined by $\|F\| = \|F\|_\infty + \|F'\|_\infty$. It is easy to see that $C^1(\mathbb{T}) \otimes C(Y) \subset C^1(\mathbb{T}, C(Y))$ and

$$\{F(\cdot, y) : F \in C^1(\mathbb{T}, C(Y)),\ y \in Y\} \subset C^1(\mathbb{T}).$$

Let $\mathfrak{M} = \mathbb{T} \times Y$ and $D : C^1(\mathbb{T}, C(Y)) \to C(\mathfrak{M})$ be defined by $D(F)(x,y) = F'(x,y)$ for $F \in C^1(\mathbb{T}, C(Y))$. Then $\|F'\|_\infty = \|D(F)\|_\infty$ for $F \in C^1(\mathbb{T}, C(Y))$. Then the conditions from ① through ②(3) of definition 4.4 are satisfied for $(\mathbb{T}, C(Y), C^1(\mathbb{T}), C^1(\mathbb{T}, C(Y))$.

Corollary 4.18. *Let Y_j be a compact Hausdorff space for $j = 1, 2$. The norm $\|F\|$ of $F \in C^1([0,1], C(Y_j))$ is defined by $\|F\| = \|F\|_\infty + \|F'\|_\infty$. Suppose that $U : C^1([0,1], C(Y_1)) \to C^1([0,1], C(Y_2))$ is a map. Then U is a surjective isometry if and only if there exists $h \in C(Y_2)$ such that $|h| = 1$ on Y_2, a continuous map $\varphi : [0,1] \times Y_2 \to [0,1]$ such that for each $y \in Y_2$ we have $\varphi(x,y) = x$ for every $x \in [0,1]$ or $\varphi(x,y) = 1 - x$ for every $x \in [0,1]$, and a homeomorphism $\tau : Y_2 \to Y_1$ which satisfy that*

$$U(F)(x,y) = h(y)F(\varphi(x,y), \tau(y)), \qquad (x,y) \in [0,1] \times Y_2$$

for every $F \in C^1([0,1], C(Y_1))$.

Proof. Suppose that $U : C^1([0,1], C(Y_1)) \to C^1([0,1], C(Y_2))$ is a surjective isometry. Then by Theorem 4.5 there exists $h \in C(Y_2)$ such that $|h| = 1$ on Y_2, a continuous map $\varphi : [0,1] \times Y_1 \to [0,1]$ such that $\varphi(\cdot, y) : [0,1] \to [0,1]$ is a homeomorphism for each $y \in Y_2$, and a homeomorphism $\tau : Y_2 \to Y_1$ which satisfy

$$U(F)(x,y) = h(y)F(\varphi(x,y), \tau(y)), \qquad (x,y) \in [0,1] \times Y_2 \tag{5.6}$$

for every $F \in C^1([0,1], C(Y_1))$. We only need to prove that, for every $y \in Y_2$ $\varphi(x,y) = x$ for every $x \in [0,1]$ or $\varphi(x,y) = 1 - x$ for every $x \in [0,1]$. Let $F_0 \in C^1([0,1], C(Y_1))$ be defined by $F_0(x,y) = x$ for every $(x,y) \in [0,1] \times Y_1$. Then we have $F_0' = 1$ on $[0,1] \times Y_1$ and $\|F_0\| = \|F_0\|_\infty + \|F_0'\|_\infty = 2$. By (5.6) we have $U(F_0)(x,y) = h(y)\varphi(x,y)$ for every $(x,y) \in [0,1] \times Y_2$. Since $U(F_0)$ is continuously differentiable we infer that φ is continuously differentiable and that $U(F_0)'(x,y) = h(y)\varphi'(x,y)$ for every $(x,y) \in [0,1] \times Y_2$. By (5.6) we infer that $\|U(F_0)\|_\infty = \|F_0\|_\infty$, hence $\|U(F_0)'\|_\infty = \|F_0'\|_\infty$ since U is an isometry with respect to $\| \cdot \|$. As $|h| = 1$ on Y_2 we see that

$$|\varphi'(x,y)| \leq \|U(F_0)'\|_\infty = \|F_0'\|_\infty = 1$$

for every $(x,y) \in [0,1] \times Y_2$. We prove that $|\varphi'(x,y)| = 1$ for every $(x,y) \in [0,1] \times Y_2$. Suppose contrary that there exists $(x_0, y_0) \in [0,1] \times Y_2$ with $|\varphi'(x_0, y_0)| < 1$. As $\varphi(\cdot, y_0) : [0,1] \to [0,1]$ is a homeomorphism we infer that $|\varphi(1, y_0) - \varphi(0, y_0)| = 1$. As $\varphi(\cdot, y_0)$ is continuously differentiable we have

$$1 = |\varphi(1, y_0) - \varphi(0, y_0)| = |\int_0^1 \varphi'(x, y_0) dx| \leq \int_0^1 |\varphi'(x, y_0)| dx.$$

Since $|\varphi(x,y)| \leq 1$ and $|\varphi'(x_0, y_0)| < 1$ we have

$$\int_0^1 |\varphi'(x, y_0)| dx < 1,$$

which is a contradiction. Hence we have that $|\varphi'(x,y)| = 1$ for every $(x,y) \in [0,1] \times Y_2$. Let $y_1 \in Y_2$ be arbitrary. As $\varphi'(\cdot, y_1)$ is continuous on $[0,1]$ and $|\varphi'(\cdot, y_1)| = 1$ on $[0,1]$ we have that $\varphi'(\cdot, y_1) = 1$ on $[0,1]$ or $\varphi'(\cdot, y_1) = -1$ on $[0,1]$ since φ' is real-valued with $|\varphi'| = 1$ on a connected space $[0,1]$. It follows by a simple calculation that $\varphi(x, y_1) = x$ for every $x \in [0,1]$ or $\varphi(x, y_1) = 1 - x$ for every $x \in [0,1]$ since $\varphi(\cdot, y_1)$ is a bijection between $[0,1]$.

Suppose conversely that there exists $h \in C(Y_2)$ such that $|h| = 1$ on Y_2, a continuous map $\varphi : [0,1] \times Y_2 \to [0,1]$ such that for each $y \in Y_2$

$\varphi(x,y)=x$ for every $x\in[0,1]$ or $\varphi(x,y)=1-x$ for every $x\in[0,1]$, and a homeomorphism $\tau:Y_2\to Y_1$ which satisfy that

$$U(F)(x,y)=h(y)F(\varphi(x,y),\tau(y)),\qquad (x,y)\in[0,1]\times Y_2$$

for every $F\in C^1([0,1],C(Y_1))$. It is straight forward to check that $\|U(F)\|_\infty=\|F\|_\infty$. Let $y\in Y_2$ be arbitrary. By a simple calculation we infer that $|U(F)'(x,y)|=|F'(x,\tau(y))|$ for every $x\in[0,1]$ or $|U(F)'(x,y)|=|F'(1-x,\tau(y))|$ for every $x\in[0,1]$ for each $y\in Y_2$ and $F\in C^1([0,1],C(Y_1))$. As τ is a surjection, we have $\|U(F)'\|_\infty=\|F'\|_\infty$ for every $F\in C^1([0,1],C(Y_1))$. To prove that U is surjective, let $F\in C^1([0,1],C(Y_2))$ be an arbitrary map. Put $G(x',y')=\overline{h(\tau^{-1}(y'))}F(\varphi(x',\tau^{-1}(y')),\tau^{-1}(y'))$, $(x',y')\in[0,1]\times Y_1$. It is easy to see that $G\in C^1([0,1],C(Y_1))$. As $\varphi(x,y)=x$ or $1-x$ depending on $y\in Y_2$ we see by a simple calculation that $\varphi(\varphi(x,y),y)=x$ for every $(x,y)\in[0,1]\times Y_2$. Then we have

$$\begin{aligned}(U(G))(x,y)&=h(y)G(\varphi(x,y),\tau(y))\\&=h(y)\overline{h(\tau^{-1}(\tau(y)))}F(\varphi(\varphi(x,y),\tau^{-1}(\tau(y)),\tau^{-1}(\tau(y)))\\&=F(\varphi(\varphi(x,y),y)=F(x,y),\quad (x,y)\in[0,1]\times Y_2\end{aligned}$$

It follows that U is a surjective isometry from $C^1([0,1],C(Y_1))$ onto $C^1([0,1],C(Y_2))$. □

Note that if Y_j is a singleton in Corollary 4.18, then $C^1([0,1],C(Y_j))$ is $C^1([0,1],\mathbb{C})$. The corresponding result on isometries was given by Rao and Roy [**104**].

COROLLARY 4.19. *Let Y_j be a compact Hausdorff space for $j=1,2$. The norm $\|F\|$ of $F\in C^1(\mathbb{T},C(Y_j))$ is defined by $\|F\|=\|F\|_\infty+\|F'\|_\infty$. Suppose that $U:C^1(\mathbb{T},C(Y_1))\to C^1(\mathbb{T},C(Y_2))$ is a map. Then U is a surjective isometry if and only if there exists $h\in C(Y_2)$ such that $|h|=1$ on Y_2, a continuous map $\varphi:\mathbb{T}\times Y_2\to\mathbb{T}$ and a continuous map $u:Y_2\to\mathbb{T}$ such that for every $y\in Y_2$ $\varphi(z,y)=u(y)z$ for every $z\in\mathbb{T}$ or $\varphi(z,y)=u(y)\bar{z}$ for every $z\in\mathbb{T}$, and a homeomorphism $\tau:Y_2\to Y_1$ which satisfy that*

$$U(F)(z,y)=h(y)F(\varphi(z,y),\tau(y)),\qquad (z,y)\in\mathbb{T}\times Y_2$$

for every $F \in C^1(\mathbb{T}, C(Y_1))$.

Proof. Suppose that $U : C^1(\mathbb{T}, C(Y_1)) \to C^1(\mathbb{T}, C(Y_2))$ is a surjective isometry. Then by Theorem 4.5 there exists $h \in C(Y_2)$ such that $|h| = 1$ on Y_2, a continuous map $\varphi : \mathbb{T} \times Y_1 \to \mathbb{T}$ such that $\varphi(\cdot, y) : \mathbb{T} \to \mathbb{T}$ is a homeomorphism for each $y \in Y_2$, and a homeomorphism $\tau : Y_2 \to Y_1$ which satisfy

$$U(F)(z, y) = h(y)F(\varphi(z, y), \tau(y)), \qquad (z, y) \in \mathbb{T} \times Y_2 \tag{5.7}$$

for every $F \in C^1(\mathbb{T}, C(Y_1))$. We prove that for every $y \in Y_2$ there corresponds $u(y) \in \mathbb{T}$ such that $\varphi(z, y) = u(y)z$ for every $z \in \mathbb{T}$ or $\varphi(z, y) = u(y)\bar{z}$ for every $z \in \mathbb{T}$. Let $F_0 \in C^1(\mathbb{T}, C(Y_1))$ be defined as $F_0(z, y) = z$ for every $(z, y) \in \mathbb{T} \times Y_1$. Then by (5.7) we have $U(F_0)(z, y) = h(y)\varphi(z, y)$. As $|h| = 1$ on Y_2 we have that $\varphi = \bar{h}U(F_0) \in C^1(\mathbb{T}, C(Y_2))$. We also have $\|F_0\|_\infty = 1$ and $\|F_0'\|_\infty = 1$, hence $\|F_0\| = 2$. By (5.7) we have $\|U(F_0)\|_\infty = 1$. Since $\|U(F_0)\| = \|F_0\|$, we infer that $\|U(F_0)'\|_\infty = \|F_0'\|_\infty$, where

$$U(F_0)'(z, y) = h(y)\varphi'(z, y), \quad (z, y) \in \mathbb{T} \times Y_2$$

as $U(F_0) = h\varphi$. Thus

$$\|\varphi'\|_\infty = \|U(F_0)'\|_\infty = \|F_0'\|_\infty = 1.$$

It follows that $|\varphi'(z, y)| \leq 1$ for every $(z, y) \in \mathbb{T} \times Y_2$. Define $u : Y_2 \to \mathbb{T}$ by $u(y) = \varphi(1, y)$. Then u is continuous since φ is continuous on $\mathbb{T} \times Y_2$. We also have that $|u(y)| = |\varphi(1, y)| = 1$. As $\varphi(\cdot, y)$ is a bijection from $\mathbb{T}$ onto itself, we have $\varphi(\mathbb{T} \setminus \{1\}, y) = \mathbb{T} \setminus \{u(y)\}$. Hence the map

$$t \mapsto -i \operatorname{Log} \overline{u(y)}\varphi(e^{it}, y)$$

is well defined from $(0, 2\pi)$ onto $(0, 2\pi)$, where Log denotes the principal value of the logarithm. As $\varphi(\cdot, y)$ is continuously differentiable, the above map has a natural extension $\mathcal{L} : [0, 2\pi] \to [0, 2\pi]$ (defining by $\mathcal{L}(0) = 0$ and $\mathcal{L}(2\pi) = 2\pi$, or $\mathcal{L}(0) = 2\pi$ and $\mathcal{L}(2\pi) = 0$, $\mathcal{L}(t) = -i \operatorname{Log} \overline{u(y)}\varphi(e^{it}, y)$ for $0 < t < 2\pi$), which is continuously differentiable. By a simple calculation we have

$$\mathcal{L}'(t) = \frac{\varphi'(e^{it}, y)e^{it}}{\varphi(e^{it}, y)}, \quad t \in [0, 2\pi].$$

Hence $|\mathcal{L}'(t)| \leq 1$ for every $t \in [0, 2\pi]$ since $|\varphi'(z, y)| \leq 1$ for every $(z, y) \in \mathbb{T} \times Y_2$. As in the same way as in the proof of Corollary 4.18 we have that $\mathcal{L}' = 1$ on $[0, 2\pi]$ or $\mathcal{L}' = -1$ on $[0, 2\pi]$. It follows that $\overline{u(y)}\varphi(e^{it}, y) = e^{it}$ for every $t \in [0, 2\pi]$ or $\overline{u(y)}\varphi(e^{it}, y) = e^{-it}$ for every $t \in [0, 2\pi]$. Hence $\varphi(z, y) = u(y)z$ for every $z \in \mathbb{T}$ or $\varphi(z, y) = u(y)\bar{z}$ for every $z \in \mathbb{T}$.

Suppose conversely that there exists $h \in C(Y_2)$ such that $|h| = 1$ on Y_2, a continuous map $\varphi : \mathbb{T} \times Y_2 \to \mathbb{T}$ and a continuous map $u : Y_2 \to \mathbb{T}$ such that $\varphi(z, y) = u(y)z$ for every $z \in \mathbb{T}$ or $\varphi(z, y) = u(y)\bar{z}$ for every $z \in \mathbb{T}$, and a homeomorphism $\tau : Y_2 \to Y_1$ which satisfy that

$$U(F)(z, y) = h(y)F(\varphi(z, y), \tau(y)), \qquad (z, y) \in \mathbb{T} \times Y_2 \tag{5.8}$$

for every $F \in C^1(\mathbb{T}, C(Y_1))$. By the hypotheses on φ and τ we infer that $(z, y) \mapsto (\varphi(z, y), \tau(y))$ gives a homeomorphism from $\mathbb{T} \times Y_2$ onto $\mathbb{T} \times Y_1$. As $|h| = 1$ on Y_2 we infer that $\|F\|_\infty = \|U(F)\|_\infty$ for every $F \in C^1(\mathbb{T}, C(Y_1))$. By (5.8) we have

$$U(F)'(z, y) = h(y)F'(\varphi(z, y), \tau(y))\varphi'(z, y), \quad (z, y) \in \mathbb{T} \times Y_2$$

for every $F \in C^1(\mathbb{T}, C(Y_1))$. As $\varphi'(z, y) = u(y)$ on $\mathbb{T} \times Y_2$ or $\varphi'(z, y) = -u(y)\bar{z}^2$ on $\mathbb{T} \times Y_2$ we infer that

$$\|U(F)'\|_\infty = \|hF'(\varphi, \tau)\varphi'\|_\infty = \|F'\|_\infty.$$

It follows that U is an isometry. It is not difficult to prove that U is a surjection. We conclude that U is a surjective isometry. $\square$

CHAPTER 5

Hermitian operators on commutative Banach algebras

1. Introduction to Hermitian operators

Let E be a Banach space. Recall that T is a Hermitian operator if $[Tx, x] \in \mathbb{R}$ for any $x \in E$, where $[\cdot, \cdot]$ is a semi-inner product on E, compatible with the norm. Detailed definition and facts about Hermitian operators are in Section 2. It is well known that $T \in \mathbf{B}(E)$ is a Hermitian operator if and only if $\|\exp(itT)\| = 1$ for every $t \in \mathbb{R}$ if and only if $\exp(itT)$ is an isometry for every $t \in \mathbb{R}$ (see Theorem 1.10 ([**30**, Theorem 5.2.6])). Hence $T \in \mathbf{B}(E)$ is a Hermitian operator on E if and only if T is Hermitian as an element in $\mathbf{B}(E)$.

Recall that a uniform algebra A on a compact Hausdorff space Y is a closed subalgebra of $C(Y)$ which contains the constants and separates the points of Y. The supremum norm on a set K is denoted by $\|\cdot\|_{\infty(K)}$.

Let (X, d) be a compact metric space. For a uniform algebra A, the Banach algebra $\mathrm{Lip}(X, A)$ is called a Lipschitz algebra with values in a uniform algebra A. Recall that for $F \in \mathrm{Lip}(X, A)$

$$\begin{aligned} L(F) &= \sup\left\{\frac{\|F(x) - F(y)\|_{\infty(Y)}}{d(x, y)} : x, y \in X, x \neq y\right\} \\ &= \sup\left\{\frac{|(F(x))(z) - (F(y))(z)|}{d(x, y)} : x, y \in X, x \neq y,\ z \in Y\right\}. \end{aligned}$$

As usual we may suppose that $F \in \mathrm{Lip}(X, A)$ is a complex valued function on $X \times Y$ in the way that $F(x, y) = (F(x))(y)$; $\mathrm{Lip}(X, A) \subset C(X \times Y)$. For a Lipschitz function $h \in \mathrm{Lip}(X)$ and an $f \in A$ we define the tensor product $(h \otimes f) \in \mathrm{Lip}(X, A)$ of h and f in the way that $(h \otimes f)(x, y) = h(x)f(y)$ for every pair $x \in X$ and $y \in Y$.

Fleming and Jamison [**29**] investigated Hermitian operators on the algebras of vector-valued continuous maps [**29**]. Botelho, Jamison, Jiménez-Vargas and Villegas-Vallecillos [**16**] studied Hermitian operators on scalar valued Lipschitz algebras. Hermitian operators on a Banach space $\mathrm{Lip}(X, E)$ with the norm $\|\cdot\|_{\max}$ of Lipschitz maps with values in a Banach space E are studied by the same authors in [**15**]. In particular, in [**15**] a characterization of a Hermitian operators on $\mathrm{Lip}(X, E)$ with the norm $\|\cdot\|_{\max}$ is given for an arbitrary Banach space E where X is a compact 2-connected metric space (cf. [**15**, Theorem 2.4]).

In this chapter we characterize Hermitian operators on a Banach algebra of Lipschitz maps with values in a uniform algebra. We do not need to assume that X is 2-connected. Applying this characterization we give a form of a surjective unital isometry on $\mathrm{Lip}(X, C(Y))$. As is expected an isometry is an algebra isomorphism and is represented by a composition operator by a self-homeomorphism on $X \times Y$. It is interesting to note that this self-homeomorphism H has a peculiar form in the sense that $H(x, y) = (\varphi(x, y), \tau(y))$ for $(x, y) \in X \times Y$ with continuous functions $\varphi : X \times Y$ and $\tau : Y \to Y$. This kind of peculiar homomorphisms are recently investigated in Chapters 2, 3 and 4 and we call it BJ type (see Definition 2.20). Note that isometries between $\mathrm{Lip}(X, E)$ with the norm $\|\cdot\|_{\max}$ are investigated in [**8, 103**].

2. Results and Proofs

Let $\mathfrak{B}$ be a unital Banach algebra and $a \in \mathfrak{B}$. The corresponding multiplication operator $M_a : \mathfrak{B} \to \mathfrak{B}$ is defined by the left multiplication; i.e., $b \mapsto ab$, $b \in \mathfrak{B}$. It is clear that M_a is a bounded operator on $\mathfrak{B}$ for every $a \in \mathfrak{B}$.

PROPOSITION 5.1. *Let $\mathfrak{B}$ be a unital Banach algebra. The element $a \in \mathfrak{B}$ is Hermitian if and only if the corresponding multiplication operator M_a is a Hermitian operator on $\mathfrak{B}$.*

PROOF. Suppose that $a \in \mathfrak{B}$ is Hermitian. Then we have $\|\exp(ita)\|_{\mathfrak{B}} = 1$ for any $t \in \mathbb{R}$. Let $t \in \mathbb{R}$ be arbitrary. For every $b \in \mathfrak{B}$ we obtain

$$\|(\exp(itM_a))(b)\|_{\mathfrak{B}} = \|(\exp(ita))b\|_{\mathfrak{B}} \le \|\exp(ita)\|_{\mathfrak{B}}\|b\|_B = \|b\|_{\mathfrak{B}}.$$

It follows that

$$\|\exp(itM_a)\| \leq 1. \tag{2.1}$$

On the other hand, we get

$$\|(\exp(itM_a))(\mathbf{1})\|_{\mathfrak{B}} = \|\exp ita\|_{\mathfrak{B}} = 1. \tag{2.2}$$

In view of the inequality (2.1) and (5.10), we have

$$\|\exp(itM_a)\| = 1$$

for any $t \in \mathbb{R}$. Therefore the bounded linear operator M_a on $\mathfrak{B}$ is a Hermitian operator.

We now prove the converse. Suppose that the map M_a is a Hermitian operator. Then $\exp(itM_a)$ is a linear isometry on $\mathfrak{B}$ for every $t \in \mathbb{R}$. It follows that

$$1 = \|\mathbf{1}\|_{\mathfrak{B}} = \|(\exp(itM_a))(\mathbf{1})\|_{\mathfrak{B}} = \|\exp(ita)\|_{\mathfrak{B}}$$

for any $t \in \mathbb{R}$. This implies that $a \in \mathrm{Her}(\mathfrak{B})$. □

LEMMA 5.2. *Let $\mathfrak{B}$ be a unital Banach algebra and T a Hermitian operator on $\mathfrak{B}$. Then $T(\mathbf{1}) \in \mathrm{Her}(\mathfrak{B})$.*

PROOF. For any $f \in \mathfrak{B}^*$ with $\|f\| = f(\mathbf{1}) = 1$, we define $\Phi_f : \boldsymbol{B}(\mathfrak{B}) \to \mathbb{C}$ by

$$\Phi_f(S) = f(S(\mathbf{1})), \quad S \in \boldsymbol{B}(\mathfrak{B}),$$

where $\boldsymbol{B}(\mathfrak{B})$ denotes the algebra of all bounded linear operators on $\mathfrak{B}$. We infer by a simple calculation that Φ_f is a bounded linear functional on $\boldsymbol{B}(\mathfrak{B})$ and satisfies $\|\Phi_f\| = \Phi_f(I) = 1$. Since T is a Hermitian operator on $\mathfrak{B}$, $T \in \mathrm{Her}(\boldsymbol{B}(\mathfrak{B}))$. This implies

$$f(T(\mathbf{1})) = \Phi_f(T) \in \mathbb{R}$$

for any $f \in \mathfrak{B}^*$ with $\|f\| = f(\mathbf{1}) = 1$. We conclude that $T(\mathbf{1})$ is a Hermitian element of $\mathfrak{B}$. □

PROPOSITION 5.3. *Let $\mathfrak{B}$ be a unital semi-simple commutative Banach algebra. Suppose that $T : \mathfrak{B} \to \mathfrak{B}$ is a bounded linear operator. Then the following are equivalent.*

(1) $T = M_{T(\mathbf{1})}$,
(2) $\exp(it(T - M_{T(\mathbf{1})}))$ *is multiplicative for every $t \in \mathbb{R}$.*

PROOF. Suppose that $T = M_{T(\mathbf{1})}$. Then $\exp(it(T - M_{T(\mathbf{1})})) = I$, the identity operator for every $t \in \mathbb{R}$. Thus (2) holds.

Suppose that (2) holds. Set $T - M_{T(\mathbf{1})} = H$ and $U_t = \exp(itH)$, $t \in \mathbb{R}$. Then U_t is multiplicative for every $t \in \mathbb{R}$; i.e., $U_t(ab) = U_t(a)U_t(b)$ for every pair $a, b \in \mathfrak{B}$. We have

$$(U_t - I)/t = iH + \sum_{n=2}^{\infty} \frac{t^{n-1}(iH)^n}{n!},$$

hence for any $|t| \le 1$

$$\left\| \frac{U_t - I}{t} - iH \right\| \le \sum_{n=2}^{\infty} \frac{|t|^{n-1}\|H\|^n}{n!} \le |t| \sum_{n=2}^{\infty} \frac{\|H\|^n}{n!} \le |t| e^{\|H\|} \to 0$$

as $t \to 0$. We conclude that

$$iH = \lim_{t \to 0} \frac{U_t - I}{t},$$

where the limit is taken with the metric induced by the operator norm. We prove that H is a derivation. For every pair $a, b \in \mathfrak{B}$ we have

$$\left(\frac{U_t - I}{t} \right)(ab) \to iH(ab)$$

as $t \to 0$ and

$$\begin{aligned}\left(\frac{U_t - I}{t} \right)(ab) &= \frac{U_t(a)U_t(b) - ab}{t} \\ &= \frac{U_t(a)(U_t(b) - b) + (U_t(a) - a)b}{t} \to a(iH(b)) + iH(a)b\end{aligned}$$

as U_t is multiplicative. We have that

$$H(ab) = aH(b) + H(a)b, \quad a, b \in \mathfrak{B},$$

that is, H is a derivation on $\mathfrak{B}$, which is also bounded by the definition of H. As $\mathfrak{B}$ is a unital semi-simple commutative Banach algebra, a theorem of Singer and Wermer [**114**] asserts that $H = 0$. Thus (1) holds. □

THEOREM 5.4. *Let $\mathfrak{B}$ be a unital semi-simple commutative Banach algebra and $T \in \boldsymbol{B}(\mathfrak{B})$. Suppose that every surjective unital linear isometry is multiplicative. Then the following are equivalent.*

(1) *T is a Hermitian operator.*
(2) *$T(\mathbf{1})$ is a Hermitian element of $\mathfrak{B}$ and $T = M_{T(\mathbf{1})}$.*

PROOF. Suppose that T is a Hermitian operator. Then $T(\mathbf{1}) \in \mathrm{Her}(\mathfrak{B})$ by Lemma 5.2. Thus $\|\exp(itT(\mathbf{1}))\|_{\mathfrak{B}} = 1$ for every $t \in \mathbb{R}$. Then

$$\|\exp(itM_{T(\mathbf{1})})\| = \|\exp(itT(\mathbf{1}))\|_{\mathfrak{B}} = 1$$

since $\exp(itM_{T(\mathbf{1})}) = M_{\exp(itT(\mathbf{1}))}$ by a simple calculation and the operator norm of M_a coincides with the norm of a for every $a \in \mathfrak{B}$. It follows that $M_{T(\mathbf{1})}$ is a Hermitian operator. Hence $T - M_{T(\mathbf{1})}$ is Hermitian. Thus $\exp(it(T - M_{T(\mathbf{1})}))$ is an isometry for every $t \in \mathbb{R}$ by a characterization of Hermitian operators. For every $t \in \mathbb{R}$ the operator $(\exp(it(T - M_{T(\mathbf{1})}))$ is surjective. As $(T - M_{T(\mathbf{1})})(\mathbf{1}) = 0$, $\exp(it(T - M_{T(\mathbf{1})}))$ is unital. By the assumption we have that $\exp(it(T - M_{T(\mathbf{1})}))$ is multiplicative . Hence by Proposition 5.3 we infer that $T = M_{T(\mathbf{1})}$.

Suppose that (2) holds. Then by Proposition 5.1 T is a Hermitian operator. □

3. An application of Theorem 5.4

Now, we characterize the Hermitian elements in a uniform algebra.

PROPOSITION 5.5. *Let A be a uniform algebra on a compact Hausdorff space Y. Then $f \in A$ is Hermitian in A if and only if $f \in A \cap C_{\mathbb{R}}(Y)$.*

PROOF. Let $f \in A$ be Hermitian in A. Then $\|\exp(itf)\|_{\infty(Y)} = 1$ for every $t \in \mathbb{R}$. We prove that f is a real-valued function on Y. Suppose that there exists $y \in Y$with the imaginary part $\mathrm{Im}\, f(y) \neq 0$. If $\mathrm{Im}\, f(y) > 0$ (resp. $\mathrm{Im}\, f(y) < 0$), then $|\exp(-if(y))| > 1$ (resp. $|\exp(if(y))| > 1$) which is contradictory to the fact that $\|\exp(itf)\|_{\infty(Y)} = 1$ for every $t \in \mathbb{R}$. Thus we have f is a real-valued function.

We now prove the converse. Let $f \in A \cap C_{\mathbb{R}}(Y)$. It follows immediately that for every $t \in \mathbb{R}$, $\|\exp itf\|_{\infty(Y)} = 1$. This implies that f is Hermitian in A. □

3.1. Hermitian operators on $\mathrm{Lip}(X, A)$**.** We exhibit a characterization of Hermitian elements of Lipschitz algebra with values in a uniform algebra.

PROPOSITION 5.6. *Let X be a compact metric space and A a uniform algebra on a compact Hausdorff space Y. Then $F \in \mathrm{Lip}(X, A)$ is Hermitian if and only if $F = \mathbf{1} \otimes f$ for $f \in A \cap C_{\mathbb{R}}(Y)$.*

PROOF. Let F be a Hermitian element in $\mathrm{Lip}(X, A)$. We have

$$\|\exp(itF)\|_{\infty(X\times Y)} + L(\exp(itF)) = \|\exp(itF)\|_{\Sigma} = 1, \quad t \in \mathbb{R}. \tag{3.1}$$

Hence $\|\exp(itF)\|_{\infty(X\times Y)} \leq 1$ for every $t \in \mathbb{R}$. Then, in a similar way as in the proof of Proposition 5.5 we have that F is real-valued and $\|\exp(itF)\|_{\infty(X\times Y)} = 1$. Hence $L(\exp(itF)) = 0$ which means that $F = \mathbf{1} \otimes f$ for an $f \in A$. As F is real-valued, we have at once that $f \in A \cap C_{\mathbb{R}}(Y)$, that is, f is a Hermitian element in A.

A Hermitian element f in A is a real-valued function by Proposition 5.5, hence $F = \mathbf{1} \otimes f \in \mathrm{Lip}(X, A)$ is a real-valued function, $\|\exp(itF)\|_{\infty(X\times Y)} = 1$ for every $t \in \mathbb{R}$. As $F = \mathbf{1}\otimes f$, $L(\exp(itF)) = 0$. Thus $\|\exp(itF)\|_L = 1$, which forces that F is a Hermitian element in $\mathrm{Lip}(X, A)$. □

As a corollary of Theorem in [**47**] we have the following.

COROLLARY 5.7. *Let X_j be a compact metric space and A_j a uniform algebra for $j = 1, 2$. If U is a linear isometry from $\mathrm{Lip}(X_1, A_1)$ onto $\mathrm{Lip}(X_2, A_2)$ with $U(\mathbf{1}) = \mathbf{1}$, then U is also an isometry with respect to the supremum norm.*

We point out that the term $-\pi/2$ and $\pi/2$ which appear in the formulae (7) and (8) in the proof of [**47**, Theorem] read as $3\pi/4$ and $\pi/4$, respectively. Hatori, Jiménez-Vargas and Villegas-Vallecillos has essentially given a revision of the proof of [**47**, Theorem] in the proof of Proposition 7 in [**36**].

THEOREM 5.8. *Let X be a compact metric space and A a uniform algebra on a compact Hausdorff space Y. Then T is a Hermitian operator on $\mathrm{Lip}(X, A)$ if and only if there exists a real-valued function $f \in A$ such that*

$$T = M_{\mathbf{1}\otimes f}.$$

PROOF. Suppose that $f \in A \cap C_{\mathbb{R}}(Y)$. Due to Proposition 5.6 we infer that $\mathbf{1} \otimes f$ is a Hermitian element in $\mathrm{Lip}(X, A)$. Then by Proposition 5.1, we have that $M_{\mathbf{1}\otimes f}$ is a Hermitian operator on $\mathrm{Lip}(X, A)$.

Suppose that T is a Hermitian operator on $\mathrm{Lip}(X, A)$. Then by Lemma 5.2, $T(\mathbf{1})$ is a Hermitian element in $\mathrm{Lip}(X, A)$. Then by Proposition 5.6, there will be an $f \in A \cap C_{\mathbb{R}}(Y)$, a Hermitian element in A, such that $T(\mathbf{1}) = \mathbf{1} \otimes f$. We show that every surjective unital linear isometry from $\mathrm{Lip}(X, A)$ onto itself is multiplicative. It will follow that $T = M_{T(\mathbf{1})}$ by Theorem 5.4. Now we prove that every surjective unital linear isometry from $\mathrm{Lip}(X, A)$ onto itself is multiplicative. To do so, let $U : \mathrm{Lip}(X, A) \to \mathrm{Lip}(X, A)$ be a surjective unital linear isometry. Then by Corollary 5.7 U is extended to a unique surjective unital linear isometry U^∞ from the uniform closure $\overline{\mathrm{Lip}(X, A)}$ of $\mathrm{Lip}(X, A)$ in $C(X \times Y)$ onto itself. As $\overline{\mathrm{Lip}(X, A)}$ is a uniform algebra on $X \times Y$ we have by a theorem of Nagasawa [**94**] that U^∞ is a composition operator defined by a self-homeomorphism between the maximal ideal space of $\overline{\mathrm{Lip}(X, A)}$. Hence U^∞ is multiplicative, and so is U. $\square$

3.2. Hermitian operators on $C^1([0, 1], A)$. We exhibit a characterization of Hermitian elements of continuously differential maps on $[0, 1]$ with values in a uniform algebra. A proof is similar to the case of the Lipschitz algebras.

PROPOSITION 5.9. *Let A be a uniform algebra on a compact Hausdorff space Y. Then $F \in C^1([0, 1], A)$ is Hermitian if and only if $F = \mathbf{1} \otimes f$ for $f \in A \cap C_{\mathbb{R}}(Y)$.*

PROOF. Let F be a Hermitian element in $C^1([0, 1], A)$. We have

$$\|\exp(itF)\|_{\infty([0,1]\times Y)} + \|\exp(itF))'\|_\infty = \|\exp(itF)\|_\Sigma = 1, \quad t \in \mathbb{R}. \tag{3.2}$$

Hence $\|\exp(itF)\|_{\infty([0,1]\times Y)} \leq 1$ for every $t \in \mathbb{R}$. Then we have that F is real-valued and $\|\exp(itF)\|_{\infty([0,1]\times Y)} = 1$. Hence $\|\exp(itF)'\|_\infty = 0$ which means that $F = \mathbf{1} \otimes f$ for an $f \in A$. We get f is a real-valued continuous function of A since F is real-valued map. We have f is a Hermitian element in A. Let $f \in A \cap C_{\mathbb{R}}(Y)$, then $F = \mathbf{1} \otimes f \in$

$C^1([0,1],A)$ is a real-valued function such that

$$\|\exp(itF)\|_{\infty([0,1]\times Y)} = 1$$

for every $t \in \mathbb{R}$ and

$$\|\exp(itF)'\|_{\infty} = 0.$$

We obtain $\|\exp(itF)\|_{\Sigma} = 1$, which implies F is a Hermitian element in $C^1([0,1],A)$. □

For the case of $C^1([0,1],A)$, we get the following corollary by theorem in [**47**].

COROLLARY 5.10. *Let A_j be a uniform algebra for $j = 1, 2$. If U is a linear isometry from $C^1([0,1],A_1)$ onto $C^1([0,1],A_2)$ with $U(\mathbf{1}) = \mathbf{1}$, then U is also an isometry with respect to the supremum norm.*

The following is a characterization of a Hermitian operator on $C^1([0,1],A)$ for a uniform algebra A.

THEOREM 5.11. *Let X be a compact metric space and A a uniform algebra on a compact Hausdorff space Y. Then T is a Hermitian operator on* $\mathrm{Lip}(X,A)$ *if and only if there exists a real-valued function $f \in A$ such that*

$$T = M_{\mathbf{1}\otimes f}.$$

The proof is similar to the case of $\mathrm{Lip}(X,A)$. Thus we omit the detail of proof.

PROOF. Suppose that $f \in A \cap C_{\mathbb{R}}(Y)$. Then $\mathbf{1} \otimes f$ is a Hermitian element in $C^1([0,1],A)$ by Proposition 5.9. This implies that $M_{\mathbf{1}\otimes f}$ is a Hermitian operator on $C^1([0,1],A)$.

Suppose that T is a Hermitian operator on $C^1([0,1],A)$. Then by Lemma 5.2, $T(\mathbf{1})$ is a Hermitian element in $C^1([0,1],A)$. Then by Proposition 5.9, there will be an $f \in A \cap C_{\mathbb{R}}(Y)$, a Hermitian element in A, such that $T(\mathbf{1}) = \mathbf{1} \otimes f$. We see that $\overline{C^1([0,1],A)}$,which is uniform closure of $C^1([0,1],A)$ is uniform algebra on $[0,1] \times Y$. This means that every surjective unital linear isometry from $C^1([0,1],A)$ onto itself is multiplicative by applying a theorem of Nagasawa [**94**]. Thus by Theorem 5.4, we have that $T = M_{\mathbf{1}\otimes f}$. □

4. Surjective linear isometries

In the Chapter 4, We have already characterized a surjective unital linear isometry from $\mathrm{Lip}(X, C(Y))$ onto itself (see Corollary 4.14). The purpose in this Section is to prove the characterization of surjective unital linear isometry from $\mathrm{Lip}(X, C(Y))$ onto itself by applying theory of Lumer's method. Lumer initiated the study of isometries on function spaces in terms of Hermitian operators [**76, 78**]. Hermitian operators are intrinsically related to surjective isometries. Let T be a Hermitian operator and U a surjective isometry on normed spaces. Then UTU^{-1} is a Hermitian operator by Theorem 5.12. It is essentially described in the proof of [**78**, Theorem 10].

THEOREM 5.12. *Let N be a normed linear space and let U be a linear operator from N into itself. Then U is an isometry if and only if there is a semi-inner product $[\cdot, \cdot]$ compatible with the norm, satisfying $[x, \alpha y] = \bar{\alpha}[x, y]$ for every $x, y \in N$ and $\alpha \in \mathbb{C}$, such that $[Uv_1, Uv_2] = [v_1, v_2]$.*

This argument has been used extensively by many authors in a variety of settings. For example, Fleming and Jamison proved that a unital surjective linear isometry is a composition operator on the space of continuous maps with values in certain Banach spaces in [**29**]. The method is called Lumer's method (see [**30**]).

4.1. Surjective linear isometries on $\mathrm{Lip}(X, C(Y))$. Suppose that U is a surjective linear isometry from $\mathrm{Lip}(X, C(Y))$ onto itself such that $U(\mathbf{1}) = \mathbf{1}$. Applying Corollary 5.7, U is extended to a surjective isometry (with respect to the supremum norm) $\tilde{U}$ from $C(X \times Y)$ onto itself. Then the Banach-Stone theorem asserts that there exists a self-homeomorphism $H : X \times Y \to X \times Y$ such that $\tilde{U}(F) = F \circ H$ for every $F \in C(X \times Y)$. As H is represented as $H(x, y) = (\varphi_1(x, y), \varphi_2(x, y))$ for $(x, y) \in X \times Y$, where $\varphi_1 : X \times Y \to X$ and $\varphi_2 : X \times Y \to Y$ are continuous maps. It follows that $U(F)(x, y) = U(\varphi_1(x, y), \varphi_2(x, y))$ for every $F \in \mathrm{Lip}(X, C(Y))$ and $(x, y) \in X \times Y$. In particular, we see that U is an isomorphism from $\mathrm{Lip}(X, C(Y))$ onto itself. Then by Theorem 2.3, the map φ_2 depends only on the second valuable y if X

is connected. In this section we prove that it is the case without the hypothesis that X is connected.

Suppose that $U : \mathrm{Lip}(X, C(Y)) \to \mathrm{Lip}(X, C(Y))$ is a surjective isometry. Let $[\cdot, \cdot]$ be a semi-inner product compatible with the norm such that $[U(F_1), U(F_2)] = [F_1, F_2]$ for every $F_j \in \mathrm{Lip}(X, C(Y))$. Such a semi-inner product exists by Theorem 5.12. For any $h \in \mathrm{Her}(C(Y)) = C_{\mathbb{R}}(Y)$ we have

$$[UM_{\mathbf{1}\otimes h}U^{-1}G, G] = [UM_{\mathbf{1}\otimes h}U^{-1}G, UU^{-1}G] = [M_{\mathbf{1}\otimes h}U^{-1}G, U^{-1}G] \in \mathbb{R}$$

for every $G \in \mathrm{Lip}(X, C(Y))$. Hence $UM_{\mathbf{1}\otimes h}U^{-1}$ is a Hermitian operator on $\mathrm{Lip}(X, C(Y))$. Then by Theorem 5.8 there exists $h' \in \mathrm{Her}(C(Y))$ such that $UM_{\mathbf{1}\otimes h}U^{-1} = M_{\mathbf{1}\otimes h'}$. It follows that we can define a map $\psi : \mathrm{Her}(C(Y)) \to \mathrm{Her}(C(Y))$ by

$$UM_{\mathbf{1}\otimes h}U^{-1} = M_{\mathbf{1}\otimes \psi(h)}.$$

LEMMA 5.13. *The map $\psi : \mathrm{Her}(C(Y)) \to \mathrm{Her}(C(Y))$ is a real algebra isomorphism.*

PROOF. For any $h' \in \mathrm{Her}(C(Y))$ we have that $U^{-1}M_{\mathbf{1}\otimes h'}U$ is also a Hermitian operator on $\mathrm{Lip}(X, C(Y))$ by a similar argument as above. Therefore, there exists $h \in \mathrm{Her}(C(Y))$ such that $U^{-1}M_{\mathbf{1}\otimes h'}U = M_{\mathbf{1}\otimes h}$. This implies that

$$UM_{\mathbf{1}\otimes h}U^{-1} = U(U^{-1}M_{\mathbf{1}\otimes h'}U)U^{-1} = M_{\mathbf{1}\otimes h'},$$

hence we conclude that $\psi(h) = h'$. As h' is arbitrary, we have that ψ is surjective.

Let h_1 and h_2 be Hermitian elements with $h_1 \neq h_2$. Due to the definition of ψ, we have

$$M_{\mathbf{1}\otimes \psi(h_1)} = UM_{\mathbf{1}\otimes h_1}U^{-1} \neq UM_{\mathbf{1}\otimes h_2}U^{-1} = M_{\mathbf{1}\otimes \psi(h_2)}.$$

Thus, this implies that $\psi(h_1) \neq \psi(h_2)$. Thus ψ is injective.

We prove that ψ is multiplicative. Let $h_1, h_2 \in \mathrm{Her}(C(Y))$ be arbitrary. Then $UM_{\mathbf{1}\otimes (h_1h_2)}U^{-1} = M_{\mathbf{1}\otimes \psi(h_1h_2)}$. On the other hand we have

$$\begin{aligned} UM_{\mathbf{1}\otimes(h_1h_2)}U^{-1} &= UM_{\mathbf{1}\otimes h_1}M_{\mathbf{1}\otimes h_2}U^{-1} \\ &= (UM_{\mathbf{1}\otimes h_1}U^{-1})(UM_{\mathbf{1}\otimes h_2}U^{-1}) = M_{\mathbf{1}\otimes\psi(h_1)}M_{\mathbf{1}\otimes\psi(h_2)} \\ &= M_{\mathbf{1}\otimes(\psi(h_1)\phi(h_2))} \end{aligned}$$

Thus we have $\psi(h_1h_2) = \psi(h_1)\psi(h_2)$ for an arbitrary pair of $h_1, h_2 \in \mathrm{Her}(C(Y))$; that is, ψ is multiplicative. We see in a similar way that ψ is real-linear. □

We define the map $\tilde{\psi}$ from $C(Y)$ onto $C(Y)$ by

$$\tilde{\psi}(h_1 + ih_2) = \psi(h_1) + i\psi(h_2)$$

for $h_1, h_2 \in \mathrm{Her}(C(Y)) = C_{\mathbb{R}}(Y)$. It is easy to see that $\tilde{\psi}$ is a complex algebra isomorphism from $C(Y)$ onto itself. As $\tilde{\psi} : C(Y) \to C(Y)$ is an algebra isomorphism there exists a homeomorphism $\tau : Y \to Y$ such that

$$\tilde{\psi}(h)(y) = h(\tau(y))$$

for any $h \in C(Y)$ and $y \in Y$. Applying the isomorphism $\tilde{\psi}$ and a homeomorphism τ we have the following. Note that by a simple calculation we have $UM_{\mathbf{1}\otimes h}U^{-1} = M_{\mathbf{1}\otimes\tilde{\psi}(h)}$ for every $h \in C(Y)$.

Theorem 5.14. *Let X be a compact metric space and Y a compact Hausdorff space. Then U is a linear isometry from* $\mathrm{Lip}(X, C(Y))$ *onto itself such that $U(\mathbf{1}) = \mathbf{1}$ if and only if there exist a continuous map $\varphi : X \times Y \to X$ such that $\varphi(\cdot, y) : X \to X$ is a surjective isometry for each $y \in Y$, and a homeomorphism $\tau : Y \to Y$ which satisfy that*

$$UF(x, y) = F(\varphi(x, y), \tau(y)) \quad x \in X, y \in Y \tag{4.1}$$

for every $F \in \mathrm{Lip}(X, C(Y))$.

Proof. Suppose that $\varphi : X \times Y \to X$ is a continuous map such that $\varphi(\cdot, y) : X \to X$ is a surjective isometry for each $y \in Y$ and $\tau : Y \to Y$ is a homeomorphism. Then the map H' from $X \times Y$ into itself defined by $H'(x, y) = (\varphi(x, y), \tau(y))$, $x \in X, y \in Y$ is continuous. As τ is a bijection on Y and $\varphi(\cdot, y)$ is a bijection on X for every $y \in Y$, we infer that H' is a bijection. As $X \times Y$ is a compact Hausdorff space, we have that the continuous bijection H' is a homeomorphism.

Let $F \in \mathrm{Lip}(X, C(Y))$ be arbitrary. We suppose in a usual way that $\mathrm{Lip}(X, C(Y)) \subset C(X \times Y)$; $[F(x)](y) = F(x, y)$, $(x, y) \in X \times Y$. Define $UF : X \times Y \to \mathbb{C}$ by $UF(x, y) = F \circ H'(x, y) = F(\varphi(x, y), \tau(y))$ for every pair $x \in X, y \in Y$. As H' is continuous we have $UF \in C(X \times Y)$. We prove that $UF \in \mathrm{Lip}(X, C(Y))$. Applying properties of φ and τ we have by a simple calculation that

$$\begin{aligned} L(F) = \sup_{x_1, x_2} \frac{\|F(x_1) - F(x_2)\|_{\infty(Y)}}{d(x_1, x_2)} &= \sup_{x_1, x_2, y} \frac{|F(x_1, y) - F(x_2, y)|}{d(x_1, x_2)} \\ &= \sup_{x_1, x_2, y} \frac{|F(\varphi(x_1, y), \tau(y)) - F(\varphi(x_2, y), \tau(y))|}{d(\varphi(x_1, y), \varphi(x_2, y))} \\ &= \sup_{x_1, x_2, y} \frac{|F(\varphi(x_1, y), \tau(y)) - F(\varphi(x_2, y), \tau(y))|}{d(x_1, x_2)} = L(UF). \end{aligned}$$

As $L(F) = L(UF)$ we infer that $UF \in \mathrm{Lip}(X, C(Y))$. Therefore the map $F \mapsto UF$ is well defined from $\mathrm{Lip}(X, C(Y))$ into itself. As H' is a surjection, we have that $\|F\|_{\infty(X \times Y)} = \|UF\|_{\infty(X \times Y)}$. Thus U is an isometry with respect to the norm $\|\cdot\|_L$. We prove that U is surjective. Let $(x, y) \in X \times Y$ be arbitrary. As τ is a bijection on Y and $\varphi(\cdot, \tau^{-1}(y))$ is a bijection on X, there is a unique $x' \in X$ with $\varphi(x', \tau^{-1}(y)) = x$. Define $\varphi' : X \times Y \to X$ by $\varphi'(x, y) = x'$. Then φ' is continuous on $X \times Y$ such that $\varphi'(\cdot, y)$ is a surjective isometry on X. Furthermore we infer that $H'^{-1} = (\varphi', \tau^{-1})$. Let $F \in \mathrm{Lip}(X, C(Y))$ be arbitrary. In a similar way as above, we have $F \circ H'^{-1} \in \mathrm{Lip}(X, C(Y))$. On the other hand we infer that $U(F \circ H'^{-1}) = F$. As F is arbitrary we have that U is surjective.

Suppose that U is a surjective isometry from $\mathrm{Lip}(X, C(Y))$ onto itself such that $U\mathbf{1} = \mathbf{1}$. Corollary 5.7 implies that U is an isometry from $(\mathrm{Lip}(X, C(Y)), \|\cdot\|_{\infty(X \times Y)})$ onto itself. We note that $\mathrm{Lip}(X, C(Y))$ is uniformly dense in $C(X \times Y)$ by the Stone-Weierstrass theorem. This shows that we have $U^\infty : C(X \times Y) \to C(X \times Y)$ which is a unique extension of U, and U^∞ is a unital linear isometry with the supremum norm. By the Banach-Stone theorem, we conclude that U^∞ is an algebra isomorphism, so is U. Let $y \in Y$ arbitrary. We define a map $\widetilde{U^y} : \mathrm{Lip}(X) \to \mathrm{Lip}(X)$ by $\widetilde{U^y}(f) = U(f \otimes \mathbf{1})(\cdot, y)$ for each $f \in \mathrm{Lip}(X)$. Then $\widetilde{U^y}$ is a unital homomorphism. By [**111**, Theorem

5.1] there exists a Lipschitz map $\varphi(\cdot, y)$ from X into itself such that $\widetilde{U^y}(f)(x) = f(\varphi(x, y))$ for every $f \in \text{Lip}(X)$ and $x \in X$. For every $f \in \text{Lip}(X)$, $h \in C(Y)$, and $(x, y) \in X \times Y$, we have

$$\begin{aligned}
(U(f \otimes h))(x, y) &= (U(M_{\mathbf{1} \otimes h} f \otimes \mathbf{1}))(x, y) = (UM_{\mathbf{1} \otimes h} U^{-1} U(f \otimes \mathbf{1}))(x, y) \\
&= (M_{\mathbf{1} \otimes \tilde{\psi}(h)} U(f \otimes \mathbf{1}))(x, y) = ((\mathbf{1} \otimes \tilde{\psi}(h)) U(f \otimes \mathbf{1}))(x, y) \\
&= (\mathbf{1} \otimes \tilde{\psi}(h))(x, y)(U(f \otimes \mathbf{1}))(x, y) = (\mathbf{1} \otimes \tilde{\psi}(h))(x, y)\widetilde{U^y}(f)(x) \\
&= h(\tau(y)) f(\varphi(x, y)) = f \otimes h(\varphi(x, y), \tau(y)).
\end{aligned}$$

Hence

$$(U(\sum_i f_i \otimes h_i))(x, y) = \sum_i f_i \otimes h_i(\varphi(x, y), \tau(y)) \tag{4.2}$$

for every $\sum_i f_i \otimes h_i \in \text{Lip}(X) \otimes C(Y)$, the algebraic tensor product of $\text{Lip}(X)$ and $C(Y)$. Let $F \in \text{Lip}(X, C(Y))$ be arbitrary. Then by the Stone-Weierstrass theorem there exists a sequence $\{F_n\} \in \text{Lip}(X) \otimes C(Y)$ such that $\|F - F_n\|_{\infty(X \times Y)} \to 0$ as $n \to \infty$. By Corollary 5.7, we obtain that U is an isometry with respect to the metric induced by the supremum norm. Thus we have

$$\|U(F_n) - U(F)\|_{\infty(X \times Y)} = \|F_n - F\|_{\infty(X \times Y)} \to 0$$

as $n \to \infty$. Hence $(U(F_n))(x, y) \to (U(F))(x, y)$ as $n \to \infty$. As $\|F_n - F\|_{\infty(X \times Y)} \to 0$, we obtain by (4.2) that

$$(U(F_n))(x, y) = F_n(\varphi(x, y), \tau(y)) \to F(\varphi(x, y), \tau(y))$$

as $n \to \infty$. We conclude that $(U(F))(x, y) = F(\varphi(x, y), \tau(y))$.

As in the same way as above, there exists a homeomorphism $\tau' : Y \to Y$ and a Lipschitz map $\varphi'(\cdot, y) : X \to X$ for each $y \in Y$ such that $(U^{-1}(G))(x, y) = G(\varphi'(x, y), \tau'(y))$ for every $G \in \text{Lip}(X, C(Y))$ and every $(x, y) \in X \times Y$. As

$$\begin{aligned}
G(x, y) = (U(U^{-1}(G)))(x, y) &= (U^{-1}(G))(\varphi(x, y), \tau(y)) \\
&= G(\varphi'(\varphi(x, y), \tau(y)), \tau'(\tau(y))),
\end{aligned}$$

$$\begin{aligned}
G(x, y) = (U^{-1}(U(G)))(x, y) &= (U(G))(\varphi'(x, y), \tau'(y)) \\
&= G(\varphi(\varphi'(x, y), \tau'(y)), \tau(\tau'(y)))
\end{aligned}$$

for every $G \in \mathrm{Lip}(X, C(Y))$ and $(x,y) \in X \times Y$, we have that $\tau' = \tau^{-1}$. We also get $x = \varphi'(\varphi(x,y), \tau(y))$ and $x = \varphi(\varphi'(x,y), \tau'(y))$ for every $(x,y) \in X \times Y$. As τ is a bijection and $\tau'^{-1} = \tau$ we have $x = \varphi(\varphi'(x, \tau(y)), y)$ for every $(x,y) \in X \times Y$. It follows that $\varphi(\cdot, y)^{-1} = \varphi'(\cdot, \tau(y))$ for every $y \in Y$. In particular, $\varphi(\cdot, y)$ is a lipeomorphism from X onto itself.

Let $z \in X$ be arbitrary. Set $f_z : X \to \mathbb{C}$ by $f_z(x) = d(x,z)$ for $x \in X$. The Lipschitz constant of f_z is 1. Hence $L(f_z \otimes \mathbf{1}) = 1$ for every $z \in X$. We infer that $f_z \in \mathrm{Lip}(X)$. As U is an isometry with respect $\|\cdot\|_L$ and $\|\cdot\|_{\infty(X\times Y)}$ simultaneously, we infer that $L(U(F)) = L(F)$ for every $F \in \mathrm{Lip}(X, C(Y))$. Let $y \in Y$ and $x_1, x_2 \in X$ be arbitrary. Then we have

$$\begin{aligned}
d(\varphi(x_1,y), \varphi(x_2,y)) &= |f_{\varphi(x_2,y)}(\varphi(x_1,y)) - f_{\varphi(x_2,y)}(\varphi(x_2,y))| \\
&= |(f_{\varphi(x_2,y)} \otimes \mathbf{1})(\varphi(x_1,y), \tau(y)) - (f_{\varphi(x_2,y)} \otimes \mathbf{1})(\varphi(x_2,y), \tau(y))| \\
&= |(U(f_{\varphi(x_2,y)} \otimes \mathbf{1}))(x_1,y) - (U(f_{\varphi(x_2,y)} \otimes \mathbf{1}))(x_2,y)| \\
&\leq \|(U(f_{\varphi(x_2,y)} \otimes \mathbf{1}))(x_1) - (U(f_{\varphi(x_2,y)} \otimes \mathbf{1}))(x_2)\|_{\infty(Y)} \\
&\leq L(U(f_{\varphi(x_2,y)} \otimes \mathbf{1}))d(x_1,x_2) = L(f_{\varphi(x_2,y)} \otimes \mathbf{1})d(x_1,x_2) = d(x_1,x_2).
\end{aligned}$$

We conclude that $d(\varphi(x_1,y), \varphi(x_2,y)) \leq d(x_1,x_2)$ for every pair $x_1, x_2 \in X$ and $y \in Y$. In the same way we have $d(\varphi'(x_1,y), \varphi'(x_2,y)) \leq d(x_1,x_2)$ for every pair $x_1, x_2 \in X$ and $y \in Y$. Since $\varphi(\cdot, y) = \varphi(\cdot, \tau^{-1}(y))^{-1}$ we have that $d(\varphi(x_1, \tau^{-1}(y))^{-1}, \varphi(x_2, \tau^{-1}(y))^{-1}) \leq d(x_1,x_2)$ for every pair $x_1, x_2 \in X$ and $y \in Y$. As τ^{-1} is bijective we have that $d(\varphi(x_1,y)^{-1}, \varphi(x_2,y)^{-1}) \leq d(x_1,x_2)$, hence $d(x_1,x_2) \leq d(\varphi(x_1,y), \varphi(x_2,y))$ for every pair $x_1, x_2 \in X$ and $y \in Y$. It follows that $d(x_1,x_2) = d(\varphi(x_1,y), \varphi(x_2,y))$ for every pair $x_1, x_2 \in X$ and $y \in Y$; $\varphi(\cdot, y)$ is an isometry for every $y \in Y$. Finally we prove φ is continuous on $X \times Y$. By Banach-Stone theorem, there exists a self-homeomorphism $H = (\varphi_1, \varphi_2) : X \times Y \to X \times Y$ such that

$$UF(x,y) = F \circ H(x,y) = F(\varphi_1(x,y), \varphi_2(x,y))$$

for every $F \in \mathrm{Lip}(X, C(Y))$, $(x,y) \in X \times Y$. Note that $\varphi_1 : X \times Y \to X$ is continuous since H is a homeomorphism. We have

$$F(\varphi_1(x,y), \varphi_2(x,y)) = F(\varphi(x,y), \tau(y))$$

for every pair $x \in X$ and $y \in Y$, and $F \in \mathrm{Lip}(X, C(Y))$. As $\mathrm{Lip}(X, C(Y))$ separates the points of $X \times Y$ we have that $\varphi = \varphi_1$. Hence φ is continuous on $X \times Y$.This completes the proof. □

4.2. Surjective linear isometries on $C^1([0,1], C(Y))$. Suppose that $U : C^1([0,1], C(Y)) \to C^1([0,1], C(Y))$ is a surjective linear isometry. By a similar argument of subsection 4.1 and applying Theorem 5.11, we get the following results. (We omit proofs). We can define a map $\psi : \mathrm{Her}(C(Y)) \to \mathrm{Her}(C(Y))$ by

$$UM_{\mathbf{1}\otimes h}U^{-1} = M_{\mathbf{1}\otimes\psi(h)}.$$

Then the map $\psi : \mathrm{Her}(C(Y)) \to \mathrm{Her}(C(Y))$ is a real algebra isomorphism. We define the map $\tilde{\psi}$ from $C(Y)$ onto $C(Y)$ by

$$\tilde{\psi}(h_1 + ih_2) = \psi(h_1) + i\psi(h_2)$$

for $h_1, h_2 \in \mathrm{Her}(C(Y)) = C_{\mathbb{R}}(Y)$. We have $\tilde{\psi}$ is a complex algebra isomorphism from $C(Y)$ onto itself. As $\tilde{\psi} : C(Y) \to C(Y)$ is an algebra isomorphism there exists a homeomorphism $\tau : Y \to Y$ such that

$$\tilde{\psi}(h)(y) = h(\tau(y))$$

for any $h \in C(Y)$ and $y \in Y$. Due to the isomorphism $\tilde{\psi}$ and a homeomorphism τ, we obtain that

$$UM_{\mathbf{1}\otimes h}U^{-1} = M_{\mathbf{1}\otimes\tilde{\psi}(h)}$$

for every $h \in C(Y)$. We now get the following characterization of a surjective linear isometry on $C^1([0,1], C(Y))$, which is a weaker statement than Corollary 4.18, but we prove it by Lumer's method.

THEOREM 5.15. *Let Y be a compact Hausdorff space. Then U is a surjective linear isometry from $C^1([0,1], C(Y))$ onto itself such that $U(\mathbf{1}) = \mathbf{1}$ if and only if there exist a continuous map $\varphi : [0,1] \times Y \to [0,1]$ such that for each $y \in Y$ continuous map $\varphi(\cdot, y) : [0,1] \to [0,1]$ with $\varphi(x, y) = x$ for every $x \in [0,1]$ or $\varphi(x, y) = 1 - x$ for every $x \in [0,1]$ and a homeomorphism $\tau : Y \to Y$ which satisfy that*

$$UF(x, y) = F(\varphi(x, y), \tau(y)) \quad x \in X, y \in Y \tag{4.3}$$

for every $F \in C^1([0,1], C(Y))$.

Proof. We can easy to see that if U is the form of the equation (4.3), U is a surjective linear isometry on $C^1([0,1], C(Y))$ with $\|\cdot\|_\Sigma$.

Suppose that U is a surjective isometry from $\mathrm{Lip}(X, C(Y))$ onto itself such that $U\mathbf{1} = \mathbf{1}$. Corollary 5.10 implies that U is an isometry from $(C^1([0,1], C(Y)), \|\cdot\|_{\infty([0,1]\times Y)})$ onto itself. We note that $C^1([0,1], C(Y))$ is uniformly dense in $C([0,1]\times Y)$ by the Stone-Weierstrass theorem. This shows that we have $U^\infty : C([0,1]\times Y) \to C([0,1]\times Y)$ which is a unique extension of U, and U^∞ is a unital linear isometry with the supremum norm. By the Banach-Stone theorem, we conclude that U^∞ is an algebra isomorphism, so is U. Let $y \in Y$ arbitrary. We define a map $\widetilde{U^y} : C^1([0,1]) \to C^1([0,1])$ by $\widetilde{U^y}(f) = U(f \otimes \mathbf{1})(\cdot, y)$ for each $f \in C^1([0,1])$. Then $\widetilde{U^y}$ is a unital homomorphism. There exists a continuous $\varphi(\cdot, y)$ from X into itself such that $\widetilde{U^y}(f)(x) = f(\varphi(x,y))$ for every $f \in C^1([0,1])$ and $x \in [0,1]$. For every $f \in C^1([0,1])$, $h \in C(Y)$, and $(x,y) \in [0,1]\times Y$, we have

$$
\begin{aligned}
(U(f\otimes h))(x,y) &= (U(M_{\mathbf{1}\otimes h} f \otimes \mathbf{1}))(x,y) \\
&= (UM_{\mathbf{1}\otimes h}U^{-1}U(f\otimes \mathbf{1}))(x,y) = (M_{\mathbf{1}\otimes\tilde{\psi}(h)}U(f\otimes\mathbf{1}))(x,y) \\
&= ((\mathbf{1}\otimes\tilde{\psi}(h))U(f\otimes\mathbf{1}))(x,y) = (\mathbf{1}\otimes\tilde{\psi}(h))(x,y)(U(f\otimes\mathbf{1}))(x,y) \\
&= (\mathbf{1}\otimes\tilde{\psi}(h))(x,y)\widetilde{U^y}(f)(x) = h(\tau(y))f(\varphi(x,y)) \\
&= f\otimes h(\varphi(x,y), \tau(y)).
\end{aligned}
$$

Hence

$$
(U(\sum_i f_i \otimes h_i))(x,y) = \sum_i f_i \otimes h_i(\varphi(x,y), \tau(y)) \tag{4.4}
$$

for every $\sum_i f_i \otimes h_i \in C^1([0,1]) \otimes C(Y)$, the algebraic tensor product of $C^1([0,1])$ and $C(Y)$. Let $F \in C^1([0,1], C(Y))$ be arbitrary. Then by the Stone-Weierstrass theorem there exists a sequence $\{F_n\} \in C^1([0,1]) \otimes C(Y)$ such that $\|F - F_n\|_{\infty(X\times Y)} \to 0$ as $n \to \infty$. By Corollary 5.10, we obtain that U is an isometry with respect to the metric induced by the supremum norm. Thus we have

$$
\|U(F_n) - U(F)\|_{\infty([0,1]\times Y)} = \|F_n - F\|_{\infty([0,1]\times Y)} \to 0
$$

as $n \to \infty$. Hence $(U(F_n))(x,y) \to (U(F))(x,y)$ as $n \to \infty$. As $\|F_n - F\|_{\infty([0,1]\times Y)} \to 0$, we obtain by (4.4) that

$$(U(F_n))(x,y) = F_n(\varphi(x,y), \tau(y)) \to F(\varphi(x,y), \tau(y))$$

as $n \to \infty$. We conclude that $(U(F))(x,y) = F(\varphi(x,y), \tau(y))$.

Although it does not complete the proof, we omit the rest of the proof since it is a routine argument to verify that $\varphi : [0,1] \times Y \to [0,1]$ is continuous and for each $y \in Y$ continuous map $\varphi(\cdot, y) : [0,1] \to [0,1]$ with $\varphi(x,y) = x$ for every $x \in [0,1]$ or $\varphi(x,y) = 1 - x$ for every $x \in [0,1]$. $\square$

CHAPTER 6

Hermitian operators on non-commutative Banach algebras

A surjective linear isometry on $\mathrm{Lip}(X,E)$, where E is a unital commutative C^*-algebra is of the canonical form, see Theorem 4.14. But the general case for a unital C^*-algebra without assuming commutativity remains open. A main difficulty relies on a lack of the complete characterization of the extreme points of the unit ball of the dual space of $\mathrm{Lip}(X,E)$ for a unital C^*-algebra E.

On the other hand, Lumer initiated a study of isometries on function spaces in terms of Hermitian operators [**76, 78**]. Hermitian operators are intrinsically related to surjective isometries. The method is called Lumer's method (see [**30**]).

In this chapter, we deal with the case where E is a Banach space of a finite dimension. We first show that Hermitian operators on $\mathrm{Lip}(X,E)$ are composition operators. Then Lumer's method as in [**29**] applies to characterizing unital surjective complex isometries on algebras of Lipschitz maps with values in $M_n(\mathbb{C})$ with the sum norm, where the norm on $M_n(\mathbb{C})$ is the operator norm (spectral norm).

In [**13**], Botelho and Jamison gave a representation for algebra homomorphisms between algebras of Lipschitz maps with values in $M_n(\mathbb{C})$.

1. Hermitian operators between Banach algebras with the values in a finite dimensional Banach space

1.1. Hermitian operators on $\mathrm{Lip}(X,E)$**.** Let B be a Banach algebra. Let $[\cdot,\cdot]$ be a semi-inner product on B, compatible with the norm. We recall that a bounded operator T is Hermitian if $[Tx,x]_B \in \mathbb{R}$ for any $x \in B$. Several equivalent conditions for Hermitian operators are exhibited in [**30**, Theorem 5.2.6]. In the rest of this chapter E denotes a finite dimensional Banach space.

For any $f \in \mathrm{Lip}(X)$ and $e \in E$, we define $f \otimes e : X \to E$ by

$$(f \otimes e)(x) = f(x)e.$$

By a simple calculation, we have $f \otimes e \in \mathrm{Lip}(X, E)$ such that

$$\|f \otimes e\|_\infty = \|f\|_\infty \|e\|_E$$

and

$$L(f \otimes e) = L(f)\|e\|_E.$$

This implies that $\|f \otimes e\|_\Sigma = \|f\|_\Sigma \|e\|_E$. This definition for $\otimes$ is understood to mean that $f \otimes e$ is an element of the algebraic tensor product space $\mathrm{Lip}(X) \otimes E$ with the crossnorm. Since E is finite dimensional, we have the following lemma:

LEMMA 6.1. *Let X be a compact metric space and $(E, \|\cdot\|_E)$ a finite dimensional Banach space. Then*

$$\mathrm{Lip}(X) \otimes E = \mathrm{Lip}(X, E).$$

PROOF. First we prove that $\mathrm{Lip}(X, E) \subset \mathrm{Lip}(X) \otimes E$. Let $\{e_i\}_{i=1}^n$ be a basis for E as a linear space and $1 \le j \le n$.

We define a linear operator $\Pi_j : E \to \mathbb{C}$ by

$$\Pi_j(\Sigma_{i=1}^n \alpha_i e_i) = \alpha_j, \quad \Sigma_{i=1}^n \alpha_i e_i \in E.$$

Then Π_j is bounded. In fact, as E is finite dimensional, any norm on E is equivalent, hence the original norm $\|\cdot\|_E$ and the norm $\|\cdot\|_1$ defined by

$$\|\Sigma_{i=1}^n \alpha_i e_i\|_1 := \Sigma_{i=1}^n |\alpha_i|$$

for $\Sigma_{i=1}^n \alpha_i e_i \in E$ is equivalent. Hence there exists $K > 0$ such that

$$|\Pi_j(\Sigma_{i=1}^n \alpha_i e_i)| = |\alpha_j| \le \Sigma_{i=1}^n |\alpha_i| \le K \|\Sigma_{i=1}^n \alpha_i e_i\|_E.$$

Thus the operator $\Pi_j : E \to \mathbb{C}$ is bounded. For any $F \in \mathrm{Lip}(X, E)$, define a function $\widetilde{\Pi}_j(F) : X \to \mathbb{C}$ by

$$\widetilde{\Pi}_j(F)(x) := \Pi_j(F(x)), \quad x \in X.$$

We have

$$|\widetilde{\Pi}_j(F)(x) - \widetilde{\Pi}_j(F)(y)| = |\Pi_j(F(x) - F(y))| \\ \leq K\|F(x) - F(y)\|_E \leq KL(F)d(x,y).$$

By the definition of a Lipschitz map we have that $\widetilde{\Pi}_j(F) \in \mathrm{Lip}(X)$. Furthermore, we have by a simple calculation that

$$F = \Sigma_{i=1}^n \widetilde{\Pi}_i(F) \otimes e_i \in \mathrm{Lip}(X) \otimes E.$$

As $F \in \mathrm{Lip}(X, E)$ is arbitrary, we see that

$\mathrm{Lip}(X, E) \subset \mathrm{Lip}(X) \otimes E$.

The opposite inclusion $\mathrm{Lip}(X) \otimes E \subset \mathrm{Lip}(X, E)$ is obvious. Thus we obtain the equality

$$\mathrm{Lip}(X) \otimes E = \mathrm{Lip}(X, E).$$

□

The following is the main result in this Section.

THEOREM 6.2. *Let X be a compact metric space and E a Banach space of a finite dimension. Then T is a Hermitian operator on* $\mathrm{Lip}(X, E)$ *if and only if there exists a Hermitian operator $\phi : E \to E$ such that*

$$TF(x) = \phi(F(x)), \quad F \in \mathrm{Lip}(X, E), \quad x \in X. \tag{1.1}$$

We make use of Lemma 6.3 to prove Theorem 6.2.

LEMMA 6.3. *Let T be a Hermitian operator on* $\mathrm{Lip}(X, E)$. *Then*

$$T(\mathbf{1} \otimes e) \in \mathbf{1} \otimes E$$

for any $e \in E$.

PROOF. Let $\widetilde{X} = \{(x, y) \in X^2; x \neq y\}$. We denote by E_1 the unit ball of E, and by E_1^* the unit ball of the dual space E^*. Let $\beta(\widetilde{X} \times E_1^*)$ be the Stone-Čech compactification of $\widetilde{X} \times E_1^*$. For any $F \in \mathrm{Lip}(X, E)$, we denote by $\widetilde{F} : \beta(\widetilde{X} \times E_1^*) \to \mathbb{C}$ the unique continuous extension of the bounded continuous function $((x, y), e^*) \mapsto e^*(\frac{F(x)-F(y)}{d(x,y)})$ on $\widetilde{X} \times E_1^*$.

Then we have

$$\begin{aligned}
\|\widetilde{F}\|_\infty &= \sup_{\xi\in\beta(\widetilde{X}\times E_1^*)} \left|\widetilde{F}(\xi)\right| \\
&= \sup_{((x,y),e^*)\in\widetilde{X}\times E_1^*} \left|e^*\left(\frac{F(x)-F(y)}{d(x,y)}\right)\right| \qquad (1.2)\\
&= \sup_{(x,y)\in\widetilde{X}} \left\|\frac{F(x)-F(y)}{d(x,y)}\right\|_E = L(F)
\end{aligned}$$

for all $F\in \mathrm{Lip}(X,E)$. We define a map

$$\Gamma : \mathrm{Lip}(X,E) \to C(X\times\beta(\widetilde{X}\times E_1^*)\times E_1, E)$$

given by

$$\Gamma(F)(x,\xi,e) = F(x) + \widetilde{F}(\xi)e$$

for all $F\in \mathrm{Lip}(X,E)$ and $(x,\xi,e)\in X\times\beta(\widetilde{X}\times E_1^*)\times E_1$. By (1.2), Γ is a linear isometric embedding. For any $G\in \mathrm{Lip}(X,E)$, we define a set P_G as follows;

$$P_G = \{t\in X\times\beta(\widetilde{X}\times E_1^*)\times E_1; \|\Gamma(G)(t)\|_E = \|\Gamma(G)\|_\infty = \|G\|_\Sigma\}.$$

As E is a finite dimensional Banach space, E_1 is compact with norm topology on E. This implies $X\times\beta(\widetilde{X}\times E_1^*)\times E_1$ is also compact. We conclude that $P_G\neq\emptyset$. Choose a choice function

$$\Psi : \mathrm{Lip}(X,E)\to X\times\beta(\widetilde{X}\times E_1^*)\times E_1$$

such that $\Psi(G)\in P_G$ for every $G\in \mathrm{Lip}(X,E)$. Such a function Ψ exists by the axiom of choice. We now prove that we can define a semi-inner product on $\mathrm{Lip}(X,E)$ by Ψ. Let $[\cdot,\cdot]_E$ on E be a semi-inner product which is compatible with the norm of E. Define a map $[\cdot,\cdot]_{\Psi L} : \mathrm{Lip}(X,E)\times \mathrm{Lip}(X,E)\to\mathbb{C}$ given by

$$[F,G]_{\Psi L} = [\Gamma(F)(\Psi(G)), \Gamma(G)(\Psi(G))]_E, \quad F,G\in \mathrm{Lip}(X,E). \qquad (1.3)$$

By a routine argument we deduce that $[\cdot,\cdot]_{\Psi L}$ is a semi-inner product on $\mathrm{Lip}(X,E)$ compatible with the norm $\|\cdot\|_\Sigma$. Let $e\in E$. We prove $T(\mathbf{1}\otimes e)\in\mathbf{1}\otimes E$. If $e=0$, then $T(\mathbf{1}\otimes e) = T(0) = 0 = \mathbf{1}\otimes 0$, hence the conclusion holds. Suppose that $0\neq e$. Fix $x'\in X$, $(x,y)\in\widetilde{X}$ and

$e^* \in E_1^*$. Let $\theta \in [0, 2\pi)$, we obtain

$$\begin{aligned}
&\Gamma(\mathbf{1} \otimes e)(x', ((x,y), e^{i\theta}e^*), e) \\
&= (\mathbf{1} \otimes e)(x') + e^{i\theta}e^* \left(\frac{(\mathbf{1} \otimes e)(x) - (\mathbf{1} \otimes e)(y)}{d(x,y)} \right) e \qquad (1.4) \\
&= e + 0e = e.
\end{aligned}$$

This implies that

$$\|\Gamma(\mathbf{1} \otimes e)(x', ((x,y), e^{i\theta}e^*), e)\|_E = \|\mathbf{1} \otimes e\|_\Sigma.$$

Thus we get $(x', ((x,y), e^{i\theta}e^*), e) \in P_{\mathbf{1} \otimes e}$. Choose a choice function $\Psi_\theta : \mathrm{Lip}(X, E) \to X \times \beta(\widetilde{X} \times E_1^*) \times E_1$ such that

$$\Psi_\theta(\mathbf{1} \otimes e) = (x', ((x,y), e^{i\theta}e^*), e)$$

and define a semi-inner product $[\cdot, \cdot]_{\Psi_\theta L}$ on $\mathrm{Lip}(X, E)$ in the manner as in (1.3). Since T is a Hermitian operator, we have

$$[T(\mathbf{1} \otimes e), \mathbf{1} \otimes e]_{\Psi_\theta L} \in \mathbb{R}.$$

By (1.4), it follows that

(1.5)

$$\begin{aligned}
\mathbb{R} \ni\ & [T(\mathbf{1} \otimes e), \mathbf{1} \otimes e]_{\Psi_\theta L} \\
&= [\Gamma(T(\mathbf{1} \otimes e))(\Psi_\theta(\mathbf{1} \otimes e)), \Gamma(\mathbf{1} \otimes e)(\Psi_\theta(\mathbf{1} \otimes e))]_E \\
&= [T(\mathbf{1} \otimes e)(x') + e^{i\theta}e^* \left(\frac{T(\mathbf{1} \otimes e)(x) - T(\mathbf{1} \otimes e)(y)}{d(x,y)} \right) e, e]_E \\
&= [T(\mathbf{1} \otimes e)(x'), e]_E + e^{i\theta}e^* \left(\frac{T(\mathbf{1} \otimes e)(x) - T(\mathbf{1} \otimes e)(y)}{d(x,y)} \right) [e, e]_E \\
&= [T(\mathbf{1} \otimes e)(x'), e]_E + e^{i\theta}e^* \left(\frac{T(\mathbf{1} \otimes e)(x) - T(\mathbf{1} \otimes e)(y)}{d(x,y)} \right) \|e\|_E^2.
\end{aligned}$$

As $e \neq 0$, we see that $\|e\|_E^2 > 0$. Since $\theta \in [0, 2\pi)$ is arbitrary, it must be

$$e^* \left(\frac{T(\mathbf{1} \otimes e)(x) - T(\mathbf{1} \otimes e)(y)}{d(x,y)} \right) = 0 \qquad (1.6)$$

for any $e^* \in E_1^*$. This implies

$$\frac{T(\mathbf{1} \otimes e)(x) - T(\mathbf{1} \otimes e)(y)}{d(x,y)} = 0$$

for any $(x,y) \in \widetilde{X}$. Thus we deduce

$$L(T(\mathbf{1} \otimes e)) = 0.$$

Therefore, there exists $e_0 \in E$ such that $T(\mathbf{1} \otimes e) = \mathbf{1} \otimes e_0$. □

Due to Lemma 6.3 we define the map $\phi : E \to E$ by $\phi(e) = e_0$, where e_0 is the corresponding element in E for $e \in E$, that is,

$$T(\mathbf{1} \otimes e) = \mathbf{1} \otimes \phi(e) \tag{1.7}$$

for each $e \in E$. Since T is a bounded linear operator, so is ϕ. Equations (1.5) and (1.6) imply

$$\begin{aligned}
\mathbb{R} \ni &[T(\mathbf{1} \otimes e), \mathbf{1} \otimes e]_{\Psi_\theta L} \\
&= [\Gamma(T(\mathbf{1} \otimes e))(\Psi_\theta(\mathbf{1} \otimes e)), \Gamma(\mathbf{1} \otimes e)(\Psi_\theta(\mathbf{1} \otimes e))]_E \\
&= \left[T(\mathbf{1} \otimes e)(x') + e^{i\theta} e^* \left(\frac{T(\mathbf{1} \otimes e)(x) - T(\mathbf{1} \otimes e)(y)}{d(x,y)}\right) e, e\right]_E \qquad (1.8) \\
&= [T(\mathbf{1} \otimes e)(x'), e]_E + 0[e,e]_E \\
&= [T(\mathbf{1} \otimes e)(x'), e]_E.
\end{aligned}$$

Due to the definition of ϕ, we have

$$[\phi(e), e]_E \in \mathbb{R}$$

for any $e \in E$. Thus, ϕ is a Hermitian operator on E.

We are now ready to prove Theorem 6.2.

Proof of Theorem 6.2. Suppose that $\phi : E \to E$ is a Hermitian operator and T is of the form described as (1.1) in the statement of Theorem 6.2; for any $F \in \mathrm{Lip}(X, E)$ and $x \in X$,

$$(TF)(x) = \phi(F(x)).$$

To prove that T is a Hermitian, we apply the fact that T is a Hermitian if and only if e^{itT} is a surjective isometry for every $t \in \mathbb{R}$, see [**30**, Theorem 5.2.6]. Let $t \in \mathbb{R}$. By the definition of T, we have

$$e^{itT} F(x) = e^{it\phi}(F(x))$$

for any $F \in \mathrm{Lip}(X, E)$ and $x \in X$. Since ϕ is Hermitian on E, we have

$$\|e^{itT} F(x)\|_E = \|e^{it\phi}(F(x))\|_E = \|F(x)\|_E$$

and

$$\begin{aligned}
\|e^{itT} F(x) - e^{itT} F(y)\|_E &= \|e^{it\phi}(F(x)) - e^{it\phi}(F(y))\|_E \\
&= \|e^{it\phi}(F(x) - F(y))\|_E = \|F(x) - F(y)\|_E
\end{aligned}$$

for any $x, y \in X$. It follows that

$$\|e^{itT}F\|_\infty = \|F\|_\infty$$

and

$$L(e^{itT}F) = L(F).$$

Thus we deduce

$$\|e^{itT}F\|_\Sigma = \|F\|_\Sigma$$

for any $F \in \mathrm{Lip}(X, E)$. This implies that e^{itT} is a surjective isometry for every $t \in \mathbb{R}$. Thus we have T is a Hermitian operator.

We now prove the converse. Suppose that $T : \mathrm{Lip}(X, E) \to \mathrm{Lip}(X, E)$ is a Hermitian operator. Let ϕ be the operator defined by (1.7). In a similar way to the first part of the proof, an operator from $\mathrm{Lip}(X, E)$ into itself given by $F \mapsto \phi \circ F$ for any $F \in \mathrm{Lip}(X, E)$ is a Hermitian operator. Hence we can define a Hermitian operator $T_0 : \mathrm{Lip}(X, E) \to \mathrm{Lip}(X, E)$ by

$$(T_0F)(x) = (TF)(x) - \phi(F(x))$$

for all $F \in \mathrm{Lip}(X, E)$ and $x \in X$. Let $e \in E$ with $\|e\|_E = 1$. We define a map $S_e : \mathrm{Lip}(X) \to \mathrm{Lip}(X)$ by

$$S_e(f)(x) = [T_0(f \otimes e)(x), e]_E, \quad f \in \mathrm{Lip}(X),\ x \in X,$$

where $[\cdot, \cdot]_E$ is a semi-inner product on E compatible with the norm which satisfies $[e_1, \lambda e_2]_E = \bar{\lambda}[e_1, e_2]_E$ for any $e_i \in E$ and $\lambda \in \mathbb{C}$ (such a semi-inner product always exists [**30**, p. 10]). Then S_e is a linear map. We get

$$|S_e(f)(x)| = |[T_0(f \otimes e)(x), e]_E| \leq \|T_0(f \otimes e)(x)\|_E$$

and

$$\begin{aligned} |S_e(f)(x) - S_e(f)(y)| &= |[T_0(f \otimes e)(x), e]_E - [T_0(f \otimes e)(y), e]_E| \\ = |[T_0(f \otimes e)(x) - T_0(f \otimes e)(y), e]_E| &\leq \|T_0(f \otimes e)(x) - T_0(f \otimes e)(y)\|_E \\ &\leq L(T_0(f \otimes e))d(x, y) \end{aligned}$$

for any $f \in \mathrm{Lip}(X)$ and $x, y \in X$. Thus we deduce

$$\begin{aligned}\|S_e(f)\|_\Sigma &= \|S_e(f)\|_\infty + L(S_e(f)) \\ &\le \|T_0(f\otimes e)\|_\infty + L(T_0(f\otimes e)) = \|T_0(f\otimes e)\|_\Sigma \\ &\le \|T_0\|\|f\otimes e\|_\Sigma = \|T_0\|\|f\|_\Sigma\end{aligned}$$

for each $f \in \mathrm{Lip}(X)$. It follows that $\|S_e\| \le \|T_0\|$, i.e., S_e is a bounded operator on $\mathrm{Lip}(X)$. Let $t \in \mathbb{R}$. We have

$$\begin{aligned}(I + itS_e)(\mathbf{1})(x) &= \mathbf{1} + it[T_0(\mathbf{1}\otimes e)(x), e]_E \\ &= \mathbf{1} + it[T(\mathbf{1}\otimes e)(x) - \phi((\mathbf{1}\otimes e)(x)), e]_E = 1 + it[0, e]_E = 1\end{aligned}$$

for any $x \in X$. This implies that

$$1 \le \|I + itS_e\|. \tag{1.9}$$

On the other hand, let $f \in \mathrm{Lip}(X)$. We obtain for any $x, y \in X$,

$$\begin{aligned}&|(I + itS_e)(f)(x)| = |f(x) + it[T_0(f\otimes e)(x), e]_E| \\ &= |f(x)[e, e]_E + it[T_0(f\otimes e)(x), e]_E| = |[(f\otimes e)(x) + itT_0(f\otimes e)(x), e]_E| \\ &\le \|(f\otimes e + itT_0(f\otimes e))(x)\|_E \le \|(I + itT_0)(f\otimes e)\|_\infty\end{aligned}$$

and

$$\begin{aligned}&|(I + itS_e)(f)(x) - (I + itS_e)(f)(y)| \\ &= |f(x)[e, e]_E + it[T_0(f\otimes e)(x), e]_E - f(y)[e, e]_E - it[T_0(f\otimes e)(y), e]_E| \\ &= |[(f\otimes e)(x) + itT_0(f\otimes e)(x), e]_E - [(f\otimes e)(y) + itT_0(f\otimes e)(y), e]_E| \\ &= |[(I + itT_0)(f\otimes e)(x) - (I + itT_0)(f\otimes e)(y), e]_E| \\ &\le \|(I + itT_0)(f\otimes e)(x) - (I + itT_0)(f\otimes e)(y)\|_E \\ &\le L((I + itT_0)(f\otimes e))d(x, y).\end{aligned}$$

Therefore, we get

$$\begin{aligned}\|(I + itS_e)(f)\|_\Sigma &= \|(I + itS_e)(f)\|_\infty + L((I + itS_e)(f)) \\ &\le \|(I + itT_0)(f\otimes e)\|_\infty + L((I + itT_0)(f\otimes e)) \\ &= \|(I + itT_0)(f\otimes e)\|_\Sigma \\ &\le \|I + itT_0\|\|f\otimes e\|_\Sigma = \|I + itT_0\|\|f\|_\Sigma\end{aligned}$$

for any $f \in \mathrm{Lip}(X)$. We conclude that

$$\|I + itS_e\| \le \|I + itT_0\|. \tag{1.10}$$

Since T_0 is a Hermitian operator on $\mathrm{Lip}(X, E)$, we have

$$\|I + itT_0\| = 1 + o(t)$$

by [**30**, Theorem 5.2.6]. Combining equation (1.9) with equation (1.10), we see that

$$1 \leq \|I + itS_e\| \leq \|I + itT_0\| = 1 + o(t).$$

This implies that $S_e : \mathrm{Lip}(X) \to \mathrm{Lip}(X)$ is a Hermitian operator. By [**16**, Theorem 3.1.] we have S_e is a real multiple of the identity. Since $S_e(\mathbf{1})(x) = [T_0(\mathbf{1} \otimes e)(x), e] = 0$, we deduce

$$S_e(f)(x) = 0f(x) = 0 \quad f \in \mathrm{Lip}(X), \quad x \in X.$$

Therefore, we have

$$[T_0(f \otimes e)(x), e]_E = 0$$

for all $f \in \mathrm{Lip}(X)$ and $x \in X$. As $e \in E$ with $\|e\|_E = 1$ is arbitrary, we obtain

$$[T_0(f \otimes e)(x), e]_E = 0, \quad e \in E, \quad f \in \mathrm{Lip}(X, E), \quad x \in X. \tag{1.11}$$

We now prove that $T_0 = 0$. Let $f \in \mathrm{Lip}(X)$ and $x \in X$. Then we define a map $S_{fx} : E \to E$ such that

$$S_{fx}(e) = T_0(f \otimes e)(x)$$

for any $e \in E$. It is easy to check that S_{fx} is linear because of linearity of T_0. In addition, we have

$$\begin{aligned}\|S_{fx}(e)\|_E &= \|T_0(f \otimes e)(x)\|_E \\ &\leq \|T_0(f \otimes e)\|_\Sigma \leq \|T_0\|\|f \otimes e\|_\Sigma = \|T_0\|\|f\|_\Sigma\|e\|_E\end{aligned}$$

for any $e \in E$. We deduce that S_{fx} is a bounded operator. Moreover, by (1.11) we have

$$[S_{fx}(e), e]_E = [T_0(f \otimes e)(x), e]_E = 0$$

for all $e \in E$. Applying [**76**, Theorem 5], we have

$$T_0(f \otimes e)(x) = S_{fx}(e) = 0, \quad e \in E.$$

As $f \in \mathrm{Lip}(X)$ and $x \in X$ be chosen arbitrary, we conclude

$$T_0(f \otimes e)(x) = 0 \tag{1.12}$$

for any $f \in \mathrm{Lip}(X)$, $e \in E$ and $x \in X$. Suppose that $\dim E = n$. Then, by Lemma 7, for any $F \in \mathrm{Lip}(X, E)$, F is represented by the following;

$$F = \sum_{k=1}^{n} f_k \otimes e_k$$

with some $f_k \in \mathrm{Lip}(X)$ and $e_k \in E$ for $k = 1, \dots, n$. It follows by (1.12) that

$$(T_0 F)(x) = T_0(\sum_{k=1}^{n} f_k \otimes e_k)(x) = \sum_{k=1}^{n} T_0(f_k \otimes e_k)(x) = 0 \tag{1.13}$$

for any $x \in X$. We recall the definition of the Hermitian operator $T_0 : \mathrm{Lip}(X, E) \to \mathrm{Lip}(X, E)$ is defined by

$$(T_0 F)(x) = (TF)(x) - \phi(F(x)), \quad x \in X$$

for every $F \in \mathrm{Lip}(X, E)$. Applying (1.13), we have

$$(TF)(x) - \phi(F(x)) = 0,$$

that is,

$$(TF)(x) = \phi(F(x))$$

for any $F \in \mathrm{Lip}(X, E)$ and $x \in X$. □

1.2. Hermitian operators on $C^1([0,1], E)$. Although the proof is similar to the case of $\mathrm{Lip}(X, E)$, we exhibit it. As we show the definition of algebraic tensor product space in Section 2, we have $f \otimes e : [0,1] \to E$ by

$$(f \otimes e)(x) = f(x)e, \quad x \in [0,1],$$

for any $f \in C^1([0,1])$ and $e \in E$ too. We also obtain the following Lemma since E is of a finite dimension.

LEMMA 6.4. *Let $(E, \|\cdot\|_E)$ be a finite dimensional Banach space. Then*

$$C^1([0,1]) \otimes E = C^1([0,1], E).$$

The proof is similar to that for Lemma 7, we omit it. The following is the main result in this Section.

THEOREM 6.5. *Let E be a Banach space of a finite dimension. Then T is a Hermitian operator on $C^1([0,1],E)$ if and only if there exists a Hermitian operator $\phi : E \to E$ such that*

$$TF(x) = \phi(F(x)), \quad F \in C^1([0,1],E), \quad x \in [0,1]. \tag{1.14}$$

We make use of Lemma 6.6 to prove Theorem 6.5.

LEMMA 6.6. *Let T be a Hermitian operator on $C^1([0,1],E)$. Then*

$$T(\mathbf{1} \otimes e) \in \mathbf{1} \otimes E$$

for any $e \in E$.

PROOF. We denote by E_1 the unit ball of E, and by E_1^* the unit ball of the dual space E^*. For any $F \in C^1([0,1],E)$, we define $\widetilde{F} : [0,1] \times E_1^* \to \mathbb{C}$ by

$$\widetilde{F}(x, e^*) = e^*(F'(x))$$

for any $x \in [0,1]$ and $e^* \in E_1^*$.

Then we have $\|\widetilde{F}\|_\infty = \|F'\|_\infty$ immediately for any $F \in C^1([0,1],E)$. We define a map

$$\Gamma : C^1([0,1],E) \to C([0,1] \times [0,1] \times E_1^* \times E_1, E)$$

given by

$$\Gamma(F)(x, \xi, e) = F(x) + \widetilde{F}(\xi)e$$

for all $F \in C^1([0,1],E)$ and $(x, \xi, e) \in [0,1] \times ([0,1] \times E_1^*) \times E_1$, where $\xi \in [0,1] \times E_1^*$. We see that Γ is a linear isometric embedding. For any $G \in C^1([0,1],E)$, we define a set P_G as follows;

$$P_G = \{t \in [0,1] \times ([0,1] \times E_1^*) \times E_1; \|\Gamma(G)(t)\|_E = \|\Gamma(G)\|_\infty = \|G\|_\Sigma\}.$$

As E is a finite dimensional Banach space, E_1 is compact with norm topology on E. Applying Banach- Alaoglu Theorem, we get $[0,1] \times ([0,1] \times E_1^*) \times E_1$ is also compact. We conclude that $P_G \neq \emptyset$. By the axiom of choice, we can define a choice function

$$\Psi : C^1([0,1],E) \to [0,1] \times ([0,1] \times E_1^*) \times E_1$$

such that $\Psi(G) \in P_G$ for every $G \in C^1([0,1],E)$. In addition, a choice function Ψ enable us to define a semi-inner product on $C^1([0,1],E)$ as follows;

(1.15)

$$[F,G]_{\Psi s} = [\Gamma(F)(\Psi(G)), \Gamma(G)(\Psi(G))]_E, \quad F,G \in C^1([0,1],E),$$

where $[\cdot,\cdot]_E$ is a a semi-inner product which is compatible with the norm of E. We get $[\cdot,\cdot]_{\Psi s}$ is a semi-inner product on $C^1([0,1],E)$ compatible with the norm $\|\cdot\|_\Sigma$.

Let $e \in E$. We prove $T(\mathbf{1}\otimes e) \in \mathbf{1}\otimes E$. If $e=0$, then $T(\mathbf{1}\otimes e) = T(0) = 0 = \mathbf{1}\otimes 0$, hence the conclusion holds. Suppose that $0 \neq e$. Fix $x', x \in [0,1]$ and $e^* \in E_1^*$. Let $\theta \in [0,2\pi)$, we obtain

$$\begin{aligned} &\Gamma(\mathbf{1}\otimes e)(x',x,e^{i\theta}e^*,e) \\ &= (\mathbf{1}\otimes e)(x') + e^{i\theta}e^*((\mathbf{1}\otimes e)'(x)) \\ &= e + 0e = e. \end{aligned} \tag{1.16}$$

This implies that

$$\|\Gamma(\mathbf{1}\otimes e)(x',x,e^{i\theta}e^*,e)\|_E = \|\mathbf{1}\otimes e\|_\Sigma.$$

Thus we get $(x',x,e^{i\theta}e^*,e) \in P_{\mathbf{1}\otimes e}$. Choose a choice function $\Psi_\theta : C^1([0,1],E) \to [0,1]\times([0,1]\times E_1^*)\times E_1$ such that

$$\Psi_\theta(\mathbf{1}\otimes e) = (x',x,e^{i\theta}e^*,e)$$

and define a semi-inner product $[\cdot,\cdot]_{\Psi_\theta s}$ on $C^1([0,1],E)$ in the manner as in (1.15). Since T is a Hermitian operator, we have

$$[T(\mathbf{1}\otimes e), \mathbf{1}\otimes e]_{\Psi_\theta s} \in \mathbb{R}.$$

By (1.16), it follows that

$$\begin{aligned} \mathbb{R} \ni\ &[T(\mathbf{1}\otimes e), \mathbf{1}\otimes e]_{\Psi_\theta s} \\ &= [\Gamma(T(\mathbf{1}\otimes e))(\Psi_\theta(\mathbf{1}\otimes e)), \Gamma(\mathbf{1}\otimes e)(\Psi_\theta(\mathbf{1}\otimes e))]_E \\ &= [T(\mathbf{1}\otimes e)(x') + e^{i\theta}e^*(T(\mathbf{1}\otimes e)'(x))e, e]_E \\ &= [T(\mathbf{1}\otimes e)(x'), e]_E + e^{i\theta}e^*(T(\mathbf{1}\otimes e)'(x))[e,e]_E \end{aligned} \tag{1.17}$$

As $e \neq 0$, we see that $\|e\|_E^2 > 0$. Since $\theta \in [0,2\pi)$ is arbitrary, it must be

$$e^*(T(\mathbf{1}\otimes e)'(x)) = 0 \tag{1.18}$$

for any $e^* \in E_1^*$. This implies $T(\mathbf{1} \otimes e)'(x) = 0$ for any $x \in [0,1]$. we obtain $T(\mathbf{1} \otimes e)' = 0$. Therefore, there exists $e_0 \in E$ such that $T(\mathbf{1} \otimes e) = \mathbf{1} \otimes e_0$. □

Due to Lemma 6.6 we define the map $\phi : E \to E$ by $\phi(e) = e_0$, where e_0 is the corresponding element in E for $e \in E$, that is,

$$T(\mathbf{1} \otimes e) = \mathbf{1} \otimes \phi(e) \tag{1.19}$$

for each $e \in E$. Since T is a bounded linear operator, so is ϕ. Equations (1.17) and (1.18) imply

$$\begin{aligned} \mathbb{R} \ni\ & [T(\mathbf{1} \otimes e), \mathbf{1} \otimes e]_{\Psi_\theta s} \\ &= [\Gamma(T(\mathbf{1} \otimes e))(\Psi_\theta(\mathbf{1} \otimes e)), \Gamma(\mathbf{1} \otimes e)(\Psi_\theta(\mathbf{1} \otimes e))]_E \\ &= [T(\mathbf{1} \otimes e)(x') + e^{i\theta} e^*(T(\mathbf{1} \otimes e)'(x))e, e]_E \\ &= [T(\mathbf{1} \otimes e)(x'), e]_E + 0[e,e]_E \\ &= [T(\mathbf{1} \otimes e)(x'), e]_E. \end{aligned} \tag{1.20}$$

Due to the definition of ϕ, we have

$$[\phi(e), e]_E \in \mathbb{R}$$

for any $e \in E$. Thus, ϕ is a Hermitian operator on E.

We are now ready to prove Theorem 6.5.

Proof of Theorem 6.5. Suppose that $\phi : E \to E$ is a Hermitian operator and T is of the form described as (1.14) in the statement of Theorem 6.5; for any $F \in C^1([0,1], E)$ and $x \in [0,1]$,

$$(TF)(x) = \phi(F(x)).$$

To prove that T is a Hermitian, we apply the fact that T is a Hermitian if and only if e^{itT} is a surjective isometry for every $t \in \mathbb{R}$, see [**30**, Theorem 5.2.6]. Let $t \in \mathbb{R}$. By the definition of T, we have

$$e^{itT} F(x) = e^{it\phi}(F(x))$$

for any $F \in C^1([0,1], E)$ and $x \in X$. Since ϕ is Hermitian on E, we have

$$\|e^{itT} F(x)\|_E = \|e^{it\phi}(F(x))\|_E = \|F(x)\|_E$$

and

$$\|e^{itT} F'(x)\|_E = \|e^{it\phi}(F'(x))\|_E = \|F'(x)\|_E,$$

for any $x \in [0,1]$. It follows that

$$\|e^{itT}F\|_\infty = \|F\|_\infty$$

and

$$\|e^{itT}F'\|_\infty = \|F'\|_\infty$$

Thus we deduce

$$\|e^{itT}F\|_\Sigma = \|F\|_\Sigma$$

for any $F \in C^1([0,1],E)$. This implies that e^{itT} is a surjective isometry for every $t \in \mathbb{R}$. Thus we have T is a Hermitian operator.

We now prove the converse. Suppose that $T : C^1([0,1],E) \to C^1([0,1],E)$ is a Hermitian operator. Let ϕ be the operator defined by (1.19). Since above argument, we can define a Hermitian operator $T_0 : C^1([0,1],E) \to C^1([0,1],E)$ by

$$(T_0F)(x) = (TF)(x) - \phi(F(x))$$

for all $F \in C^1([0,1],E)$ and $x \in [0,1]$. Let $e \in E$ with $\|e\|_E = 1$. We define a linear map $S_e : C^1([0,1]) \to C^1([0,1])$ by

$$S_e(f)(x) = [T_0(f \otimes e)(x), e]_E, \quad f \in C^1([0,1]),\ x \in [0,1],$$

where $[\cdot,\cdot]_E$ is a semi-inner product on E compatible with the norm which satisfies $[e_1, \lambda e_2]_E = \bar{\lambda}[e_1, e_2]_E$ for any $e_i \in E$ and $\lambda \in \mathbb{C}$We get

$$|S_e(f)(x)| = |[T_0(f \otimes e)(x), e]_E| \leq \|T_0(f \otimes e)(x)\|_E$$

and

$$|(S_e(f))'(x)| = |[T_0((f\otimes e)')(x), e]_E| \leq \|T_0(f\otimes e)'(x)\|_E \leq \|(Tb(f\otimes e))'\|_\infty$$

for any $f \in C^1([0,1])$ and $x \in [0,1]$. Thus we deduce

$$\begin{aligned}\|S_e(f)\|_\Sigma &= \|S_e(f)\|_\infty + \|(S_e(f))'\|_\infty \\ &\leq \|T_0(f \otimes e)\|_\infty + \|T_0(f \otimes e)'\|_\infty = \|T_0(f \otimes e)\|_\Sigma \\ &\leq \|T_0\|\|f \otimes e\|_\Sigma = \|T_0\|\|f\|_\Sigma\end{aligned}$$

for each $f \in C^1([0,1])$. It follows that $\|S_e\| \leq \|T_0\|$, i.e., S_e is a bounded operator on $C^1([0,1])$. Let $t \in \mathbb{R}$. We have

$$\begin{aligned}(I + itS_e)(\mathbf{1})(x) &= \mathbf{1} + it[T_0(\mathbf{1} \otimes e)(x), e]_E \\ &= \mathbf{1} + it[T(\mathbf{1} \otimes e)(x) - \phi((\mathbf{1} \otimes e)(x)), e]_E = 1 + it[0, e]_E = 1\end{aligned}$$

for any $x \in X$. This implies that

$$1 \leq \|I + itS_e\|. \tag{1.21}$$

On the other hand, let $f \in C^1([0,1])$. We obtain for any $x, y \in X$,

$$\begin{aligned}
&|(I + itS_e)(f)(x)| = |f(x) + it[T_0(f \otimes e)(x), e]_E| \\
&= |f(x)[e,e]_E + it[T_0(f \otimes e)(x), e]_E| = |[(f \otimes e)(x) + itT_0(f \otimes e)(x), e]_E| \\
&\qquad \leq \|(f \otimes e + itT_0(f \otimes e))(x)\|_E \leq \|(I + itT_0)(f \otimes e)\|_\infty,
\end{aligned}$$

for any $x \in [0,1]$. In addition, we have for any $x, y \in [0,1]$

$$\begin{aligned}
&|(I + itS_e)(f)(x) - (I + itS_e)(f)(y)| \\
&\quad = \ |f(x)[e,e]_E + it[T_0(f \otimes e)(x), e]_E - f(y)[e,e]_E - it[T_0(f \otimes e)(y), e]_E| \\
&\quad = \ |[(f \otimes e)(x) + itT_0(f \otimes e)(x), e]_E - [(f \otimes e)(y) + itT_0(f \otimes e)(y), e]_E| \\
&\quad = \ |[(I + itT_0)(f \otimes e)(x) - (I + itT_0)(f \otimes e)(y), e]_E| \\
&\quad \leq \ \|(I + itT_0)(f \otimes e)(x) - (I + itT_0)(f \otimes e)(y)\|_E.
\end{aligned}$$

Thus we obtain

$$\|(I + itS_e)(f)'\|_\infty \leq \|(I + itT_0)(f \otimes e)'\|_\infty.$$

Therefore, we get

$$\begin{aligned}
\|(I + itS_e)(f)\|_\Sigma &= \|(I + itS_e)(f)\|_\infty + \|(I + itS_e)(f)'\|_\infty \\
&\leq \|(I + itT_0)(f \otimes e)\|_\infty + \|(I + itT_0)(f \otimes e)'\|_\infty \\
&= \|(I + itT_0)(f \otimes e)\|_\Sigma \\
&\leq \|I + itT_0\| \|f \otimes e\|_\Sigma = \|I + itT_0\| \|f\|_\Sigma
\end{aligned}$$

for any $f \in C^1([0,1])$. We conclude that

$$\|I + itS_e\| \leq \|I + itT_0\|. \tag{1.22}$$

Since T_0 is a Hermitian operator on $C^1([0,1], E)$, we have

$$\|I + itT_0\| = 1 + o(t)$$

by [**30**, Theorem 5.2.6]. Combining equation (1.21) with equation (1.22), we see that

$$1 \leq \|I + itS_e\| \leq \|I + itT_0\| = 1 + o(t).$$

This implies that $S_e : C^1([0,1]) \to C^1([0,1])$ is a Hermitian operator. It is easy to see that S_e is a real multiple of the identity. Since $S_e(\mathbf{1})(x) = [T_0(\mathbf{1} \otimes e)(x), e] = 0$, we deduce

$$S_e(f)(x) = 0f(x) = 0 \quad f \in C^1([0,1]), \quad x \in [0,1].$$

Therefore, we have

$$[T_0(f \otimes e)(x), e]_E = 0$$

for all $f \in C^1([0,1])$ and $x \in [0,1]$. As $e \in E$ with $\|e\|_E = 1$ is arbitrary, we obtain

$$[T_0(f \otimes e)(x), e]_E = 0, \quad e \in E, \quad f \in C^1([0,1]), \quad x \in [0,1]. \tag{1.23}$$

We now prove that $T_0 = 0$. Let $f \in C^1([0,1])$ and $x \in [0,1]$. Then we define a map $S_{fx} : E \to E$ such that

$$S_{fx}(e) = T_0(f \otimes e)(x)$$

for any $e \in E$. It is easy to check that S_{fx} is linear because of linearity of T_0. In addition, we have

$$\begin{aligned} \|S_{fx}(e)\|_E &= \|T_0(f \otimes e)(x)\|_E \\ &\leq \|T_0(f \otimes e)\|_\Sigma \leq \|T_0\|\|f \otimes e\|_\Sigma = \|T_0\|\|f\|_\Sigma\|e\|_E \end{aligned}$$

for any $e \in E$. We deduce that S_{fx} is a bounded operator. Moreover, by (1.23) we have

$$[S_{fx}(e), e]_E = [T_0(f \otimes e)(x), e]_E = 0$$

for all $e \in E$. Applying [**76**, Theorem 5], we have

$$T_0(f \otimes e)(x) = S_{fx}(e) = 0, \quad e \in E.$$

As $f \in C^1([0,1])$ and $x \in [0,1]$ be chosen arbitrary, we conclude

$$T_0(f \otimes e)(x) = 0 \tag{1.24}$$

for any $f \in C^1([0,1])$, $e \in E$ and $x \in [0,1]$. Suppose that $\dim E = n$. Then, by Lemma 7, for any $F \in C^1([0,1], E)$, F is represented by the following;

$$F = \sum_{k=1}^{n} f_k \otimes e_k$$

with some $f_k \in C^1([0,1])$ and $e_k \in E$ for $k = 1, \dots, n$. It follows by (1.24) that

$$(T_0F)(x) = T_0(\sum_{k=1}^{n} f_k \otimes e_k)(x) = \sum_{k=1}^{n} T_0(f_k \otimes e_k)(x) = 0 \tag{1.25}$$

for any $x \in [0,1]$. We recall the definition of the Hermitian operator $T_0 : C^1([0,1], E) \to C^1([0,1], E)$ is defined by

$$(T_0F)(x) = (TF)(x) - \phi(F(x)), \quad x \in X$$

for every $F \in C^1([0,1], E)$. Applying (1.25), we have

$$(TF)(x) - \phi(F(x)) = 0,$$

that is,

$$(TF)(x) = \phi(F(x))$$

for any $F \in C^1([0,1], E)$ and $x \in [0,1]$. □

2. Surjective linear isometries between Banach algebras with the values in $M_n(\mathbb{C})$

We denote the Banach algebra of complex matrices of order n by $M_n(\mathbb{C})$. The metric we consider on $M_n(\mathbb{C})$ is the metric inherited from the operator norm (spectral norm). In this section we study surjective isometries with respect to the norm $\|\cdot\|_L$ between Banach algebras of Lipschitz maps that take values in $M_n(\mathbb{C})$.

We employ Lumer's method involving the notion of Hermitian operators. Theorem 6.2 implies that Hermitian operators on $\mathrm{Lip}(X, M_n(\mathbb{C}))$ are characterized via Hermitian operators on $M_n(\mathbb{C})$. We say that a bounded operator D on a unital C^*-algebra $\mathcal{A}$ is a $*$-derivation if

$$\begin{aligned} D(ab) &= D(a)b + aD(b), \\ D(a^*) &= D(a)^* \end{aligned} \tag{2.1}$$

for every pair $a, b \in \mathcal{A}$. Note that the definition of $*$-derivation on a unital C^*-algebra in [**113**] differs from the above definition (2.1). In fact, due to the definition of Sinclair in [**113**] a bounded operator δ on $\mathcal{A}$ is a $*$-derivation if $\delta(ab) = \delta(a)b + a\delta(b)$ and $\delta(a^*) = -\delta(a)^*$ hold for every pair $a, b \in \mathcal{A}$. Hence a bounded operator D on $\mathcal{A}$ is $*$-derivation in this dissertation if and only if iD is a $*$-derivation in the sense of Sinclair.

For each $a \in \mathcal{A}$, a left multiplication operator $M_a : \mathcal{A} \to \mathcal{A}$ is defined by $M_a b = ab$ for every $b \in \mathcal{A}$.

THEOREM 6.7 (Sinclair [**113**]). *Let $\mathcal{A}$ be a unital C^*-algebra. A bounded operator T on $\mathcal{A}$ is a Hermitian operator if and only if there exist a Hermitian element $h \in \mathcal{A}$ and a $*$-derivation D on $\mathcal{A}$ such that $T = M_h + iD$.*

It is well known that an operator $D : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is a $*$-derivation if and only if there exists $B \in M_n(\mathbb{C})$ with $B^* = -B$ such that

$$D(A) = BA - AB, \quad A \in M_n(\mathbb{C}).$$

Hence we deduce the following characterization of Hermitian operators on $M_n(\mathbb{C})$. We denote the set of all Hermitian matrices of $M_n(\mathbb{C})$ by $\mathrm{Her}(M_n(\mathbb{C}))$.

THEOREM 6.8. *A linear operator T on $M_n(\mathbb{C})$ is a Hermitian operator if and only if there exist $H \in \mathrm{Her}(M_n(\mathbb{C}))$ and a $*$-derivation $D : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ such that*

$$T(A) = M_H(A) + iD(A), \quad A \in M_n(\mathbb{C}).$$

In particular, there exists $B \in M_n(\mathbb{C})$ with $B^* = -B$ such that

$$D(A) = BA - AB, \quad A \in M_n(\mathbb{C}).$$

2.1. Surjective linear isometries on $\mathrm{Lip}(X, M_n(\mathbb{C}))$. The following is the main result in this subsection.

THEOREM 6.9. *Let X_j be a compact metric space for $j = 1, 2$. Then $U : \mathrm{Lip}(X_1, M_n(\mathbb{C})) \to \mathrm{Lip}(X_2, M_n(\mathbb{C}))$ is a surjective linear isometry such that $U(\mathbf{1}) = \mathbf{1}$ if and only if there exists a unitary matrix $V \in M_n(\mathbb{C})$, and a surjective isometry $\varphi : X_2 \to X_1$, such that*

$$(UF)(x) = VF(\varphi(x))V^{-1}, \quad F \in \mathrm{Lip}(X_1, M_n(\mathbb{C})),\ x \in X_2$$

or

$$(UF)(x) = VF^t(\varphi(x))V^{-1}, \quad F \in \mathrm{Lip}(X_1, M_n(\mathbb{C})),\ x \in X_2,$$

where $F^t(y)$ *denote transpose of* $F(y)$ *for* $y \in X_1$.

One of Schur's theorem asserts that a map $U : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is a surjective isometry if and only if there exist unitary matrices V_1 and V_2 such that either $U(A) = V_1 A V_2$ for every $A \in M_n(\mathbb{C})$, or $U(A) = V_1 A^t V_2$, for every $A \in M_n(\mathbb{C})$ holds (see[**109**]). In order to prove Theorem 6.9 we apply Schur's theorem and several lemmas, we assume that X denotes a compact metric space.

DEFINITION 6.10. For any $H \in \mathrm{Her}(M_n(\mathbb{C}))$, we define a multiplication operator $M_{\mathbf{1}\otimes H} : \mathrm{Lip}(X, M_n(\mathbb{C})) \to \mathrm{Lip}(X, M_n(\mathbb{C}))$ by

$$M_{\mathbf{1}\otimes H}(F) = (\mathbf{1} \otimes H)F, \quad F \in \mathrm{Lip}(X, M_n(\mathbb{C})).$$

For any $*$-derivation $D : M_n(\mathbb{C}) \to M_n(\mathbb{C})$, we define a map $\widehat{D}$: $\mathrm{Lip}(X, M_n(\mathbb{C})) \to \mathrm{Lip}(X, M_n(\mathbb{C}))$ by

$$\widehat{D}(F)(x) = D(F(x)), \quad F \in \mathrm{Lip}(X, M_n(\mathbb{C})), \quad x \in X.$$

Using Theorem 6.2 and Theorem 6.8, we prove the following characterization of Hermitian operators on $\mathrm{Lip}(X, M_n(\mathbb{C}))$.

LEMMA 6.11. *Suppose that* $T : \mathrm{Lip}(X, M_n(\mathbb{C})) \to \mathrm{Lip}(X, M_n(\mathbb{C}))$ *is a map. Then* T *is a Hermitian operator if and only if there exists* $H \in \mathrm{Her}(M_n(\mathbb{C}))$ *and a* $*$*-derivation* D *on* $M_n(\mathbb{C})$ *such that*

$$T = M_{\mathbf{1}\otimes H} + i\widehat{D}. \tag{2.2}$$

PROOF. Suppose that T is a Hermitian operator on $\mathrm{Lip}(X, M_n(\mathbb{C}))$. Theorem 6.2 implies the existence of a Hermitian operator $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ such that

$$TF(x) = \phi(F(x)), \quad F \in \mathrm{Lip}(X, M_n(\mathbb{C})), \quad x \in X.$$

In addition, by Theorem 6.8, there exist $H \in \mathrm{Her}(M_n(\mathbb{C}))$ and $*$-derivation D on $M_n(\mathbb{C})$ such that

$$\phi(A) = M_H(A) + iD(A), \quad A \in M_n(\mathbb{C}).$$

By Definition 6.10, this implies for any $F \in \mathrm{Lip}(X, M_n(\mathbb{C}))$, we have

$$\begin{aligned}(TF)(x) = \phi(F(x)) &= (M_H + iD)(F(x)) \\ &= M_H(F(x)) + iD(F(x)) \\ &= H(F(x)) + i\widehat{D}(F)(x) = (M_{\mathbf{1}\otimes H} + i\widehat{D})(F)(x).\end{aligned}$$

Thus we conclude that

$$T = M_{\mathbf{1}\otimes H} + i\widehat{D}.$$

We prove that the converse holds. Suppose that an operator T satisfies the condition (2.7). Then, we get

$$TF(x) = (M_{\mathbf{1}\otimes H} + i\widehat{D})(F)(x) = (M_H + iD)(F(x))$$

for all $F \in \mathrm{Lip}(X, M_n(\mathbb{C}))$ and $x \in X$. Applying Theorem 6.8, $M_H + iD$ is a Hermitian operator on $M_n(\mathbb{C})$. By Theorem 6.2, we conclude that T is a Hermitian operator on $\mathrm{Lip}(X, M_n(\mathbb{C}))$. $\square$

By the definition of a Hermitian operator we immediately get the following proposition. A proof is omitted.

PROPOSITION 6.12. *Let $\mathcal{B}_j$ be a Banach algebra for $j = 1, 2$. Suppose that U is a surjective linear isometry from $\mathcal{B}_1$ onto $\mathcal{B}_2$ and T is a Hermitian operator on $\mathcal{B}_1$. Then the map UTU^{-1} is a Hermitian operator on $\mathcal{B}_2$.*

LEMMA 6.13. *For any $H \in \mathrm{Her}(M_n(\mathbb{C}))$, there exists $H_0 \in \mathrm{Her}(M_n(\mathbb{C}))$ such that*

$$U(\mathbf{1} \otimes H) = \mathbf{1} \otimes H_0.$$

In particular, if $H = \mathbf{1}$, the identity matrix, then $H_0 = \mathbf{1}$.

PROOF. Let $H \in \mathrm{Her}(M_n(\mathbb{C}))$. Lemma 6.11 shows that $M_{\mathbf{1}\otimes H}$ is a Hermitian operator on $\mathrm{Lip}(X_1, M_n(\mathbb{C}))$. By Proposition 6.12, we have that $UM_{\mathbf{1}\otimes H}U^{-1}$ is a Hermitian operator on $\mathrm{Lip}(X_2, M_n(\mathbb{C}))$. Lemma 6.11 implies the existence of $H_0 \in \mathrm{Her}(M_n(\mathbb{C}))$ and a $*$-derivation D on $M_n(\mathbb{C})$ such that

$$(UM_{\mathbf{1}\otimes H}U^{-1})(F)(x) = (M_{\mathbf{1}\otimes H_0} + i\widehat{D})(F)(x) = H_0(F(x)) + iD(F(x))$$

for all $F \in \mathrm{Lip}(X_2, M_n(\mathbb{C}))$ and $x \in X_2$. In particular, when $F = \mathbf{1}$, we get

$$(UM_{\mathbf{1}\otimes H}U^{-1})(\mathbf{1})(x) = (UM_{\mathbf{1}\otimes H}\mathbf{1})(x) = U(\mathbf{1} \otimes H)(x)$$

and

$$H_0(\mathbf{1}(x)) + iD(\mathbf{1}(x)) = H_0 + i0 = H_0.$$

Thus, we have

$$U(\mathbf{1}\otimes H)(x) = H_0, \quad x \in X_2.$$

Hence $U(\mathbf{1}\otimes H) = \mathbf{1}\otimes H_0$. In particular, if $H = \mathbf{1}$, then we have $H_0 = \mathbf{1}$ by the hypothesis $U(\mathbf{1}) = \mathbf{1}$. This completes the proof. □

By Lemma 6.13, we define a map $\psi_0 : \mathrm{Her}(M_n(\mathbb{C})) \to \mathrm{Her}(M_n(\mathbb{C}))$ by

$$U(\mathbf{1}\otimes H) = \mathbf{1}\otimes \psi_0(H).$$

LEMMA 6.14. *The map ψ_0 is a real linear isometry from $\mathrm{Her}(M_n(\mathbb{C}))$ onto itself such that $\psi_0(\mathbf{1}) = \mathbf{1}$.*

PROOF. For any $H_1 \in \mathrm{Her}(M_n(\mathbb{C}))$, we have that $U^{-1}M_{\mathbf{1}\otimes H_1}U$ is a Hermitian operator on $\mathrm{Lip}(X_1, M_n(\mathbb{C}))$. Lemma 6.11 implies the existence of $H_2 \in \mathrm{Her}(M_n(\mathbb{C}))$ and a $*$-derivation D_0 on $M_n(\mathbb{C})$ such that

$$U^{-1}M_{\mathbf{1}\otimes H_1}U = M_{\mathbf{1}\otimes H_2} + i\widehat{D_0}.$$

Hence we get $M_{\mathbf{1}\otimes H_2} = U^{-1}M_{\mathbf{1}\otimes H_1}U - i\widehat{D_0}$. Then we obtain

$$\begin{aligned} UM_{\mathbf{1}\otimes H_2}U^{-1}(\mathbf{1}) &= U(U^{-1}M_{\mathbf{1}\otimes H_1}U - i\widehat{D_0})U^{-1}(\mathbf{1}) \\ &= M_{\mathbf{1}\otimes H_1}(\mathbf{1}) - U(i\widehat{D_0}(\mathbf{1})) \\ &= \mathbf{1}\otimes H_1 - iU(0) \\ &= \mathbf{1}\otimes H_1. \end{aligned}$$

It follows that $U(\mathbf{1}\otimes H_2) = \mathbf{1}\otimes H_1$, and we have $\psi_0(H_2) = H_1$. As $H_1 \in \mathrm{Her}(M_n(\mathbb{C}))$ is arbitrary, we get that ψ_0 is surjective.

We prove that ψ_0 is an isometry. For any $H \in \mathrm{Her}(M_n(\mathbb{C}))$, we get

$$\begin{aligned} \|\psi_0(H)\|_{M_n(\mathbb{C})} = \|\mathbf{1}\otimes\psi_0(H)\|_\Sigma &= \|U(\mathbf{1}\otimes H)\|_\Sigma \\ &= \|\mathbf{1}\otimes H\|_\Sigma = \|H\|_{M_n(\mathbb{C})}. \end{aligned}$$

Thus, we have ψ_0 is an isometry. By the definition of ψ_0, we infer that $\psi_0(\mathbf{1}) = \mathbf{1}$. By a simple calculation, we see that ψ_0 is real linear. □

For any $A \in M_n(\mathbb{C})$, there exists $H_1, H_2 \in \mathrm{Her}(M_n(\mathbb{C}))$ such that $A = H_1 + iH_2$. Applying this decomposition, we define a map $\psi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ as follows;

$$\psi(A) = \psi(H_1 + iH_2) = \psi_0(H_1) + i\psi_0(H_2).$$

Due to the definition for ψ, we have

$$\begin{aligned} U(\mathbf{1}\otimes A) &= U(\mathbf{1}\otimes (H_1+iH_2)) \\ &= U(\mathbf{1}\otimes H_1)+iU(\mathbf{1}\otimes H_2) \\ &= \mathbf{1}\otimes \psi_0(H_1)+i\mathbf{1}\otimes\psi_0(H_2) \\ &= \mathbf{1}\otimes(\psi_0(H_1)+i\psi_0(H_2)) \\ &= \mathbf{1}\otimes\psi(A), \end{aligned} \tag{2.3}$$

for any $A\in M_n(\mathbb{C})$.

LEMMA 6.15. *The map ψ is a complex linear isometry from $M_n(\mathbb{C})$ onto itself such that $\psi(\mathbf{1})=\mathbf{1}$. There exists a unitary matrix $V\in M_n(\mathbb{C})$ such that*

$$\psi(A)=VAV^{-1},\quad A\in M_n(\mathbb{C})$$

or

$$\psi(A)=VA^tV^{-1},\quad A\in M_n(\mathbb{C}).$$

PROOF. The equation (2.3) and Lemma 6.14 imply that ψ is a complex linear isometry with $\psi(\mathbf{1})=\mathbf{1}$. We show that ψ is surjective. For any $A\in M_n(\mathbb{C})$, there exists $H_1,\ H_2\in \mathrm{Her}(M_n(\mathbb{C}))$ such that $A=H_1+iH_2$. By Lemma 6.14, $\psi_0:\mathrm{Her}(M_n(\mathbb{C}))\to\mathrm{Her}(M_n(\mathbb{C}))$ is surjective, there exist $H_1',H_2'\in\mathrm{Her}(M_n(\mathbb{C}))$ such that $\psi_0(H_1')=H_1$ and $\psi_0(H_2')=H_2$. We define $A'=H_1'+iH_2'\in M_n(\mathbb{C})$. This implies that

$$\psi(A')=\psi_0(H_1')+i\psi_0(H_2')=H_1+iH_2=A.$$

As $A\in M_n(\mathbb{C})$ is arbitrary, we have that ψ is surjective. Applying Schur's theorem in [**109**], there exist unitary matrices V and W such that

$$\psi(A)=VAW,\quad A\in M_n(\mathbb{C})$$

or

$$\psi(A)=VA^tW,\quad A\in M_n(\mathbb{C}).$$

As $\psi(\mathbf{1})=\mathbf{1}$ we have that $W=V^{-1}$, hence we get the desired formulae. □

LEMMA 6.16. *For any* $f \in \mathrm{Lip}(X_1)$*, there exists* $g \in \mathrm{Lip}(X_2)$ *such that*

$$U(f \otimes \mathbf{1}) = g \otimes \mathbf{1}$$

PROOF. For any $B \in M_n(\mathbb{C})$ with $B^* = -B$, we define a $*$-derivation D on $M_n(\mathbb{C})$ by

$$D(A) = BA - AB, \quad A \in M_n(\mathbb{C}).$$

Note that Lemma 6.11 shows that the map $i\widehat{D} : \mathrm{Lip}(X_2, M_n(\mathbb{C})) \to \mathrm{Lip}(X_2, M_n(\mathbb{C}))$ defined by

$$(i\widehat{D})(F)(x) = iD(F(x)) \quad F \in \mathrm{Lip}(X_2, M_n(\mathbb{C})), \quad x \in X_2,$$

is a Hermitian operator on $\mathrm{Lip}(X_2, M_n(\mathbb{C}))$. Since the map U is an isometry, $U^{-1}i\widehat{D}U$ is a Hermitian operator. Lemma 6.11 implies the existence of $H \in \mathrm{Her}(M_n(\mathbb{C}))$ and $*$-derivation D' on $M_n(\mathbb{C})$ such that

$$U^{-1}i\widehat{D}U = M_{\mathbf{1}\otimes H} + i\widehat{D'}.$$

In addition, there exists $C \in M_n(\mathbb{C})$ with $C^* = -C$ such that $D'(A) = CA - AC$ for every $A \in M_n(\mathbb{C})$. On the other hand, since we assume that $U(\mathbf{1}) = \mathbf{1}$, we have

$$(U^{-1}i\widehat{D}U)(\mathbf{1}) = i(U^{-1}\widehat{D}U)(\mathbf{1}) = iU^{-1}\widehat{D}(\mathbf{1}) = iU^{-1}(0) = 0.$$

Thus we deduce that

$$\begin{aligned} 0 = (U^{-1}i\widehat{D}U)(\mathbf{1}) &= (M_{\mathbf{1}\otimes H} + i\widehat{D'})(\mathbf{1}) \\ &= \mathbf{1} \otimes H + i\widehat{D'}(\mathbf{1}) = \mathbf{1} \otimes H + i0 = \mathbf{1} \otimes H. \end{aligned}$$

We conclude that $U^{-1}i\widehat{D}U = i\widehat{D'}$. Let $f \in \mathrm{Lip}(X_1)$. We have

$$\begin{aligned} (U^{-1}i\widehat{D}U)(f \otimes \mathbf{1})(x) &= i\widehat{D'}(f \otimes \mathbf{1})(x) \\ &= iD'(f(x)\mathbf{1}) = i(Cf(x)\mathbf{1} - f(x)\mathbf{1}C) \\ &= i(f(x)C - f(x)C)\mathbf{1} = 0 \end{aligned}$$

for any $x \in X_1$. This implies that

$$(U^{-1}i\widehat{D}U)(f \otimes \mathbf{1}) = 0. \tag{2.4}$$

Note that for any $x \in X_2$,

$$\widehat{D}(U(f \otimes \mathbf{1}))(x) = D(U(f \otimes \mathbf{1})(x)) = BU(f \otimes \mathbf{1})(x) - U(f \otimes \mathbf{1})(x)B.$$

Thus, we have

$$\widehat{D}(U(f\otimes\mathbf{1})) = \mathbf{1}\otimes BU(f\otimes\mathbf{1}) - U(f\otimes\mathbf{1})\mathbf{1}\otimes B.$$

Therefore, we get

$$\begin{aligned}(U^{-1}i\widehat{D}U)(f\otimes\mathbf{1}) &= U^{-1}(i\widehat{D}U(f\otimes\mathbf{1}))\\ &= iU^{-1}(\mathbf{1}\otimes BU(f\otimes\mathbf{1}) - U(f\otimes\mathbf{1})\mathbf{1}\otimes B).\end{aligned}\tag{2.5}$$

Combining equation (2.4) with equation (2.5), we infer that

$$U^{-1}(\mathbf{1}\otimes BU(f\otimes\mathbf{1}) - U(f\otimes\mathbf{1})\mathbf{1}\otimes B) = 0.$$

As U^{-1} is injective, we deduce that

$$\mathbf{1}\otimes BU(f\otimes\mathbf{1}) = U(f\otimes\mathbf{1})\mathbf{1}\otimes B.\tag{2.6}$$

Notice that $B\in M_n(\mathbb{C})$ with $B^* = -B$ is arbitrary. For any $A\in M_n(\mathbb{C})$, there exist $B_1, B_2\in M_n(\mathbb{C})$ which satisfy $B_k^* = -B_k$ for $k = 1,2$ and $A = -iB_1 + B_2$. Then we have by the equation (2.6) that

$$\begin{aligned}\mathbf{1}\otimes AU(f\otimes\mathbf{1}) &= \mathbf{1}\otimes(-iB_1+B_2)U(f\otimes\mathbf{1})\\ &= -i\mathbf{1}\otimes B_1U(f\otimes\mathbf{1}) + \mathbf{1}\otimes B_2U(f\otimes\mathbf{1})\\ &= -iU(f\otimes\mathbf{1})\mathbf{1}\otimes B_1 + U(f\otimes\mathbf{1})\mathbf{1}\otimes B_2\\ &= U(f\otimes\mathbf{1})\mathbf{1}\otimes(-iB_1+B_2) = U(f\otimes\mathbf{1})\mathbf{1}\otimes A.\end{aligned}$$

We infer that for every $x\in X_2$, we have

$$AU(f\otimes\mathbf{1})(x) = U(f\otimes\mathbf{1})(x)A$$

for any $A\in M_n(\mathbb{C})$. Since $U(f\otimes\mathbf{1})(x)$ is commutative with any matrices, we have $U(f\otimes\mathbf{1})(x)$ is a scalar multiple of the identity matrix. It follows that there exists $g(x)\in\mathbb{C}$ such that

$$U(f\otimes\mathbf{1})(x) = g(x)\mathbf{1}.$$

Since $U(f\otimes\mathbf{1})\in \mathrm{Lip}(X_2, M_n(\mathbb{C}))$, we get $g\in\mathrm{Lip}(X_2)$ and

$$U(f\otimes\mathbf{1}) = g\otimes\mathbf{1}.$$

This completes the proof. □

LEMMA 6.17. *There exists a surjective isometry $\varphi : X_2\to X_1$ such that*

$$U(f\otimes\mathbf{1})(x) = f(\varphi(x))\otimes\mathbf{1}$$

for all $f\in\mathrm{Lip}(X_1)$ and $x\in X_2$.

Proof. By Lemma 6.16, we define a map $P_U : \mathrm{Lip}(X_1) \to \mathrm{Lip}(X_2)$ by

$$U(f \otimes \mathbf{1}) = P_U(f) \otimes \mathbf{1}, \quad f \in \mathrm{Lip}(X_1).$$

Let $g \in \mathrm{Lip}(X_2)$. Applying a similar argument to Lemma 6.16 for U^{-1} instead of U, there exists $f \in \mathrm{Lip}(X_1)$ such that $U^{-1}(g \otimes \mathbf{1}) = f \otimes \mathbf{1}$. Since

$$P_U(f) \otimes \mathbf{1} = U(f \otimes \mathbf{1}) = U(U^{-1}(g \otimes \mathbf{1})) = g \otimes \mathbf{1},$$

we have $P_U(f) = g$. Thus we have P_U is surjective. In addition, we get

$$\|P_U(f)\|_\Sigma = \|U(f \otimes \mathbf{1})\|_\Sigma = \|f \otimes \mathbf{1}\|_\Sigma = \|f\|_\Sigma,$$

for all $f \in \mathrm{Lip}(X_1)$. It is easy to see that P_U is complex linear. Hence we conclude that P_U is a linear isometry from $\mathrm{Lip}(X_1)$ onto $\mathrm{Lip}(X_2)$. When $X_1 = X_2$, Theorem 2.1 in [**16**] asserts that there exists a surjective isometry $\varphi : X_2 \to X_1$ such that $P_U(f) = f \circ \varphi$ for every $f \in \mathrm{Lip}(X_1)$. For the convenience of the readers, we exhibit a proof for the general case which is a little bit different from the case of $X_1 = X_2$. As is pointed out in [**16**], the algebra $\mathrm{Lip}(X_j)$ is a *regular subspace* of $C(X_j)$, and the norm $\|\cdot\|_L$ is a *natural norm* in the sense of Jarosz [**47**]. Then by Theorem in [**47**], P_U is a surjective isometry from $(\mathrm{Lip}(X_1), \|\cdot\|_\infty)$ onto $(\mathrm{Lip}(X_2), \|\cdot\|_\infty)$. The Stone-Weierstrass theorem asserts that $\mathrm{Lip}(X_j)$ is uniformly dense in $C(X_j)$. Then P_U is extended to a surjective isometry $\overline{P_U}$ from $C(X_1)$ onto $C(X_2)$. Then by the Banach-Stone theorem, there exists a homeomorphism $\varphi : X_2 \to X_1$ such that $\overline{P_U}(f) = f \circ \varphi$, $f \in C(X_1)$. Hence we have that $P_U(f) = f \circ \varphi$ for every $f \in \mathrm{Lip}(X_1)$. The rest is a routine argument to prove that φ is an isometry since P_U preserves two norms $\|\cdot\|_L$ and $\|\cdot\|_\infty$ respectively (see the proof of [**16**, Theorem 2.1]). A proof is rather simple by applying [**48**, Example 8], while the statement is just confirmed by Corollary 4.15. Thus we obtain that

$$U(f \otimes \mathbf{1})(x) = P_U(f)(x) \otimes \mathbf{1} = f(\varphi(x)) \otimes \mathbf{1}, \quad f \in \mathrm{Lip}(X_1),\ x \in X_2.$$

□

We now give a proof of Theorem 6.9.

Proof of Theorem 6.9. A proof of the sufficient part of Theorem 6.9 is rather simple and is omitted. We prove the converse implication. For any $H \in \mathrm{Her}(M_n(\mathbb{C}))$, there exists $\psi_0(H) \in \mathrm{Her}(M_n(\mathbb{C}))$ and $*$-derivation D on $M_n(\mathbb{C})$ such that

$$UM_{\mathbf{1}\otimes H}U^{-1} = M_{\mathbf{1}\otimes\psi_0(H)} + i\widehat{D}.$$

Let $f \in \mathrm{Lip}(X_1)$. By Lemma 6.17, we see that $U(f \otimes \mathbf{1})(x) \in \mathbb{C}\mathbf{1}$ for every $x \in X_2$. Thus we have that

$$\begin{aligned} U(f \otimes H)(x) &= U(M_{\mathbf{1}\otimes H}(f \otimes \mathbf{1}))(x) = UM_{\mathbf{1}\otimes H}U^{-1}U(f \otimes \mathbf{1})(x) \\ &= (M_{\mathbf{1}\otimes\psi_0(H)} + i\widehat{D})(U(f \otimes \mathbf{1}))(x) \\ &= M_{\mathbf{1}\otimes\psi_0(H)}(U(f \otimes \mathbf{1}))(x) + i\widehat{D}(U(f \otimes \mathbf{1}))(x) \\ &= \psi_0(H)(U(f \otimes \mathbf{1})(x)) = f(\varphi(x))\psi_0(H) \end{aligned}$$

for any $x \in X_2$. For any $A \in M_n(\mathbb{C})$, there exist $H_1, H_2 \in \mathrm{Her}(M_n(\mathbb{C}))$ such that $A = H_1 + iH_2$ and we get

$$\begin{aligned} U(f \otimes A)(x) &= U(f \otimes (H_1 + iH_2))(x) \\ &= U(f \otimes H_1)(x) + iU(f \otimes H_2)(x) \\ &= f(\varphi(x))\psi_0(H_1) + if(\varphi(x))\psi_0(H_2) \\ &= f(\varphi(x))\psi(A) \\ &= \psi((f \otimes A)(\varphi(x))) \end{aligned}$$

for any $f \in \mathrm{Lip}(X_1)$ and any $x \in X_2$. By Lemma 7, for every $F \in \mathrm{Lip}(X_1, M_n(\mathbb{C}))$, F is represented by $F = \sum_{k=1}^{m} f_k \otimes A_k$ with some $f_k \in \mathrm{Lip}(X_1)$ and $A_k \in M_n(\mathbb{C})$ for $k = 1, \ldots, m$. Thus we deduce that

$$\begin{aligned} U(F)(x) &= U(\sum_{k=1}^{m} f_k \otimes A_k)(x) \\ &= \sum_{k=1}^{m} U(f_k \otimes A_k)(x) = \sum_{k=1}^{m} \psi((f_k \otimes A_k)(\varphi(x))) \\ &= \psi(\sum_{k=1}^{m} (f_k \otimes A_k)(\varphi(x))) = \psi(F(\varphi(x))) \end{aligned}$$

for any $x \in X_2$. By Lemma 6.15, this would yield the desired conclusion. □

2.2. Surjective linear isometries on $C^1([0,1], M_n(\mathbb{C}))$. The following is the main result in this subsection.

THEOREM 6.18. *We have $U : C^1([0,1], M_n(\mathbb{C})) \to C^1([0,1], M_n(\mathbb{C}))$ is a surjective linear isometry such that $U(\mathbf{1}) = \mathbf{1}$ if and only if there exists a unitary matrix $V \in M_n(\mathbb{C})$, and a continuous function $\varphi : [0,1] \to [0,1]$, where $\varphi(x) = x$ for any $x \in [0,1]$ or $\varphi(x) = 1 - x$ for any $x \in [0,1]$, such that*

$$(UF)(x) = VF(\varphi(x))V^{-1}, \quad F \in C^1([0,1], M_n(\mathbb{C})),\ x \in [0,1]$$

or

$$(UF)(x) = VF^t(\varphi(x))V^{-1}, \quad F \in C^1([0,1], M_n(\mathbb{C})),\ x \in [0,1],$$

where $F^t(y)$ denote transpose of $F(y)$ for $y \in [0,1]$.

We need some lemmas and Schur's theorem to prove Theorem 6.18. We define the multiplication operator and $\widehat{D}$ on $C^1([0,1], M_n(\mathbb{C}))$.

DEFINITION 6.19. For any $H \in \mathrm{Her}(M_n(\mathbb{C}))$, we define a multiplication operator $M_{\mathbf{1}\otimes H} : C^1([0,1], M_n(\mathbb{C})) \to C^1([0,1], M_n(\mathbb{C}))$ by

$$M_{\mathbf{1}\otimes H}(F) = (\mathbf{1} \otimes H)F, \quad F \in C^1([0,1], M_n(\mathbb{C})).$$

For any $*$-derivation $D : M_n(\mathbb{C}) \to M_n(\mathbb{C})$, we define a map $\widehat{D} : C^1([0,1], M_n(\mathbb{C})) \to C^1([0,1], M_n(\mathbb{C}))$ by

$$\widehat{D}(F)(x) = D(F(x)), \quad F \in C^1([0,1], M_n(\mathbb{C})), \quad x \in X.$$

Using Theorem 6.5 and Theorem 6.8, we also prove the following characterization of Hermitian operators on $C^1([0,1], M_n(\mathbb{C}))$ too.

LEMMA 6.20. *Suppose that $T : C^1([0,1], M_n(\mathbb{C})) \to C^1([0,1], M_n(\mathbb{C}))$ is a map. Then T is a Hermitian operator if and only if there exists $H \in \mathrm{Her}(M_n(\mathbb{C}))$ and a $*$-derivation D on $M_n(\mathbb{C})$ such that*

$$T = M_{\mathbf{1}\otimes H} + i\widehat{D}. \tag{2.7}$$

Now the reader will have no trouble verifying Lemma 6.20, we omit the proof. Recall that by Lemma 6.13, we define a map $\psi_0 : \mathrm{Her}(M_n(\mathbb{C})) \to \mathrm{Her}(M_n(\mathbb{C}))$ by

$$U(\mathbf{1} \otimes H) = \mathbf{1} \otimes \psi_0(H).$$

It is easy to see that the map ψ_0 is a real linear isometry from $\mathrm{Her}(M_n(\mathbb{C}))$ onto itself such that $\psi_0(\mathbf{1}) = \mathbf{1}$ (see Lemma 6.14). In addition, every $A \in M_n(\mathbb{C})$ is represented by $A = H_1 + iH_2$ with $H_1, H_2 \in \mathrm{Her}(M_n(\mathbb{C}))$. Applying this decomposition, we define a map $\psi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ as follows;

$$\psi(A) = \psi(H_1 + iH_2) = \psi_0(H_1) + i\psi_0(H_2).$$

Due to the definition for ψ, we have $U(\mathbf{1} \otimes A) = \mathbf{1} \otimes \psi(A)$ for any $A \in M_n(\mathbb{C})$. By Lemma 6.15, we see the form of ψ.

LEMMA 6.21. *For any $f \in C^1([0,1])$, there exists $g \in C^1([0,1])$ such that*

$$U(f \otimes \mathbf{1}) = g \otimes \mathbf{1}$$

PROOF. For any $B \in M_n(\mathbb{C})$ with $B^* = -B$, we define a $*$-derivation D on $M_n(\mathbb{C})$ by

$$D(A) = BA - AB, \quad A \in M_n(\mathbb{C}).$$

Note that Lemma 6.20 shows that the map $i\widehat{D} : C^1([0,1], M_n(\mathbb{C})) \to C^1([0,1], M_n(\mathbb{C}))$ defined by

$$(i\widehat{D})(F)(x) = iD(F(x)) \quad F \in C^1([0,1], M_n(\mathbb{C})), \quad x \in [0,1],$$

is a Hermitian operator on $C^1([0,1], M_n(\mathbb{C}))$. Since the map U is an isometry, $U^{-1}i\widehat{D}U$ is a Hermitian operator. Lemma 6.20 implies the existence of $H \in \mathrm{Her}(M_n(\mathbb{C}))$ and $*$-derivation D' on $M_n(\mathbb{C})$ such that

$$U^{-1}i\widehat{D}U = M_{\mathbf{1}\otimes H} + i\widehat{D'}.$$

By a similar way with Lemma 6.16, we can prove that for any $f \in C^1([0,1])$ and $x \in [0,1]$,

$$(U^{-1}i\widehat{D}U)(f \otimes \mathbf{1}) = 0. \tag{2.8}$$

We have

$$\widehat{D}(U(f \otimes \mathbf{1})) = \mathbf{1} \otimes BU(f \otimes \mathbf{1}) - U(f \otimes \mathbf{1})\mathbf{1} \otimes B.$$

Therefore, we get

$$\begin{aligned}(U^{-1}i\widehat{D}U)(f \otimes \mathbf{1}) &= U^{-1}(i\widehat{D}U(f \otimes \mathbf{1})) \\ &= iU^{-1}(\mathbf{1} \otimes BU(f \otimes \mathbf{1}) - U(f \otimes \mathbf{1})\mathbf{1} \otimes B).\end{aligned} \tag{2.9}$$

Combining equation (2.8) with equation (2.9), we infer that

$$U^{-1}(\mathbf{1}\otimes BU(f\otimes\mathbf{1}) - U(f\otimes\mathbf{1})\mathbf{1}\otimes B) = 0.$$

As U^{-1} is injective, we deduce that

$$\mathbf{1}\otimes BU(f\otimes\mathbf{1}) = U(f\otimes\mathbf{1})\mathbf{1}\otimes B. \tag{2.10}$$

Notice that $B \in M_n(\mathbb{C})$ with $B^* = -B$ is arbitrary. This implies that for every $x \in [0,1]$, we have

$$AU(f\otimes\mathbf{1})(x) = U(f\otimes\mathbf{1})(x)A$$

for any $A \in M_n(\mathbb{C})$. It follows that there exists $g(x) \in \mathbb{C}$ such that

$$U(f\otimes\mathbf{1})(x) = g(x)\mathbf{1}.$$

Since $U(f\otimes\mathbf{1}) \in C^1([0,1], M_n(\mathbb{C}))$, we get $g \in C^1([0,1])$ and

$$U(f\otimes\mathbf{1}) = g\otimes\mathbf{1}.$$

This completes the proof. □

LEMMA 6.22. *There exists a surjective isometry $\varphi : [0,1] \to [0,1]$, which is $\varphi(x) = x$ for any $x \in [0,1]$ or $\varphi(x) = 1-x$ for any $x \in [0,1]$, such that*

$$U(f\otimes\mathbf{1})(x) = f(\varphi(x))\otimes\mathbf{1}$$

for all $f \in C^1([0,1])$ and $x \in [0,1]$.

PROOF. By Lemma 6.21, we define a map $P_U : C^1([0,1]) \to C^1([0,1])$ by

$$U(f\otimes\mathbf{1}) = P_U(f)\otimes\mathbf{1}, \quad f \in C^1([0,1]).$$

Let $g \in C^1([0,1])$. Applying a similar argument to Lemma 6.21 for U^{-1} instead of U, there exists $f \in C^1([0,1])$ such that $U^{-1}(g\otimes\mathbf{1}) = f\otimes\mathbf{1}$. Since

$$P_U(f)\otimes\mathbf{1} = U(f\otimes\mathbf{1}) = U(U^{-1}(g\otimes\mathbf{1})) = g\otimes\mathbf{1},$$

we have $P_U(f) = g$. Thus we have P_U is surjective. In addition, we get

$$\|P_U(f)\|_\Sigma = \|U(f\otimes\mathbf{1})\|_\Sigma = \|f\otimes\mathbf{1}\|_\Sigma = \|f\|_\Sigma,$$

for all $f \in C^1([0,1])$. It is easy to see that P_U is complex linear. Hence we conclude that P_U is a linear isometry from $C^1([0,1])$ onto $C^1([0,1])$.

Applying [**104**, Theorem 4.1], we get there exists $\varphi : [0,1] \to [0,1]$ with $\varphi(x) = x$ or $\varphi(x) = 1 - x$ such that

$$P_U(f)(x) = f(\varphi(x)), \quad f \in C^1([0,1])$$

Thus we obtain that

$$U(f \otimes \mathbf{1})(x) = P_U(f)(x) \otimes \mathbf{1} = f(\varphi(x)) \otimes \mathbf{1}, \quad f \in C^1([0,1]),\ x \in [0,1].$$

□

We now give a proof of Theorem 6.18.

Proof of Theorem 6.18. We omit a proof of the sufficiency of Theorem 6.18. We prove the converse implication. For any $H \in \mathrm{Her}(M_n(\mathbb{C}))$, there exists $\psi_0(H) \in \mathrm{Her}(M_n(\mathbb{C}))$ and $*$-derivation D on $M_n(\mathbb{C})$ such that

$$UM_{\mathbf{1}\otimes H}U^{-1} = M_{\mathbf{1}\otimes\psi_0(H)} + i\widehat{D}.$$

Let $f \in C^1([0,1])$. By Lemma 6.22, we see that $U(f \otimes \mathbf{1})(x) \in \mathbb{C}\mathbf{1}$ for every $x \in [0,1]$. Thus we have that

$$\begin{aligned}
U(f \otimes H)(x) &= U(M_{\mathbf{1}\otimes H}(f \otimes \mathbf{1}))(x) = UM_{\mathbf{1}\otimes H}U^{-1}U(f \otimes \mathbf{1})(x) \\
&= (M_{\mathbf{1}\otimes\psi_0(H)} + i\widehat{D})(U(f \otimes \mathbf{1}))(x) \\
&= M_{\mathbf{1}\otimes\psi_0(H)}(U(f \otimes \mathbf{1}))(x) + i\widehat{D}(U(f \otimes \mathbf{1}))(x) \\
&= \psi_0(H)(U(f \otimes \mathbf{1})(x)) = f(\varphi(x))\psi_0(H)
\end{aligned}$$

for any $x \in [0,1]$. For any $A \in M_n(\mathbb{C})$, there exist $H_1, H_2 \in \mathrm{Her}(M_n(\mathbb{C}))$ such that $A = H_1 + iH_2$ and we get

$$\begin{aligned}
U(f \otimes A)(x) &= U(f \otimes (H_1 + iH_2))(x) \\
&= U(f \otimes H_1)(x) + iU(f \otimes H_2)(x) \\
&= f(\varphi(x))\psi_0(H_1) + if(\varphi(x))\psi_0(H_2) \\
&= f(\varphi(x))\psi(A) \\
&= \psi((f \otimes A)(\varphi(x)))
\end{aligned}$$

for any $f \in C^1([0,1])$ and any $x \in [0,1]$. By Lemma 6.4, for every $F \in C^1([0,1], M_n(\mathbb{C}))$, F is represented by $F = \sum_{k=1}^m f_k \otimes A_k$ with some $f_k \in C^1([0,1])$ and $A_k \in M_n(\mathbb{C})$ for $k = 1, \ldots, m$. Thus we deduce that

$$\begin{aligned}
U(F)(x) &= U(\sum_{k=1}^{m} f_k \otimes A_k)(x) \\
&= \sum_{k=1}^{m} U(f_k \otimes A_k)(x) = \sum_{k=1}^{m} \psi((f_k \otimes A_k)(\varphi(x))) \\
&= \psi(\sum_{k=1}^{m} (f_k \otimes A_k)(\varphi(x))) = \psi(F(\varphi(x)))
\end{aligned}$$

for any $x \in [0,1]$. By Lemma 6.15, this would yield the desired conclusion. $\square$

CHAPTER 7

Tensor products of uniform algebras and C^*-algebras

1. Preliminary

In this chapter, X is a compact Hausdorff space and E is a complex Banach space. The space of all E-valued continuous maps on X, with the supremum norm, is denoted by $C(X, E)$. Recall that A is a uniform algebra on X if A is a uniformly closed subalgebra of $C(X)$ which separates the points of X and contains constant functions. A uniform algebra A on X is called *natural* if the canonical embedding $x \mapsto \delta_x$ from X into the maximal ideal space $\mathcal{M}(A)$ of A is surjective, where δ_x denotes the point evaluation at x. Hence the maximal ideal space of a natural uniform algebra on X is identified with X itself. For a uniform algebra A, the Gelfand transform $A \to \hat{A} \subset C(\mathcal{M})$ is an isometric algebra isomorphism into $\hat{A}$. Identifying A with its Gelfand transform $\hat{A}$ we may suppose that a uniform algebra is defined on the maximal ideal space. Under this identification, the statement " a uniform algebra on the maximal ideal space is natural" makes sense. The algebraic tensor product of A and E over $\mathbb{C}$ is denoted by $A \otimes E$. The injective tensor product of A and E is denoted by $\overline{A \otimes E}$. The canonical embedding $A \otimes E \subset C(X, E)$ allows us to identify $\overline{A \otimes E}$ with the uniform closure of $A \otimes E$ in $C(X, E)$ in the way that an element $F \in \overline{A \otimes E}$ is regarded as a continuous map $F : X \to E$. By the standard argument using the partition of unity we have $\overline{C(X) \otimes E} = C(X, E)$. Throughout the dissertation, the operator norm of a bounded linear operator from a Banach space into a Banach space is denoted by $\| \cdot \|_{op}$.

For a complex Banach space E with the norm $\| \cdot \|_E$, a complex-valued function $[\cdot, \cdot]_E : E \times E \to \mathbb{C}$ is a semi-inner product compatible with the norm of E, see Definition 1.8.There may be several (actually infinitely many) semi-inner products compatible with the norm as is observed below. For a vector $e \in E$, let

$$\Pi_e = \{e^* \in E^* : \|e^*\|_{op} = \|e\|_E, e^*(e) = \|e\|_E^2\},$$

the set of all dual maps of e. Proposition 1.9 shows it is not empty by the Hahn-Banach theorem. Suppose that a map $\mathfrak{I} : E \to \cup_{e\in E}\Pi_e$ satisfies that $\mathfrak{I}(e) \in \Pi_e$ for every $e \in E$. Then the function $[\cdot,\cdot] : E \times E \to \mathbb{C}$ defined by $[a, e] = [\mathfrak{I}(e)](a)$ for $a \in E$ is a semi-inner product on E compatible with the norm. Such a map $\mathfrak{I}$ exists by the axiom of choice. Conversely, for a semi-inner product $[\cdot,\cdot]_E$ on E compatible with the norm, the functional $e^* : E \to \mathbb{C}$ defined by $e^*(a) = [a, e]_E$ $(a \in E)$ is an element in Π_e.

Recall that $\mathbf{B}(E)$ is the Banach algebra of all bounded linear operators on E with the operator norm and let $[\cdot,\cdot]$ be a semi-inner product on E compatible with the norm. An operator $T \in \mathbf{B}(E)$ is called a *Hermitian* operator if $[T(x), x]$ is real for all x in E. It is important to note that the definition does not depend on the choice of semi-inner product; in fact it is known that, for an operator $T \in \mathbf{B}(E)$, $[T(x), x]_E$ is real for every x in E for a semi-inner product $[\cdot,\cdot]_E$ if and only if $[T(x), x]$ is real for every x in E for *any* semi-inner product $[\cdot,\cdot]$ on E (see [**6**, pp. 5,6]). Since a semi-inner product is a linear functional, we see that an operator $T : E \to E$ is Hermitian if and only if $e^*(T(e))$ is real for every pair $e \in E$ and $e^* \in \Pi_e$.

These notions of semi-inner products and Hermitian operators were introduced by Lumer in [**76**] in the study of isometries on certain Orlicz space [**76, 77, 78**]. The method he applied in [**77, 78**] to describe isometries is now called *Lumer's method*.

Vidav [**115**] called an element a in a unital Banach algebra $\mathfrak{B}$ *Hermitian* if $\|\mathbf{1} + ita\|_{\mathfrak{B}} = 1 + o(t)$ for t real, where $\mathbf{1}$ is the unit of $\mathfrak{B}$. It is known [**30**, Theorem 6.2.1] that $a \in \mathfrak{B}$ is Hermitian if and only if $\|\exp(ita)\|_{\mathfrak{B}} = 1$ for every real number t. The set of all Hermitian elements in $\mathfrak{B}$ is a real Banach space and is denoted by $\mathrm{Her}(\mathfrak{B})$. This implies that $T \in \mathbf{B}(E)$ is a Hermitian operator if and only if T is a Hermitian element in the Banach algebra $\mathbf{B}(E)$ applying Theorem 1.10. Hence the set $\mathrm{Her}(\mathbf{B}(E))$ is precisely the set of all Hermitian operators on E, and an operator $T \in \mathbf{B}(E)$ is Hermitian if and only if $\|I + itT\|_{op} = 1 + o(t), t \in \mathbb{R}$. If $\mathfrak{B}$ is a unital C^*-algebra, then $\mathrm{Her}(\mathfrak{B})$ is the self-adjoint part of $\mathfrak{B}$. In this chapter, we study Hermitian operators on $\overline{A \otimes E}$ and surjective isometries between these spaces by applying Lumer's method. Fleming and Jamison [**29**, Theorem 4] proved that a Hermitian operator on $C(X, E)$ has a specific form (see Corollary 7.2), the proof of which heavily depends on a property of $C(X)$ that a general uniform algebra does not have. We prove that a Hermitian operator on $\overline{A \otimes E}$ has the same form as the one demonstrated by Fleming and Jamison, while our proof is rather different from theirs. An application is a demonstration of the Banach-Stone property of unital factor C^*-algebras. Following Cambern [**27**] we say that a Banach space E has the *Banach-Stone property* if every surjective isometry $U : C(X_1, E) \to C(X_2, E)$ admits a homeomorphism $\varphi : X_2 \to X_1$ and a strongly continuous family $\{V_y\}_{y \in X_2}$ of surjective isometries from E onto itself such that

$$[U(F)](y) = V_y(F(\varphi(y))), \qquad F \in C(X_1, E),\ y \in X_2.$$

Here we say that a map $\phi : X_2 \to S \subset \mathbf{B}(E)$ is strongly continuous, or the famliy $\{\phi(x)\}_{x \in X_2}$ is strongly continuous, if ϕ is continuous with respect to the relative topology on S induced by the strong operator topology on $\mathbf{B}(E)$, that is, the coarsest topology such that the map the map $X_2 \to E, x \mapsto [\phi(x)](e)$, is continuous for each $e \in E$. Fleming and Jamison applied [**29**, Theorem 4] to prove in [**29**, Theorem 9] that E has the Banach-Stone property if E has the one-dimensional centralizer, a result by Behrends [**4, 5**]. As an application of Theorem 7.1, we characterize unital surjective isometries $\overline{A \otimes E_1} \to \overline{A \otimes E_2}$ and establish in Corollary 7.4 that a unital factor C^*-algebra has the Banach-Stone property. Since a unital factor C^*-algebra has the one-dimensional centralizer, Corollary 7.4 follows from [**29**, Theorem 9], yet we believe that our proof is simpler than the previous proofs.

2. Results of Hermitian operators on $\overline{A \otimes E}$

Let X be a compact Hausdorff space and E a complex Banach space. Let $\phi : X \to \mathbf{B}(E)$ be a strongly continuous map and $F \in$

$C(X,E)$. We prove that the map $[x \mapsto [\phi(x)](F(x))]$ is a continuous map from X into E. First, for each $e \in E$ the map from X into E defined by $[x \mapsto [\phi(x)](e)]$ is continuous since ϕ is strongly continuous. Then the map $\|\phi(\cdot)e\|_E : X \to \mathbb{R}$ defined by $[x \mapsto \|[\phi(x)](e)\|_E]$ is continuous, hence it is bounded since X is compact. Applying the uniform boundedness principle for the family $\{\phi(x)\}_{x\in X}$, we have $\sup_{x\in X} \|\phi(x)\|_{op} = M < \infty$. Let $x_0 \in X$ and $\{x_\alpha\}$ is a net in X which converges to x_0. Then

$$\begin{aligned}
&\|[\phi(x_0)](F(x_0)) - [\phi(x_\alpha)](F(x_\alpha))\|_E \\
&\quad \le \|[\phi(x_0) - \phi(x_\alpha)](F(x_0))\|_E + \|[\phi(x_\alpha)](F(x_0) - F(x_\alpha))\|_E \\
&\qquad \le \|[\phi(x_0) - \phi(x_\alpha)](F(x_0))\|_E + M\|F(x_0) - F(x_\alpha)\|_E \to 0
\end{aligned}$$

as $x_\alpha \to x_0$. It follows that the map $[x \to [\phi(x)](F(x))]$ is continuous. We define an operator $C_\phi : C(X,E) \to C(X,E)$ by

$$[C_\phi(F)](x) = [\phi(x)](F(x)), \quad F \in C(X,E),\ x \in X.$$

Note that C_ϕ is a bounded operator since

$$\|C_\phi(F)\|_\infty = \sup_{x\in X} \|[C_\phi(F)](x)\|_E = \sup_{x\in X} \|[\phi(x)](F(x))\|_E \le M\|F\|_\infty.$$

The following is a generalization of Theorem 4 in [**29**] for $C(X)$ to a uniform algebra A.

THEOREM 7.1. *Let A be a natural uniform algebra on a compact Hausdorff space X and E a complex Banach space. For a bounded linear operator $T : \overline{A \otimes E} \to \overline{A \otimes E}$, the following conditions are equivalent.*

(i) *The operator T is a Hermitian operator.*
(ii) *There exists a strongly continuous map $\phi : X \to \mathrm{Her}(\mathbf{B}(E))$ such that $C_\phi(\overline{A \otimes E}) \subset \overline{A \otimes E}$ and $C_\phi|_{\overline{A\otimes E}} = T$.*

In this case $\|T\|_{op} = \sup_{x\in X} \|\phi(x)\|_{op}$.

As a consequence of Theorem 7.1 we have a slightly stronger version of [**29**, Theorem 4]. It gives us a characterization of Hermitian operators on $C(X,E)$.

COROLLARY 7.2. *Let X be a compact Hausdorff space and E a complex Banach space. Then a bounded linear operator $T : C(X, E) \to C(X, E)$ is a Hermitian operator if and only if there exists a strongly continuous map $\phi : X \to \mathbf{B}(E)$ such that $\phi(x) : E \to E$ is a Hermitian operator for every $x \in X$ which satisfies*

$$[T(F)](x) = [\phi(x)](F(x)), \qquad x \in X$$

for every $F \in C(X, E)$. In this case $\|T\|_{op} = \sup_{x \in X} \|\phi(x)\|_{op}$.

PROOF. Suppose that $T : C(X, E) \to C(X, E)$ is a Hermitian operator. Then by Theorem 7.1 there exists a map $\phi : X \to \mathbf{B}(E)$, continuous with respect to the strong operator topology on $\mathbf{B}(E)$, with $\phi(x) : E \to E$ being a Hermitian operator for every $x \in X$, which satisfies

$$[T(F)](x) = [\phi(x)](F(x)), \qquad x \in X$$

for every $F \in C(X, E)$.

Conversely each operator of the above form is Hermitian by Theorem 7.1 The equality $\|T\|_{op} = \sup_{x \in X} \|\phi(x)\|_{op}$ also follows from Theorem 7.1. □

3. Proofs of results of Hermitian operators

Proof of Theorem 7.1. Suppose that T is a Hermitian operator. Let $x \in X$. Define $\phi(x) : E \to E$ by

$$[\phi(x)](e) = [T(\mathbf{1} \otimes e)](x), \qquad e \in E.$$

Then $\phi(x)$ is a bounded operator. We have

$$\|\phi(x)\|_{op} \leq \|T\|_{op} \tag{3.1}$$

since

$$\begin{aligned} \|[\phi(x)](e)\|_E &= \|[T(\mathbf{1} \otimes e)](x)\|_E \\ &\leq \|T(\mathbf{1} \otimes e)\|_\infty \leq \|T\|_{op} \|\mathbf{1} \otimes e\|_\infty = \|T\|_{op} \|e\|_E \end{aligned}$$

for every $e \in E$. We prove that $\phi(x)$ is Hermitian. Let $e \in E$ be an arbitrary element and $e^* \in \Pi_e$. Let $\delta_x^E : \overline{A \otimes E} \to E$ be the point evaluation defined by $\delta_x^E(F) = F(x)$, $F \in \overline{A \otimes E}$. Put $\theta : \overline{A \otimes E} \to \mathbb{C}$ by $\theta = e^* \circ \delta_x^E$. For $F \in \overline{A \otimes E}$, we have

$$|\theta(F)| = |e^* \circ \delta_x^E(F)| \leq \|e^*\|_{op}\|F(x)\|_E \leq \|e^*\|_{op}\|F\|_\infty = \|e\|_E\|F\|_\infty$$

since $e^* \in \Pi_e$. Hence $\|\theta\|_{op} \leq \|e\|_E$. On the other hand we have

$$|\theta(\mathbf{1} \otimes e)| = |e^*(e)| = \|e\|_E^2 = \|e\|_E\|\mathbf{1} \otimes e\|_\infty,$$

since $\|e\|_E = \|\mathbf{1} \otimes e\|_\infty$. We infer that

$$\|\theta\|_{op} = \|\mathbf{1} \otimes e\|_\infty, \quad |\theta(\mathbf{1} \otimes e)| = \|\mathbf{1} \otimes e\|_\infty^2,$$

hence $\theta \in \Pi_{\mathbf{1}\otimes e}$. Since T is Hermitian, we have $\theta(T(\mathbf{1} \otimes e)) \in \mathbb{R}$ for every $e \in E$. Hence we have

$$e^*([\phi(x)](e)) = e^*([T(\mathbf{1} \otimes e)](x)) = \theta(T(\mathbf{1} \otimes e)) \in \mathbb{R}.$$

Thus we have that $\phi(x)$ is Hermitian.

We prove that the map $\phi : X \to \mathbf{B}(E)$ is strongly continuous. Let $e \in E$ be an arbitrary element. By the definition of ϕ, $[\phi(x)](e) = [T(\mathbf{1} \otimes e)](x)$ for every $x \in X$. As $T(\mathbf{1} \otimes e) : X \to E$ is continuous we have that $x \mapsto [\phi(x)](e)$ is a continuous map from X into E, which means the strong continuity of ϕ.

By a simple calculation C_ϕ is complex-linear and

$$\begin{aligned}\|[C_\phi(F)](x)\|_E &= \|[\phi(x)](F(x))\|_E \\ &\leq \|\phi(x)\|_{op}\|F(x)\|_E \leq \|T\|_{op}\|F\|_\infty, \quad F \in C(X, E).\end{aligned}$$

Hence C_ϕ is a bounded complex-linear operator. We prove that $C_\phi(\overline{A \otimes E}) \subset \overline{A \otimes E}$. To prove it, we observe that $\overline{A \otimes E}$ is an A-module. Indeed, let $f \in A$ and $F \in \overline{A \otimes E}$. Define $f \cdot F$ by $(f \cdot F)(x) = f(x)F(x)$, $x \in X$. If $F = \sum f_i \otimes e_i \in A \otimes E$, then

$$(f \cdot F)(x) = f(x) \sum f_i(x)e_i = \sum (f(x)f_i(x))e_i, \quad x \in X.$$

Thus we have that $f \cdot F \in A \otimes E$. Suppose that $F \in \overline{A \otimes E}$. Then there exists a sequence $\{F_n\}$ in $A \otimes E$ such that $\|F_n - F\|_\infty \to 0$ as $n \to \infty$. Then $\|f \cdot F_n - f \cdot F\|_\infty \leq \|f\|_\infty\|F_n - F\|_\infty$ assures that $f \cdot F \in \overline{A \otimes E}$ since $f \cdot F_n \in A \otimes E$ for every positive integer n. Thus $\overline{A \otimes E}$ is an A-module.

We prove $C_\phi(\mathbf{1} \otimes e) \in \overline{A \otimes E}$ for every $e \in E$. Let $e \in E$. By the definition of C_ϕ we have

$$[C_\phi(\mathbf{1} \otimes e)](x) = [\phi(x)]((\mathbf{1} \otimes e)(x)) = [\phi(x)](e) = [T(\mathbf{1} \otimes e)](x)$$

for every $x \in X$. Hence $C_\phi(\mathbf{1} \otimes e) = T(\mathbf{1} \otimes e) \in \overline{A \otimes E}$. By the definition of C_ϕ, we have for $f \in A$,

$$\begin{aligned} [C_\phi(f \cdot F)](x) &= [\phi(x)]((f \cdot F)(x)) = [\phi(x)](f(x)F(x)) \\ &= f(x)[\phi(x)](F(x)) = f(x)[C_\phi(F)](x), \quad x \in X \end{aligned}$$

since $f(x)$ is a complex number and $\phi(x)$ is a complex-linear map for every $x \in X$. Thus we have

$$C_\phi(f \cdot F) = f \cdot C_\phi(F)$$

for every $f \in A$ and $F \in \overline{A \otimes E}$. Therefore

$$C_\phi(f \otimes e) = C_\phi(f \cdot (\mathbf{1} \otimes e)) = f \cdot C_\phi(\mathbf{1} \otimes e) \in \overline{A \otimes E}$$

for every pair $f \in A$ and $e \in E$. As C_ϕ is a bounded complex-linear map, we have that

$$C_\phi(\overline{A \otimes E}) \subset \overline{A \otimes E}.$$

Let Φ be the restriction of C_ϕ to $\overline{A \otimes E}$: $\Phi = C_\phi|_{\overline{A \otimes E}} : \overline{A \otimes E} \to \overline{A \otimes E}$. We prove that Φ is a Hermitian operator. Let $F \in \overline{A \otimes E}$. Since X is compact, there exists $x_F \in X$ such that $\|F(x_F)\|_E = \|F\|_\infty$. Choose any $\beta_F \in \Pi_{F(x_F)}$. Define $F^* : \overline{A \otimes E} \to \mathbb{C}$ by $F^* = \beta_F \circ \delta_{x_F}^{\overline{A \otimes E}}$. It is a routine argument to prove that $F^* \in \Pi_F$. Then we have that

$$F^*(\Phi(F)) = \beta_F\left([\Phi(F)]\,(x_F)\right) = \beta_F\left([\phi(x_F)](F(x_F))\right) \in \mathbb{R}$$

since $\beta_F \in \Pi_{F(x_F)}$ and $\phi(x_F)$ is a Hermitian operator. This holds for every $F \in \overline{A \otimes E}$, hence Φ is Hermitian.

Define a bounded linear Hermitian operator J by $J = T - \Phi$ and we are to prove $J = 0$ which implies

$$[T(F)](x) = [\phi(x)](F(x)), \qquad x \in X$$

for every $F \in \overline{A \otimes E}$, the desired conclusion in (ii). To prove $J = 0$, it is enough to show that $J(f \otimes e) = 0$ for every $f \in A$ and every

$e \in E$, because $A \otimes E$ is dense in $\overline{A \otimes E}$. For $e \in E$ and $e^* \in \Pi_e$, define $S_e : A \to C(X)$ by

$$[S_e(f)](x) = e^* \left([J(f \otimes e)](x)\right), \quad f \in A, \; x \in X. \tag{3.2}$$

As J is complex-linear, so is S_e. We prove that S_e is a Hermitian operator, and $S_e(A) \subset A$, and then conclude $S_e = 0$ by appealing to [7, Theorem 4]. It is enough to prove these for $e \in E$ with $\|e\|_E = 1$. Suppose that $e \in E$ with $\|e\|_E = 1$ and $e^* \in \Pi_e$. First we prove that S_e is a bounded operator. Since $\|e^*\|_{op} = \|e\|_E = 1$, we have

$$\begin{aligned} |[S_e(f)](x)| &\leq \|e^*\|_{op} \, \|[J(f \otimes e)](x)\|_E \\ &\leq \|J(f \otimes e)\|_\infty \leq \|J\|_{op} \|f \otimes e\|_\infty = \|J\|_{op} \|f\|_\infty \end{aligned}$$

for every pair $f \in A$ and $x \in X$. Hence

$$\|S_e(f)\|_\infty \leq \|J\|_{op} \|f\|_\infty, \quad f \in A,$$

so that S_e is bounded. To prove that $S_e(A) \subset A$, we show that, for each $F \in \overline{A \otimes E}$, the function

$$X \ni x \mapsto e^*(F(x))$$

belongs to A. Suppose that $F = \sum f_i \otimes e_i \in A \otimes E$. Then for every $x \in X$ we have

$$e^*(F(x)) = \sum e^*(e_i) f_i(x) = \left(\sum e^*(e_i) f_i\right)(x)$$

Hence the function

$$X \ni x \mapsto e^*(F(x)) \tag{3.3}$$

belongs to A for $F = \sum f_i \otimes e_i$. Suppose that $F \in \overline{A \otimes E}$, in general. Then there exists a sequence $\{F_n\}$ in $A \otimes E$ such that $\|F_n - F\|_\infty \to 0$ as $n \to \infty$. Then

$$\begin{aligned} \sup_{x \in X} |e^*(F(x)) - e^*(F_n(x))| &= \sup_{x \in X} |e^*(F(x) - F_n(x))| \\ &\leq \sup_{x \in X} \|F(x) - F_n(x)\|_E = \|F - F_n\|_\infty. \end{aligned}$$

Since the function $[[x \mapsto e^*(F_n(x))]$ belongs to A by (3.3) and A is uniformly closed , we infer that the function

$$X \ni x \mapsto e^*(F(x))$$

belongs to A. Applying this to $J(f\otimes e) \in \overline{A\otimes E}$ for an arbitrary $f\in A$ we have

$$[S_e(f)] = \Big[X \ni x \mapsto e^*\big([J(f\otimes e)](x)\big)\Big] \in A$$

for every $f\in A$. Thus $S_e(A)\subset A$.

To prove that S_e is a Hermitian operator, we show that

$$\|I + itS_e\|_{op} = 1 + o(t), \qquad t\in\mathbb{R},$$

where $I : A\to A$ is the identity operator (see Section 1). Let $f\in A$ and $x\in X$. Then using $e^*(e) = \|e\| = 1$ we have

$$\begin{aligned}
|[(I+itS_e)(f)](x)| &= |f(x) + it[S_e(f)](x)| \\
&= |f(x)e^*(e) + ite^*\left([J(f\otimes e)](x)\right)| \\
&= |e^*\left(f(x)e + it[J(f\otimes e)](x)\right)| \\
&= |e^*\left((f\otimes e + it[J(f\otimes e)])(x)\right)| \\
&\le \|[(I+itJ)(f\otimes e)](x)\|_E \|e^*\|_{op} \\
&\le \|(I+itJ)(f\otimes e)\|_\infty \\
&\le \|I+itJ\|_{op}\|f\otimes e\|_\infty \\
&= \|I+itJ\|_{op}\|f\|_\infty.
\end{aligned}$$

It follows from this that

$$\|I+itS_e\|_{op} \le \|I+itJ\|_{op}, \qquad t\in\mathbb{R}. \tag{3.4}$$

Since J is Hermitian, $\|I+itJ\|_{op} = 1+o(t)$ for every $t\in\mathbb{R}$. As $(I+itS_e)(1) = 1$, we infer that $1 \le \|I+itS_e\|_{op}$. Thus by (3.4) we have that

$$\|I+itS_e\|_{op} = 1+o(t),\ t\in\mathbb{R}$$

hence S_e is Hermitian. hence S_e is a Hermitian operator on the uniform algebra A. Therefore we obtain by Theorem 5.8 that

$$S_e = M_{S_e(\mathbf{1})}, \tag{3.5}$$

where $M_{S_e(\mathbf{1})}$ is the left multiplication operator by the element $S_e(\mathbf{1})$. Note that $J(\mathbf{1}\otimes e)=0$ since

$$\begin{aligned}[J(\mathbf{1}\otimes e)](x) &= [T(\mathbf{1}\otimes e)](x)-[\Phi(\mathbf{1}\otimes e)](x)\\ &= [T(\mathbf{1}\otimes e)](x)-[\phi(x)]((\mathbf{1}\otimes e)(x))\\ &= [T(\mathbf{1}\otimes e)](x)-[\phi(x)](e)=0, \quad x\in X.\end{aligned}$$

Hence we have that $[S_e(\mathbf{1})](x)=e^*([J(\mathbf{1}\otimes e)](x))=0$ for every $x\in X$, that is, $S_e(\mathbf{1})=0$. By (3.5) we have that $S_e=0$. It follows by a simple calculation due to the definition (3.2) of S_e that $S_e=0$ for any $e\in E$ and $e^*\in\Pi_e$.

Let $f\in A$ and $x\in X$ be an arbitrary pair. Consider the map $[J(f\otimes\cdot)](x):E\to E$. Then the inequality

$$\begin{aligned}\|[J(f\otimes e)](x)\|_E &\le \|J(f\otimes e)\|_\infty\\ &\le \|J\|_{op}\|f\otimes e\|_\infty=\|J\|_{op}\|f\|_\infty\|e\|_E, \quad e\in E\end{aligned}$$

assures that $[J(f\otimes\cdot)](x)$ is bounded. Then Since

$$e^*\left([J(f\otimes e)](x)\right)=e^*\left([S_e(f)](x)\right)=0$$

for every $e\in E$ and every $e^*\in\Pi_e$, we conclude $[J(f\otimes\cdot)](x)=0$ on E by a theorem of Lumer [**76**, Theorem 5].Thus we have $[J(f\otimes e)](x)=0$ for every $f\in A$, $x\in X$ and $e\in E$, which implies $J(f\otimes e)=0$ and thus $J=0$ on $A\otimes E$ by the complex-linearity of J. Since $A\otimes E$ is dense in $\overline{A\otimes E}$, the continuity of J yields $J=0$ on $\overline{A\otimes E}$. It follows that $T=\Phi$ and

$$[T(F)](x)=[\phi(x)](F(x)), \qquad x\in X$$

for every $F\in\overline{A\otimes E}$. We have that (ii) holds.

Suppose conversely that (ii) holds. By the hypothesis $T=C_\phi|_{\overline{A\otimes E}}$ is a bounded linear operator. We prove that T is a Hermitian operator. For $F\in\overline{A\otimes E}$, let $P_F=\{x\in X:\|F(x)\|_E=\|F\|_\infty\}$. Then the family $\{P_F\}_{F\in\overline{A\otimes E}}$ consists of non-empty compact sets. By the axiom of choice we get a subset $\{x_F\}_{F\in\overline{A\otimes E}}$ of X such that $x_F\in P_F$ for every $F\in\overline{A\otimes E}$. Let $[\cdot,\cdot]_E$ be a semi-inner product on E compatible with the norm $\|\cdot\|_E$. Put $[\cdot,\cdot]_\infty$ as

$$[G,F]_\infty=[G(x_F),F(x_F)]_E, \qquad F,G\in\overline{A\otimes E}.$$

It is easy to see that $[\cdot,\cdot]_\infty$ is a semi-inner product on $\overline{A \otimes E}$ compatible with the norm $\|\cdot\|_\infty$. Since $\phi(x_F)$ is a Hermitian operator, we have

(3.6)
$$[T(F), F]_\infty = [[T(F)](x_F), F(x_F)]_E = [[\phi(x_F)](F(x_F)), F(x_F)]_E \in \mathbb{R}$$

for every $F \in \overline{A \otimes E}$. Since $[\cdot,\cdot]_\infty$ is a semi-inner product on $\overline{A \otimes E}$, we see that T is a Hermitian operator on $\overline{A \otimes E}$ by (3.6). We finish the proof of Theorem 7.1.

□

REMARK. For a map $\phi : X \to \mathrm{Her}(\mathbf{B}(E))$, the operator $C_\phi : C(X,E) \to C(X,E)$ may not preserve the subspace $\overline{A \otimes E}$. In fact, let A be the disc algebra on the closed unit disk Δ in the complex plane $\mathbb{C}$ and $E = \mathbb{C}$. Then $\overline{A \otimes E}$ is identified with A. Let $r : \Delta \to \mathbb{R}$ be a continuous map. Let $\phi(x)$ be the multiplication operator on $\mathbb{C}$ defined by $[\phi(x)](z) = r(x)z$, $z \in \mathbb{C}$ for $x \in \Delta$. Then $\phi : \Delta \to \mathbf{B}(\mathbb{C})$ is a strongly continuous map and $\phi(x)$ is a Hermitian operator for every $x \in \Delta$. On the other hand $[T(f)](x) = [\phi(x)](f(x))$ is well defined operator on A only if ϕ is a constant function; if ϕ is not a constant function, then the operator T given by $T(f)(x) = [\phi(x)] = r(x)f(x)$ fails to satisfy $T(A) \subset A$ since r is not an analytic function.

4. Results of Isometries on $\overline{A \otimes E}$

In the rest of this chapter, we study surjective unital isometries from $\overline{A_1 \otimes E_1}$ onto $\overline{A_2 \otimes E_2}$ for a uniform algebra A_j and a unital factor C^*-algebra E_j for $j = 1, 2$. As a corollary of Theorem 7.3 we describe in Corollary 7.4 the form of a surjective isometries from $C(X_1, E_1)$ onto $C(X_2, E_2)$. This gives an alternative and simple proof that a unital factor C^*-algebra has the Banach-Stone property (cf. [**4, 5, 29**]). Let $B(E_1, E_2)$ denote the Banach algebra of all bounded linear operators from E_1 into E_2. We say that a map $\phi : X_2 \to B(E_1, E_2)$ is strongly continuous, or the famliy $\{\phi(x)\}_{x \in X}$ is strongly continuous, if ϕ is continuous with respect to the strong operator topology on $B(E_1, E_2)$, that is, for every $e \in E_1$ the map $X_2 \to E_2$ defined by $x \mapsto [\phi(x)](e)$ is continuous. For a strongly continuous map

$V : X_2 \to \mathbf{B}(E_1, E_2)$ and a continuous map $\varphi : X_2 \to X_1$, we define an operator $C_{V,\varphi} : C(X_1, E_1) \to C(X_2, E_2)$ by

$$[C_{V,\varphi}(F)](y) = V_y(F(\varphi(y))), \quad F \in C(X_1, E_1),\ y \in X_2,$$

where we denote $V_y = V(y)$. Since V is strongly continuous, applying the uniform boundedness principle in a similar way to the one indicated at the beginning of Section 2 we see that $C_{V,\varphi} : C(X_1, E_1) \to C(X_2, E_2)$ is indeed a bounded linear operator.

THEOREM 7.3. *For $j = 1, 2$, let A_j be a natural uniform algebra on X_j, and E_j a unital factor C^*-algebra. For a bounded linear operator $U : \overline{A_1 \otimes E_1} \to \overline{A_2 \otimes E_2}$ the following conditions are equivalent.*

(i) *The operator U is a surjective isometry such that $U(\mathbf{1}) = \mathbf{1}$.*
(ii) *There exists a homeomorphism $\varphi : X_2 \to X_1$ and a strongly continuous map $V : X_2 \to \mathbf{B}(E_1, E_2)$ such that*
  (ii.1) *each V_y is a Jordan $*$-isomorphism,*
  (ii.2) *$C_{V,\varphi}(\overline{A_1 \otimes E_1}) = \overline{A_2 \otimes E_2}$, and*
  (ii.3) *$U = C_{V,\varphi}|_{\overline{A_1 \otimes E_1}}$.*

A well known theorem of Kadison [**62**] states that the class of Jordan $*$-isomorphisms between unital C^*-algebra is precisely the class of unital surjective isometries. Recall that a unital C^*-algebra E is called a *factor* provided that the center of E coincides with $\mathbb{C}\mathbf{1}$.

COROLLARY 7.4. *Let X_j be a compact Hausdorff space, and E_j a unital factor C^*-algebra for $j = 1, 2$. Then a bounded linear operator $U : C(X_1, E_1) \to C(X_2, E_2)$ is a surjective isometry if and only if there exist*

(i) *a homeomorphism $\varphi : X_2 \to X_1$,*
(ii) *a strongly continuous family $\{V_y\}_{y \in X_2}$ of Jordan $*$-isomorphisms from E_1 onto E_2, and*
(iii) *a unitary element $u \in C(X_2, E_2)$*

such that

$$[U(F)](y) = uV_y(F(\varphi(y))), \qquad F \in C(X_1, E_1),\ y \in X_2. \tag{4.1}$$

Note that the hypothesis that E_j is a factor is essential. There exists compact Hausdorff spaces Y_1 and Y_2 which are not homeomorphic, while $Y_1 \times [0,1]$ and $Y_2 \times [0,1]$ are homeomorphic (see [**5**, fig.16]). Since $C(Y_j, C([0,1])) = C(Y_j \times [0,1])$, the spaces $C(Y_1, C([0,1])$ and $C(Y_2, C([0,1]))$ are isometric, but Y_1 is not homeomorphic to Y_2. Here the unital commutative C^*-algebra $C([0,1])$ is not a factor. As is previously mentioned, Corollary 7.4 implies that a unital factor C^*-algebra has the Banach-Stone property.

Proof of Corollary 7.4. Suppose that $U : C(X_1, E_1) \to C(X_2, E_2)$ is a surjective isometry. Note that $C(X_j, E_j)$ is a unital C^*-algebra. By a theorem of Kadison [**62**] on isometries on C^*-algebras, $U(\mathbf{1})$ is a unitary element in $C(X_2, E_2)$. Hence $U_0 = U(\mathbf{1})^* U$ is a unital surjective isometry from $C(X_1, E_1)$ onto $C(X_2, E_2)$. Applying Theorem 7.3 to U_0, we have

$$[U_0(F)](y) = V_y(F(\varphi(y))), \qquad F \in C(X_1, E_1),\ y \in X_2,$$

for a homeomorphism $\varphi : X_2 \to X_1$ and a strongly continuous family $\{V_y\}_{y \in X_2}$ of Jordan $*$-isomorphisms from E_1 onto E_2. It follows that U has the form as is described in (4.1) with $u = U(\mathbf{1})$.

Conversely Theorem 7.3 asserts that every operator U of the form (4.1) is a surjective isometry. □

5. Proofs of results of isometries

Throughout this section A_j is a natural uniform algebra on a compact Hausdorff space X_j, hence the maximal ideal space of A_j is canonically identified with X_j, E_j is a unital factor C^*-algebra for $j = 1, 2$, and $U : \overline{A_1 \otimes E_1} \to \overline{A_2 \otimes E_2}$ is a surjective linear isometry such that $U(\mathbf{1}) = \mathbf{1}$. Note that $e \in E_j$ is a self-adjoint if and only if e is a Hermitian element, that is $\|\exp(ite)\|_E = 1$. Thus the real space of all Hermitian element in E_j coincides with the self-adjoint part of E_j and is denoted by $\mathrm{Her}(E_j)$.

Lemma 7.5. *For every $e \in \mathrm{Her}(E_1)$ and $y \in X_2$, we have that $[U(1 \otimes e)](y) \in \mathrm{Her}(E_2)$.*

PROOF. Let $e \in \mathrm{Her}(E_1)$. For the

$$[M_{\mathbf{1}\otimes e}(F)](x) = [(\mathbf{1}\otimes e)F](x) = eF(x) = M_e(F(x)), \quad x \in X_1. \tag{5.1}$$

As e is a Hermitian element, the multiplication operator $M_e : E_1 \to E_1$ is a Hermitian operator [**40**, Proposition 1]. Define the constant map $\phi : X_1 \to B(E_1)$ by $\phi(x) = M_e$ for every $x \in X_1$ which is strongly continuous. By (5.1) we infer that

$$[\phi(\cdot)](F(\cdot)) = M_{\mathbf{1}\otimes e}(F) \in \overline{A_1 \otimes E_1}$$

for every $F \in \overline{A_1 \otimes E_1}$. Then by Theorem 7.1 the map $M_{\mathbf{1}\otimes e} : \overline{A_1 \otimes E_1} \to \overline{A_1 \otimes E_1}$ is Hermitian. By Theorem 5.2.6 in [**30**], $\| \exp(itM_{\mathbf{1}\otimes e})\|_{op} = 1$ for every $t \in \mathbb{R}$. Since $U : \overline{A_1 \otimes E_1} \to \overline{A_2 \otimes E_2}$ is a surjective isometry we infer that

$$\begin{aligned}\| \exp(itUM_{\mathbf{1}\otimes e}U^{-1})\|_{op} &= \|U(\exp(itM_{\mathbf{1}\otimes e}))U^{-1}\|_{op} \\ &= \| \exp(itM_{\mathbf{1}\otimes e})\|_{op} = 1\end{aligned}$$

for every $t \in \mathbb{R}$. Then by [**30**, Theorem 6.2.1] we have that $UM_{\mathbf{1}\otimes e}U^{-1} : \overline{A_2 \otimes E_2} \to \overline{A_2 \otimes E_2}$ is a Hermitian operator. Then by Theorem 7.1, there exists a family $\{\tilde{\phi}(y) : E_2 \to E_2\}_{y\in X_2}$ of Hermitian operators such that

$$\left[\left(UM_{\mathbf{1}\otimes e}U^{-1}\right)(\mathbf{1})\right](y) = [\tilde{\phi}(y)](\mathbf{1}(y)).$$

Note that $\mathbf{1}(y)$ is the identity element in E_2 for every $y \in X_2$ since $\mathbf{1}$ is the identity element in $\overline{A_2 \otimes E_2}$, which allows us to write $\mathbf{1}$ as $\mathbf{1}(y)$. By a theorem of Sinclair [**113**, Remark 3.5] about the representation of a Hermitian operator on a unital C^*-algebra, there exists an $h_y^e \in \mathrm{Her}(E_2)$ and a $*$-derivation[1] $D_y : E_2 \to E_2$ such that

$$[\tilde{\phi}(y)](e) = h_y^e e + iD_y(e), \quad e \in E_2.$$

As $D_y(\mathbf{1}) = 0$ we have

$$\left[\left(UM_{\mathbf{1}\otimes e}U^{-1}\right)(\mathbf{1})\right](y) = h_y^e, \qquad y \in X_2.$$

On the other hand, since $U(\mathbf{1}) = \mathbf{1}$ we have

$$\left[\left(UM_{\mathbf{1}\otimes e}U^{-1}\right)(\mathbf{1})\right](y) = \left[\left(UM_{\mathbf{1}\otimes e}\right)(\mathbf{1})\right](y) = [U(\mathbf{1}\otimes e)](y).$$

[1]Note that a $*$-derivation in [**113**] is a derivation D such that $D(a^*) = -D(a)^*$ for every a. Our $*$-derivation is a derivation D such that $D(a^*) = D(a)^*$ for every a.

Hence we conclude that

$$[U(\mathbf{1}\otimes e)](y) = h_y^e \in \mathrm{Her}(E_2)$$

for every $y \in X_2$. □

LEMMA 7.6. *We have* $U(A_1 \otimes \mathbf{1}) = A_2 \otimes \mathbf{1}$.

PROOF. We prove $U(A_1\otimes\mathbf{1}) \subset A_2\otimes\mathbf{1}$. Repeating the argument by replacing U with U^{-1}, we will have the inclusion $U^{-1}(A_2\otimes\mathbf{1}) \subset A_1\otimes\mathbf{1}$. With both inclusions it will follow that $U(A_1 \otimes \mathbf{1}) = A_2 \otimes \mathbf{1}$.

Let $f \in A_1$. We prove that $U(f\otimes\mathbf{1}) \in A_2\otimes\mathbf{1}$. Let $b \in E_2$ such that $b^* = -b$. Define a bounded operator $D_b : E_2 \to E_2$ by $D_b(a) = ba - ab$, for $a \in E_2$. As $b^* = -b$, the operator D_b is a $*$-derivation. Define $\tilde{D}_b : \overline{A_2 \otimes E_2} \to \overline{A_2 \otimes E_2}$ by

$$\tilde{D}_b(F) = (\mathbf{1}\otimes b)F - F(\mathbf{1}\otimes b), \qquad F \in \overline{A_2 \otimes E_2}.$$

By some calculation we have that

$$\begin{aligned}(5.2)\qquad \left[\left(i\tilde{D}_b(F)\right)\right](y) &= \left[i\left((\mathbf{1}\otimes b)F - F(\mathbf{1}\otimes b)\right)\right](y)\\ &= i(bF(y) - F(y)b) = (iD_b(F(y))), \quad F \in \overline{A_2\otimes E_2},\ y \in X_2.\end{aligned}$$

Since D_b is a $*$-derivation, we infer that iD_b is a Hermitian operator by a theorem of Sinclair [**113**, Remark 3.5]. We use an argument in the proof of Lemma 7.5 to see that $i\tilde{D}_b$ is a Hermitian operator. In fact, define the constant, and thus a strongly continuous map $\phi : X_2 \to \mathbf{B}(E_2)$ by $\phi(y) = iD_b$, a Hermitian operator, for every $y \in X_2$. Then the operator $C_\phi : \overline{A_2 \otimes E_2} \to \overline{A_2 \otimes E_2}$ defined by $[C_\phi(F)](x) = [\phi(x)](F(x)) = iD_b(F(x))$ is well defined and is a Hermitian operator by Theorem 7.1. On the other hand, the equality $iD_b(F(x)) = [i\tilde{D}_b(F)](x)$ holds by (5.2), hence we have that $C_\phi = i\tilde{D}_b$. Thus the map $i\tilde{D}_b : \overline{A_2\otimes E_2} \to \overline{A_2 \otimes E_2}$ is a Hermitian operator.

Since $U : \overline{A_1 \otimes E_1} \to \overline{A_2 \otimes E_2}$ is a surjective isometry, we have that $U^{-1}i\tilde{D}_bU : \overline{A_1 \otimes E_1} \to \overline{A_1 \otimes E_1}$ is a Hermitian operator. Then by Theorem 7.1, for every $x \in X_1$, there is a Hermitian operator $\phi'(x) : E_1 \to E_1$ such that

$$\left[\left(U^{-1}i\tilde{D}_bU\right)(F)\right](x) = [\phi'(x)](F(x)), \qquad F \in \overline{A_1 \otimes E_1}.$$

By a theorem of Sinclair [**113**, Remark 3.5], there exists an $h'_x \in \mathrm{Her}(E_1)$ and a $*$-derivation $D'_x : E_1 \to E_1$ such that

$$[\phi'(x)](e) = h'_x e + iD'_x(e), \qquad e \in E_1.$$

It follows that

(5.3)
$$\left[\left(U^{-1} i\tilde{D}_b U\right)(F)\right](x) = h'_x F(x) + iD'_x(F(x)), \quad F \in \overline{A_1 \otimes E_1},\ x \in X_1.$$

In particular, we have

$$\left[\left(U^{-1} i\tilde{D}_b U\right)(\mathbf{1})\right](x) = h'_x + iD'_x(\mathbf{1}) = h'_x, \quad x \in X_1.$$

Since $U(\mathbf{1}) = \mathbf{1}$ we infer that

$$\left[\left(U^{-1} i\tilde{D}_b U\right)(\mathbf{1})\right](x) = \left[U^{-1}\left(i((\mathbf{1} \otimes b)\mathbf{1} - \mathbf{1}(\mathbf{1} \otimes b))\right)\right](x) = 0, \quad x \in X_1.$$

Hence we have that $h'_x = 0$ for every $x \in X_1$. Thus we have by (5.3) that

$$\left[\left(U^{-1} i\tilde{D}_b U\right)(F)\right](x) = (iD'_x)(F(x)), \quad F \in \overline{A_1 \otimes E_1},\ x \in X_1. \tag{5.4}$$

Applying (5.4) to $F = f \otimes \mathbf{1}$ we get

$$\left[\left(U^{-1} i\tilde{D}_b U\right)(f \otimes \mathbf{1})\right](x) = (iD'_x)((f \otimes \mathbf{1})(x)) = iD'_x(f(x)\mathbf{1}) = if(x)D'_x(\mathbf{1}) = 0$$

for every $x \in X_1$. Hence we have $(U^{-1} i\tilde{D}_b U)(f \otimes \mathbf{1}) = 0$, so that $i\tilde{D}_b(U(f \otimes \mathbf{1})) = 0$. Hence

$$(\mathbf{1} \otimes b)(U(f \otimes \mathbf{1}) - (U(f \otimes \mathbf{1}))(\mathbf{1} \otimes b) = 0.$$

Thus

$$b[U(f \otimes \mathbf{1})](y) = [U(f \otimes \mathbf{1})](y)b \tag{5.5}$$

for every $y \in X_2$, where b is an arbitrary element in E_2 with $b^* = -b$. We show that (5.5) holds for any $a \in E_2$. In fact, let $a \in E_2$ be an arbitrary element and put $b_1 = \frac{a-a^*}{2}$ and $b_2 = \frac{a+a^*}{2i}$. Then $b_j^* = -b_j$ for $j = 1, 2$ and $a = b_1 + ib_2$. By (5.5) we have

$$b_j[U(f \otimes \mathbf{1})](y) = [U(f \otimes \mathbf{1})](y)b_j, \qquad y \in X_2,$$

for $j = 1, 2$, which, together with $a = b_1 + ib_2$, implies

$$a[U(f \otimes \mathbf{1})](y) = [U(f \otimes \mathbf{1})](y)a, \qquad y \in X_2.$$

As $a \in E_2$ is arbitrary, we have that $(U(f \otimes \mathbf{1}))(y)$ is in the center of E_2 for each $y \in X_2$. Since E_2 is a factor, we infer that $(U(f \otimes \mathbf{1}))(y) \in \mathbb{C}\mathbf{1}$. Thus a function $g : X_2 \to \mathbb{C}$ is defined in the way

$$(U(f \otimes \mathbf{1}))(y) = g(y)\mathbf{1}. \tag{5.6}$$

We prove that $g \in A_2$. As $U(f \otimes \mathbf{1}) \in \overline{A_2 \otimes E_2}$, there exists a sequence $\{F_n = \sum f_{n,j} e_{n,j} \in A_2 \otimes E_2\}$ such that $\|F_n - U(f \otimes \mathbf{1})\|_\infty \to 0$ as $n \to \infty$. For every $y \in X_2$ we have

$$\|F_n - U(f \otimes \mathbf{1})\|_\infty \geq \|F_n(y) - g(y)\mathbf{1}\|_{E_2} = \left\| \sum f_{n,j}(y) e_{n,j} - g(y)\mathbf{1} \right\|_{E_2}.$$

Choose a linear functional $\psi \in \Pi_{\mathbf{1}} \subset E_2^*$. By the definition of $\Pi_{\mathbf{1}}$ we infer that $\|\psi\|_{op} = \psi(\mathbf{1}) = \mathbf{1}$. Then

$$\begin{aligned} \left| \sum f_{n,j}(y)\psi(e_{n,j}) - g(y) \right| &= \left| \sum f_{n,j}(y)\psi(e_{n,j}) - g(y)\psi(\mathbf{1}) \right| \\ = \left| \psi \left(\sum f_{n,j}(y) e_{n,j} - g(y)\mathbf{1} \right) \right| &\leq \left\| \sum f_{n,j}(y) e_{n,j} - g(y)\mathbf{1} \right\|_{E_2} \\ &\leq \|F_n - U(f \otimes \mathbf{1})\|_\infty. \end{aligned}$$

It follows that

$$\begin{aligned} \left\| \sum \psi(e_{n,j}) f_{n,j} - g \right\|_{\infty(X_2)} &= \sup_{y \in X_2} \left| \sum f_{n,j}(y)\psi(e_{n,j}) - g(y) \right| \\ &\leq \|F_n - U(f \otimes \mathbf{1})\|_\infty \to 0 \end{aligned}$$

as $n \to \infty$. Since $\sum \psi(e_{n,j}) f_{n,j} \in A_2$, we conclude that $g \in A_2$. By (5.6) we have that $U(f \otimes \mathbf{1}) = g \otimes \mathbf{1} \in A_2 \otimes \mathbf{1}$. □

We prove now that (i) implies (ii). Suppose that (i) holds. Let $c : A_2 \otimes \mathbf{1} \to A_2$ be the isomorphism defined by $c(g \otimes \mathbf{1}) = g$ for $g \in A_2$. Define a map $N : A_1 \to A_2$ by $f \mapsto c([U(f \otimes \mathbf{1})])$ for $f \in A_1$. Then by Lemma 7.6 N is well defined surjection. As U is a linear map such that $U(\mathbf{1}) = \mathbf{1}$, N is a surjective linear isometry such that $N(\mathbf{1}) = \mathbf{1}$. By the representation theorem of isometries between uniform algebras by Nagasawa [**94**], N is an algebra isomorphism. Hence by Gelfand theory there exists a homeomorphism $\varphi : X_2 \to X_1$ between maximal ideal spaces such that

$$N(f) = f \circ \varphi, \qquad f \in A_1.$$

Therefore we have

$$[U(f \otimes \mathbf{1})](y) = [N(f)]\,(y)\mathbf{1} = f(\varphi(y))\mathbf{1}, \qquad y \in X_2.$$

Let $e \in \mathrm{Her}(E_1)$. We see that $UM_{\mathbf{1}\otimes e}U^{-1}$, unitarily conjugate to the Hermitian operator $M_{\mathbf{1}\otimes e}$, is a Hermitian operator (see the proof of Lemma 7.5)

Then we see by Theorem 7.1 that there is a strongly continuous map $\phi : X_2 \to \mathbf{B}(E_2)$ such that $\phi(y) : E_2 \to E_2$ is Hermitian for every $y \in X_2$ which satisfies

$$[UM_{\mathbf{1}\otimes e}U^{-1}(F)](y) = [\phi(y)](F(y)).$$

By a theorem of Sinclair [**113**, Remark 3.5] there exists $h_y^e \in \mathrm{Her}(E_2)$ and a $*$-derivation D_y such that $\phi(y) = M_{h_y^e} + iD_y$. For each $y \in X_2$, we define a map

$$\psi_y : \mathrm{Her}(E_1) \to \mathrm{Her}(E_2)$$

by $\psi_y(e) = h_y^e$. As we assume $U(\mathbf{1}) = \mathbf{1}$,

$$\begin{aligned}[U(\mathbf{1} \otimes e)](y) &= [UM_{\mathbf{1}\otimes e}U^{-1}(\mathbf{1})](y) \\ &= [\phi(y)](\mathbf{1}) = M_{h_y^e}(\mathbf{1}) + iD_y(\mathbf{1}) = h_y^e,\end{aligned}$$

hence $\psi_y(e) = [U(\mathbf{1} \otimes e)](y)$. For every $f \in A_1$ and $y \in X_2$ we have

$$\begin{aligned}&\big(U(f \otimes e)\big)(y) = \big(UM_{\mathbf{1}\otimes e}U^{-1}\big)\,(U(f \otimes \mathbf{1}))\,(y) \\ &= \big(\phi(y)\big)\Big(\big(U(f \otimes \mathbf{1})\big)(y)\Big) = h_y^e\big(U(f \otimes \mathbf{1})\big)(y) + iD_y\Big(\big(U(f \otimes \mathbf{1})\big)(y)\Big) \\ &= h_y^e\Big(\big(f(\varphi(y))\big)\mathbf{1}\Big) + iD_y\Big(\big(f(\varphi(y))\big)\mathbf{1}\Big) \\ &= h_y^e\Big(\big(f(\varphi(y))\big)\mathbf{1}\Big) + if(\varphi(y))D_y(\mathbf{1}) \\ &= h_y^e\Big(\big(f(\varphi(y))\big)\mathbf{1}\Big) = \psi_y(e)\Big(\big(f(\varphi(y))\big)\mathbf{1}\Big)\end{aligned}$$

Let $V_y : E_1 \to E_2$ be defined by

$$V_y(e) = \psi_y(\mathrm{Re}\, e) + i\psi_y(\mathrm{Im}\, e), \qquad e \in E_1,$$

where $\mathrm{Re}\, e = (e + e^*)/2$ and $\mathrm{Im}\, e = (e - e^*)/2i$. Then V_y is a complex-linear map. As $U(\mathbf{1}) = \mathbf{1}$, we have for every $y \in X_2$ that

$$V_y(\mathbf{1}) = \psi_y(\mathbf{1}) = U(\mathbf{1} \otimes \mathbf{1})(y) = \mathbf{1},$$

hence $V_y(\mathbf{1}) = \mathbf{1}$. We have for $f \in A_1$, $e \in E_1$ and $y \in X_2$ that

$$\begin{aligned}\big(U(f \otimes e)\big)(y) &= \big(U(f \otimes \mathrm{Re}\, e)\big)(y) + i\big(U(f \otimes \mathrm{Im}\, e)\big)(y) \\ &= \psi_y(\mathrm{Re}\, e)\Big(\big(f(\varphi(y))\big)\mathbf{1}\Big) + i\psi_y(\mathrm{Im}\, e)\Big(\big(f(\varphi(y))\big)\mathbf{1}\Big) \\ &= V_y(e)\Big(\big(f(\varphi(y))\big)\mathbf{1}\Big) = V_y\Big(\big(f(\varphi(y))\big)e\Big)\mathbf{1} \\ &= V_y\Big(\big(f(\varphi(y))\big)e\Big) = V_y\Big(\big(f \otimes e\big)(\varphi(y))\Big).\end{aligned}$$

Hence we infer that

$$\begin{aligned}(5.7) \quad (U(F))(y) &= \left(U\left(\sum f_j \otimes e_j\right)\right)(y) \\ &= V_y\Big(\left(\sum f_j \otimes e_j\right)(\varphi(y))\Big) = V_y(F(\varphi(y)))\end{aligned}$$

for $F = \sum f_j \otimes e_j \in A_1 \otimes E_1$. Then

$$\begin{aligned}\|V_y(e)\|_{E_2} &= \big\|V_y\big((\mathbf{1} \otimes e)(\varphi(y)\big)\big\|_{E_2} = \|\big(U(\mathbf{1} \otimes e)\big)(y)\|_{E_2} \\ &\leq \|U(\mathbf{1} \otimes e)\|_\infty = \|\mathbf{1} \otimes e\|_\infty = \|e\|_{E_1}\end{aligned}$$

holds for every $e \in E_1$, so that,

$$(5.8) \qquad \|V_y(e)\|_{E_2} \leq \|e\|_{E_1}, \quad e \in E_1.$$

Since $V_y(\mathbf{1}) = \mathbf{1}$, $\|V_y\|_{op} = 1$ by (5.8). Let $F \in \overline{A_1 \otimes E_1}$. Then there exists an $F_n \in A_1 \otimes E_1$ such that $\|F - F_n\|_\infty \to 0$, $(n \to \infty)$. By (5.7) we have for every $y \in X_2$ that

$$\begin{aligned}(5.9) \quad (U(F))(y) &= \lim_{n\to\infty} (U(F_n))(y) \\ &= \lim_{n\to\infty} \Big(V_y\big(F_n(\varphi(y))\big)\Big) = V_y\big(F(\varphi(y))\big).\end{aligned}$$

The strong continuity of the family $\{V_y\}_{y \in X_2}$ is proved as follows. Letting $F = \mathbf{1} \otimes e$ for any $e \in E_1$, we have by (5.7) that

$$[U(\mathbf{1} \otimes e)](y) = V_y(e)$$

for every $y \in X_2$. Since $U(\mathbf{1} \otimes e) \in C(X_2, E_2)$, the map $X_2 \to E_2$ defined by

$$X_2 \ni y \mapsto [U(\mathbf{1} \otimes e)](y) = V_y(e)$$

is continuous. As $e \in E_1$ is arbitrary, the family $\{V_y\}_{y\in X_2}$ is strongly continuous.

We prove that $V_y : E_1 \to E_2$ is surjective for every $y \in X_2$. Let $e \in E_2$. Since U is a surjection there is an $F \in \overline{A_1 \otimes E_1}$ such that $U(F) = \mathbf{1} \otimes e$. Thus we have that

$$e = (1 \otimes e)(y) = \big(U(F)\big)(y) = V_y\big(F(\varphi(y))\big), \quad y \in X_2,$$

which proves that V_y is surjective.

We prove that V_y is an isometry for every $y \in X_2$. Repeating the same argument by replacing U with the inverse $U^{-1} : \overline{A_2 \otimes E_2} \to \overline{A_1 \otimes E_1}$, we see that there exist a homeomorphism $\varphi' : X_1 \to X_2$ and a strong continuous family $\{V'_x\}_{x\in X_1}$ of surjective bounded operators from E_2 onto E_1 with the operator norm 1 such that

$$\big(U^{-1}(G)\big)(x) = V'_x\big(G(\varphi'(x))\big), \quad G \in \overline{A_2 \otimes E_2},\ x \in X_1. \tag{5.10}$$

Let $e \in E_1$. Applying (5.10) to $G = U(1 \otimes e) \in \overline{A_2 \otimes E_2}$, and applying (5.9) we have

$$\begin{aligned} e = (\mathbf{1} \otimes e)(x) &= \Big(U^{-1}\big(U(\mathbf{1} \otimes e)\big)\Big)(x) \\ &= V'_x\Big(\big(U(\mathbf{1} \otimes e)\big)(\varphi'(x))\Big) = V'_x\Big(V_{\varphi'(x)}\big((\mathbf{1} \otimes e)(\varphi(\varphi'(x)))\big)\Big) \\ &= \big(V'_x \circ V_{\varphi'(x)}\big)(e), \quad x \in X_1. \end{aligned}$$

As φ' is bijective we see that

$$e = V'_{\varphi'^{-1}(y)} \circ V_y(e)$$

for every $y \in X_2$. Therefore

$$\|e\|_{E_1} = \|V'_{\varphi'^{-1}(y)} \circ V_y(e)\|_{E_1} \le \|V'_{\varphi'^{-1}(y)}\|_{op}\|V_y(e)\|_{E_2}.$$

As $\|V'_{\varphi'^{-1}(y)}\|_{op} \le 1$, we have

$$\|e\|_{E_1} \le \|V_y(e)\|_{E_2}$$

holds for every $e \in E_1$. Together with (5.8) we have

$$\|e\|_{E_1} = \|V_y(e)\|_{E_2}$$

for every $e \in E_1$. As V_y is complex-linear we conclude that V_y is an isometry.

We prove the converse implication. Suppose that the hypotheses of (ii) are satisfied. Since $V_y : E_1 \to E_2$ is a Jordan $*$-isomorphism, it is a surjective isometry by a theorem of Kadison [**62**] for every $y \in X_2$. Hence we have

$$\begin{aligned}\|[U(F_1)](y) - [U(F_2)](y)\|_{E_2} &= \|[C_{V,\varphi}(F_1)](y) - [C_{V,\varphi}(F_2)](y)\|_{E_2} \\ &= \|V_y\big(F_1(\varphi(y)) - F_2(\varphi(y))\big)\|_{E_2} = \|F_1(\varphi(y)) - F_2(\varphi(y))\|_{E_1}\end{aligned}$$

for every $y \in X_2$ and every pair F_1 and F_2 in $\overline{A_1 \otimes E_1}$. Since $\varphi : X_2 \to X_1$ is a surjection we infer that

$$\|U(F_1) - U(F_2)\|_\infty = \|F_1 - F_2\|_\infty$$

for every pair F_1 and F_2 in $\overline{A_1 \otimes E_1}$, $U : \overline{A_1 \otimes E_1} \to \overline{A_2 \otimes E_2}$ is an isometry.

CHAPTER 8

Local maps

1. Introduction to local maps in isometry groups

This chapter is a contribution to the study of the algebraic reflexivity of the surjective linear isometry group of algebras of Lipschitz maps. In addition, there are other important classes of maps which deserve attention by following a similar approach, as for example, groups of surjective linear isometries on spaces of vector-valued Lipschitz maps.

Botelho and Jamison [**10**] investigate algebraic reflexivity of surjective linear isometry groups on $(\mathrm{Lip}(X, E), \|\cdot\|_{\max})$ under some hypotheses on X and E by applying a characterization due to Jiménez-Vargas and Villegas-Vallecillos [**54**] of linear isometries between $\mathrm{Lip}(X, E)$ with the max norm $\|\cdot\|_{\max}$.

In the case of $E = \mathbb{C}$, in [**51**] Jiménez-Vargas, Morales Campoy and Villegas-Vallecillos proved that isometry groups on complex-valued Lipschitz functions are algebraically reflexive (they apply [**48**, Example 8], which is established in Corollary 4.15). We also characterized unital surjective linear isometries on $\mathrm{Lip}(X, M_n(\mathbb{C}))$ with respect to the sum norm, where $M_n(\mathbb{C})$ is a Banach algebra of complex matrices of degree n with operator norm $\|\cdot\|$ in Theorem 6.2. The purpose of this chapter is to investigate the algebraic reflexivity of the groups of surjective linear isometries on spaces of vector-valued Lipschitz maps $\mathrm{Lip}(X, E)$ for $E = C(Y),\ M_n(\mathbb{C})$ for details see sections 2 and 3, respectively.

Let A_i be a complex Banach space for $i = 1, 2$. Denote by $\mathcal{B}(A_1, A_2)$ the set of all bounded linear operators from A_1 into A_2. The subset $\mathcal{S} \subset \mathcal{B}(A_1, A_2)$ is called algebraically reflexive if the implication

$$T \in \mathcal{B}(A_1, A_2), \quad Tf \in \mathcal{S}f \ (\forall f \in A_1) \Longrightarrow T \in \mathcal{S}$$

holds. We review the definition of locally surjective linear isometry (resp. locally unital surjective linear isometry). A bounded linear operator $T : A_1 \to A_2$ a locally surjective linear isometry (resp. locally unital surjective linear isometry) if for every $f \in A_1$, there exists a surjective linear isometry (resp. unital surjective linear isometry) $T_f : A_1 \to A_2$ such that $Tf = T_f f$. Thus if every locally surjective linear isometry (resp. locally unital surjective linear isometry) is surjective then the group of surjective linear isometries (resp. unital surjective linear isometry) is algebraically reflexive.

Let K be a compact Hausdorff space. We consider a linear subspace B of $C(K)$, which separates the points of K and contains the constants. We denote the Choquet boundary for B by $\mathrm{Ch}(B)$.

2. The group of surjective isometries on a Banach algebra of Lipschitz maps whose values are in a unital commutative C^*- algebra

Li, Peralta, Wang and Wang has proved Theorem 8.1 by applying a generalization of Gleason-Kahane-Żelazko theorem in [**75**, Theorem 2.5 (b)] under the additional hypothesis that a surjective linear isometry from $\mathrm{Lip}(X_1)$ onto $\mathrm{Lip}(X_2)$ is canonical. By Corollary 4.15 we do not need to assume the hypothesis. We exhibit a simple proof of Theorem 8.1 by applying the original Gleason-Kahane-Żelazko theorem.

THEOREM 8.1. *Let X_i be a compact metric space for $i = 1, 2$. The group of all surjective linear isometries from* $\mathrm{Lip}(X_1)$ *onto* $\mathrm{Lip}(X_2)$ *is algebraically reflexive.*

PROOF. Let Ψ be a locally surjective linear isometry from $\mathrm{Lip}(X_1)$ onto $\mathrm{Lip}(X_2)$. By applying Corollary 4.15, there exists $\alpha_1 \in \mathbb{C}$ with $|\alpha_1| = 1$ and surjective isometry $\varphi_1 : X_2 \to X_1$ such that

$$\Psi(\mathbf{1})(x) = \Psi_1(\mathbf{1})(x) = \alpha_1 \mathbf{1}(\varphi_1(x)) = \alpha_1$$

for every $x \in X_2$. Considering $\overline{\alpha_1}\Psi$ instead of Ψ, without loss of generality, we may assume $\Psi(1) = 1$. For any $g \neq 0 \in \mathrm{Lip}(X_1)$, we have

$$\Psi(g) = \Psi_g(g) = \alpha_g g \circ \varphi_g,$$

where $\alpha_g \in \mathbb{C}$ with $|\alpha_g| = 1$ and φ_g is a surjective isometry from X_2 onto X_1. There exists $x_0 \in X_1$ such that $|g(x_0)| = \|g\|_\infty$. Put $g(x_0) = \lambda$. Since $g \neq 0$, we have $\lambda \neq 0$. We define $g' \in \mathrm{Lip}(X_1)$ by $g' = g + \lambda 1$. There exists $\alpha_{g'} \in \mathbb{C}$ with $|\alpha_{g'}| = 1$ and a surjective isometry $\varphi_{g'}$ from X_2 onto X_1 such that

$$\begin{aligned}\Psi(g') &= \alpha_{g'} g' \circ \varphi_{g'} = \alpha_{g'}(g + \lambda 1) \circ \varphi_{g'} \\ &= \alpha_{g'} g \circ \varphi_{g'} + \alpha_{g'} \lambda 1.\end{aligned}$$

In addition, we have

$$\begin{aligned}\Psi(g') &= \Psi(g + \lambda 1) \\ &= \Psi(g) + \Psi(\lambda 1) = \alpha_g g \circ \varphi_g + \lambda 1.\end{aligned}$$

Thus we have

$$\alpha_{g'} g \circ \varphi_{g'} + \alpha_{g'} \lambda 1 = \alpha_g g \circ \varphi_g + \lambda 1. \tag{2.1}$$

As $\varphi_{g'}$ is surjective, there exists $x_1 \in X_2$ such that $\varphi_{g'}(x_1) = x_0$. By (2.1) and $\lambda = g(x_0)$, we have

$$\alpha_{g'} \lambda + \alpha_{g'} \lambda = \alpha_g g(\varphi_g(x_1)) + \lambda. \tag{2.2}$$

Since $\|g \circ \varphi_g\|_\infty = \|g\|_\infty = |\lambda|$, $|\alpha_{g'}| = 1$ and $|\alpha_g| = 1$, we get $|g(\varphi_g(x_1))| = |\lambda|$. Since we have

$$(2\alpha_{g'} - 1)\lambda = \alpha_g g(\varphi_g(x_1))$$

by (2.2) we obtain

$$|2\alpha_{g'} - 1||\lambda| = |\alpha_g g(\varphi_g(x_1))| = |\lambda|.$$

As $\lambda \neq 0$, we get $|2\alpha_{g'} - 1| = 1$, hence $\alpha_{g'} = 1$. Thus the equation (2.1) shows that

$$g \circ \varphi_{g'} = \alpha_g g \circ \varphi_g = \Phi(g).$$

For any $x \in X_2$, $\Psi(g)(x) = g(\varphi_{g'}(x)) \in \sigma(g)$, where $\sigma(g)$ denote the spectrum of g. By the Gleason-Kahane-Żelazko theorem, we have Ψ is multiplicative. This implies that $\Psi : \mathrm{Lip}(X_1) \to \mathrm{Lip}(X_2)$ is an algebra homomorphism with $\Psi(1) = 1$. By [**111**, Theorem 5.1], there is a Lipschitz map $\varphi : X_1 \to X_2$ such that

$$\Psi(g)(x) = g(\varphi(x)), \quad x \in X_2 \tag{2.3}$$

for every $g \in \mathrm{Lip}(X_1)$.

We show that φ is surjective. Suppose that φ is not surjective. Then there exists $x_0 \in X_1 \setminus \varphi(X_2)$. Let

$$\delta = d(x_0, \varphi(X_2)) = \inf_{x \in \varphi(X_2)} d(x_0, x).$$

We define a function $g_{x_0,\delta}$ on X_1 by

$$g_{x_0,\delta}(x) = \max\{1 - \frac{d(x_0, x)}{\delta}, 0\}$$

for $x \in X_1$. By a simple calculation, we have $g_{x_0,\delta} \in \mathrm{Lip}(X_1)$. In addition, for every $x \in \varphi(X_2)$, we obtain $d(x_0, x) \geq \delta$. This implies $\frac{d(x_0,x)}{\delta} \geq 1$. Thus we have

$$g_{x_0,\delta}(x) = 0, \quad x \in \varphi(X_2).$$

This shows that

$$\Psi(g_{x_0,\delta}) = g_{x_0,\delta} \circ \varphi = 0. \tag{2.4}$$

On the other hand, since Ψ is a locally surjective linear isometry, we have

$$\begin{aligned} \Psi(g_{x_0,\delta}) &= \Psi_{g_{x_0,\delta}}(g_{x_0,\delta}) \\ &= \alpha_{g_{x_0,\delta}} g_{x_0,\delta} \circ \varphi_{g_{x_0,\delta}}, \end{aligned} \tag{2.5}$$

where $\alpha_{g_{x_0,\delta}} \in \mathbb{C}$ with $|\alpha_{g_{x_0,\delta}}| = 1$ and a surjective isometry $\varphi_{g_{x_0,\delta}} : X_2 \to X_1$. Thus, there exists $x_0' \in X_2$ such that $\varphi_{g_{x_0,\delta}}(x_0') = x_0$. Taking $x_0' \in X_2$ in (2.5), we obtain

$$\begin{aligned} \Psi(g_{x_0,\delta})(x_0') &= \alpha_{g_{x_0,\delta}} g_{x_0,\delta}(\varphi_{g_{x_0,\delta}}(x_0')) \\ &= \alpha_{g_{x_0,\delta}} g_{x_0,\delta}(x_0) = \alpha_{g_{x_0,\delta}}. \end{aligned}$$

This contradicts the equation (2.4). Hence, φ is surjective.

We show that φ is an isometry from X_2 onto X_1. Let $x_0 \in X_2$. We define a Lipschitz function g' on X_1 by

$$g'(x) = d(x, \varphi(x_0))$$

for all $x \in X_1$. As Ψ is a locally surjective isometry, for every $z \in X_2$, we have

$$\begin{aligned} d(\varphi(z), \varphi(x_0)) &= g'(\varphi(z)) \\ &= (\Psi g')(z) = \Psi_{g'}(g')(z) = \alpha_{g'} g'(\varphi_{g'}(z)) = \alpha_{g'} d(\varphi_{g'}(z), \varphi(x_0)), \end{aligned} \tag{2.6}$$

where $\alpha_{g'} \in \mathbb{C}$ with $|\alpha_{g'}| = 1$ and $\varphi_{g'} : X_2 \to X_1$ is a surjective isometry. As $d(\cdot,\cdot) \geq 0$, we get $\alpha_g = 1$. Taking $z = x_0$ in (2.6), we have

$$0 = d(\varphi(x_0), \varphi(x_0)) = g'(\varphi(x_0)) \\ = \alpha_{g'} d(\varphi_{g'}(x_0), \varphi(x_0)) = d(\varphi_{g'}(x_0), \varphi(x_0)). \tag{2.7}$$

As z and x_0 are arbitrary, we conclude that φ is an isometry by (2.6) and (2.7). It follows that Ψ as defined by (2.3) is a surjective linear isometry. This completes the proof. □

THEOREM 8.2. *Let X_i be a compact metric space and Y_i a compact Hausdorff space for $i = 1, 2$. If the group of all surjective linear isometries from $C(Y_1)$ onto $C(Y_2)$ is algebraically reflexive, then the group of all surjective linear isometries from* $\mathrm{Lip}(X_1, C(Y_1))$ *onto* $\mathrm{Lip}(X_2, C(Y_2))$ *is algebraically reflexive.*

PROOF. Let T be a locally surjective linear isometry from $\mathrm{Lip}(X_1, C(Y_1))$ onto $\mathrm{Lip}(X_2, C(Y_2))$. By Corollary 4.14, for every $F \in \mathrm{Lip}(X_1, C(Y_1))$, there exists $h_F \in C(Y_2)$ with $|h_F| = 1$ on Y_2, a continuous map $\varphi_F : X_2 \times Y_2 \to X_1$ such that $\varphi_F(\cdot, y) : X_2 \to X_1$ is a surjective isometry for every $y \in Y_2$ and a homeomorphism $\tau_F : Y_2 \to Y_1$ which satisfy that

$$T(F)(x, y) = h_F(y) F(\varphi_F(x, y), \tau_F(y)), \quad (x, y) \in X_2 \times Y_2 \tag{2.8}$$

for every $F \in \mathrm{Lip}(X_1, C(Y_1))$. Taking $F = 1$ in (2.8), we get

$$T(\mathbf{1})(x, y) = h_1(y) \mathbf{1}(\varphi_1(x, y), \tau_1(y)) = h_1(y).$$

Considering $\overline{h_1} T$ instead of T we may assume without loss of generality that $T(\mathbf{1}) = \mathbf{1}$. In addition, for every $f \in C(Y_1)$, we have

$$T(\mathbf{1} \otimes f)(x, y) = h_{\mathbf{1}\otimes f}(y) 1 \otimes f(\varphi_{\mathbf{1}\otimes f}(x, y), \tau_{\mathbf{1}\otimes f}(y)) \\ = h_{\mathbf{1}\otimes f}(y) f(\tau_{\mathbf{1}\otimes f}(y)), \quad (x, y) \in X_2 \times Y_2$$

We emphasize that the value $T(1 \otimes f)(x, y)$ does not depend on $x \in X_2$. Thus, we define $\Phi : C(Y_1) \to C(Y_2)$ by

$$\Phi(f)(y) = T(\mathbf{1} \otimes f)(x, y) = h_{\mathbf{1}\otimes f}(y) f(\tau_{\mathbf{1}\otimes f}(y))$$

for every $f \in C(Y_1)$ and $y \in Y_2$. We prove that Φ is linear. Since T is linear, we have

$$\begin{aligned}\Phi(f+g)(y) &= T(1 \otimes (f+g))(x,y) = T(\mathbf{1} \otimes f + \mathbf{1} \otimes g)(x,y) \\ &= T(\mathbf{1} \otimes f)(x,y) + T(\mathbf{1} \otimes g)(x,y) = \Phi(f)(y) + \Phi(g)(y),\end{aligned}$$

and

$$\begin{aligned}\Phi(\lambda f)(y) = T(\mathbf{1} \otimes \lambda f)(x,y) &= T(\lambda(\mathbf{1} \otimes f))(x,y) \\ &= \lambda T(\mathbf{1} \otimes f)(x,y) = \lambda \Phi(f)(y)\end{aligned}$$

for every $f, g \in C(Y_1)$, $\lambda \in \mathbb{C}$, $x \in X_2$ and $y \in Y_2$. Thus Φ is linear. Let $f \in C(Y_1)$. We define $\Phi_f : C(Y_1) \to C(Y_2)$ by

$$\Phi_f(g)(y) = h_{\mathbf{1}\otimes f}(y) g(\tau_{\mathbf{1}\otimes f}(y)), \quad y \in Y_2$$

for every $g \in C(Y_1)$. We get Φ_f is a surjective linear isometry by the Banach-Stone theorem. Thus Φ is a linear map and for every $f \in C(Y_1)$ there exists a surjective linear isometry Φ_f such that $\Phi(f) = \Phi_f(f)$, that is, Φ is a locally surjective linear isometry from $C(Y_1)$ onto $C(Y_2)$. The assumption that the group of all surjective linear isometries from $C(Y_1)$ onto $C(Y_2)$ is algebraically reflexive ensures that Φ is a surjective linear isometry from $C(Y_1)$ onto $C(Y_2)$. The Banach-Stone theorem asserts that there exists a homeomorphism $\tau : Y_2 \to Y_1$ and $h \in C(Y_2)$ with $|h| = 1$ on Y_2 such that

$$T(\mathbf{1} \otimes f)(x,y) = \Phi(f)(y) = h(y) f(\tau(y))$$

for all $f \in C(Y_1)$ and $(x,y) \in X_2 \times Y_2$. Moreover by the assumption that $T(\mathbf{1}) = \mathbf{1}$, we have $h(y) = 1$ for every $y \in Y_2$. This implies

$$T(\mathbf{1} \otimes f)(x,y) = f(\tau(y)) \tag{2.9}$$

for every $f \in C(Y_1)$ and $(x,y) \in X_2 \times Y_2$.

Let $g \in \mathrm{Lip}(X_1)$. Substituting $F = g \otimes \mathbf{1}$ in (2.8), we get

$$\begin{aligned}T(g \otimes \mathbf{1})(x,y) &= h_{g\otimes\mathbf{1}}(y) g \otimes \mathbf{1}(\varphi_{g\otimes\mathbf{1}}(x,y), \tau_{g\otimes\mathbf{1}}(y)) \\ &= h_{g\otimes\mathbf{1}}(y) g(\varphi_{g\otimes\mathbf{1}}(x,y)), \quad (x,y) \in X_2 \times Y_2\end{aligned}$$

for every $g \in \mathrm{Lip}(X_1)$. Fix $y \in Y_2$. We define $\Psi : \mathrm{Lip}(X_1) \to \mathrm{Lip}(X_2)$ by

$$\Psi(g)(x) = T(g \otimes \mathbf{1})(x,y) = h_{g\otimes\mathbf{1}}(y) g(\varphi_{g\otimes\mathbf{1}}(x,y))$$

for every $g \in \mathrm{Lip}(X_1)$ and $x \in X_2$. Since $\varphi_{g\otimes\mathbf{1}}(\cdot, y) : X_2 \to X_1$ is a surjective linear isometry and $|h_{g\otimes\mathbf{1}}(y)| = 1$, the map Ψ is a locally surjective linear isometry. By Theorem 8.1, Ψ is a surjective linear isometry from $\mathrm{Lip}(X_f)$ onto $\mathrm{Lip}(X_2)$. By Corollary 15 in [**42**] there exist a surjective isometry $\varphi(\cdot, y) : X_2 \to X_1$ and a complex number $\alpha(y)$ with unit modulus such that

$$T(g \otimes \mathbf{1})(x, y) = \Psi(g)(x) = \alpha(y)g(\varphi(x, y))$$

for every $g \in \mathrm{Lip}(X_1)$ and $x \in X_2$. In addition by the assumption that $T(\mathbf{1}) = \mathbf{1}$, we deduce

$$\alpha(y) = \alpha(y)\mathbf{1}(\varphi(x, y)) = T(\mathbf{1})(x, y) = \mathbf{1}$$

for every $y \in Y_2$. This shows that

$$T(g \otimes \mathbf{1})(x, y) = g(\varphi(x, y)) \tag{2.10}$$

for every $(x, y) \in X_2 \times Y_2$. By (2.8), we have

$$\|T(F)\|_\infty = \|F\|_\infty$$

for all $F \in \mathrm{Lip}(X_1, C(Y_1))$. For any $i = 1, 2$, $\mathrm{Lip}(X_i, C(Y_i))$ is a linear subspace of $C(X_i \times Y_i)$ which separates the points of $X_i \times Y_i$ and contains the constant functions. Applying Theorem 1 in Novinger [**97**], since $T(\mathbf{1}) = \mathbf{1}$, there exists a surjective continuous map
$\phi : \mathrm{Ch}(T(\mathrm{Lip}(X_1, C(Y_1)))) \to \mathrm{Ch}(\mathrm{Lip}(X_1, C(Y_1)))$ such that

$$T(F) = F \circ \phi \tag{2.11}$$

for every $F \in \mathrm{Lip}(X_1, C(Y_1))$, where $\mathrm{Ch}(\cdot)$ denotes the Choquet boundary (cf. [**97, 102**]). We show that $\mathrm{Ch}(T(\mathrm{Lip}(X_1, C(Y_1)))) = X_2 \times Y_2$. Let $(x_0, y_0) \in X_2 \times Y_2$. Suppose that μ is a probability regular Borel measure on $X_2 \times Y_2$ such that $F(x_0, y_0) = \int_{X_2\times Y_2} F d\mu$ for each $F \in T(\mathrm{Lip}(X_1, C(Y_1)))$. We prove $\mu(S) = 1$ for every Borel set S in $X_2 \times Y_2$ which contains (x_0, y_0). It will follow that μ is the Dirac measure at (x_0, y_0), and $(x_0, y_0) \in \mathrm{Ch}(T(\mathrm{Lip}(X_1, C(Y_1))))$ by [**102**, Proposition 6.2]. Let S be an arbitrary Borel set of $X_2 \times Y_2$ which contains (x_0, y_0). As μ is a regular measure on a compact set, we may suppose that S is an open set. Under the natural projection π of $X_2 \times Y_2$ onto Y_2, we have $\pi(S)$ is an open set of Y_2. Since $y_0 \in \pi(S)$, there exists $f \in C(Y_2)$ such that $|f(y_0)| = \|f\|_\infty$ and $|f| < \|f\|_\infty$ in $Y_2 \setminus \pi(S)$. We denote $\theta \in [0, 2\pi)$ by $f(y_0) = e^{i\theta}|f(y_0)|$. Moreover we define $g \in \mathrm{Lip}(X_2)$ by $g(x) = \max\{1 - d(x, x_0), 0\}$. We define $F \in \mathrm{Lip}(X_2, C(Y_2))$ by $F = e^{i\theta} g \otimes \mathbf{1} + \mathbf{1} \otimes f$. By (2.9) and (2.10), since $T(e^{i\theta} g \circ \varphi^{-1} \otimes \mathbf{1} + \mathbf{1} \otimes f \circ \tau^{-1}) = F$, we get $F \in T(\mathrm{Lip}(X_1, C(Y_1)))$. We have

$$\|F\|_\infty = \|e^{i\theta} g \otimes \mathbf{1} + \mathbf{1} \otimes f\|_\infty \leq \|g\|_\infty + \|f\|_\infty = 1 + \|f\|_\infty,$$

and

$$|F(x_0, y_0)| = |e^{i\theta} g(x_0) + f(y_0)| = |e^{i\theta} g(x_0) + e^{i\theta}|f(y_0)|| = 1 + \|f\|_\infty.$$

Thus we get

$$\|F\|_\infty = 1 + \|f\|_\infty = |F(x_0, y_0)|.$$

In addition, if $|e^{i\theta} g(x) + f(y)| = |F(x, y)| = 1 + \|f\|_\infty$ then we have $x = x_0$ and $y \in \pi(S)$, thus $(x, y) \in \{x_0\} \times \pi(S) \subset S$. Therefore we get

$$(x_0, y_0) \in \{(x, y) \in X_2 \times Y_2; |F(x, y)| = \|F\|_\infty\} \subset S.$$

Suppose that $\mu(X_2 \times Y_2 \setminus S) \neq 0$. Then

$$\begin{aligned}
\|F\|_\infty = |F(x_0, y_0)| = &| \int_{X_2 \times Y_2} F d\mu| \leq \int_S |F| d\mu + \int_{X_2 \times Y_2 \setminus S} |F| d\mu \\
&\leq \|F\|_\infty \mu(S) + \int_{X_2 \times Y_2 \setminus S} |F| d\mu \\
&< \|F\|_\infty \mu(S) + \|F\|_\infty \mu(X_2 \times Y_2 \setminus S) = \|F\|_\infty,
\end{aligned}$$

since

$$\int_{X_2 \times Y_2 \setminus S} |F| d\mu < \|F\|_\infty \mu(X_2 \times Y_2 \setminus S)$$

by the assumption $\mu(X_2 \times Y_2 \setminus S) \neq 0$. We arrive at a contradiction. This shows that $\mu(X_2 \times Y_2 \setminus S) = 0$ and $\mu(S) = 1$. We obtain that every Borel set which contains (x_0, y_0) has measure 1. By Proposition 6.2 in [**102**], (x_0, y_0) is in the Choquet boundary of $T(\mathrm{Lip}(X_1, C(Y_1)))$. Thus we conclude that $\mathrm{Ch}(T(\mathrm{Lip}(X_1, C(Y_1)))) = X_2 \times Y_2$. By (2.11), T is an algebra homomorphism. By (2.9) and (2.10), we have

$$\begin{aligned}
T(g \otimes f)(x, y) &= T(g \otimes \mathbf{1})(x, y) T(\mathbf{1} \otimes f)(x, y) \\
&= g(\varphi(x, y)) f(\tau(y)) \\
&= (g \otimes f)(\varphi(x, y), \tau(y))
\end{aligned}$$

for every $g \in \mathrm{Lip}(X_1)$, $f \in C(Y_1)$ and $(x,y) \in X_2 \times Y_2$. For any $F \in \mathrm{Lip}(X_1) \otimes C(Y_1)$, F is represented as follows:

$$F = \Sigma_{i=1}^{n} g_i \otimes f_i$$

with some $g_i \in \mathrm{Lip}(X_1)$ and $f_i \in C(Y_1)$ for $n \in \mathbb{N}$. Thus we have

$$\begin{aligned} T(F)(x,y) &= \Sigma_{i=1}^{n} T(g_i \otimes f_i)(x,y) \\ &= \Sigma_{i=1}^{n} (g_i \otimes f_i)(\varphi(x,y), \tau(y)) \\ &= F(\varphi(x,y), \tau(y)) \end{aligned}$$

for every $(x,y) \in X_2 \times Y_2$. Since T is isometry with $\|\cdot\|_\infty$ and $\mathrm{Lip}(X_i, C(Y_i))$ is a subset of the closure of $\mathrm{Lip}(X_i) \otimes C(Y_i)$ with $\|\cdot\|_\infty$, we conclude

$$T(F)(x,y) = F(\varphi(x,y), \tau(y))$$

for $F \in \mathrm{Lip}(X_1, C(Y_1))$ and $(x,y) \in X_2 \times Y_2$ in the same way as in [**98**, Theorem 1]. Finally we prove that the map $\varphi : X_2 \times Y_2 \to X_1$ is continuous. Let $(x_0, y_0) \in X_2 \times Y_2$. We define $g \in \mathrm{Lip}(X_1)$ by

$$g(z) = d(z, \varphi(x_0, y_0))$$

for every $z \in X_1$. We have

$$\begin{aligned} d(\varphi(x,y), \varphi(x_0,y_0)) &= g(\varphi(x,y)) - g(\varphi(x_0,y_0)) \\ &= T(g \otimes \mathbf{1})(x,y) - T(g \otimes \mathbf{1})(x_0,y_0). \end{aligned}$$

Since $T(g \otimes \mathbf{1}) \in \mathrm{Lip}(X_2, C(Y_2)) \subset C(X_2 \times Y_2)$, for every $\epsilon > 0$, there exists G which is a neighborhood of $(x_0, y_0) \in X_2 \times Y_2$ such that if $(x,y) \in G$ then we have

$$|T(g \otimes \mathbf{1})(x,y) - T(g \otimes \mathbf{1})(x_0,y_0)| < \epsilon.$$

Hence we have

$$d(\varphi(x,y), \varphi(x_0,y_0)) = |T(g \otimes \mathbf{1})(x,y) - T(g \otimes \mathbf{1})(x_0,y_0)| < \epsilon.$$

It follows that $\varphi : X_2 \times Y_2 \to X_1$ is continuous. By Corollary 4.14, we have T is a surjective linear isometry. $\square$

Molnár and Zalar [**92**, Theorem 2.2] proved that if Ω is a first countable compact Hausdorff space, then the group of all surjective isometries of $C(\Omega)$ onto itself is algebraically reflexive.

3. The group of surjective isometries on a Banach algebra of Lipschitz maps whose values are in $M_n(\mathbb{C})$

Let $M_n(\mathbb{C})$ be the C^*-algebra of all $n \times n$ matrices with the complex entries. If Φ is a linear isometry on $M_n(\mathbb{C})$, then Φ is injective. Since $M_n(\mathbb{C})$ is a finite dimensional vector space, we have Φ is also surjective. Hence the group of all surjective complex linear isometries on $M_n(\mathbb{C})$ is reflexive.

THEOREM 8.3. *Let X_i be a compact metric space for $i = 1, 2$. The group of all unital surjective linear isometries from* $\mathrm{Lip}(X_1, M_n(\mathbb{C}))$ *onto* $\mathrm{Lip}(X_2, M_n(\mathbb{C}))$ *is algebraically reflexive.*

PROOF. Let $T : \mathrm{Lip}(X_1, M_n(\mathbb{C})) \to \mathrm{Lip}(X_2, M_n(\mathbb{C}))$ be a locally unital surjective linear isometry. For any $A \in M_n(\mathbb{C})$, by Theorem 6.9,

$$T(\mathbf{1} \otimes A)(x) = T_{\mathbf{1}\otimes A}(\mathbf{1} \otimes A)(x) = \Phi_{\mathbf{1}\otimes A}(A),$$

where $\Phi_{\mathbf{1}\otimes A}$ is a surjective linear isometry on $M_n(\mathbb{C})$. We define $\Phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ by $\Phi(A) = \Phi_{\mathbf{1}\otimes A}(A)$ for every $A \in M_n(\mathbb{C})$. It follows that Φ is a locally unital surjective linear isometry on $M_n(\mathbb{C})$. Then $\Phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is a surjective linear isometry such that

$$T(\mathbf{1} \otimes A)(x) = \Phi_{\mathbf{1}\otimes A}(A) = \Phi(A) \tag{3.1}$$

for every $x \in X_2$ and $A \in M_n(\mathbb{C})$. Let $E_{ij} \in M_n(\mathbb{C})$ denote the matrix with all 0 except at the position ij where it should be 1 for every $1 \leq i, j \leq n$. The collection of $\{E_{ij}\}_{1\leq i,j\leq n}$ is a basis for $M_n(\mathbb{C})$ as a linear space. Since Φ is a surjective linear isometry, we have that $\{\Phi(E_{ij})\}_{1\leq i,j\leq n}$ is a basis for $M_n(\mathbb{C})$, too. Let $k_1, k_2 \in \{1, \cdots, n\}$ and $x \in X_2$. For any $g \in \mathrm{Lip}(X_1)$, since $T(g \otimes E_{k_1k_2})(x) \in M_n(\mathbb{C})$, there exists complex numbers $\{\lambda^{ij}_{k_1k_2}(x)\}_{1\leq i,j\leq n}$ such that

$$T(g \otimes E_{k_1k_2})(x) = \Sigma_{i,j}\lambda^{ij}_{k_1k_2}(x)\Phi(E_{ij}). \tag{3.2}$$

Let $i, j \in \{1, \cdots, n\}$. We define the map $\delta^{ij}_{(k_1k_2,x)} : \mathrm{Lip}(X_1) \to \mathbb{C}$ by

$$\delta^{ij}_{(k_1k_2,x)}(g) = \lambda^{ij}_{k_1k_2}(x)$$

for every $g \in \mathrm{Lip}(X_1)$. By (3.2), it follows that

$$T(g \otimes E_{k_1k_2})(x) = \Sigma_{i,j}\delta^{ij}_{(k_1k_2,x)}(g)\Phi(E_{ij}). \tag{3.3}$$

For any $g_1, g_2 \in \text{Lip}(X_1)$, we have

$$T((g_1+g_2)\otimes E_{k_1k_2})(x) = \Sigma_{i,j}\delta^{ij}_{(k_1k_2,x)}(g_1+g_2)\Phi(E_{ij})$$

and

$$\begin{aligned}T((g_1+g_2)\otimes E_{k_1k_2})(x) &= T(g_1\otimes E_{k_1k_2})(x) + T(g_2\otimes E_{k_1k_2})(x)\\ &= \Sigma_{i,j}\delta^{ij}_{(k_1k_2,x)}(g_1)\Phi(E_{ij}) + \Sigma_{i,j}\delta^{ij}_{(k_1k_2,x)}(g_2)\Phi(E_{ij})\\ &= \Sigma_{i,j}(\delta^{ij}_{(k_1k_2,x)}(g_1) + \delta^{ij}_{(k_1k_2,x)}(g_2))\Phi(E_{ij}).\end{aligned}$$

As $\{\Phi(E_{ij})\}_{1\le i,j\le n}$ is a basis, we obtain

$$\delta^{ij}_{(k_1k_2,x)}(g_1+g_2) = \delta^{ij}_{(k_1k_2,x)}(g_1) + \delta^{ij}_{(k_1k_2,x)}(g_2)$$

for every $i, j \in \{1, \cdots, n\}$. For any $\lambda \in \mathbb{C}$ and $g \in \text{Lip}(X_1)$, we have

$$T(\lambda g\otimes E_{k_1k_2})(x) = \Sigma_{i,j}\delta^{ij}_{(k_1k_2,x)}(\lambda g)\Phi(E_{ij})$$

and

$$\begin{aligned}T(\lambda g\otimes E_{k_1k_2})(x) &= \lambda T(g\otimes E_{k_1k_2})(x)\\ &= \lambda\Sigma_{i,j}\delta^{ij}_{(k_1k_2,x)}(g)\Phi(E_{ij}) = \Sigma_{i,j}\lambda\delta^{ij}_{(k_1k_2,x)}(g)\Phi(E_{ij})\end{aligned}$$

This shows that

$$\delta^{ij}_{(k_1k_2,x)}(\lambda g) = \lambda\delta^{ij}_{(k_1k_2,x)}(g)$$

for $i, j \in \{1, \cdots, n\}$. Since T is a locally unital surjective linear isometry, for every $F \in \text{Lip}(X_1, M_n(\mathbb{C}))$ there exist a surjective linear isometry Φ_F on $M_n(\mathbb{C})$ and a surjective isometry $\varphi_F : X_2 \to X_1$ such that

$$T(F) = T_F(F) = \Phi_F(F\circ\varphi_F).$$

Therefore we have

$$\|TF\|_\infty = \|\Phi_F(F\circ\varphi_F)\|_\infty = \|F\circ\varphi_F\|_\infty = \|F\|_\infty.$$

If follows that T is an isometry with $\|\cdot\|_\infty$. Moreover since $M_n(\mathbb{C})$ is of a finite dimension, every norm on $M_n(\mathbb{C})$ is equivalent to each other. Thus for $g \in \text{Lip}(X_1)$ we have

$$\begin{aligned}|\delta^{ij}_{(k_1k_2,x)}(g)| \le \Sigma_{i,j}|\delta^{ij}_{(k_1k_2,x)}(g)| &\le M\|T(g\otimes E_{k_1k_2})(x)\|\\ &\le M\|T(g\otimes E_{k_1k_2})\|_\infty = M\|g\otimes E_{k_1k_2}\|_\infty\\ &\le M\|g\otimes E_{k_1k_2}\|_\Sigma = M\|g\|_\Sigma\|E_{k_1k_2}\|,\end{aligned}$$

for some $M > 0$. Thus $\delta^{ij}_{(k_1k_2,x)}$ is a bounded linear functional on $\text{Lip}(X_1)$. Let $g \in \text{Lip}(X_1)$ be an invertible. Then we have $g(x) \neq 0$

for every $x \in X_1$. We shall prove that $\delta^{k_1k_2}_{(k_1k_2,x)}(g) \neq 0$. Aiming for a contradiction, suppose that $\delta^{k_1k_2}_{(k_1k_2,x)}(g) = 0$. Let $1 \leq i, j \leq n$ with $i \neq k_1$ and $j \neq k_2$ and complex numbers α_{ij}, we define $F \in \mathrm{Lip}(X_1, M_n(\mathbb{C}))$ by

$$F = g \otimes E_{k_1k_2} + \Sigma_{i\neq k_1, j\neq k_2}\alpha_{ij} 1 \otimes E_{ij}.$$

Since $\{E_{ij}\}_{1\leq i,j\leq n}$ is a basis, we have $F(x) \neq 0$ for all $x \in X_1$. Thus for every $x \in X_2$ we have

$$T(F)(x) = \Phi_F(F(\varphi_F(x))) \neq 0,$$

since Φ_F is a surjective linear isometry on $M_n(\mathbb{C})$ and $\varphi_F : X_2 \to X_1$ is a surjective isometry. On the other hand, the hypothesis $\delta^{k_1k_2}_{(k_1k_2,x)}(g) = 0$ implies that $T(g\otimes E_{k_1k_2})(x)$ is a linear combination of the set $\{\Phi(E_{ij})\}_{i\neq k_1, j\neq k_2}$. Therefore, there exist complex numbers α_{ij} for $1 \leq i, j \leq n$ with $i \neq k_1$ and $j \neq k_2$ such that

$$T(g \otimes E_{k_1k_2})(x) = \Sigma_{i\neq k_1, j\neq k_2}\alpha_{ij}\Phi(E_{ij}) = \Sigma_{i\neq k_1, j\neq k_2}\alpha_{ij}T(1 \otimes E_{ij})(x).$$

This implies that

$$T(g \otimes E_{k_1k_2} - \Sigma_{i\neq k_1, j\neq k_2}\alpha_{ij} \otimes E_{ij})(x) = 0.$$

We arrive at a contradiction. It follows that $\delta^{k_1k_2}_{(k_1k_2,x)}(g) \neq 0$. In addition by (3.2), we have $T(\mathbf{1} \otimes E_{k_1k_2})(x) = \Phi(E_{k_1k_2})$. It follows that $\delta^{k_1k_2}_{(k_1k_2,x)}(\mathbf{1}) = 1$. The well known Gleason-Kahane-Żelazko theorem asserts that $\delta^{k_1k_2}_{(k_1k_2,x)}$ is multiplicative. Since the maximal ideal space of $\mathrm{Lip}(X_1)$ is homeomorphic to X_1, there exists $\varphi_{k_1k_2}(x) \in X_1$ such that $\delta^{k_1k_2}_{(k_1k_2,x)}(g) = g(\varphi_{k_1k_2}(x))$. We define $G_1 \in \mathrm{Lip}(X_1, M_n(\mathbb{C}))$ by

$$G_1 = g \otimes E_{k_1k_2} - \Sigma_{i,j}\delta^{ij}_{(k_1k_2,x)}(g)1 \otimes E_{ij}$$

By (3.2), we obtain

$$\begin{aligned} T(G_1)(x) &= T(g \otimes E_{k_1k_2} - \Sigma_{i,j}\delta^{ij}_{(k_1k_2,x)}(g)\mathbf{1} \otimes E_{ij})(x) \\ &= T(g \otimes E_{k_1k_2})(x) - \Sigma_{i,j}\delta^{ij}_{(k_1k_2,x)}(g)T(\mathbf{1} \otimes E_{ij})(x) \\ &= T(g \otimes E_{k_1k_2})(x) - \Sigma_{i,j}\delta^{ij}_{(k_1k_2,x)}(g)\Phi(E_{ij}) \\ &= 0. \end{aligned}$$

Since T is a locally unital surjective linear isometry, there exist a surjective linear isometry Φ_{G_1} on $M_n(\mathbb{C})$ and a surjective isometry

$\varphi_{G_1} : X_2 \to X_1$ such that

$$\begin{aligned} 0 &= T(G_1)(x) \\ &= \Phi_{G_1}(g \otimes E_{k_1k_2} - \Sigma_{i,j}\delta^{ij}_{(k_1k_2,x)}(g)\mathbf{1} \otimes E_{ij})(\varphi_{G_1}(x)). \end{aligned}$$

Since Φ_{G_1} is an isometry, we have

$$(g \otimes E_{k_1k_2} - \Sigma_{i,j}\delta^{ij}_{(k_1k_2,x)}(g)\mathbf{1} \otimes E_{ij})(\varphi_{G_1}(x)) = 0.$$

As $\{E_{ij}\}_{i,j}$ is a basis on $M_n(\mathbb{C})$, this implies that

$$g(\varphi_{G_1}(x)) - \delta^{k_1k_2}_{(k_1k_2,x)}(g) = 0$$

and

$$\delta^{ij}_{(k_1k_2,x)}(g) = 0, \quad i \neq k_1, j \neq k_2. \tag{3.4}$$

By (3.3) and (3.4) for every $g \in \mathrm{Lip}(X_1)$, we get

$$T(g \otimes E_{k_1k_2})(x) = \delta^{k_1k_2}_{(k_1k_2,x)}(g)\Phi(E_{k_1k_2}) = g(\varphi_{k_1k_2}(x))\Phi(E_{k_1k_2}).$$

As k is arbitrary,

$$T(g \otimes E_{k_1k_2})(x) = g(\varphi_{k_1k_2}(x))\Phi(E_{k_1k_2})$$

for every $k_1, k_2 \in \{1, \cdots, n\}$. We now prove that $\varphi_{k_1k_2} = \varphi_{l_1l_2}$ even if $k_1 \neq l_1$ or $k_2 \neq l_2$. Suppose, towards a contradiction, that there exist $x \in X_2$ and $k_1, k_2, l_1, l_2 \in \{1, \cdots, n\}$ such that

$$\varphi_{k_1k_2}(x) \neq \varphi_{l_1l_2}(x).$$

We define $g_{k_1k_2} \in \mathrm{Lip}(X_1)$ and $g_{l_1l_2} \in \mathrm{Lip}(X_1)$by

$$g_{k_1k_2}(z) = d(z, \varphi_{k_1k_2}(x))$$

and

$$g_{l_1l_2}(z) = d(z, \varphi_{l_1l_2}(x)).$$

In addition, we define $G \in \mathrm{Lip}(X_1, M_n(\mathbb{C}))$ by

$$G = g_{k_1k_2} \otimes E_{k_1k_2} + g_{l_1l_2} \otimes E_{l_1l_2}.$$

We have

$$\begin{aligned} T(G)(x) &= T(g_{k_1k_2} \otimes E_{k_1k_2} + g_{l_1l_2} \otimes E_{l_1l_2})(x) \\ &= g_{k_1k_2}(\varphi_{k_1k_2}(x))\Phi(E_{k_1k_2}) + g_{l_1l_2}(\varphi_{l_1l_2}(x))\Phi(E_{l_1l_2}) \\ &= 0. \end{aligned}$$

As T is a locally unital surjective linear isometry, there exist a surjective linear isometry Φ_G on $M_n(\mathbb{C})$ and a surjective isometry $\varphi_G : X_2 \to X_1$ such that

$$T(G) = \Phi_G(G \circ \varphi_G).$$

We define $z = \varphi_G(x)$, then

$$0 = T(G)(x) = \Phi_G(G(\varphi_G(x))) = \Phi_G(G(z)).$$

Since Φ_G is a surjective linear isometry on $M_n(\mathbb{C})$, we obtain $G(z) = 0$. Therefore we have

$$g_{k_1k_2}(z)E_{k_1k_2} + g_{l_1l_2}(z)E_{l_1l_2} = 0.$$

Thus we obtain

$$g_{k_1k_2}(z) = g_{l_1l_2}(z) = 0,$$

in contradiction with the definition of $g_{k_1k_2}$ and $g_{l_1l_2}$ and the hypotheses of $\varphi_{k_1k_2}(x) \neq \varphi_{l_1l_2}(x)$. This implies that

$$\varphi_{k_1k_2} = \varphi_{l_1l_2}$$

for $k_1, k_2, l_1, l_2 \in \{1, \cdots, n\}$. Thus we define $\varphi : X_2 \to X_1$ by

$$\varphi(x) = \varphi_{k_1k_2}(x)$$

for every $k_1, k_2 \in \{1 \cdots, n\}$. Therefore we have

$$T(g \otimes E_{k_1k_2})(x) = g(\varphi(x))\Phi(E_{k_1k_2})$$

for every $k_1, k_2 \in \{1 \cdots, n\}$ and $x \in X_2$. By Lemma 7, for every $F \in \mathrm{Lip}(X_1, M_n(\mathbb{C}))$, F is represented as follows:

$$F = \Sigma_{i,j} g_{ij} \otimes E_{ij}$$

with some $g_{ij} \in \mathrm{Lip}(X_1)$. Thus we have

$$\begin{aligned}(TF)(x) = T(\Sigma_{i,j} g_{ij} \otimes E_{ij})(x) &= \Sigma_{i,j} T(g_{ij} \otimes E_{ij})(x) \\ = \Sigma_{i,j} g_{ij}(\varphi(x))\Phi(E_{ij}) &= \Phi(\Sigma_{i,j} g_{ij} \otimes E_{ij}(\varphi(x))) \\ &= \Phi(F(\varphi(x)))\end{aligned}$$

for every $x \in X_2$. Since $T(\mathbf{1}) = \mathbf{1}$, we conclude that $\varphi : X_2 \to X_1$ is a surjective isometry in the same way as in proof for Theorem 8.1. By Theorem 6.9, we have that T is a unital surjective linear isometry from $\mathrm{Lip}(X_1, M_n(\mathbb{C}))$ onto $\mathrm{Lip}(X_2, M_n(\mathbb{C}))$. □

CHAPTER 9

2-local isometry with W_j

1. Introduction to 2-local isometries

Motivated by the paper by Kowalski and Słodkowski [**69**], the concept of 2-locality was introduced by Šemrl,who obtained the first results on 2-local automorphisms and 2-local derivations [**110**]. Molnár [**87**] studied 2-local isometries on operator algebras. Given a metric space $\mathfrak{M}_j$ for $j = 1, 1$ an isometry from $\mathfrak{M}_1$ into $\mathfrak{M}_2$ is a distance preserving map. The set of all surjective isometries from $\mathfrak{M}_1$ onto $\mathfrak{M}_2$ is denoted by $\mathrm{Iso}(\mathfrak{M}_1, \mathfrak{M}_2)$. We say a map $T : \mathfrak{M}_1 \to \mathfrak{M}_2$ is 2-local in $\mathrm{Iso}(\mathfrak{M}_1, \mathfrak{M}_2)$ if for every pair $x, y \in \mathfrak{M}_1$ there exists a surjective isometry $T_{x,y} \in \mathrm{Iso}(\mathfrak{M}_1, \mathfrak{M}_2)$ such that

$$T(x) = T_{x,y}(x) \text{ and } T(y) = T_{x,y}(y).$$

In this case we say that T is a 2-local isometry. It is obvious by the definition that a 2-local isometry is in fact an isometry, which needs not be surjective. Hence a 2-local isometry T belongs to $\mathrm{Iso}(\mathfrak{M}_1, \mathfrak{M}_2)$ if T is surjective. We say that $\mathrm{Iso}(\mathfrak{M}_1, \mathfrak{M}_2)$ is 2-locally reflexive if every 2-local isometry belongs to $\mathrm{Iso}(\mathfrak{M}_1, \mathfrak{M}_2)$.

If $\mathfrak{M}_j$ is a Banach space, linearity of the maps are subjects of consideration. Let $\mathrm{Iso}_{\mathbb{C}}(\mathfrak{M}_1, \mathfrak{M}_2)$ denote the set of all surjective *complex-linear* isometries. There exists a extensive literature on 2-local isometries in $\mathrm{Iso}_{\mathbb{C}}(\mathfrak{M}_1, \mathfrak{M}_2)$ and 2-locally reflexivity of $\mathrm{Iso}_{\mathbb{C}}(\mathfrak{M}_1, \mathfrak{M}_2)$ (see, for example, [**1, 17, 34, 38, 53, 50, 75, 87**]). Note that Hosseini [**46**] proved that a 2-local *real-linear* isometries is in fact a surjective real-linear isometry on the algebra of n-times continuously differentiable functions on the interval $[0, 1]$ with a certain norm. She also proved [**46**, Proposition 3.2] that a 2-local real-linear isometry defined on the Banach algebra $C(X)$ of all complex-valued continuous functions on a compact Hausdorff space X with additional hypotheses that it is separable and first countable is in fact a surjective real-linear isometry on $C(X)$. We note that the situation is very different for the problem of 2-local isometry.

In this dissertation we study 2-local reflexivity for $\mathrm{Iso}(\mathfrak{M}_1, \mathfrak{M}_2)$ and we consider the question whether every 2-local isometry necessarily belongs to $\mathrm{Iso}(\mathfrak{M}_1, \mathfrak{M}_2)$ where $\mathfrak{M}_j$ is a certain spaces of continuous functions.

In the remaining of this chapter E_j is a subspace of $C(X_j)$ which contains the constant functions and separates the points of X_j. For $c \in \mathbb{C}$ we write the constant function which takes the value c by c. We assume that the norm $\|\cdot\|_j$ is defined on E_j (not necessary complete) and it satisfies that $\|c\|_j = |c|$ for every $c \in \mathbb{C}$. We assume that E_j is conjugate closed in the sense that $f \in E_j$ implies $\bar{f} \in E_j$, and that $\|f\|_j = \|\bar{f}\|_j$ for every $f \in E_j$. For an $\epsilon \in \{\pm 1\}$ and $f \in E_j$, $[f]^\epsilon = f$ if $\epsilon = 1$ and $[f]^\epsilon = \bar{f}$ if $\epsilon = -1$. Let $M(E_1, E_2)$ be the set of all maps from E_1 into E_2. Note that we say a map is a surjective isometry if it is just a distance preserving map, we do not assume complex nor real linearity on it. We abbreviate $\mathrm{Iso}(E_j, E_j)$ by $\mathrm{Iso}(E_j)$. Let Π denotes a non-empty set of (not always all) homeomorphisms from E_2 onto E_1. Let

$$\begin{aligned} G_\Pi(E_1, E_2) = \{T \in M(E_1, E_2) : &\text{ there exists } \lambda \in E_2, \\ &\alpha \in \mathbb{C} \text{ of unit modulus, } \pi \in \Pi, \text{ and } \epsilon \in \{\pm 1\} \\ &\text{such that } T(f) = \lambda + \alpha [f \circ \pi]^\epsilon \text{ for every } f \in E_1\}. \end{aligned}$$

We abbreviate $G_\Pi(E_j, E_j)$ by $G_\Pi(E_j)$. We usually abbreviate $G_\Pi(E_1, E_2)$ by G_Π if E_1 and E_2 are clear from the context. Let $\mathrm{Id}_{[0,1]} = \pi_0 : [0,1] \to [0,1]$ be the identity function and $\pi_1 = 1 - \mathrm{Id}_{[0,1]}$. Put $\Pi_0 = \{\pi_0, \pi_1\}$. Kawamura, Koshimizu and Miura [**60**] (cf. [**84**]) proved that $G_{\Pi_0}(C^1[0,1]) = \mathrm{Iso}(C^1[0,1], \|\cdot\|)$ with respect to several norms including $\|\cdot\|_\Sigma$, we will show later that $G_{\Pi_0}(\mathrm{Lip}[0,1]) = \mathrm{Iso}(\mathrm{Lip}[0,1]), \|\cdot\|_{\sum})$, where $\|f\|_\Sigma = \|f\|_\infty + L(f)$ for the Lipschitz constant $L(f) = \sup_{x,y} \frac{|f(x)-f(y)|}{|x-y|}$, $f \in \mathrm{Lip}[0,1]$. Note that f' exists and

$f' \in L^\infty[0,1]$ and $\|f'\|_\infty = L(f)$ for $f \in \mathrm{Lip}[0,1]$ (cf. [**35**]). Let

$$W_j = \{f \in E_j : \text{if } S : \mathbb{C} \to \mathbb{C} \text{ is an isometry} \\ \text{and } S(f(X_j)) = f(X_j), \text{ then } S \text{ is the identity function}\}.$$

Suppose that $S : \mathbb{C} \to \mathbb{C}$ is an isometry. It is well known that there exists $a, b \in \mathbb{C}$ with $|a| = 1$ such that $S(z) = b + az$, $(z \in \mathbb{C})$ or $S(z) = b + a\bar{z}$, $(z \in \mathbb{C})$. The first case of S is a parallel translation by b if $a = 1$ and S is a rotation around $b/(1-a)$. Hence there is no fixed point if $a = 1$ and $b \neq 0$, and $b/(1-a)$ is the unique fixed point if $a \neq 1$ for the first case. For the second case, denoting one of the square root of a by $a^{\frac{1}{2}}$, S is a symmetric translation with respect to the line $t \mapsto a^{\frac{1}{2}}t + i(\mathrm{Im}(\overline{a^{\frac{1}{2}}}b)/2)$ $(t \in \mathbb{R})$ followed by the parallel translation by $\mathrm{Re}(\overline{a^{\frac{1}{2}}}b)/2)$ to the direction of $a^{\frac{1}{2}}$. Hence the fixed points exist and they are all the points on line $t \mapsto a^{\frac{1}{2}}t + i(\mathrm{Im}(\overline{a^{\frac{1}{2}}}b)/2)$ if and only if $\mathrm{Re}(\overline{a^{\frac{1}{2}}}b)/2) = 0$. We will prove that W_j for $E_j = C^1[0,1]$ is uniformly dense in $C[0,1]$ (see Proposition 9.3).

LEMMA 9.1. *If $T \in \mathfrak{M}$ is 2-local in $G_\Pi \cap \mathrm{Iso}(E_1, E_2)$, then T is an isometry with respect to the metric induced by the norm $\|\cdot\|_j$; $\|T(f) - T(g)\|_2 = \|f-g\|_1$ for every pair $f, g \in E_1$. The map T is also an isometry with respect to the supremum norm $\|\cdot\|_\infty$.*

PROOF. Let $f, g \in E_1$. Then there exists $T_{f,g} \in G_\Pi \cap \mathrm{Iso}(E_1, E_2)$ such that

$$T(f) = T_{f,g}(f), \qquad T(g) = T_{f,g}(g). \tag{1.1}$$

As $T_{f,g}$ is an isometry we have

$$\|T(f) - T(g)\|_2 = \|T_{f,g}(f) - T_{f,g}(g)\|_2 = \|f-g\|_1.$$

Thus T is an isometry. As $T_{f,g} \in G_\Pi \cap \mathrm{Iso}(E_1, E_2)$ there exists $\lambda_{f,g} \in E_2$, $\alpha_{f,g} \in \mathbb{C}$ of unit modulus, $\pi \in \Pi$ and $\epsilon_{f,g} \in \{\pm 1\}$ such that

$$T_{f,g}(h) = \lambda_{f,g} + \alpha_{f,g}[h \circ \pi]^\epsilon, \quad h \in E_1. \tag{1.2}$$

Then by (1.1) and (1.2) we observe that

$$\begin{aligned}\|T(f)-T(g)\|_\infty &= \|T_{f,g}(f)-T_{f,g}(g)\|_\infty \\ &= \|\alpha_{f,g}[f\circ\pi]^\epsilon-\alpha_{f,g}[g\circ\pi]^\epsilon\|_\infty \\ &= \|[f\circ\pi]^\epsilon-[g\circ\pi]^\epsilon\|_\infty=\|f-g\|_\infty\end{aligned}$$

since π is a surjection. Thus T is an isometry with respect to the supremum norm. □

PROPOSITION 9.2. *Suppose that $T\in\mathfrak{M}$ is 2-local in $G_\Pi\cap\mathrm{Iso}(E_1,E_2)$. Then there exists $\epsilon\in\{\pm1\}$ and $\alpha\in\mathbb{C}$ of unit modulus such that for every $f\in W_1$ there exists a homeomorphism $\pi_f\in\Pi$ such that*

$$T(f)=T(0)+\alpha[f\circ\pi_f]^\epsilon.$$

Note that if we proved that π_f did not depend on f, then the map T were surjective, hence $T\in\mathrm{Iso}(E_1,E_2)$ by the Mazur-Ulam theorem. But it is not the case in general (cf. [**38**, Theorem 2.3]); $G_\Pi\cap\mathrm{Iso}(E_1,E_2)$ needs not be 2-locally reflexive in $\mathfrak{M}$.

PROOF OF PROPOSITION 9.2. Put $T_0=T-T(0)$. We infer by a simple calculation that T_0 is 2-local in $G_\Pi\cap\mathrm{Iso}(E_1,E_2)$. Let $h\in E_1$. Since T_0 is 2-local in $G_\Pi\cap\mathrm{Iso}(E_1,E_2)$, there exist $T_{h,0}\in G_\Pi\cap\mathrm{Iso}(E_1,E_2)$, $\lambda_{h,0}\in E_2$, $\alpha_{h,0}\in\mathbb{C}$ with $|\alpha_{h,0}|=1$, a homeomorphism $\pi_{h,0}:X_2\to X_2$ and $\epsilon_{h,0}\in\{\pm1\}$ such that

$$\begin{aligned} T_0(h) &= T_{h,0}(h)=\lambda_{h,0}+\alpha_{h,0}[h\circ\pi_{h,0}]^{\epsilon_{h,0}}, \\ 0=T_0(0) &= T_{h,0}(0)=\lambda_{h,0}. \end{aligned} \tag{1.3}$$

Note that $T_{h,0}$ is represented by $T_{h,0}(\cdot)=\alpha_{h,0}[\cdot\circ\pi_{h,0}]^{\epsilon_{h,0}}$ on E_1. Hence $\lambda_{h,0}=0$ and

$$T_0(h)=\alpha_{h,0}[h\circ\pi_{h,0}]^{\epsilon_{h,0}}. \tag{1.4}$$

In particular, if $h=c\in\mathbb{C}$, then

$$T_0(c)=\alpha_{c,0}[c]^{\epsilon_{h,0}}.$$

Thus we obtain $T_0(\mathbb{C})\subset\mathbb{C}$. For every pair $c,d\in\mathbb{C}$, here exists $T_{c,d}\in G_\Pi\cap\mathrm{Iso}(E_1,E_2)$ such that

$$T_0(c)=T_{c,d}(c),\quad T_0(d)=T_{c,d}(d).$$

As $T_{c,d} \in \mathrm{Iso}(E_1, E_2)$ we infer that

$$|T_0(c) - T_0(d)| = \|T_0(c) - T_0(d)\|_2 = \|T_{c,d}(c) - T_{c,d}(d)\|_2 = \|c - d\|_1 = |c - d|.$$

As c, d are arbitrary, we have that $T_0|_{\mathbb{C}} : \mathbb{C} \to \mathbb{C}$ is an isometry. Applying a well known result about the form of an isometry on $\mathbb{C}$, there exists an $\alpha \in \mathbb{C}$ such that

$$T_0(z) = \alpha z, \quad z \in \mathbb{C}, \tag{1.5}$$

or

$$T_0(z) = \alpha \bar{z}, \quad z \in \mathbb{C}. \tag{1.6}$$

Put $T_1 = \bar{\alpha}[T_0]^\epsilon$, where $\epsilon = 1$ if (1.5) holds and $\epsilon = -1$ if (1.6) holds. Since E_2 is conjugate closed, T_1 is well defined in the second case. Since $\|f\|_2 = \|\bar{f}\|_2$ for every $f \in E_2$, it is a routine work to see that T_1 is 2-local in $G_\Pi \cap \mathrm{Iso}(E_1, E_2)$. By the definition of T_1 we infer that $T_1(z) = z$ for every $z \in \mathbb{C}$. We will prove that for every $f \in W_1$ there exists a homeomorphism $\pi_f : X_2 \to X_1$ such that

$$T_1(f) = f \circ \pi_f.$$

If it is proved, then we have

$$T(f) = T(0) + \alpha[f \circ \pi_f]^\epsilon,$$

the desired form.

Let $f \in W_1$ and $c \in \mathbb{C}$. As T_1 is 2-local in $G_\Pi \cap \mathrm{Iso}(E_1, E_2)$, there exists $\lambda_{f,c} \in E_2$ and $\alpha_{f,c} \in \mathbb{C}$ of modulus 1, a homeomorphism $\pi_{f,c} : X_2 \to X_1$, and $\epsilon_{f,c} \in \{\pm 1\}$ such that

$$T_0(f) = T_{f,c}(f) = \lambda_{f,c} + \alpha_{f,c}[f \circ \pi_{f,c}]^{\epsilon_{f,c}}, \tag{1.7}$$

$$c = T_0(c) = \lambda_{f,c} + \alpha_{f,c}[c]^{\epsilon_{f,c}}. \tag{1.8}$$

By (1.8) we infer that $\lambda_{f,c}$ is a constant. By comparing (1.4) for f with (1.7) we get

$$\alpha_{f,0}[f \circ \pi_{f,0}]^{\epsilon_{f,0}} = \lambda_{f,c} + \alpha_{f,c}[f \circ \pi_{f,c}]^{\epsilon_{f,c}},$$

hence

$$f \circ \pi_{f,0} = [\overline{\alpha_{f,0}}\lambda_{f,c} + \overline{\alpha_{f,0}}\alpha_{f,c}[f \circ \pi_{f,c}]^{\epsilon_{f,c}}]^{\epsilon_{f,0}}. \tag{1.9}$$

Considering the range of the both side of (1.9) we get

$$f(X_1) = [\overline{\alpha_{f,0}}\lambda_{f,c} + \overline{\alpha_{f,0}}\alpha_{f,c}[f(X_1)]^{\epsilon_{f,c}}]^{\epsilon_{f,0}}. \tag{1.10}$$

We have four possibility depending on (i)$\epsilon_{f,0} = 1$ and $\epsilon_{f,c} = 1$; (ii)$\epsilon_{f,0} = 1$ and $\epsilon_{f,c} = -1$; (iii)$\epsilon_{f,0} = -1$ and $\epsilon_{f,c} = -1$; (iv)$\epsilon_{f,0} = -1$ and $\epsilon_{f,c} = 1$. Then we have that at least one of the following (i) through (iv) holds.

(i) $f(X_1) = \overline{\alpha_{f,0}}\lambda_{f,c} + \overline{\alpha_{f,0}}\alpha_{f,c}f(X_1)$;

(ii) $f(X_1) = \overline{\alpha_{f,0}}\lambda_{f,c} + \overline{\alpha_{f,0}}\alpha_{f,c}\overline{f(X_1)}$;

(iii) $f(X_1) = \alpha_{f,0}\overline{\lambda_{f,c}} + \alpha_{f,0}\overline{\alpha_{f,c}}f(X_1)$;

(iv) $f(X_1) = \alpha_{f,0}\overline{\lambda_{f,c}} + \alpha_{f,0}\overline{\alpha_{f,c}}\overline{f(X_1)}$.

corresponding to the cases (i) through (iv) respectively. Let $S_j : \mathbb{C} \to \mathbb{C}$ be defined by $S_1(z) = \overline{\alpha_{f,0}}\lambda_{f,c} + \overline{\alpha_{f,0}}\alpha_{f,c}z$, $z \in \mathbb{C}$; $S_2(z) = \overline{\alpha_{f,0}}\lambda_{f,c} + \overline{\alpha_{f,0}}\alpha_{f,c}\overline{z}$, $z \in \mathbb{C}$; $S_3(z) = \alpha_{f,0}\overline{\lambda_{f,c}} + \alpha_{f,0}\overline{\alpha_{f,c}}z$, $z \in \mathbb{C}$; $S_4(z) = \alpha_{f,0}\overline{\lambda_{f,c}} + \alpha_{f,0}\overline{\alpha_{f,c}z}$, $z \in \mathbb{C}$. Then S_j is an isometry on $\mathbb{C}$ for $j = 1, 2, 3, 4$. Using S_j we rewrite (i) by $f(X_1) = S_1(f(X_1))$, (ii) by $f(X_1) = S_2(f(X_1))$, (iii) by $f(X_1) = S_3(f(X_1))$ and (iv) by $f(X_1) = S_4(f(X_1))$. As $f \in W_1$ we see that (ii) and (iv) do not occur. We have $\overline{\alpha_{f,0}}\lambda_{f,c} = 0$ and $\overline{\alpha_{f,0}}\alpha_{f,c} = 1$ for the case (i). Hence $\lambda_{f,c} = 0$ and $\alpha_{f,0} = \alpha_{f,c}$ in this case. In the same way we have $\lambda_{f,0} = 0$ and $\alpha_{f,0} = \alpha_{f,c}$ for the case (iii). We conclude that only (i) or (iii) occur, and in any case

$$\lambda_{f,c} = 0 \text{ and } \alpha_{f,0} = \alpha_{f,c}.$$

To prove $\alpha_{f,0} = 1$, put $c = 1$. Then (i) or (iii) occur. If (i) occurs, then

$$1 = T_0(\mathbf{1}) = T_{f,\mathbf{1}}(\mathbf{1}) = \alpha_{f,\mathbf{1}}\mathbf{1} = \alpha_{f,0}\mathbf{1}.$$

If (iii) occurs, then

$$1 = T_0(\mathbf{1}) = T_{f,\mathbf{1}}(\mathbf{1}) = \alpha_{f,\mathbf{1}}\bar{\mathbf{1}} = \alpha_{f,0}\mathbf{1}.$$

It follows that

$$\alpha_{f,0} = 1. \tag{1.11}$$

Next we prove that (iii) does not occur for any $c \in \mathbb{C}$ and $\epsilon_{f,0} = 1$. Suppose that (iii) occurs for some $c_0 \in \mathbb{C}$. Then we have

$$c_0 = T_0(c) = T_{f,c_0}(c_0) = \alpha_{f,c_0}\overline{c_0} = \alpha_{f,0}\overline{c_0} = \overline{c_0}.$$

Hence c_0 is a real number. We also have

$$T_0(f) = T_{f,c_0}(f) = \overline{f \circ \pi_{f,c_0}} \tag{1.12}$$

since $\alpha_{f,c_0} = \alpha_{f,0} = 1$. On the other hand, (iii) does not occur for c_0+i since c_0+i is not a real number. Thus (i) occurs and

$$T_0(f) = T_{f,c_0+i}(f) = f \circ \pi_{f,c_0+i}. \tag{1.13}$$

It follows by (1.12) and (1.13) that

$$\overline{f(X_1)} = f(X_1),$$

which is a contradiction since $f \in W_1$. We conclude that (iii) does not occur for any $c \in \mathbb{C}$. It follows that only (i) occurs, hence we have

$$\epsilon_{f,0} = 1. \tag{1.14}$$

Then by (1.4) for $h = f$ we have

$$T_0(f) = T_{f,0}(f) = f \circ \pi_{f,0}.$$

Letting $\pi_f = \pi_{f,0}$ we have the conclusion. □

2. Spaces of continuous functions on $[0,1]$

In this chapter A_j ($j = 1,2$) is a subspace of $C[0,1]$ and a superspace of $C^1[0,1]$, the space of all complex-valued continuously differentiable functions on the interval $[0,1]$. We assume that A_j is conjugate-closed in the sense that $f \in A_j$ implies $\bar{f} \in A_j$. Suppose that a norm $\|\cdot\|_j$ such that $|c| = \|c\|_j$ for every $c \in \mathbb{C}$ is defined on A_j. We assume that $\|f\|_j = \|\bar{f}\|_j$ for every $f \in A_j$. We do not assume the completeness of $\|\cdot\|_j$. The space A_j satisfies the conditions for E_j in the previous section. The difference between A_j and E_j is that A_j is defined on $[0,1]$ and we assume that $C^1[0,1] \subset A_j$. The spaces $(C^1[0,1], \|\cdot\|_\Sigma)$, $(C^1[0,1], \|\cdot\|_M)$, $(\mathrm{Lip}[0,1], \|\cdot\|_\Sigma)$, $(\mathrm{Lip}[0,1], \|\cdot\|_M)$ and $(C[0,1], \|\cdot\|_\infty)$ are typical examples of A_j. Recall that π_0 is the identity function on $[0,1]$ and $\pi_1 = 1-\pi_0$, $\Pi_0 = \{\pi_0, \pi_1\}$. Kawamura, Koshimizu and Miura [**60**] studies the space $C^1[0,1]$ with a variety of norms including $\|\cdot\|_\Sigma$

and $\|\cdot\|_M$. Recall that

$$\begin{aligned}W_j = \{f \in A_j : &\text{ if } S : \mathbb{C} \to \mathbb{C} \text{ is an isometry} \\ &\text{and } S(f(X_j)) = f(X_j), \text{ then } S \text{ is the identity function}\}.\end{aligned}$$

Put $P = \{p + iq : p$ and q are polynomials of real-coefficients$\}$. Many polynomials are in W_1, but some are not. For example $(t - \frac{1}{2})^4 + i(t - \frac{1}{2})^3 \notin W_1$. We do no know if $P \cap W_1$ is uniformly closed in P or not. We have that following. Let $cl(\cdot)$ denote the uniform closure on $[0, 1]$.

PROPOSITION 9.3. *We have $P \subset cl(W_1)$. Hence $cl(W_1) = C[0, 1]$.*

PROOF. Let $f = p + iq \in P$ and $\varepsilon > 0$. If p is not a constant, then put $p_\varepsilon = p$. If p is a constant, then put $p_\varepsilon = p + \varepsilon\pi_0$, where π_0 is the identity function on $[0, 1]$. Let l be any positive integer greater than both of the degree of p and q. Put $q_\varepsilon = q + \varepsilon\pi_0^l$. Then p_ε is not a constant and there is no pair of complex numbers c and d such that $p_\varepsilon = cq_\varepsilon + d$ since the degree of the each side of the equation is different. We prove that $p_\varepsilon + iq_\varepsilon \in cl(W_1)$. Then $p + iq \in cl(W_1)$ follows since $p_\varepsilon + iq_\varepsilon$ uniformly converges on $[0, 1]$ to $p + iq$ as $\varepsilon \to 0$.

Since p_ε is a non-constant polynomial, there exists a positive integer m_0 such that $p'_\varepsilon(\frac{1}{m}) \neq 0$ for every $m \geq m_0$. Let $m \geq m_0$. Put

$$f_m(t) = \begin{cases} iw\left(\frac{1}{m} - t\right) + \left(p'_\varepsilon\left(\frac{1}{m}\right) + iq'_\varepsilon\left(\frac{1}{m}\right)\right)\left(t - \frac{1}{m}\right) & \\ \qquad\qquad + p_\varepsilon\left(\frac{1}{m}\right) + iq_\varepsilon\left(\frac{1}{m}\right), & 0 \leq t \leq \frac{1}{m}, \\ (p_\varepsilon + q_\varepsilon)(t), & \frac{1}{m} \leq t \leq 1, \end{cases}$$

where

$$w(t) = \begin{cases} 0, & t = 0 \\ t^3 \sin\frac{1}{t}, & 0 < t \leq 1 \end{cases}$$

Then $f_m \in C^1[0, 1]$ for every $m \geq m_0$. It is a routine work to prove that f_m converges uniformly to $p + iq$ on $[0, 1]$ and a proof is omitted. We prove that $f_m \in W_1$ for every $m \geq m_0$.

Let K be a real number. We look at the number of the points t on $[0, 1]$, where $f_m(t)$ is a tangent point of a tangent line of $f_m([0, 1])$ whose slope is K. The curve $f_m([0, 1])$ has a tangent line of the slope

K at the tangent point $f_m(t)$ if and only if

$$K = \lim_{\delta\to 0} \frac{\operatorname{Im} f_m(t+\delta) - \operatorname{Im} f_m(t)}{\operatorname{Re} f_m(t+\delta) - \operatorname{Re} f_m(t)}.$$

Suppose that $0 \le t \le \frac{1}{m}$. Since $\operatorname{Re} f_m(t) = p'_\varepsilon\left(\frac{1}{m}\right)\left(t - \frac{1}{m}\right) + p_\varepsilon\left(\frac{1}{m}\right)$ and $\operatorname{Im} f_m(t) = w\left(\frac{1}{m} - t\right) + q'_\varepsilon\left(\frac{1}{m}\right)\left(t - \frac{1}{m}\right) + q_\varepsilon\left(\frac{1}{m}\right)$, we have

$$\lim_{\delta\to 0} \frac{\operatorname{Im} f_m(t+\delta) - f_m(t)}{\operatorname{Re} f_m(t+\delta) - \operatorname{Re} f_m(t)} = \frac{-w'\left(\frac{1}{m} - t\right) + q'_\varepsilon\left(\frac{1}{m}\right)}{p'_\varepsilon\left(\frac{1}{m}\right)}, \quad 0 \le t \le \frac{1}{m}.$$

Hence the curve $f_m([0,1])$ has a tangent line of the slope K at the tangent point $f_m(t)$ for $0 \le t \le \frac{1}{m}$ if and only if

$$\frac{-w'\left(\frac{1}{m} - t\right) + q'_\varepsilon\left(\frac{1}{m}\right)}{p'_\varepsilon\left(\frac{1}{m}\right)} = K. \tag{2.1}$$

If $K \neq \frac{q'_\varepsilon\left(\frac{1}{m}\right)}{p'_\varepsilon\left(\frac{1}{m}\right)}$, the number of such points $0 \le t \le \frac{1}{m}$ is at most finite. (The reason is as follows. Suppose that $\frac{q'_\varepsilon\left(\frac{1}{m}\right)}{p'_\varepsilon\left(\frac{1}{m}\right)} \neq K = \frac{-w'\left(\frac{1}{m}-t\right)+q'_\varepsilon\left(\frac{1}{m}\right)}{p'_\varepsilon\left(\frac{1}{m}\right)}$. Then

$$w'\left(\frac{1}{m} - t\right) = p'_\varepsilon\left(\frac{1}{m}\right)\left(\frac{q'_\varepsilon\left(\frac{1}{m}\right)}{p'_\varepsilon\left(\frac{1}{m}\right)} - K\right). \tag{2.2}$$

On the other hand, a simple calculation shows that

$$\left|w'\left(\frac{1}{m} - t\right)\right| \le 4\left(\frac{1}{m} - t\right). \tag{2.3}$$

We have $p'_\varepsilon\left(\frac{1}{m}\right)\left(\frac{q'_\varepsilon\left(\frac{1}{m}\right)}{p'_\varepsilon\left(\frac{1}{m}\right)} - K\right) \neq 0$ sincce $\frac{q'_\varepsilon\left(\frac{1}{m}\right)}{p'_\varepsilon\left(\frac{1}{m}\right)} \neq K$. By (2.3) there is no $t \le \frac{1}{m}$ with

$$\frac{1}{m} - t < \frac{1}{4}\left|p'_\varepsilon\left(\frac{1}{m}\right)\left(\frac{q'_\varepsilon\left(\frac{1}{m}\right)}{p'_\varepsilon\left(\frac{1}{m}\right)} - K\right)\right|$$

such that (2.2) holds. It is easy to see that the number of $t \ge 0$ with

$$\frac{1}{4}\left|p'_\varepsilon\left(\frac{1}{m}\right)\left(\frac{q'_\varepsilon\left(\frac{1}{m}\right)}{p'_\varepsilon\left(\frac{1}{m}\right)} - K\right)\right| \le \frac{1}{m} - t$$

such that (2.2) holds is at most finite.) On the other hand if $K = \frac{q'_\varepsilon\left(\frac{1}{m}\right)}{p'_\varepsilon\left(\frac{1}{m}\right)}$, then by (2.1) we infer that $w'\left(\frac{1}{m} - t\right) = 0$. By a calculation, for every positive integer k there exists a unique $k\pi < s_k < k\pi + \pi/2$ such that

$w'\left(\frac{1}{s_k}\right) = 0$. Letting $t_k = \frac{1}{m} - \frac{1}{s_k}$ we have $w'\left(\frac{1}{m} - t_k\right) = 0$. Thus $K = \frac{q_\varepsilon'\left(\frac{1}{m}\right)}{p_\varepsilon'\left(\frac{1}{m}\right)}$ for $0 \le t < \frac{1}{m}$ if and only if $t = t_k$ for some positive integer k. As $w'(0) = 0$, we see that $w'\left(\frac{1}{m} - t\right) = \frac{q_\varepsilon'\left(\frac{1}{m}\right)}{p_\varepsilon'\left(\frac{1}{m}\right)}$ if $t = \frac{1}{m}$. We conclude that the set of the all points in $f_m([0, \frac{1}{m}])$ at which $f_m([0,1])$ has a tangent line with the slope $\frac{q_\varepsilon'\left(\frac{1}{m}\right)}{p_\varepsilon'\left(\frac{1}{m}\right)}$ is $\{f_m(t_n)\}_{n\ge N} \cup \left\{f_m\left(\frac{1}{m}\right)\right\}$, where $N = \min\left\{k : \frac{1}{m} > t_k\right\}$.

Suppose that $\frac{1}{m} < t \le 1$. We have $\operatorname{Re} f_m(t) = p_\varepsilon(t)$ and $\operatorname{Im} f_m(t) = q_\varepsilon(t)$. Therefore we have

$$\frac{\operatorname{Im} f_m(t+\delta) - \operatorname{Im} f_m(t)}{\operatorname{Re} f_m(t+\delta) - \operatorname{Re} f_m(t)} = \frac{q_\varepsilon(t+\delta) - q_\varepsilon(t)}{p_\varepsilon(t+\delta) - p_\varepsilon(t)}.$$

Hence the curve $f_m([0,1])$ has a tangent line of the slope K at $f_m(t)$ for $\frac{1}{m} < t \le 1$ if and only if

$$\lim_{\delta\to 0} \frac{q_\varepsilon(t+\delta) - q_\varepsilon(t)}{p_\varepsilon(t+\delta) - p_\varepsilon(t)} = K.$$

If $p_\varepsilon(t) \neq 0$, then $\frac{q_\varepsilon'(t)}{p_\varepsilon'(t)} = K$. The number of such points $t \in \left(\frac{1}{m}, 1\right]$ is at most finite. Suppose not. Then $q_\varepsilon'(t) = Kp_\varepsilon'(t)$ for infinitely many t, hence $q_\varepsilon' = Kp_\varepsilon'$ on the interval $\left([\frac{1}{m}, 1\right]$ since p_ε' and q_ε' are polynomials. It follows that $q_\varepsilon = Kp_\varepsilon + c$ for some $c \in \mathbb{C}$, which contradicts to our hypothesis on p_ε and q_ε. We obtain that the number of $t \in \left(\frac{1}{m}, 1\right]$ such that $\frac{q_\varepsilon'(t)}{p_\varepsilon'(t)} = K$ is at most finite. The number of $t \in \left(\frac{1}{m}, 1\right]$ such that $p_\varepsilon'(t) = 0$ is at most finite since p_ε is a polynomial. We conclude that the number of point t such that $f_m([0,1])$ has a tangent line of the slope K at $f_m(t)$ is at most finite. In a way similar we see that the number of $t \in \left(\frac{1}{m}, 1\right]$ such that $f_m([0,1])$ has a tangent line at $f_m(t)$ which is parallel to the imaginary axis is at most finite. If $p_\varepsilon'(t) \neq 0$, then $f_m([0,1])$ has a tangent line which is not parallel to the imaginal axis. Hence if $f_m([0,1])$ has a tangent line with a tangent point at $f_m(t)$ which is parallel to the imaginary axis, then $p_\varepsilon'(t) = 0$. Thus the number of such points is at most finite.

We conclude that for a real number $K \neq \frac{q_\varepsilon'\left(\frac{1}{m}\right)}{p_\varepsilon'\left(\frac{1}{m}\right)}$ the number of $t \in [0,1]$ such that $f_m([0,1])$ has a tangent line of the slope K at $f_m(t)$ is at most finite; the number of $t \in [0,1]$ such that $f_m([0,1])$ has a tangent line which is parallel to the imaginal axis at $f_m(t)$ is at most finite; the number of $t \in [0,1]$ such that $f_m([0,1])$ has a tangent line of the slope $\frac{q'_\varepsilon\left(\frac{1}{m}\right)}{p'_\varepsilon\left(\frac{1}{m}\right)}$ at $f_m(t)$ is countable, the number of such $t \in \left(\frac{1}{m}, 1\right]$ is at most finite, say $\{t_{-n}\}_{n=1}^{l}$, there is a sequence $\{t_k\}_{k\geq N}$ in $[0, \frac{1}{m})$ of such points that converges to $\frac{1}{m}$. Denote them as $\{t_k\} = \{t_k\}_{k\geq N} \cup \{t_\infty = \frac{1}{m}\} \cup \{t_{-n}\}_{n=1}^{l}$.

Suppose that $S : \mathbb{C} \to \mathbb{C}$ is an isometry such that $S(f_m([0,1]) = f_m([0,1])$. We prove that S is the identity so that $f_m \in W$. Since there are $a, b \in \mathbb{C}$ with $|a| = 1$ such that $S(z) = b + az$, $z \in \mathbb{C}$ or $S(z) = b + a\bar{z}$, $z \in \mathbb{C}$, $S(\ell_1)$ and $S(\ell_2)$ are parallel for every pair of parallel lines ℓ_1 and ℓ_2 in $\mathbb{C}$. Thus the parallel tangent line at $\{t_k\}$ are translated by S as a parallel tangent line. Hence we get

$$\{S(f_m(t_k))\} = \{f_m(t_k)\}.$$

As S is an isometry the unique cluster points t_∞ of $\{t_k\}$ translates to t_∞ by S; $S(t_\infty) = t_\infty$. As $\{f_m(t_{-k})\}_{k=1}^{l}$ is discrete, there is a positive integer M such that $\{S(f_m(t_k))\}_{k\geq M} \subset \{f_m(t_k)\}_{k\geq N}$. Hence if $n \geq M$, then there is an $n_1 \geq N$ such that

$$|f_m(t_\infty) - f_m(t_n)| = |S(f_m(t_\infty)) - S(f_m(t_n))| = |f_m(t_\infty) - f_m(t_{n_1})|.$$

If $n, n_1 \geq N$ and $n \neq n_1$, then by the definition of t_n and t_{n_1} we have

$$|f_m(t_\infty) - f_m(t_n)| \neq |f_m(t_\infty) - f_m(t_{n_1})|.$$

It follows that $t_n = t_{n_1}$. Since $S : \mathbb{C} \to \mathbb{C}$ is an isometry, the set of the fixed point of S is empty or a singleton or points on a straight line if S is not the identity. As $\{f_m(t_n)\}_{n\geq M}$ is a set of fixed points which are not on the line since $w\left(\frac{1}{m} - t_n\right) > 0$ when n is an even number and $w\left(\frac{1}{m} - t_n\right) < 0$ when n is an odd number. Thus we conclude that S is an isometry.

By the Weierstrass approximation theorem we see that $cl(W_1) = C[0,1]$. □

THEOREM 9.4. *$G_{\Pi_0} \cap \mathrm{Iso}(A_1, A_2)$ is 2-locally reflexive in $M(A_1, A_2)$.*

PROOF. Suppose that $T \in M(A_1, A_2)$ is 2-local in $G_{\Pi_0} \cap \mathrm{Iso}(A_1, A_2)$. Then by Proposition 9.2 there exist an $\alpha \in \mathbb{C}$ of unit modulus and

$\epsilon \in \{\pm 1\}$ which satisfies that for every f in W_1, there exists a $\pi_f \in \Pi_0$ such that

$$T(f) = T(0) + \alpha[f \circ \pi_f]^{\epsilon}.$$

We prove that π_f is independent of $f \in W_1$. Let $T_1 \in M(A_1, A_2)$ be defined by

$$T_1(h) = [\bar{\alpha}(T(h) - T(0))]^{\epsilon}, \quad h \in A_1.$$

Then T_1 is 2-local in $G_{\Pi_0} \cap \mathrm{Iso}(A_1, A_2)$. As in the same way in the proof of Proposition 9.2 there exists a $T_{f,0} \in G_{\Pi_0} \cap \mathrm{Iso}(A_1, A_2)$ such that

$$T_1(f) = T_{f,0}(f) = f \circ \pi_f \tag{2.4}$$

for every $f \in W_1$. Let $\varepsilon > 0$ is given. Then $g_\varepsilon = \pi_0 + i\varepsilon\pi_0^2 \in W_1$. Hence there exists $T_{g_\varepsilon,0} \in G_{\Pi_0} \cap \mathrm{Iso}(A_1, A_2)$ and $\pi_\varepsilon \in \Pi_0$ such that

$$T_1(g_\varepsilon) = T_{g_{\varepsilon,0}}(g_\varepsilon) = g_\varepsilon \circ \pi_\varepsilon. \tag{2.5}$$

Note that $T_{g_{\varepsilon,0}}(h) = h \circ \pi_\varepsilon$ for every $h \in A_1$ by the proof of Proposition 9.2. (In fact, due to the note just after (1.3) we have $T_{g_\varepsilon,0}(h) = \alpha_{g_{\varepsilon,0}}[h \circ \pi_{g_{\varepsilon,0}}]^{\epsilon_{g_{\varepsilon,0}}}$ for $h \in A_1$. As $g_\varepsilon \in W_1$, we have by (1.11) and (1.14) and letting $\pi_{g_{\varepsilon,0}} = \pi_\varepsilon$ we have $T_{g_{\varepsilon,0}}(h) = h \circ \pi_\varepsilon$ for every $h \in A_1$.) We prove that there exists an $\varepsilon > 0$ such that $\pi_\varepsilon = \pi_{\varepsilon'}$ for every $0 < \varepsilon, \varepsilon' < \varepsilon_0$. Suppose not. Then there exist sequences $\{\varepsilon_n\}$ and $\{\varepsilon'_n\}$ of positive real numbers which converge to 0 respectively such that $\pi_{\varepsilon_n} \neq \pi_{\varepsilon'_n}$ for every n. By Lemma 9.1 T_1 is a isometry with respect to $\|\cdot\|_\infty$, hence we infer that

$$\|T_1(g_{\varepsilon_n}) - T_1(\varepsilon'_n)\|_\infty = \|g_{\varepsilon_n} - g_{\varepsilon'_n}\|_\infty = \|\varepsilon_n - \varepsilon'_n| \to 0$$

as $n \to \infty$. On the other hand, as $\pi_{\varepsilon_n} \neq \pi_{\varepsilon'_n}$ for every n we have

$$\|T_1(g_{\varepsilon_n}) - T_1(\varepsilon'_n)\|_\infty = \|g_{\varepsilon_n} \circ \pi_{\varepsilon_n} - g_{\varepsilon'_n} \circ \pi_{\varepsilon'_n}\|_\infty \geq \|\pi_{\varepsilon_n} - \pi_{\varepsilon'_n}\|_\infty - \varepsilon_n - \varepsilon'_n \to 1$$

as $n \to \infty$, which is a contradiction proving $\pi_\varepsilon = \pi_{\varepsilon'}$ for every $0 < \varepsilon, \varepsilon' < \varepsilon_0$ for some positive ε_0. Put the common π_ε as π. Letting $\varepsilon \to 0$ in (2.5) we get

$$T_1(\pi_0) = \pi_0 \circ \pi.$$

We prove that

$$T_1(f) = f \circ \pi \tag{2.6}$$

for every $f \in W$. We show a proof for the case where $\pi = \pi_0$. A proof for the case where $\pi = \pi_1$ is similar, and is omitted.

If $T_1(f) = f$ for every $f \in W_1$ is proved, then it turns out that T_1 is a surjective isometry. The reason is as follows. For a sufficiently small positive ε, we have proved $\pi_\varepsilon = \pi_0$ since we assume $\pi = \pi_0$ in (2.6). Then $T_{g_{\varepsilon,0}} = T_1$ on W_1. Proposition 9.3 asserts that W_1 is uniformly dense in $C[0,1]$, hence in A_1. As T_1 is continuous with respect to $\|\cdot\|_\infty$ by Lemma 9.1, we conclude that $T_1 = T_{g_{\varepsilon},0}$ on A_1. Since T_{g_ε} is a surjective isometry we conclude that T_1 is a surjective isometry. We prove that $T_1(f) = f$ for every $f \in W_1$. To prove it, suppose that there exists a $f_0 \in W_1$ such that $T_1(f_0) \neq f_0$. Then by (2.4) we have

$$T_1(f_0) = f_0 \circ \pi_1. \tag{2.7}$$

As T_1 is 2-local in $G_{\Pi_0} \cap \mathrm{Iso}(A_1, A_2)$, there exists a $\lambda_{f_0,\pi_0} \in A_2$, $\alpha_{f_0,\pi_0} \in \mathbb{C}$ of unit modulus and $\epsilon_{f_0,\pi_0} \in \{\pm 1\}$ such that one of

$$\begin{aligned} f_0 \circ \pi_1 = T_1(f_0) &= \lambda_{f_0,\pi_0} + \alpha_{f_0,\pi_0}[f_0]^{\epsilon_{f_0,\pi_0}}, \\ \pi_0 = T_1(\pi_0) &= \lambda_{f_0,\pi_0} + \alpha_{f_0,\pi_0}\pi_0 \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} f_0 \circ \pi_1 = T_1(f_0) &= \lambda_{f_0,\pi_0} + \alpha_{f_0,\pi_0}[f_0 \circ \pi_1]^{\epsilon_{f_0,\pi_0}}, \\ \pi_0 = T_1(\pi_0) &= \lambda_{f_0,\pi_0} + \alpha_{f_0,\pi_0}\pi_1 \end{aligned} \tag{2.9}$$

holds. Thus

$$\lambda_{f_0,\pi_0}(t) = (1 - \alpha_{f_0,\pi_0})t, \quad t \in [0,1] \tag{2.10}$$

when (2.8) occurs and

$$\lambda_{f_0,\pi_0}(t) = (1 + \alpha_{f_0,\pi_0})t - \alpha_{f_0,\pi_0}, \quad t \in [0,1] \tag{2.11}$$

when (2.9) occurs.

We will prove that both of (2.8) and (2.9) are impossible. Suppose that (2.8) occurs. Rewriting the first equation of (2.8) using (2.10) we get

(2.12)

$$f_0(1-t) = (T_1(f_0))(t) = (1 - \alpha_{f_0,\pi_0})t + \alpha_{f_0,\pi_0}[f_0(t)]^{\epsilon_{f_0,\pi_0}}, \quad t \in [0,1]$$

Suppose that $\alpha_{f_0,\pi_0} = 1$. Then

$$f_0(1-t) = f_0(t), \qquad t \in [0,1] \tag{2.13}$$

or

$$f_0(1-t) = \overline{f_0(t)}, \qquad t \in [0,1]. \tag{2.14}$$

If (2.13) holds, then $(T_1(f_0))(t) = f_0(1-t) = f_0(t)$, $t \in [0,1]$ by (2.7), which is against to our choice of f_0. Thus (2.13) does not hold. Suppose that (2.14) holds. Then $f_0([0,1]) = \overline{f_0([0,1])}$ holds, which means that $f_0 \notin W_1$. Thus (2.14) does not hold. It follows that $\alpha_{f_0,\pi_0} \neq 1$. Suppose that $\varepsilon_{f_0,\pi_0} = 1$ for (2.12). Then we have

$$f_0(1-t) = (1-\alpha_{f_0,\pi_0})t + \alpha_{f_0,\pi_0} f_0(t), \quad t \in [0,1]. \tag{2.15}$$

Changing $1-t$ by t we have

$$f_0(t) = (1-\alpha_{f_0,\pi_0})(1-t) + \alpha_{f_0,\pi_0} f_0(1-t), \quad t \in [0,1]. \tag{2.16}$$

Applying (2.15) we have

$$f_0(t) = (1-\alpha_{f_0,\pi_0})(1-t)+\alpha_{f_0,\pi_0}\left((1-\alpha_{f_0,\pi_0})t + \alpha_{f_0,\pi_0} f_0(t)\right), \quad t \in [0,1]. \tag{2.17}$$

As $\alpha_{f_0,\pi_0} \neq 1$ we infer that

$$(1+\alpha_{f_0,\pi_0}) f_0(t) = 1 - (1-\alpha_{f_0,\pi_0})t, \qquad t \in [0,1]. \tag{2.18}$$

If $\alpha_{f_0,\pi_0} = -1$, then we have that $0 = 1-2t$ for every $t \in [0,1]$, which is a contradiction. Hence $\alpha_{f_0,\pi_0} \neq -1$. Then by (2.18) we have

$$f_0(t) = \frac{1}{(1+\alpha_{f_0,\pi_0})} - \frac{(1-\alpha_{f_0,\pi_0})}{(1+\alpha_{f_0,\pi_0})}t, \qquad t \in [0,1].$$

Hence $f_0 \in W_1$, which is a contradiction. Suppose that $\epsilon_{f_0,\pi_0} = -1$ for (2.12). Then we have

$$f_0(1-t) = (1-\alpha_{f_0,\pi_0})t + \alpha_{f_0,\pi_0} \overline{f_0(t)}, \quad t \in [0,1]. \tag{2.19}$$

Substituting $1-t$ by t in (2.19), we have

$$f_0(t) = (1-\alpha_{f_0,\pi_0})(1-t) + \alpha_{f_0,\pi_0} \overline{f_0(1-t)}, \quad t \in [0,1]. \tag{2.20}$$

Substituting (2.19) in (2.20) we get

$$f_0(t) = (1-\alpha_{f_0,\pi_0})(1-t) + \alpha_{f_0,\pi_0} \overline{(1-\alpha_{f_0,\pi_0})t + \alpha_{f_0,\pi_0} \overline{f_0(t)}}, \quad t \in [0,1]. \tag{2.21}$$

Hence we get

$$0=\left(\alpha_{f_0,\pi_0}\overline{(1-\alpha_{f_0,\pi_0})}-(1-\alpha_{f_0,\pi_0})\right)t+(1-\alpha_{f_0,\pi_0}),\qquad t\in[0,1].$$

We get $\alpha_{f_0,\pi_0}=1$, which contradicts to $\alpha_{f_0,\pi_0}\neq 1$. We conclude that (2.8) does not occur.

Suppose that (2.9) holds. Rewriting (2.9) by applying (2.11) we get

$$f_0(1-t)=(T_1(f_0))(t)=(1+\alpha_{f_0,\pi_0})t-\alpha_{f_0,\pi_0}+\alpha_{f_0,\pi_0}[f_0(1-t)]^{\epsilon_{f_0,\pi_0}}. \tag{2.22}$$

Suppose that $\epsilon_{f_0,\pi_0}=1$. By (2.22) we get

$$(1-\alpha_{f_0,\pi_0})f_0(1-t)=(1+\alpha_{f_0,\pi_0})t-\alpha_{f_0,\pi_0},\qquad t\in[0,1]. \tag{2.23}$$

Then we have $0=1-2t$ for every $t\in[0,1]$ if $\alpha_{f_0,\pi_0}=1$, which is impossible, so that $\alpha_{f_0,\pi_0}\neq 1$. Then by (2.23) we get

$$f_0(1-t)=\frac{1+\alpha_{f_0,\pi_0}}{1-\alpha_{f_0,\pi_0}}t-\frac{\alpha_{f_0,\pi_0}}{1-\alpha_{f_0,\pi_0}},\qquad t\in[0,1],$$

so that

$$f_0(t)=\frac{1+\alpha_{f_0,\pi_0}}{1-\alpha_{f_0,\pi_0}}(1-t)-\frac{\alpha_{f_0,\pi_0}}{1-\alpha_{f_0,\pi_0}},\qquad t\in[0,1],$$

which is a contradiction to $f_0\in W_1$. We have that $\epsilon_{f_0,\pi_0}\neq 1$, hence $\epsilon_{f_0,\pi_0}=-1$. Then by (2.22) we get

$$f_0(1-t)=(1+\alpha_{f_0,\pi_0})t-\alpha_{f_0,\pi_0}+\alpha_{f_0,\pi_0}\overline{f_0(1-t)},\qquad t\in[0,1]. \tag{2.24}$$

Thus

$$\begin{aligned}f_0(1-t)&=(1+\alpha_{f_0,\pi_0})t-\alpha_{f_0,\pi_0}\\&\quad+\alpha_{f_0,\pi_0}\overline{((1+\alpha_{f_0,\pi_0})t-\alpha_{f_0,\pi_0}+\alpha_{f_0,\pi_0}\overline{f_0(1-t)})},\quad t\in[0,1].\end{aligned} \tag{2.25}$$

As $|\alpha_{f_0,\pi_0}|=1$ we get

$$\begin{aligned}f_0(1-t)&=(1+\alpha_{f_0,\pi_0})t-\alpha_{f_0,\pi_0}\\&\quad+\alpha_{f_0,\pi_0}\overline{(1+\alpha_{f_0,\pi_0})}t-1+f_0(1-t),\quad t\in[0,1].\end{aligned} \tag{2.26}$$

Hence

$$0=\left((1+\alpha_{f_0,\pi_0})+\alpha_{f_0,\pi_0}(1+\overline{\alpha_{f_0,\pi_0}})\right)t-(\alpha_{f_0,\pi_0}+1),\qquad t\in[0,1]. \tag{2.27}$$

Thus $\alpha_{f_0,\pi_0}=-1$. Substituting $\alpha_{f_0,\pi_0}=-1$ into (2.22) we get

$$f_0(1-t)=1-\overline{f_0(1-t)},\qquad t\in[0,1]$$

since $\epsilon_{f_0,\pi_0} = -1$. Then

$$f_0([0,1]) = 1 - \overline{f_0([0,1])},$$

which contradicts to $f_0 \in W_1$. It follows that (2.9) does not occur.

Assuming the existence of $f_0 \in W_1$ such that $T_1(f_0) \neq f_0$ we arrived at the contradiction. We conclude that $T_1(f) = f$ for every $f \in W_1$.

Let $g \in A_1$ Then by Proposition 9.3 there is a sequence $\{g_n\}$ in W_1 such that $\|g - g_n\|_\infty \to 0$ as $n \to \infty$. By the previous part of the proof we have

$$T_1(g_n) = g_n \tag{2.28}$$

for every n. By Lemma 9.1, T_1 is an symmetry with respect to $\|\cdot\|_\infty$, we have $T_1(g) = g$ by letting $n \to \infty$ for (2.28). We conclude that $T_1(g) = g$ for every $g \in A_1$ if $T_1(\pi_0) = \pi_0$. It follows that

$$T(g) = T(0) + \alpha[g]^\epsilon, \qquad g \in A_1$$

if $T_1(\pi_0) = \pi_0$. Suppose that $T_1(\pi_0) = \pi_1$. As we have already described, we see that $T_1(g) = g \circ \pi_1$ for every $g \in A_1$. Hence we have

$$T(g) = T(0) + \alpha[g \circ \pi_1]^\epsilon, \qquad g \in A_1.$$

Thus we observed that $T \in G_{\Pi_0}$. As we have already proved that T_1 is a surjective isometry from A_1 onto A_2, we see that T is also a surjective isometry. Hence we conclude that $T \in G_{\Pi_0} \cap \mathrm{Iso}(A_1, A_2)$. □

3. Applications

We apply Theorem 9.4 to get the following corollaries.

Corollary 9.5. $\mathrm{Iso}(C^1[0,1], \|\cdot\|_\Sigma)$ *is 2-local reflexive.*

Corollary 9.6. $\mathrm{Iso}(\mathrm{Lip}[0,1], \|\cdot\|_\sum)$ *is 2-locally reflexive.*

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APPENDIX A

Spherical version of the Kowalski-Słodkowski theorem

One of the basic problem in the operator theory is to find sufficient sets of conditions for linearity and multiplicativity of maps between Banach algebras. As a generalization of the Gleason-Kahane-Żelazko theorem [**33, 56, 118**], Kowalski and Słodkowski [**69**] proved the linearity and the multiplicativity of a functional Δ on a Banach algebra A under the spectral condition. Recently Li, Peralta, Wang and Wang proved interesting spherical variants of the Gleason-Kahane-Żelazko theorem, and the Kowalski-Słodkowski theorem [**75**] as follows. They [**75**] proved a spherical variant of the Kowalski-Słodkowski theorem; a 1-homogeneous functional which satisfies certain spectral condition is complex-linear.

THEOREM A.1. [**75**, Proposition 3.2] *Let A be a unital complex Banach algebra, and let $\Delta : A \to \mathbb{C}$ be a mapping which satisfies the following properties:*

(1) $\Delta(\lambda a) = \lambda\Delta(a)$ *for every* $a \in A$, $\lambda \in \mathbb{C}$
(2) $\Delta(a) - \Delta(b) \in \mathbb{T}\sigma(a - b)$, *for every* $a, b \in A$.

Then Δ is linear and there exists $\lambda_0 \in \mathbb{T}$ such that $\lambda_0\Delta$ is multiplicative.

We shall prove a generalization of a spherical variant of the Kowalski-Słodkowski theorem without hypothesis that the 1-homogeneity. This hypothesis is the same as one of the original Kowalski-Słodkowski theorem.

THEOREM A.2. *Let A be a unital Banach algebra. Suppose that a map $\Delta : A \to \mathbb{C}$ satisfies the conditions*

(a) $\Delta(0) = 0$,
(b) $\Delta(x) - \Delta(y) \in \mathbb{T}\sigma(x - y), \quad x, y \in A.$

Then Δ is a complex-linear or conjugate linear and $\overline{\Delta(\mathbf{1})}\Delta$ is multiplicative.

Fix $a \in A$, we define a map $f : \mathbb{C} \to \mathbb{C}$ by $f(\lambda) = \Delta(a+\lambda\cdot\mathbf{1})-\Delta(a)$. For any $\lambda_1, \lambda_2 \in \mathbb{C}$, we get

$$\Delta(a+\lambda_1\cdot\mathbf{1})-\Delta(a+\lambda_2\cdot\mathbf{1}) \in \mathbb{T}\sigma((\lambda_1-\lambda_2)\cdot\mathbf{1}) = (\lambda_1-\lambda_2)\mathbb{T}\sigma(\mathbf{1}) = (\lambda_1-\lambda_2)\mathbb{T},$$

by the assumption (b). Thus we have

$$\begin{aligned} |f(\lambda_1) - f(\lambda_2)| &= |\Delta(a + \lambda_1 \cdot \mathbf{1}) - \Delta(a) - \{\Delta(a + \lambda_2 \cdot \mathbf{1}) - \Delta(a)\}| \\ &= |\Delta(a + \lambda_1 \cdot \mathbf{1}) - \Delta(a + \lambda_2 \cdot \mathbf{1})| \\ &= |\lambda_1 - \lambda_2|. \end{aligned}$$

This implies that the map f is an isometry on $\mathbb{C}$. The form of an isometry on $\mathbb{C}$ is well known. Without assuming surjectivity on the isometry there exist $\alpha, \beta \in \mathbb{C}$ with $|\alpha| = 1$ such that $f(\lambda) = \beta + \lambda\alpha$ ($\lambda \in \mathbb{C}$) or $f(\lambda) = \beta + \bar{\lambda}\alpha$ ($\lambda \in \mathbb{C}$). Since

$$f(0) = \Delta(a + 0 \cdot \mathbf{1}) - \Delta(a) = \Delta(a) - \Delta(a) = 0,$$

we have

$$f(\lambda) = \lambda\alpha, \quad \lambda \in \mathbb{C},$$

or

$$f(\lambda) = \overline{\lambda}\alpha, \quad \lambda \in \mathbb{C}.$$

In addition, we have $\alpha = f(1) = \Delta(a + \mathbf{1}) - \Delta(a)$, we infer that

$$\Delta(a + \lambda \cdot \mathbf{1}) - \Delta(a) = \lambda\{\Delta(a + \mathbf{1}) - \Delta(a)\}, \quad \lambda \in \mathbb{C},$$

or

$$\Delta(a + \lambda \cdot \mathbf{1}) - \Delta(a) = \overline{\lambda}\{\Delta(a + \mathbf{1}) - \Delta(a)\}, \quad \lambda \in \mathbb{C}.$$

Let

$$A_1 = \{a \in A; \Delta(a + \lambda \cdot \mathbf{1}) - \Delta(a) = \lambda\{\Delta(a + \mathbf{1}) - \Delta(a)\}, \quad \lambda \in \mathbb{C}\}$$

and

$$A_{-1} = \{a \in A; \Delta(a + \lambda \cdot \mathbf{1}) - \Delta(a) = \overline{\lambda}\{\Delta(a + \mathbf{1}) - \Delta(a)\}, \quad \lambda \in \mathbb{C}\}.$$

For any $a \in A$, the map $\lambda \mapsto \Delta(a + \lambda \cdot \mathbf{1}) - \Delta(a)$ is an isometry on $\mathbb{C}$, we have $A = A_1 \cup A_{-1}$.

LEMMA A.3. *We have $A = A_1$ or $A = A_{-1}$.*

PROOF. We have proved that $A = A_1 \cup A_{-1}$. We prove that A_1 and A_{-1} are closed subsets of A. Let $\{a_\alpha\} \subset A_1$ be a net and $a_0 \in A$ such that $a_\alpha \to a_0$. By assumption (b), $\Delta(a_\alpha) - \Delta(a_0) \in \mathbb{T}\sigma(a_\alpha - a_0)$. Hence $|\Delta(a_\alpha) - \Delta(a_0)| \leq r(a_\alpha - a_0)$ for the spectral radius $r(\cdot)$. Since $r(\cdot) \leq \|\cdot\|$ for the original norm $\|\cdot\|$ on A, we get $\Delta(a_\alpha) - \Delta(a_0) \to 0$ as $a_\alpha \to a_0$. In the same way we have that $\Delta(a_0 + \lambda \cdot \mathbf{1}) - \Delta(a_\alpha + \lambda \cdot \mathbf{1}) \to 0$ as $a_\alpha \to a_0$ for any $\lambda \in \mathbb{C}$. Thus for any $\lambda \in \mathbb{C}$, we have

$$
\begin{aligned}
&|\Delta(a_0 + \lambda \cdot \mathbf{1}) - \Delta(a_0) - \lambda\{\Delta(a_0 + \mathbf{1}) - \Delta(a_0)\}| \\
&\qquad = |\Delta(a_0 + \lambda \cdot \mathbf{1}) - \Delta(a_0) - \{\Delta(a_\alpha + \lambda \cdot \mathbf{1}) - \Delta(a_\alpha)\} \\
&\qquad\quad + \lambda\{\Delta(a_\alpha + \mathbf{1}) - \Delta(a_\alpha)\} - \lambda\{\Delta(a_0 + \mathbf{1}) - \Delta(a_0)\}| \\
&\qquad \leq |\Delta(a_0 + \lambda \cdot \mathbf{1}) - \Delta(a_\alpha + \lambda \cdot \mathbf{1})| + |\Delta(a_0) - \Delta(a_\alpha)| \\
&\qquad\quad + |\lambda||\Delta(a_\alpha + \mathbf{1}) - \Delta(a_0 + \mathbf{1})| + |\lambda||\Delta(a_\alpha) - \Delta(a_0)| \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \to 0,
\end{aligned}
$$

as $a_\alpha \to a_0$. This implies that $\Delta(a_0 + \lambda \cdot \mathbf{1}) - \Delta(a_0) = \lambda\{\Delta(a_0 + \mathbf{1}) - \Delta(a_0)\}$ for any $\lambda \in \mathbb{C}$. Since $a_0 \in A_1$, we have A_1 is closed. We can prove that A_{-1} is also closed in the same way. In addition, suppose that $a \in A_1 \cap A_{-1}$. Then we have for any $\lambda \in \mathbb{C}$,

$$
\lambda\{\Delta(a + \mathbf{1}) - \Delta(a)\} = \Delta(a + \lambda \cdot \mathbf{1}) - \Delta(a) = \overline{\lambda}\{\Delta(a + \mathbf{1}) - \Delta(a)\}.
$$

This shows that $\Delta(a + \mathbf{1}) - \Delta(a) = 0$. On the other hand, we have

$$
\Delta(a + \mathbf{1}) - \Delta(a) \in \mathbb{T}\sigma(\mathbf{1}) = \mathbb{T}.
$$

We arrive at a contradiction. Therefore $A_1 \cap A_{-1} = \emptyset$. Since A is connected, we conclude that $A_1 = A$ or $A_{-1} = A$. □

PROOF OF THEOREM A.2. Lemma A.3 shows that one of $A = A_1$ and $A = A_{-1}$ occurs. First we take up the case $A = A_1$.

(i) We consider the case that A is separable. By the definition of A_1, for any $a \in A_1$, we get

$$
\Delta(a + \lambda \cdot \mathbf{1}) - \Delta(a) = \lambda\{\Delta(a + \mathbf{1}) - \Delta(a)\}, \quad \lambda \in \mathbb{C}. \tag{0.1}
$$

By assumption (b), it follows that

$$
|\Delta(a) - \Delta(b)| \leq \|a - b\|, \quad a, b \in A,
$$

which implies that Δ is a Lipschitz map. [**69**, Theorem 2.3] ([**75**, Theorem 3.4]) shows that Δ has real differentials except some zero set. We say that Δ has a real differential at a point of $a \in A$ if for every $x \in A$ the derivative $\Delta'_x(a) = \lim_{\mathbb{R} \ni r \to 0} \dfrac{\Delta(a+rx) - \Delta(a)}{r}$ exists and the map $(D\Delta)_a : A \to \mathbb{C}$, defined by $(D\Delta)_a(x) = \Delta'_x(a)$, is real linear and continuous. (cf.[**79, 69, 75**].) Since

$$\frac{\Delta(a+rx) - \Delta(a)}{r} \in \frac{\mathbb{T}\sigma(rx)}{r} = \frac{r\mathbb{T}\sigma(x)}{r} = \mathbb{T}\sigma(x), \quad r \in \mathbb{R} \setminus \{0\},$$

we have

$$(D\Delta)_a(x) = \lim_{\mathbb{R} \ni r \to 0} \frac{\Delta(a+rx) - \Delta(a)}{r} \in \mathbb{T}\sigma(x).$$

As $(D\Delta)_a$ is a real linear, [**75**, Lemma 3.3] implies that $(D\Delta)_a$ is a complex-linear or conjugate linear. Since $a \in A = A_1$, Δ satisfies (0.1), we have

$$\begin{aligned}(D\Delta)_a(\mathbf{1}) = \lim_{r \to 0} \frac{\Delta(a + r\mathbf{1}) - \Delta(a)}{r} &= \lim_{r \to 0} \frac{r\{\Delta(a+\mathbf{1}) - \Delta(a)\}}{r} \\ &= \Delta(a+\mathbf{1}) - \Delta(a) \in \mathbb{T}\sigma(1) = \mathbb{T},\end{aligned}$$

and

$$\begin{aligned}(D\Delta)_a(i\mathbf{1}) = \lim_{r \to 0} \frac{\Delta(a + ri\mathbf{1}) - \Delta(a)}{r} &= \lim_{r \to 0} \frac{ri\{\Delta(a+\mathbf{1}) - \Delta(a)\}}{r} \\ &= i\{\Delta(a+\mathbf{1}) - \Delta(a)\}.\end{aligned}$$

It follows that $(D\Delta)_a(i\mathbf{1}) = i(D\Delta)_a(\mathbf{1})$ and $(D\Delta)_a(\mathbf{1}) \neq 0$. We conclude that $(D\Delta)_a$ is complex-linear. We have proved that if Δ has a real differential at a point $a \in A = A_1$, then $(D\Delta)_a$ is complex-linear. We conclude that Δ is holomorphic in A by applying [**69**, Lemma 2.4]. For $a, b \in A$, we define a map $f_{a,b} : \mathbb{C} \to \mathbb{C}$ by

$$f_{a,b}(\lambda) = \Delta(\lambda a + b) - \Delta(b).$$

Since Δ is holomorphic in A, $f_{a,b}$ is entire. Moreover, we have for any $\lambda \in \mathbb{C} \setminus \{0\}$

$$\frac{f_{a,b}(\lambda)}{\lambda} = \frac{\Delta(\lambda a + b) - \Delta(b)}{\lambda} \in \frac{\mathbb{T}\sigma(\lambda a)}{\lambda} = \frac{\lambda\mathbb{T}\sigma(a)}{\lambda} = \mathbb{T}\sigma(a),$$

and

$$\left|\frac{f_{a,b}(\lambda)}{\lambda}\right| \leq \|a\|.$$

By Liouville's Theorem, there exists $M \in \mathbb{C}$ such that $f_{a,b}(\lambda) = \lambda M$ for all $\lambda \in \mathbb{C}$. As $M = f_{a,b}(1) = \Delta(a+b) - \Delta(b)$, we get

$$\Delta(\lambda a + b) - \Delta(b) = \lambda\{\Delta(a+b) - \Delta(b)\}, \quad \lambda \in \mathbb{C},$$

and

$$\Delta(\lambda a + b) = \lambda\{\Delta(a+b) - \Delta(b)\} + \Delta(b), \quad \lambda \in \mathbb{C}. \tag{0.2}$$

Taking $b = 0$ in (0.2), we have

$$\Delta(\lambda a) = \lambda\Delta(a), \quad \lambda \in \mathbb{C}, \tag{0.3}$$

by the hypothesis (a). For any $c, d \in A$, taking $a = \frac{1}{2}(c-d)$, $b = d$ and $\lambda = 2$ in (0.2), we get

$$\begin{aligned}\Delta(c) = \Delta(2a+b) &= 2\{\Delta(a+b) - \Delta(b)\} + \Delta(b) \\ &= 2\left\{\Delta\left(\frac{1}{2}(c+d)\right) - \Delta(d)\right\} + \Delta(d) \\ &= 2\Delta\left(\frac{1}{2}(c+d)\right) - \Delta(d),\end{aligned}$$

and

$$\Delta\left(\frac{1}{2}(c+d)\right) = \frac{1}{2}\Delta(c) + \frac{1}{2}\Delta(d). \tag{0.4}$$

We conclude that Δ is complex-linear by (0.3) and (0.4).

(ii) We consider the case A is not separable. For any $a, b \in A$, we can restrict Δ to subalgebra $[a, b]$ of A generated by a and b. Since $[a, b]$ is separable, we conclude $\Delta|_{[a,b]}$ is complex-linear. As a, b are chosen arbitrarily, Δ is complex-linear too.

In addition, since $\Delta(a) = \Delta(a) - \Delta(0) \in \mathbb{T}\sigma(a)$, we apply [**75**, Proposition 2.2] to conclude that $\overline{\Delta(\mathbf{1})}\delta^{ij}_{(k_1k_2,x)}$ is multiplicative.

Secondly we assume that $A = A_{-1}$. We define the map $\overline{\Delta} : A \to \mathbb{C}$ by

$$\overline{\Delta}(a) = \overline{\Delta(a)}, \quad a \in A.$$

In the case $A = A_{-1}$, Δ satisfies for any $a \in A$,

$$\Delta(a + \lambda \cdot \mathbf{1}) - \Delta(a) = \overline{\lambda}\{\Delta(a + \mathbf{1}) - \Delta(a)\}, \quad \lambda \in \mathbb{C}.$$

Thus we have

$$\overline{\Delta}(a + \lambda \cdot \mathbf{1}) - \overline{\Delta}(a) = \lambda\{\overline{\Delta}(a + \mathbf{1}) - \overline{\Delta}(a)\}, \quad \lambda \in \mathbb{C}.$$

Moreover, it is clear that $\overline{\Delta}(0) = \overline{\Delta(0)} = 0$. Therefore, the map $\overline{\Delta} : A \to \mathbb{C}$ satisfies the conditions for Δ in the case of $A = A_1$. This in turn implies that $\overline{\Delta}$ is complex-linear and $\overline{\overline{\Delta}(\mathbf{1})}\,\overline{\Delta}$ is multiplicative. Thus we conclude that Δ is conjugate linear and $\overline{\Delta(\mathbf{1})}\Delta$ is multiplicative. □

1. 2-local isometries with the spherical version of the Kowalski-Słodkowski Theorem

Applying [**75**, Proposition 3.2] (or Theorem A.1), Li, Peralta, Wang and Wang studied 2-local and weak 2-local complex-linear isometries in [**75**]. In this appendix, we apply Theorem A.2 to solve 2-local problems.

Molnár [**87**] began to study 2-local complex-linear isometries. Given a Banach space $\mathfrak{M}_j$ for $j = 1, 2$, an isometry from $\mathfrak{M}_1$ into $\mathfrak{M}_2$ is a distance preserving map. The set of all surjective complex-linear isometries from $\mathfrak{M}_1$ onto $\mathfrak{M}_2$ is denoted by $\mathrm{Iso}_{\mathbb{C}}(\mathfrak{M}_1, \mathfrak{M}_2)$. The set of all maps from $\mathfrak{M}_1$ into $\mathfrak{M}_2$ is denoted by $M(\mathfrak{M}_1, \mathfrak{M}_2)$. We say that a map $T \in M(\mathfrak{M}_1, \mathfrak{M}_2)$ is a 2-local complex-linear isometry if for every $x, y \in \mathfrak{M}_1$ there is a $T_{x,y} \in \mathrm{Iso}_{\mathbb{C}}(\mathfrak{M}_1, \mathfrak{M}_2)$ such that $T(x) = T_{x,y}(x)$ and $T(y) = T_{x,y}(y)$. Molnár [**87**] proved that a 2-local complex-linear isometry on a certain C^*-algebra is a surjective complex-linear isometry. Initiated by his result, there are a lot of studies on 2-local complex-linear isometries on operator algebras and function spaces assuring that a 2-local complex-linear isometry is in fact a surjective complex-linear isometry [**1, 17, 34, 38, 50, 53, 75, 88, 87**].

Molnár raised a problem on 2-local isometries [**89, 90**]. The set of all surjective isometries (not necessarily linear) from $\mathfrak{M}_1$ onto $\mathfrak{M}_2$ is denoted by $\mathrm{Iso}(\mathfrak{M}_1, \mathfrak{M}_2)$. We say that $T \in M(\mathfrak{M}_1, \mathfrak{M}_2)$ is a 2-local isometry or T is 2-local in $\mathrm{Iso}(\mathfrak{M}_1, \mathfrak{M}_2)$ if for every $x, y \in \mathfrak{M}_1$ there is a $T_{x,y} \in \mathrm{Iso}(\mathfrak{M}_1, \mathfrak{M}_2)$ such that

$$T(x) = T_{x,y}(x) \text{ and } T(y) = T_{x,y}(y).$$

The problem asks whether a 2-local isometry is in fact a surjective isometry or not. One may expect that the problems on 2-local *complex-linear* isometries and 2-local isometries are not so different. But the problem on 2-local isometries is very different from the one on 2-local complex-linear isometries. To clarify the situation we exhibit an example that the assumption of the linearity makes a quite big difference in the conclusion for 2-local maps. Let $\mathrm{A}(\mathbb{C},\mathbb{C}) = \{T : \mathbb{C} \to \mathbb{C}; Tx = ax + b \ (\exists a, b \in \mathbb{C})\}$. Since any map $T : \mathbb{C} \to \mathbb{C}$ is 2-local in $\mathrm{A}(\mathbb{C},\mathbb{C})$, it needs not be $T \in \mathrm{A}(\mathbb{C},\mathbb{C})$ in general. However, let $\mathrm{A}_{\mathbb{C}}(\mathbb{C},\mathbb{C}) = \{T; T \in \mathrm{A}(\mathbb{C},\mathbb{C}), \mathrm{T} \text{ is } \mathbb{C}\text{-linear}\} = \{T : \mathbb{C} \to \mathbb{C}; Tx = ax \ (\exists a \in \mathbb{C})\}$. Then we get every 2-local map in $\mathrm{A}_{\mathbb{C}}(\mathbb{C},\mathbb{C})$ is an element of $\mathrm{A}_{\mathbb{C}}(\mathbb{C},\mathbb{C})$. We can easily prove that a 2-local isometry is necessarily an isometry. What we need to prove is that a 2-local isometry is surjective. One may think that it is not a big deal, but it is not. Molnár [**89**] worked quite hard to prove that a 2-local isometry on $B(H)$ for a separable complex Hilbert space H is in fact a surjective isometry on $B(H)$. The author believes that this is the first result on the problem of 2-local isometries. Molnár asked a question whether a 2-local map in $\mathrm{Iso}(C([0,1]), C([0,1]))$ is an element in $\mathrm{Iso}(C([0,1]), C([0,1]))$ or not [**90**]. Inspired by his problem, Hatori and the author proved that a 2-local map in $\mathrm{Iso}(B, B)$ is an element of $\mathrm{Iso}(B, B)$, where B is the Banach space of all continuously differentiable functions or the Banach space of Lipschitz functions on the closed unit interval equipped with a certain norm [**42**].

The aim of this chapter is to apply a generalization of a spherical variant of the Kowalski and Słodkowski theorem exhibited in [**75**]. Then we prove that 2-local isometries on several function spaces are surjective isometries. In particular, we give an affirmative answer to the problem of Molnár (Corollary A.11). We remark that Mori [**93**] also got an affirmative answer to the problem by a different approach applying theory of operator algebras.

In this chapter, we denote the unit circle on the complex plane by $\mathbb{T} = \{z \in \mathbb{C}; |z| = 1\}$. For the simplicity of the notation we denote $[f]^1 = f$ and $[f]^{-1} = \overline{f}$, the complex-conjugate of f for any complex-valued function f. For any unital Banach algebra, **1** stands for the unity of itself. The identity map is denoted by Id.

2. Results and Proofs

In this chapter B_j is a unital semi-simple commutative Banach algebra with maximal ideal space $\mathcal{M}_j$ for $j = 1, 2$. The Gelfand transform $\hat{\cdot} : B_j \to \widehat{B_j} \subset C(\mathcal{M}_j)$ is a continuous isomorphism. Identifying B_j with $\widehat{B_j}$, we consider that B_j is a subalgebra of $C(\mathcal{M}_j)$.

We say that $f \in B_j$ is unimodular if $|f| = 1$ on $\mathcal{M}_j$. Since $\mathcal{M}_j$ is a maximal ideal space and a unimodular element f of B_j has no zeros on $\mathcal{M}_j$, $\bar{f} = 1/f \in B_j$.

An interesting generalization of the concept of 2-local maps, that is weak 2-locality, was introduced in [**24, 75**]. We define a *pointwise* 2-local map.

Definition A.4. Let $\mathcal{S} \subset M(B_1, B_2)$. We say $T \in M(B_1, B_2)$ is a pointwise 2-local in $\mathcal{S}$ if for every trio $f, g \in B_1$ and $x \in \mathcal{M}_2$ there exists $T_{f,g,x} \in \mathcal{S}$ such that

$$(T(f))\,(x) = (T_{f,g,x}(f))\,(x) \text{ and } (T(g))\,(x) = ((T_{f,g,x}(g))\,(x).$$

Note that if a map T is 2-local, then T is weak 2-local. If T is weak 2-local, then T is pointwise 2-local. We say that $T \in M(B_1, B_2)$ is a pointwise 2-local isometry if T is pointwise 2-local in $\mathrm{Iso}(B_1, B_2)$. Our interest is whether a pointwise 2-local in $\mathrm{Iso}(B_1, B_2)$ is in fact surjective isometry from B_1 onto B_2 or not. Simple examples show that a pointwise 2-local isometry need not be a surjection or an isometry. We show three of them.

- a map on $C[0, 1]$
  Let $\pi : [0, 1] \to [0, 1]$ be a continuous function such that $\pi(0) = 0$, $\pi(1) = 1$ and $0 < \pi(x) < 1$ for $x \in (0, 1)$. Put $T(f) = f \circ \pi$, $f \in C[0, 1]$. It is easy to see that T is pointwise 2-local in $\mathrm{Iso}(C[0, 1], C[0, 1])$ while it is not surjective when π is not a homeomorphism.
- a map on $C^1[0, 1]$
  With the norm $\|f\|_\Sigma = \|f\|_\infty + \|f'\|_\infty$ for $f \in C^1[0, 1]$, $C^1[0, 1]$ is a unital semi-simple commutative Banach algebra with maximal ideal space $[0, 1]$. Let $T : C^1[0, 1] \to C^1[0, 1]$ stand $T(f) = \exp(i\cdot)f$, $f \in C^1[0, 1]$. By a simple calculation we

have T is pointwise 2-local in $\mathrm{Iso}(C^1[0,1], C^1[0,1])$ while T is not an isometry since $\|\mathbf{1}\|_\Sigma = 1$ and $\|T(\mathbf{1})\|_\Sigma = 2$.

- a map on the disk algebra $A(\bar{\mathbb{D}})$.
  It is well known that the maximal ideal space of the disk algebra $A(\bar{\mathbb{D}})$ is $\bar{\mathbb{D}}$. Let $\pi_0(z) = z^2$, $z \in \bar{\mathbb{D}}$. Then the map $T : A(\bar{\mathbb{D}}) \to A(\bar{\mathbb{D}})$ defined by $T(f) = f \circ \pi_0$, $f \in A(\bar{\mathbb{D}})$. Trivially T is not surjective, hence $T \notin \mathrm{Iso}(A(\bar{\mathbb{D}}), A(\bar{\mathbb{D}}))$. On the other hand, T is pointwise 2-local in $\mathrm{Iso}(A(\bar{\mathbb{D}}), A(\bar{\mathbb{D}}))$. The reason is as follows. Let $f, g \in A(\bar{\mathbb{D}})$ and $x \in \bar{\mathbb{D}}$ be arbitrary. If $|x| = 1$, then put $\varphi_x(z) = xz$. If $|x| < 1$, then it is well known that there is a Möbius transformation φ_x such that $\varphi_x(x) = x^2$ since both of x and x^2 is in $\mathbb{D}$. Put $T_{f,g,x}(h) = h \circ \varphi_x$, $h \in A(\bar{\mathbb{D}})$. We infer by a calculation that $(T(f))(x) = (T_{f,g,x}(f))(x)$ and $(T(g))(x) = (T_{f,g,x}(g))(x)$. Thus T is pointwise 2-local in $\mathrm{Iso}(A(\bar{\mathbb{D}}), A(\bar{\mathbb{D}}))$.

It is interesting to point out that a pointwise 2-local isometry is in fact a surjective isometry for some Banach algebra (see Section 3.4). A simple example is a pointwise 2-local isometry on the annulus algebra.

- Let $0 < r < 1$ and $\Omega = \{z : r \leq |z| \leq 1\}$ be an annulus. Let $A(\Omega)$ be the algebra of all complex-valued continuous functions which is analytic on the interior of Ω. It is well known that $A(\Omega)$ is a uniform algebra on Ω whose maximal ideal space is homeomorphic to Ω. A pointwise 2-local map in $\mathrm{Iso}(A(\Omega), A(\Omega))$ is a surjective isometry. This can be proved by using the fact that a homeomorphism on Ω which is analytic on the interior is just a rotation.

Recall that for an $\epsilon \in \{\pm 1\}$ and $f \in B_j$, $[f]^\epsilon = f$ if $\epsilon = 1$ and $[f]^\epsilon = \bar{f}$ if $\epsilon = -1$.

Let

$$\begin{aligned}\mathrm{GWC} = \{T \in M(B_1, B_2);\ &\text{there exist a } \beta \in B_2,\\ &\text{an } \alpha \in B_2 \text{ with } |\alpha| = 1 \text{ on } \mathcal{M}_2,\\ &\text{a continuous map } \pi : \mathcal{M}_2 \to \mathcal{M}_1,\\ &\text{and a continuous map } \epsilon : \mathcal{M}_2 \to \{\pm 1\}\\ &\text{such that } T(f) = \beta + \alpha[f \circ \pi]^{\epsilon} \text{ for every } f \in B_1\},\end{aligned}$$

Applying Theorem A.2 we show that a pointwise 2-local map in GWC is also in GWC.

THEOREM A.5. *Suppose that* $T \in M(B_1, B_2)$ *is pointwise 2-local in* GWC. *Then there exist a continuous map* $\pi : \mathcal{M}_2 \to \mathcal{M}_1$ *and a continuous map* $\epsilon : \mathcal{M}_2 \to \{\pm 1\}$ *such that*

$$T(f) = T(0) + (T(\mathbf{1}) - T(0))[f \circ \pi]^{\epsilon}, \quad f \in B_1, \tag{2.1}$$

where $T(\mathbf{1}) - T(0)$ *is a unimodular element in* B_2. *In particular, a pointwise 2-local map in* GWC *is an element in* GWC.

Proof of Theorem A.5. Put $T_0 = T - T(0)$. We infer that $T_0(0) = 0$. Since T is pointwise 2-local in GWC, it is obvious that T_0 is also pointwise 2-local in GWC. Let $x \in \mathcal{M}_2$. There exists $\beta_{0,\mathbf{1},x}, \alpha_{0,\mathbf{1},x} \in B_2$ with $|\alpha_{0,\mathbf{1},x}| = 1$ on $\mathcal{M}_2$, a continuous map $\pi_{0,\mathbf{1},x} : \mathcal{M}_2 \to \mathcal{M}_1$ and a continuous map $\epsilon_{0,\mathbf{1},x} : \mathcal{M}_2 \to \{\pm 1\}$ such that

$$T_0(\mathbf{1})(x) = \beta_{0,\mathbf{1},x}(x) + \alpha_{0,\mathbf{1},x}(x)[\mathbf{1} \circ \pi_{0,\mathbf{1},x}]^{\epsilon_{0,\mathbf{1},x}(x)}(x) = \beta_{0,\mathbf{1},x}(x) + \alpha_{0,\mathbf{1},x}(x),$$

and

$$0 = T_0(0)(x) = \beta_{0,\mathbf{1},x}(x) + \alpha_{0,\mathbf{1},x}(x)[0 \circ \pi_{0,\mathbf{1},x}]^{\epsilon_{0,\mathbf{1},x}(x)}(x) = \beta_{0,\mathbf{1},x}(x).$$

It follows that $T_0(\mathbf{1})(x) = \alpha_{0,\mathbf{1},x}(x)$. As $x \in \mathcal{M}_2$ is arbitrary we have

$$|T_0(\mathbf{1})(x)| = 1, \quad x \in \mathcal{M}_2. \tag{2.2}$$

Hence $T_0(\mathbf{1})$ has no zeros on $\mathcal{M}_2$, so $\overline{T_0(\mathbf{1})} = T_0(\mathbf{1})^{-1} \in B_2$. We define $T_1 \in M(B_1, B_2)$ by

$$T_1 = \overline{T_0(\mathbf{1})} T_0. \tag{2.3}$$

We see that

$$T_1(0) = \overline{T_0(\mathbf{1})} T_0(0) = 0, \quad T_1(\mathbf{1}) = \overline{T_0(\mathbf{1})} T_0(\mathbf{1}) = 1 \tag{2.4}$$

by (2.2). To proceed the proof of Theorem A.5, we need some claims.

Claim 1. *There exists a map* $\pi : \mathcal{M}_2 \to \mathcal{M}_1$ *and a map* $\epsilon : \mathcal{M}_2 \to \{\pm 1\}$ *such that*

$$T_1(f) = [f \circ \pi]^{\epsilon}, \quad f \in B_1.$$

Proof. By (2.2), we infer that T_1 is pointwise 2-local in GWC. Fix $x \in \mathcal{M}_2$. We define $\Delta_x : B_1 \to \mathbb{C}$ by

$$\Delta_x(f) = (T_1(f))(x), \quad f \in B_1.$$

As T_1 is pointwise 2-local in GWC, for any $f, g \in B_1$, there exists $T_{f,g,x} \in$ GWC such that

$$\begin{aligned}\Delta_x(f) &= (T_1(f))(x)\\ &= T_{f,g,x}(f)(x) = \beta_{f,g,x}(x) + \alpha_{f,g,x}(x)[f \circ \pi_{f,g,x}]^{\epsilon_{f,g,x}(x)}(x)\end{aligned}$$

and

$$\begin{aligned}\Delta_x(g) &= (T_1(g))(x)\\ &= T_{f,g,x}(g)(x) = \beta_{f,g,x}(x) + \alpha_{f,g,x}(x)[g \circ \pi_{f,g,x}]^{\epsilon_{f,g,x}(x)}(x).\end{aligned}$$

We infer that

$$\Delta_x(f) - \Delta_x(g) = \alpha_{f,g,x}(x)[(f-g) \circ \pi_{f,g,x}]^{\epsilon_{f,g,x}(x)}(x).$$

If $x \in \epsilon_{f,g,x}^{-1}(1)$, we have

$$[(f-g) \circ \pi_{f,g,x}]^{\epsilon_{f,g,x}(x)}(x) = (f-g)(\pi_{f,g,x}(x)) \in \sigma(f-g).$$

If $x \in \epsilon_{f,g,x}^{-1}(-1)$, we have

$$[(f-g) \circ \pi_{f,g,x}]^{\epsilon_{f,g,x}(x)}(x) = \overline{(f-g)(\pi_{f,g,x}(x))} \in \mathbb{T}\sigma(f-g).$$

Therefore we get

$$\Delta_x(f) - \Delta_x(g) \in \mathbb{T}\sigma(f-g), \quad f, g \in B_1.$$

By (2.4), we have $\Delta_x(0) = T_1(0)(x) = 0$. Applying Theorem A.2, we obtain Δ_x is complex linear or conjugate linear and $\overline{\Delta_x(\mathbf{1})}\Delta_x$ is multiplicative. As $\overline{\Delta_x(\mathbf{1})} = \overline{T_1(\mathbf{1})(x)} = 1$ by (2.4), we conclude that Δ_x is multiplicative. In addition $\Delta_x(\mathbf{1}) = 1$ implies that $\Delta_x \neq 0$. Therefore for any $x \in \mathcal{M}_2$, one of the following (i) and (ii) occurs:

(i) Δ_x is a non-zero multiplicative complex-linear functional,
(ii) Δ_x is a non-zero multiplicative conjugate linear functional.

In the case (i), by Gelfand theory, there exists $\pi(x) \in \mathcal{M}_1$ such that

$$\Delta_x(f) = f(\pi(x)), \quad f \in B_1.$$

In the case (ii), $\overline{\Delta_x}$ is non-zero multiplicative complex-linear functional. Thus there exists $\pi(x) \in \mathcal{M}_1$ such that

$$\overline{\Delta_x}(f) = f(\pi(x)), \quad f \in B_1,$$

hence

$$\Delta_x(f) = \overline{f(\pi(x))}, \quad f \in B_1.$$

Recall that $\Delta_x(f) = (T_1(f))(x)$, we have

$$T_1(f)(x) = \begin{cases} f \circ \pi(x), & (\Delta_x \text{ is complex-linear}) \\ \overline{f \circ \pi}(x), & (\Delta_x \text{ is conjugate linear}). \end{cases}$$

We define a map $\epsilon : \mathcal{M}_2 \to \{\pm 1\}$ by

$$\epsilon(x) = \begin{cases} 1, & (\Delta_x \text{ is complex-linear}) \\ -1, & (\Delta_x \text{ is conjugate linear}). \end{cases} \tag{2.5}$$

Then we conclude that

$$T_1(f)(x) = [f \circ \pi]^{\epsilon(x)}(x), \quad f \in B_1,\ x \in \mathcal{M}_2.$$

□

Let

$$K_1 = \{x \in \mathcal{M}_2; \Delta_x \text{ is complex-linear}\}$$

and

$$K_{-1} = \{x \in \mathcal{M}_2; \Delta_x \text{ is conjugate linear}\}.$$

Rewriting (2.5), we have

$$\epsilon(x) = \begin{cases} 1, & (x \in K_1) \\ -1, & (x \in K_{-1}). \end{cases}$$

Claim 2. *We have* $K_1 = \{x \in \mathcal{M}_2; \Delta_x(i) = i\}$ *and* $K_{-1} = \{x \in \mathcal{M}_2; \Delta_x(i) = -i\}$. *In addition* $\mathcal{M}_2 = K_1 \cup K_{-1}$, $K_1 \cap K_{-1} = \emptyset$ *and* K_1 *and* K_{-1} *are closed subset of* $\mathcal{M}_2$.

Proof. Since for any $x \in \mathcal{M}_2$, Δ_x is complex-linear or conjugate linear, it is clear that $\mathcal{M}_2 = K_1 \cup K_{-1}$. By the definition of K_1 and $\Delta_x(\mathbf{1}) = 1$, if $x \in K_1$, then $x \in \{x \in \mathcal{M}_2; \Delta_x(i) = i\}$. Suppose that $x \in \{x \in \mathcal{M}_2; \Delta_x(i) = i\}$. Then $\Delta_x(i) = i\Delta_x(\mathbf{1})$. This implies that

$x \in K_1$. We conclude that $K_1 = \{x \in \mathcal{M}_2; \Delta_x(i) = i\}$. We can also prove that $K_{-1} = \{x \in \mathcal{M}_2; \Delta_x(i) = -i\}$ in the similar argument. Therefore it is easy to see that $K_1 \cap K_{-1} = \emptyset$. Let $\{\boldsymbol{x}_\alpha\} \subset K_1$ be a net with $x_\alpha \to x_0 \in \mathcal{M}_2$. We get

$$i = \Delta_{x_\alpha}(i) = (T_1(i))(x_\alpha) \to (T_1(i))(x_0) = \Delta_{x_0}(i).$$

This implies that $\Delta_{x_0}(i) = i$ and $x_0 \in K_1$. We have K_1 is closed in $\mathcal{M}_2$. We also get K_{-1} is closed in the same way. □

Claim 2 shows that $\epsilon : \mathcal{M}_2 \to \{\pm 1\}$ is continuous.

Claim 3. *We have* $\pi : \mathcal{M}_2 \to \mathcal{M}_1$ *is continuous.*

Proof. Let $\{\boldsymbol{x}_\alpha\} \subset \mathcal{M}_2$ be a net with $x_\alpha \to x_0 \in \mathcal{M}_2$. By Claim 2, K_1 and K_{-1} are closed and $K_1 \cap K_{-1} = \emptyset$. Thus there is no loss of generality to assume that

(i) $\{x_\alpha\} \subset K_1$ and $x_0 \in K_1$
(ii) $\{x_\alpha\} \subset K_{-1}$ and $x_0 \in K_{-1}$.

First, we consider the case (i). Then we have

$$T_1(f)(x_\alpha) \to T_1(f)(x_0), \quad f \in B_1,$$

hence

$$(f \circ \pi)(x_\alpha) \to (f \circ \pi)(x_0), \quad f \in B_1.$$

This implies that $\pi(x_\alpha) \to \pi(x_0)$ with the Gelfand topology. For the case (ii), we have

$$T_1(f)(x_\alpha) \to T_1(f)(x_0), \quad f \in B_1,$$

and

$$\overline{(f \circ \pi)(x_\alpha)} \to \overline{(f \circ \pi)(x_0)}, \quad f \in B_1.$$

Thus we get $\pi(x_\alpha) \to \pi(x_0)$ with the Gelfand topology. We conclude that π is continuous. □

Continuation of Proof of Theorem A.5. By (2.3), we get $T_0 = T_0(\mathbf{1})T_1$. As $T_0 = T - T(0)$ and Claim 1, we have

$$\begin{aligned} T(f) &= T_0(f) + T(0) \\ &= T_0(\mathbf{1})T_1(f) + T(0) \\ &= T_0(\mathbf{1})[f \circ \pi]^\epsilon + T(0), \quad f \in B_1. \end{aligned}$$

Putting $f = \mathbf{1}$, we have $T_0(\mathbf{1}) = T(\mathbf{1}) - T(0)$ and

$$T(f) = (T(\mathbf{1}) - T(0))[f \circ \pi]^{\epsilon} + T(0).$$

In addition, by (2.2), we have $|T_0(\mathbf{1})| = 1$. We obtain that $T_0(\mathbf{1}) = T(\mathbf{1}) - T(0)$ is a unimodular element in B_2. $\square$

REMARK A.6. Even though a map $T \in M(B_1, B_2)$ is a 2-local map in GWC, it is not always the case that $\pi : \mathcal{M}_2 \to \mathcal{M}_1$ is a homeomorphism. In fact, the map T_0 in [**38**, Theorem 2.3] is a 2-local automorphism, hence 2-local in $\mathrm{Iso}_{\mathbb{C}}(C(\bar{\mathcal{K}}), C(\bar{\mathcal{K}}))$. On the other hand, the corresponding continuous map is not injective, hence it is not a homeomorphism.

COROLLARY A.7. *Suppose that $T \in M(B_1, B_2)$ is a pointwise 2-local in* GWC *and T is injective. Then $\pi(\mathcal{M}_2)$ is a uniqueness set for B_1, i.e. if $g \in B_1$ and $g = 0$ on $\pi(\mathcal{M}_2)$, then $g = 0$.*

PROOF. Suppose that $g \in B_1$ and $g = 0$ on $\pi(\mathcal{M}_2)$. Substituting g in (2.1), we get

$$T(g) = T(0) + (T(\mathbf{1}) - T(0))[g \circ \pi]^{\epsilon} = T(0) + (T(\mathbf{1}) - T(0))[0]^{\epsilon} = T(0).$$

Since T is injective, we have that $g = 0$. Hence $\pi(\mathcal{M}_2)$ is a uniqueness set for B_1. $\square$

Let

$$\begin{aligned}\mathrm{WC}_{\mathbb{C}} = \{T \in M(B_1, B_2);\ &\text{there exists} \\ &\text{an } \alpha \in B_2 \text{ with } |\alpha| = 1 \text{ on } \mathcal{M}_2, \\ &\text{and a continuous map } \pi : \mathcal{M}_2 \to \mathcal{M}_1 \\ &\text{such that } T(f) = \alpha f \circ \pi \text{ for every } f \in B_1\}.\end{aligned}$$

Then $\mathrm{WC}_{\mathbb{C}}$ is a set of weighted composition operators. We see that a pointwise 2-local weighted composition operator is a weighted composition operator.

COROLLARY A.8. *Suppose that $T \in M(B_1, B_2)$ is pointwise 2-local in $\mathrm{WC}_{\mathbb{C}}$. Then $T \in \mathrm{WC}_{\mathbb{C}}$.*

Proof. Let $T \in M(B_1, B_2)$ be pointwise 2-local in $\mathrm{WC}_{\mathbb{C}}$. Since $\mathrm{WC}_{\mathbb{C}} \subset \mathrm{GWC}$, we see by Theorem A.5 that there exist a continuous map $\pi : \mathcal{M}_2 \to \mathcal{M}_1$ and a continuous map $\epsilon : \mathcal{M}_2 \to \{\pm 1\}$ such that

$$T(f) = T(0) + (T(\mathbf{1}) - T(0))[f \circ \pi]^{\epsilon}, \quad f \in B_1, \tag{2.6}$$

where $T(\mathbf{1}) - T(0)$ is a unimodular element in B_2. Since any map in $\mathrm{WC}_{\mathbb{C}}$ is complex-linear, we infer by a simple calculation that $T(0) = 0$ and T is homogeneous with respect to complex scalar. We see by (2.6) that

$$T(f) = T(\mathbf{1}) f \circ \pi, \quad f \in B_1,$$

where $T(\mathbf{1})$ is a unimodular function. Thus $T \in \mathrm{WC}_{\mathbb{C}}$. □

3. Applications

In this section we study 2-local isometries on several function spaces by applying Theorem A.5.

3.1. Uniform algebras. We say that A is a uniform algebra on X if A is a uniformly closed subalgebra of $C(X)$ which contains constant functions and separates the points of X. As the Gelfand transformation on a uniform algebra is an isometric isomorphism, a uniform algebra is isometrically isomorphic to its Gelfand transform. We may suppose that X is a subset of the maximal ideal space $\mathcal{M}_A$, and A is a uniform algebra on $\mathcal{M}_A$. The Banach algebra $C(X)$ is a uniform algebra on X whose maximal ideal space is X. By Theorem 2.1 and Corollary 3.4 in [**37**] we have the following. We denote the maximal ideal space of a uniform algebra A_j by $\mathcal{M}_j$ for $j = 1, 2$.

Theorem A.9. *Let A_j be a uniform algebra on a compact Hausdorff space X_j for $j = 1, 2$. Suppose that $U : A_1 \to A_2$ is a surjective isometry from A_1 onto A_2. Then there exists a homeomorphism $\pi : \mathcal{M}_2 \to \mathcal{M}_1$, a unimodular function $\alpha \in A_2$, and a continuous map $\epsilon : \mathcal{M}_2 \to \{\pm 1\}$ such that*

$$U(f) = U(0) + \alpha [f \circ \pi]^{\epsilon}, \quad f \in A_1. \tag{3.1}$$

If $A_j = C(X_j)$, the map U defined by (3.1) is a surjective isometry from $C(X_1)$ onto $C(X_2)$.

By Theorem A.9 we see that

$$\mathrm{Iso}(A_1, A_2) \subset \mathrm{GWC}$$

for uniform algebras A_1 and A_2. A direct consequence of Theorem A.5 we have Corollary A.10, which is a generalization of Theorem 3.10 of [**75**].

COROLLARY A.10. *Let A_j be a uniform algebra on a compact Hausdorff space X_j for $j = 1, 2$. Suppose that $T \in M(A_1, A_2)$ is pointwise 2-local in* $\mathrm{Iso}(A_1, A_2)$*. Then there exist a continuous map $\pi : \mathcal{M}_2 \to \mathcal{M}_1$ and a continuous map $\epsilon : \mathcal{M}_2 \to \{\pm 1\}$ such that*

$$T(f) = T(0) + (T(\mathbf{1}) - T(0))[f \circ \pi]^{\epsilon}, \quad f \in A_1,$$

where $T(\mathbf{1}) - T(0)$ is a unimodular function.

We have the following.

COROLLARY A.11. *Let X_j be a first countable compact Hausdorff space for $j = 1, 2$. Suppose that $T \in M(C(X_1), C(X_2))$ is 2-local in* $\mathrm{Iso}(C(X_1), C(X_2))$*. Then we have $T \in \mathrm{Iso}(C(X_1), C(X_2))$.*

PROOF. Let T be a 2-local in $\mathrm{Iso}(C(X_1), C(X_2))$. By Corollary A.10, there exist a continuous map $\pi : X_2 \to X_1$ and a continuous map $\epsilon : X_2 \to \{\pm 1\}$ such that

$$T(f) = T(0) + (T(\mathbf{1}) - T(0))[f \circ \pi]^{\epsilon}, \quad f \in C(X_1). \tag{3.2}$$

We prove π is an injection. Suppose that $y_1, y_2 \in X_2$ such that $\pi(y_1) = \pi(y_2) = x \in X_1$. Since X_1 is first countable there exists $g \in C(X_1)$ such that $g^{-1}(0) = \{x\}$. Since $T_1 = \overline{T_0(\mathbf{1})}T_0$ for $T_0 = T - T(0)$ is 2-local in $\mathrm{Iso}(C(X_1), C(X_2))$, we have

$$\begin{aligned} 0 = T_1(0) &= T_{0,g}(0) \\ &= \beta_{0,g} + \alpha_{0,g}[0 \circ \pi_{0,g}]^{\epsilon_{0,g}} = \beta_{0,g}, \end{aligned}$$

and

$$\begin{aligned} T_1(g) &= T_{0,g}(g) \\ &= \beta_{0,g} + \alpha_{0,g}[g \circ \pi_{0,g}]^{\epsilon_{0,g}}. \end{aligned}$$

Hence we see that

$$T_1(g) = \alpha_{0,g}[g \circ \pi_{0,g}]^{\epsilon_{0,g}}.$$

We have

$$(T_1(g))^{-1}(0) = (g \circ \pi_{0,g})^{-1}(0) = \pi_{0,g}^{-1}(x).$$

Since $\pi_{0,g}$ is homeomorphism, the set $\pi_{0,g}^{-1}(x)$ is a singleton. Moreover applying (3.2) we have

$$T_1(g) = [g \circ \pi]^{\epsilon}.$$

Thus we obtain

$$(T_1(g))^{-1}(0) = (g \circ \pi)^{-1}(0) = \pi^{-1}(x) \ni \{y_1, y_2\}.$$

As we have already proved that the set $(T_1(g))^{-1}(0) = \pi_{0,g}^{-1}(x)$ is a singleton, we infer that $y_1 = y_2$. Thus π is injective. Since T is a 2-local isometry, T is an isometry by the definition of a 2-local isometry. Hence T is injective. By Corollary A.7, $\pi(X_2)$ is a uniqueness set for $C(X_1)$, which is X_1 itself. As X_j is compact Hausdorff space, we infer that π is a homeomorphism. It follows that $T \in \mathrm{Iso}(C(X_1), C(X_2))$ □

Corollary A.11 gives an affirmative answer to the problem mentioned by Molnár. Mori proved the same statement in [**93**, Theorem 4.6] by a different argument applying theory of linear operators.

Next we consider the disk algebra.

Corollary A.12. *Suppose that U is a surjective isometry from the disk algebra $A(\bar{\mathbb{D}})$ onto itself. Then there exists a Möbius transformation φ on $\bar{\mathbb{D}}$ and a unimodular constant α such that*

$$U(f) = U(0) + \alpha f \circ \varphi, \quad f \in A(\bar{\mathbb{D}})$$

or

$$U(f) = U(0) + \alpha \overline{f \circ \bar{\varphi}}, \quad f \in A(\bar{\mathbb{D}}).$$

Conversely if one of the above equations holds, then U is a surjective isometry from the disk algebra onto itself.

Proof. Applying Theorem A.9 we have a homeomorphism $\pi : \bar{\mathbb{D}} \to \bar{\mathbb{D}}$, a unimodular function $\alpha \in A(\bar{\mathbb{D}})$, and a continuous map $\epsilon : \bar{\mathbb{D}} \to \{\pm 1\}$ such that

$$U(f) = U(0) + \alpha[f \circ \pi]^{\epsilon}, \quad f \in A(\bar{\mathbb{D}}). \tag{3.3}$$

Due to the maximum modulus principle for analytic functions, α is a unimodular constant. Since $\bar{\mathbb{D}}$ is connected, $\epsilon = 1$ on $\bar{\mathbb{D}}$, or $\epsilon = -1$ on $\bar{\mathbb{D}}$. Letting $f = \mathrm{Id}$, the identity function, in (3.3) we have

$$\bar{\alpha}(U(\mathrm{Id}) - U(0)) = \pi \text{ if } \epsilon = 1, \tag{3.4}$$

$$\bar{\alpha}(U(\mathrm{Id}) - U(0)) = \bar{\pi} \text{ if } \epsilon = -1. \tag{3.5}$$

Suppose that $\epsilon = 1$. Then π is analytic on $\mathbb{D}$ by (3.4). As π is a homeomorphism, we conclude that π is a Möbius transformation. In the same way, $\bar{\pi}$ is a Möbius transformation if $\epsilon = -1$. Letting $\varphi = \pi$ if $\epsilon = 1$, and $\varphi = \bar{\pi}$ if $\epsilon = -1$, φ is a Möbius transformation. It follows that

$$U(f) = U(0) + \alpha f \circ \varphi, \quad f \in A(\bar{\mathbb{D}})$$

if $\epsilon = 1$ and

$$U(f) = U(0) + \alpha \overline{f \circ \bar{\varphi}}, \quad f \in A(\bar{\mathbb{D}})$$

if $\epsilon = -1$.

The converse statement is trivial. □

By Corollary A.12 we see that

$$\mathrm{Iso}(A(\bar{\mathbb{D}}), A(\bar{\mathbb{D}})) \subset \mathrm{GWC}$$

for the disk algebra $A(\bar{\mathbb{D}})$.

Corollary A.13. *Suppose that $T \in M(A(\bar{\mathbb{D}}), A(\bar{\mathbb{D}}))$ is 2-local in* $\mathrm{Iso}(A(\bar{\mathbb{D}}), A(\bar{\mathbb{D}}))$. *Then $T \in \mathrm{Iso}(A(\bar{\mathbb{D}}), A(\bar{\mathbb{D}}))$.*

Proof. Corollary A.10 asserts that there exist a continuous map $\pi : \bar{\mathbb{D}} \to \bar{\mathbb{D}}$ and a continuous map $\epsilon : \bar{\mathbb{D}} \to \{\pm 1\}$ such that

$$T(f) = T(0) + (T(\mathbf{1}) - T(0))[f \circ \pi]^{\epsilon}, \quad f \in A(\bar{\mathbb{D}}), \tag{3.6}$$

where $T(\mathbf{1}) - T(0)$ is a unimodular function. By the same way as the proof of Corollary A.12 we see that $T(\mathbf{1}) - T(0)$ is a unimodular constant. We also see that $\epsilon = 1$ on $\bar{\mathbb{D}}$ or $\epsilon = -1$ on $\bar{\mathbb{D}}$ because $\bar{\mathbb{D}}$ is connected and ϵ is continuous. Letting $f = \mathrm{Id}$ in (3.6), we have that π is analytic on $\mathbb{D}$ if $\epsilon = 1$, and $\bar{\pi}$ is analytic on $\mathbb{D}$ if $\epsilon = -1$. Put $\varphi = \pi$ if $\epsilon = 1$ and $\varphi = \bar{\pi}$ if $\epsilon = -1$. Put $T_1 = \overline{T(\mathbf{1}) - T(0)}(T - T(0))$. Then

$$T_1(f) = f \circ \varphi, \quad f \in A(\bar{\mathbb{D}})$$

if $\epsilon = 1$, and

$$T_1(f) = \overline{f \circ \bar{\varphi}}, \quad f \in A(\bar{\mathbb{D}})$$

if $\epsilon = -1$. Since T_1 is 2-local in $\mathrm{Iso}(A(\bar{\mathbb{D}}), A(\bar{\mathbb{D}}))$, there exists a Möbius transform φ_0, $u \in A(\bar{\mathbb{D}})$, and a unimodular constant α such that

$$\varphi = T_1(\mathrm{Id}) = u + \alpha\varphi_0 \text{ and } 0 = T_1(0) = u.$$

It follows that $\varphi = \alpha\varphi_0$. As $|\alpha| = 1$, we infer that φ is a Möbius transformation on $\bar{\mathbb{D}}$. We infer by Corollary A.12 that $T \in \mathrm{Iso}(A(\bar{\mathbb{D}}), A(\bar{\mathbb{D}}))$. □

3.2. Lipschitz algebras.

Corollary A.14. *Let (X_j, d) be a compact metric space for $j = 1, 2$. Let $\|\cdot\|_j$ be any norm on $\mathrm{Lip}(X_j)$. We do not assume that $\|\cdot\|_j$ is complete. Suppose that*

$$\begin{aligned}(3.7)\quad \mathrm{Iso}((\mathrm{Lip}(X_1), \|\cdot\|_1), &(\mathrm{Lip}(X_2), \|\cdot\|_2)) \\ &= \{T \in M(\mathrm{Lip}(X_1), \mathrm{Lip}(X_2)); \\ &\textit{there exist } \beta \in \mathrm{Lip}(X_2),\ \alpha \in \mathbb{T}, \\ &\textit{a surjective isometry } \pi : X_2 \to X_1, \textit{ and } \epsilon = \pm 1 \\ &\textit{such that } T(f) = \beta + \alpha[f \circ \pi]^\epsilon \textit{ for every } f \in \mathrm{Lip}(X_1)\}.\end{aligned}$$

Suppose that $T \in M((\mathrm{Lip}(X_1), \|\cdot\|_1), (\mathrm{Lip}(X_2), \|\cdot\|_2))$ is 2-local in $\mathrm{Iso}((\mathrm{Lip}(X_1), \|\cdot\|_1), (\mathrm{Lip}(X_2), \|\cdot\|_2))$. Then $T \in \mathrm{Iso}((\mathrm{Lip}(X_1), \|\cdot\|_1), (\mathrm{Lip}(X_2), \|\cdot\|_2))$.

Proof. Suppose that T is 2-local in $\mathrm{Iso}((\mathrm{Lip}(X_1), \|\cdot\|_1), (\mathrm{Lip}(X_2), \|\cdot\|_2))$. The equality (3.7) implies that $\mathrm{Iso}((\mathrm{Lip}(X_1), \|\cdot\|_1), (\mathrm{Lip}(X_2), \|\cdot\|_2)) \subset \mathrm{GWC}$. Applying Theorem A.5, there exists a continuous map $\pi : X_2 \to X_1$ and a continuous map $\epsilon : X_2 \to \{\pm 1\}$ such that

$$(3.8)\qquad T(f) = T(0) + (T(\mathbf{1}) - T(0))[f \circ \pi]^\epsilon, \quad f \in \mathrm{Lip}(X_1).$$

Recall that $T_1 = \overline{T_0(\mathbf{1})}T_0$ for $T_0 = T - T(0)$. Since T_0 is 2-local, we have

$$T_0(\mathbf{1}) = \beta_{0,\mathbf{1}} + \alpha_{0,\mathbf{1}}[\mathbf{1} \circ \pi_{0,\mathbf{1}}]^{\epsilon_{0,\mathbf{1}}},$$

and

$$0 = T_0(0) = \beta_{0,\mathbf{1}} + \alpha_{0,\mathbf{1}}[0 \circ \pi_{0,\mathbf{1}}]^{\epsilon_{0,\mathbf{1}}} = \beta_{0,\mathbf{1}}.$$

It follows that $T(\mathbf{1}) - T(0) = T_0(\mathbf{1})$ is a unimodular constant. Thus $T_1 = \overline{T_0(\mathbf{1})}T_0$ is 2-local in $\mathrm{Iso}((\mathrm{Lip}(X_1), \|\cdot\|_1), (\mathrm{Lip}(X_2), \|\cdot\|_2))$. We get

$$\begin{aligned} 0 = T_1(0) &= T_{0,i}(0) \\ &= \beta_{0,i} + \alpha_{0,i}[0 \circ \pi_{0,i}]^{\epsilon_{0,i}} = \beta_{0,i}, \end{aligned}$$

and

$$\begin{aligned} T_1(i) &= T_{0,i}(i) \\ &= \beta_{0,i} + \alpha_{0,i}[i \circ \pi_{0,i}]^{\epsilon_{0,i}}. \end{aligned}$$

We get

$$T_1(i) = \alpha_{0,i}[i \circ \pi_{0,i}]^{\epsilon_{0,i}}.$$

Since $\alpha_{0,i}$ is a unimodular constant and $\epsilon_{0,i} = \pm 1$, so we obtain $T_1(i)$ is a constant. Moreover applying (3.8), we have

$$T_1(i) = [i \circ \pi]^{\epsilon}.$$

Thus we conclude that $\epsilon = 1$ or $\epsilon = -1$. As T is a 2-local isometry, T is an isometry, hence T is injective. Corollary A.7 asserts that $\pi(X_2)$ is a uniqueness set for $\mathrm{Lip}(X_1)$. Thus we have $\pi(X_2) = X_1$. This implies that π is surjective. Finally we shall prove that π is an isometry. Let $x_0 \in X_2$. We define a Lipschitz function g on X_1 by

$$g(x) = d(x, \pi(x_0)), \quad x \in X_1.$$

As T_1 is 2-local in $\mathrm{Iso}((\mathrm{Lip}(X_1), \|\cdot\|_1), (\mathrm{Lip}(X_2), \|\cdot\|_2))$, there exists $\alpha_{0,g} \in \mathbb{T}$ and $\pi_{0,g} : X_2 \to X_1$ is a surjective isometry such that

$$\begin{aligned} 0 = T_1(0) &= T_{0,g}(0) \\ &= \beta_{0,g} + \alpha_{0,g}[0 \circ \pi_{0,g}]^{\epsilon_{0,g}} = \beta_{0,g}, \end{aligned}$$

and

$$\begin{aligned} T_1(g) &= T_{0,g}(g) \\ &= \beta_{0,g} + \alpha_{0,g}[g \circ \pi_{0,g}]^{\epsilon_{0,g}} = \beta_{0,g} + \alpha_{0,g} g \circ \pi_{0,g}, \end{aligned}$$

because g is a real-valued function. If follows that

$$(T_1(g))(z) = \alpha_{0,g} g(\pi_{0,g}(z)), \quad z \in X_2.$$

By (3.8), for any $z \in X_2$

$$\begin{aligned} d(\pi(z), \pi(x_0)) &= [g(\pi(z))]^{\epsilon} \\ &= (T_1(g))(z) = \alpha_{0,g} g(\pi_{0,g}(z)) = \alpha_{0,g} d(\pi_{0,g}(z), \pi(x_0)). \end{aligned} \tag{3.9}$$

We may suppose that X_1 is not a singleton. (Suppose that it is so. Then X_2 is a singleton since $\pi_{0,g}$ is a surjective isometry. Then π is automatically surjective isometry.) Hence there exists $z_0 \in X_2$ such that $d(\pi_{0,g}(z_0), \pi(x_0)) \neq 0$. By (3.9) with $z = z_0$ we have

$$\alpha_{0,g} = \frac{d(\pi(z_0), \pi(x_0))}{d(\pi_{0,g}(z_0), \pi(x_0))} \geq 0,$$

we obtain $\alpha_{0,g} = 1$. Hence by (3.9) we have

$$d(\pi(z), \pi(x_0)) = d(\pi_{0,g}(z), \pi(x_0)), \quad z \in X_2. \tag{3.10}$$

Putting $z = x_0$ in (3.10), we have

$$0 = d(\pi(x_0), \pi(x_0)) = d(\pi_{0,g}(x_0), \pi(x_0)).$$

It follows $\pi_{0,g}(x_0) = \pi(x_0)$. By (3.10)

$$d(\pi(z), \pi(x_0)) = d(\pi_{0,g}(z), \pi(x_0)) = d(\pi_{0,g}(z), \pi_{0,g}(x_0)) = d(z, x_0)$$

since $\pi_{0,g}$ is an isometry. As z and x_0 are arbitrary, we conclude that π is an isometry. This completes the proof. □

For an arbitrary compact metric space X_j for $j = 1, 2$, [**42**, Theorem 6] shows that $\mathrm{Iso}((\mathrm{Lip}(X_1), \|\cdot\|_\Sigma), (\mathrm{Lip}(X_2), \|\cdot\|_\Sigma))$ fulfills the condition of Corollary A.14. Thus we have the following.

COROLLARY A.15. *Suppose that* $T \in M(\mathrm{Lip}(X_1), \mathrm{Lip}(X_2))$ *is 2-local in* $\mathrm{Iso}((\mathrm{Lip}(X_1), \|\cdot\|_\Sigma), (\mathrm{Lip}(X_2), \|\cdot\|_\Sigma))$. *Then* $T \in \mathrm{Iso}((\mathrm{Lip}(X_1), \|\cdot\|_\Sigma), (\mathrm{Lip}(X_2), \|\cdot\|_\Sigma))$.

Corollary A.15 generalizes Theorem 8 in [**42**], where the case $X_1 = X_2 = [0, 1]$ is proved.

3.3. The algebra of continuously differentiable functions. We have the following corollary.

COROLLARY A.16. *Let* $\|\cdot\|_j$ *be any norm on* $C^1([0,1])$ *for* $j=1,2$. *We do not assume that* $\|\cdot\|_j$ *is complete. Suppose that*

$$\begin{aligned}(3.11)\quad \mathrm{Iso}((C^1([0,1]),\|\cdot\|_1),(C^1([0,1]),\|\cdot\|_2)) \\ =\{T\in M(C^1([0,1]),C^1([0,1]));\\ \textit{there exist } \beta\in C^1([0,1]),\ \alpha\in\mathbb{T},\\ \pi=\mathrm{Id} \textit{ or } \pi=1-\mathrm{Id} \textit{ and } \epsilon=\pm1\\ \textit{such that } T(f)=\beta+\alpha[f\circ\pi]^{\epsilon} \textit{ for every } f\in C^1([0,1])\}.\end{aligned}$$

Suppose that $T\in M(C^1([0,1]),C^1([0,1]))$ *is 2-local in* $\mathrm{Iso}((C^1([0,1]),\|\cdot\|_1),(C^1([0,1]),\|\cdot\|_2))$. *Then* $T\in \mathrm{Iso}((C^1([0,1]),\|\cdot\|_1),(C^1([0,1]),\|\cdot\|_2))$.

PROOF. Let T be 2-local in $\mathrm{Iso}((C^1([0,1]),\|\cdot\|_1),(C^1([0,1]),\|\cdot\|_2))$. By (3.11), $\mathrm{Iso}((C^1([0,1]),\|\cdot\|_1),(C^1([0,1]),\|\cdot\|_2))\subset \mathrm{GWC}$. Theorem A.5 asserts that there exists a continuous map $\pi:[0,1]\to[0,1]$ and a continuous map $\epsilon:[0,1]\to\{\pm1\}$ such that

$$(3.12)\qquad T(f)=T(0)+(T(\mathbf{1})-T(0))[f\circ\pi]^{\epsilon},\quad f\in C^1([0,1]).$$

Since $\epsilon:[0,1]\to\{\pm1\}$ is continuous and $[0,1]$ is connected, we conclude that $\epsilon=\pm1$. As T is a 2-local isometry, we get T is an isometry. This implies that T is injective. Corollary A.7 asserts that $\pi([0,1])$ is a uniqueness set for $C^1([0,1])$, which is $[0,1]$. Thus we have π is surjective. To complete the proof we shall prove that π is an isometry. Let $x_0\in[0,1]$. We define the function $g(x)=x-\pi(x_0)\in C^1[0,1]$. Define $T_1=\overline{T_0(\mathbf{1})}T_0$ for $T_0=T-T(0)$. It is easy to see that T_0 is 2-local in $\mathrm{Iso}((C^1([0,1]),\|\cdot\|_1),(C^1([0,1]),\|\cdot\|_2))$, we have

$$T_0(\mathbf{1})=\beta_{0,\mathbf{1}}+\alpha_{0,\mathbf{1}}[\mathbf{1}\circ\pi_{0,\mathbf{1}}]^{\epsilon_{0,\mathbf{1}}},$$

and

$$0=T_0(0)=\beta_{0,\mathbf{1}}+\alpha_{0,\mathbf{1}}[0\circ\pi_{0,\mathbf{1}}]^{\epsilon_{0,\mathbf{1}}}=\beta_{0,\mathbf{1}}.$$

It follows that $T(\mathbf{1})-T(0)=T_0(\mathbf{1})$ is a unimodular constant. We have $T_1=\overline{T_0(\mathbf{1})}T_0$ is 2-local in $\mathrm{Iso}((C^1([0,1]),\|\cdot\|_1),(C^1([0,1]),\|\cdot\|_2))$. Hence we get

$$\begin{aligned}0=T_1(0)&=T_{0,g}(0)\\ &=\beta_{0,g}+\alpha_{0,g}[0\circ\pi_{0,g}]^{\epsilon_{0,g}}=\beta_{0,g},\end{aligned}$$

and

$$\begin{aligned} T_1(g) &= T_{0,g}(g) \\ &= \beta_{0,g} + \alpha_{0,g}[g \circ \pi_{0,g}]^{\epsilon_{0,g}}. \end{aligned}$$

It follows that

$$(T_1(g))(z) = \alpha_{0,g}[g \circ \pi_{0,g}]^{\epsilon_{0,g}}(z) = \alpha_{0,g}[g(\pi_{0,g}(z))]^{\epsilon_{0,g}}, \quad z \in [0,1].$$

Thus by (3.12), we have for any $z \in [0,1]$ that

$$\begin{aligned} [\pi(z) - \pi(x_0)]^{\epsilon} &= [g(\pi(z))]^{\epsilon} = (T_1(g))(z) \\ &= \alpha_{0,g}[g(\pi_{0,g}(z))]^{\epsilon_{0,g}} = \alpha_{0,g}[\pi_{0,g}(z) - \pi(x_0)]^{\epsilon_{0,g}}, \end{aligned}$$

where $\alpha_{0,g} \in \mathbb{T}$ and $\pi_{0,g} = \mathrm{Id}$ or $\pi_{0,g} = 1 - \mathrm{Id}$. Putting $z = x_0$, we have

$$0 = [\pi(x_0) - \pi(x_0)]^{\epsilon} = \alpha_{0,g}[\pi_{0,g}(x_0) - \pi(x_0)]^{\epsilon_{0,g}}.$$

It follows that $\pi_{0,g}(x_0) = \pi(x_0)$. Thus we have

$$[\pi(z) - \pi(x_0)]^{\epsilon} = \alpha_{0,g}[\pi_{0,g}(z) - \pi(x_0)]^{\epsilon_{0,g}} = \alpha_{0,g}[\pi_{0,g}(z) - \pi_{0,g}(x_0)]^{\epsilon_{0,g}},$$

and

$$|\pi(z) - \pi(x_0)| = |\pi_{0,g}(z) - \pi_{0,g}(x_0)| = |z - x_0|.$$

As z and x_0 are arbitrary, we conclude that π is an isometry. This completes the proof. □

In [**60, 84**], they gave a characterization for surjective isometries on $C^1([0,1])$ with respect to various norms. There are many norms with which the groups of surjective isometries on $C^1([0,1])$ satisfys the condition of Corollary A.16. We present one of them.

Corollary A.17. *Suppose that*

$$T \in M((C^1([0,1]), \|\cdot\|_{\Sigma}), (C^1([0,1]), \|\cdot\|_{\Sigma}))$$

and T is 2-local in $\mathrm{Iso}((C^1([0,1]), \|\cdot\|_{\Sigma}), (C^1([0,1]), \|\cdot\|_{\Sigma}))$. *We conclude that* $T \in \mathrm{Iso}((C^1([0,1]), \|\cdot\|_{\Sigma}), (C^1([0,1]), \|\cdot\|_{\Sigma}))$.

Corollary A.17 has been shown in [**42**, Theorem 9] in a different way.

3.4. The algebra $\mathrm{S}^\infty(\mathbb{D})$. Let

$$\mathrm{S}^\infty(\mathbb{D}) = \{f \in H(\mathbb{D}); f' \in H^\infty(\mathbb{D})\},$$

where $H(\mathbb{D})$ is the linear space of all analytic functions on $\mathbb{D}$ and $H^\infty(\mathbb{D})$ is the algebra of all bounded analytic functions on $\mathbb{D}$. The algebra $\mathrm{S}^\infty(\mathbb{D})$ equipped with the norm $\|f\|_\Sigma = \sup_{z\in\mathbb{D}} |f(z)| + \sup_{w\in\mathbb{D}} |f'(w)|$ for $f \in \mathrm{S}^\infty(\mathbb{D})$ is a unital semi-simple commutative Banach algebra. As is described in [**83**], $\mathrm{S}^\infty(\mathbb{D})$ coincides with the space of all Lipschitz functions in the linear space of all analytic functions on $\mathbb{D}$ and each $f \in \mathrm{S}^\infty(\mathbb{D})$ is continuously extended to the closed unit disk $\bar{\mathbb{D}}$. Hence we may suppose that $\mathrm{S}^\infty(\mathbb{D})$ is a unital subalgebra of the disk algebra on $\bar{\mathbb{D}}$. Trivially all analytic polynomials are in $\mathrm{S}^\infty(\mathbb{D})$.

THEOREM A.18. *The maximal ideal space $\mathcal{M}_\infty$ of $\mathrm{S}^\infty(\mathbb{D})$ is homeomorphic to the closed unit disk $\bar{\mathbb{D}}$.*

PROOF. For each $p \in \bar{\mathbb{D}}$, the point evaluation on $\mathrm{S}^\infty(\mathbb{D})$ which takes the value at p is a nontrivial complex homomorphism. Hence we may suppose that $\bar{\mathbb{D}} \subset \mathcal{M}_\infty$. To prove $\bar{\mathbb{D}} = \mathcal{M}_\infty$, suppose that $f_1, \ldots, f_n$ be an arbitrary finite number of functions in $\mathrm{S}^\infty(\mathbb{D})$ such that

$$\sum_{j=1}^{n} |f_j| > 0 \text{ on } \bar{\mathbb{D}}.$$

If we prove that there exist the same number of $g_1, \ldots, g_n \in \mathrm{S}^\infty(\mathbb{D})$ such that

$$\sum_{j=1}^{n} f_j g_j = 1,$$

then a general result assures that $\bar{\mathbb{D}} = \mathcal{M}_\infty$. We prove the existence of such $g_1, \ldots, g_n \in \mathrm{S}^\infty(\mathbb{D})$. It is well known that the maximal ideal space of the disk algebra $A(\bar{\mathbb{D}})$ is $\bar{\mathbb{D}}$. As $f_1, \ldots, f_n \in \mathrm{S}^\infty(\mathbb{D}) \subset A(\bar{\mathbb{D}})$, there exists $h_1, \ldots, h_n \in A(\bar{\mathbb{D}})$ such that

$$\sum_{j=1}^{n} f_j h_j = 1.$$

As functions in $A(\bar{\mathbb{D}})$ are uniformly approximated by analytic polynomials, there exists a sequence of polynomials $\{p_m^{(j)}\}_{m=1}^\infty$ such that

$\|p_m^{(j)} - h_j\|_\infty \to 0$ as $m \to \infty$ for every $j = 1, \dots, n$. Hence for sufficiently large m_0 we have

$$\left\| 1 - \sum_{j=1}^{n} f_j p_{m_0}^{(j)} \right\| < 1/2.$$

In particular, $\sum_{j=1}^n f_j p_{m_0}^{(j)}$ has no zeros on $\bar{\mathbb{D}}$. Then $1/\sum_{j=1}^n f_j p_{m_0}^{(j)} \in \mathrm{S}^\infty(\mathbb{D})$. Put $g_j = p_{m_0}^{(j)}/\sum_{j=1}^n f_j p_{m_0}^{(j)}$ for $j = 1, \dots, n$. Then $g_j \in \mathrm{S}^\infty(\mathbb{D})$ and $\sum_{j=1}^n f_j g_j = 1$ by a simple calculation. It follows that $\bar{\mathbb{D}} = \mathcal{M}_\infty$. □

Miura [**83**, Theorem1] showed the form of the surjective isometry on $\mathrm{S}^\infty(\mathbb{D})$.

Theorem A.19 (Miura [**83**]). *Suppose that $U : \mathrm{S}^\infty(\mathbb{D}) \to \mathrm{S}^\infty(\mathbb{D})$ is a surjective isometry with respect to the norm $\|\cdot\|_\Sigma$. Then there exists unimodular constants $\alpha, \lambda \in \mathbb{C}$ such that*

$$U(f) = U(0) + \alpha f(\lambda\cdot), \quad f \in \mathrm{S}^\infty(\mathbb{D})$$

or

$$U(f) = U(0) + \alpha \overline{f(\overline{\lambda\cdot})}, \quad f \in \mathrm{S}^\infty(\mathbb{D}).$$

Conversely, each of the above form is a surjective isometry from $\mathrm{S}^\infty(\mathbb{D})$ onto $\mathrm{S}^\infty(\mathbb{D})$.

As we stated in the beginning of Section 2, for some Banach algebras B_j, a pointwise 2-local map in $\mathrm{Iso}(B_1, B_2)$ is not always a surjective isometry. But applying Theorem A.5 and Theorem A.19 we deduce that a pointwise 2-local map in $\mathrm{Iso}(\mathrm{S}^\infty(\mathbb{D}), \mathrm{S}^\infty(\mathbb{D}))$ is always a surjective isometry.

Corollary A.20. *Suppose that $T \in M(\mathrm{S}^\infty(\mathbb{D}), \mathrm{S}^\infty(\mathbb{D}))$ is pointwise 2-local in $\mathrm{Iso}(\mathrm{S}^\infty(\mathbb{D}), \mathrm{S}^\infty(\mathbb{D}))$. Then $T \in \mathrm{Iso}(\mathrm{S}^\infty(\mathbb{D}), \mathrm{S}^\infty(\mathbb{D}))$.*

Proof. Let $T \in M(\mathrm{S}^\infty(\mathbb{D}), \mathrm{S}^\infty(\mathbb{D}))$ be a pointwise 2-local map in $\mathrm{Iso}(\mathrm{S}^\infty(\mathbb{D}), \mathrm{S}^\infty(\mathbb{D}))$. By Theorem A.19 $\mathrm{Iso}(\mathrm{S}^\infty(\mathbb{D}), \mathrm{S}^\infty(\mathbb{D})) \subset \mathrm{GWC}$. Then Theorem A.5 asserts that there exist a continuous map $\pi : \bar{\mathbb{D}} \to \bar{\mathbb{D}}$ and a continuous map $\epsilon : \bar{\mathbb{D}} \to \{\pm 1\}$ such that

$$T(f) = T(0) + \alpha[f \circ \pi]^\epsilon, \quad f \in \mathrm{S}^\infty(\mathbb{D}),$$

where $\alpha = T(\mathbf{1}) - T(0)$ is a unimodular constant since $T(\mathbf{1}) - T(0)$ is unimodular function and it is analytic on $\mathbb{D}$. Furthermore $\epsilon = 1$ on $\bar{\mathbb{D}}$ or $\epsilon = -1$ on $\bar{\mathbb{D}}$. Put $T_1 = \bar{\alpha}(T - T(0))$. Then

$$T_1 = f \circ \pi, \quad f \in \mathrm{S}^\infty(\mathbb{D})$$

if $\epsilon = 1$, and

$$T_1(f) = \overline{f \circ \pi}, \quad f \in \mathrm{S}^\infty(\mathbb{D})$$

if $\epsilon = -1$. Letting $f = \mathrm{Id}$, the identity function, we see that $\pi \in \mathrm{S}^\infty(\mathbb{D})$ if $\epsilon = 1$ and $\bar{\pi} \in \mathrm{S}^\infty(\mathbb{D})$ if $\epsilon = -1$. Put $\varphi = \pi$ if $\epsilon = 1$, and $\varphi = \bar{\pi}$ if $\epsilon = -1$. Then we have that $\varphi \in \mathrm{S}^\infty(\mathbb{D})$ and

$$T_1(f) = f \circ \varphi, \quad f \in \mathrm{S}^\infty(\mathbb{D})$$

if $\epsilon = 1$, and

$$T_1(f) = \overline{f \circ \bar{\varphi}}, \quad f \in \mathrm{S}^\infty(\mathbb{D})$$

if $\epsilon = -1$. In particular, we have

$$T_1(\mathrm{Id}) = \varphi \tag{3.13}$$

either for $\epsilon = 1$ and for $\epsilon = -1$. Since T_1 is pointwise 2-local in $\mathrm{Iso}(\mathrm{S}^\infty(\mathbb{D}), \mathrm{S}^\infty(\mathbb{D}))$ by the definition of T_1, for every $x \in \bar{\mathbb{D}}$ there exists $u_x \in \mathrm{S}^\infty(\mathbb{D})$ and unimodular constant α_x, λ_x such that

$$(T_1(\mathrm{Id}))(x) = u_x(x) + \alpha_x \,\mathrm{Id}(\lambda_x x)$$

and

$$0 = (T_1(0))(x) = u_x(x),$$

or

$$(T_1(\mathrm{Id}))(x) = u_x(x) + \alpha_x \overline{\mathrm{Id}(\overline{\lambda_x x})} = u_x(x) + \alpha_x \,\mathrm{Id}(\lambda_x x)$$

and

$$0 = (T_1(0))(x) = u_x(x).$$

In any case we have

$$(T_1(\mathrm{Id}))(x) = \alpha_x \lambda_x x. \tag{3.14}$$

Combining (3.13) and (3.14) we have

$$\varphi(x) = \alpha_x \lambda_x x$$

for every $x \in \bar{\mathbb{D}}$. Then we have $\varphi(0) = 0$, and $|\varphi(x)| = |x|$ for every $x \in \bar{\mathbb{D}}$. Since $\varphi : \bar{\mathbb{D}} \to \bar{\mathbb{D}}$ is analytic in $\mathbb{D}$, the Schwartz lemma asserts that there is a unimodular constant λ_0 such that

$$\varphi(x) = \lambda_0 x, \quad x \in \bar{\mathbb{D}}.$$

It follows that

$$T(f) = T(0) + (T(\mathbf{1}) - T(0))f(\lambda_0 \cdot), \quad f \in \mathrm{S}^\infty(\mathbb{D})$$

or

$$T(f) = T(0) + (T(\mathbf{1}) - T(0))\overline{f(\overline{\lambda_0 \cdot})}, \quad f \in \mathrm{S}^\infty(\mathbb{D}).$$

By Theorem A.19 we conclude that $T \in \mathrm{Iso}(\mathrm{S}^\infty(\mathbb{D}), \mathrm{S}^\infty(\mathbb{D}))$. □

4. Iso-reflexivity

Many literatures study isometries from the point of view of how they are determined by their local actions [**3, 25, 51, 85, 91, 92, 100**]. By Theorem A.5 we have that several 2-local maps are linear, hence they are local maps. In this section we prove that a local isometry in $\mathrm{Iso}_{\mathbb{C}}(B_1, B_2)$ is 2-local in $\mathrm{Iso}(B_1, B_2)$. Applying Theorem A.23 we see the reflexivity of $\mathrm{Iso}_{\mathbb{C}}(B_1, B_2)$ for several Banach spaces of continuous functions.

Definition A.21. Put

$$M_{\mathbb{C}}(B_1, B_2) = \{T \in M(B_1, B_2); T \text{ is complex-linear }\}$$

$$\mathrm{Iso}_{\mathbb{C}}(B_1, B_2) = \{T \in \mathrm{Iso}(B_1, B_2); T \text{ is complex-linear }\}.$$

Recall that $T \in M_{\mathbb{C}}(B_1, B_2)$ is local in $\mathrm{Iso}_{\mathbb{C}}(B_1, B_2)$ if for every $f \in B_1$, there exists $T_f \in \mathrm{Iso}_{\mathbb{C}}(B_1, B_2)$ such that

$$T(f) = T_f(f).$$

We say that $\mathrm{Iso}_{\mathbb{C}}(B_1, B_2)$ is iso-reflexive if every local map in $\mathrm{Iso}_{\mathbb{C}}(B_1, B_2)$ is an element in $\mathrm{Iso}_{\mathbb{C}}(B_1, B_2)$.

Proposition A.22. *Suppose that $T \in M_{\mathbb{C}}(B_1, B_2)$ is local in $\mathrm{Iso}_{\mathbb{C}}(B_1, B_2)$. Then T is 2-local in $\mathrm{Iso}(B_1, B_2)$.*

Proof. Let $f, g \in B_1$ be arbitrary. Then there exists $T_{f,g} \in \mathrm{Iso}_{\mathbb{C}}(B_1, B_2)$ such that

$$T(f-g) = T_{f,g}(f-g).$$

As T and $T_{f,g}$ are complex-linear, we have

$$T(f) - T(g) = T_{f,g}(f) - T_{f,g}(g). \tag{4.1}$$

Put

$$h_{f,g} = T(f) - T_{f,g}(f).$$

By (4.1) we have

$$T(f) = h_{f,g} + T_{f,g}(f),$$
$$T(g) = h_{f,g} + T_{f,g}(g).$$

It is easy to see that $h_{f,g} + T_{f,g}(\cdot) \in \mathrm{Iso}(B_1, B_2)$. It follows that T is 2-local in $\mathrm{Iso}(B_1, B_2)$. □

Theorem A.23. *Suppose that every 2-local map in* $\mathrm{Iso}(B_1, B_2)$ *is an element in* $\mathrm{Iso}(B_1, B_2)$*. Then* $\mathrm{Iso}_{\mathbb{C}}(B_1, B_2)$ *is iso-reflexive.*

Proof. Suppose that $T \in M_{\mathbb{C}}(B_1, B_2)$ is local in $\mathrm{Iso}_{\mathbb{C}}(B_1, B_2)$. Then by Proposition A.22, T is 2-local in $\mathrm{Iso}(B_1, B_2)$. By assumption, we have $T \in \mathrm{Iso}(B_1, B_2)$. Since T is complex-linear, we infer that $T \in \mathrm{Iso}_{\mathbb{C}}(B_1, B_2)$. □

Applying Corollaries A.11,A.13,A.15,A.17 and A.20, we see that $\mathrm{Iso}_{\mathbb{C}}(C(X_1), C(X_2))$ for first countable compact Hausdorff spaces X_1 and X_2, $\mathrm{Iso}_{\mathbb{C}}(A(\bar{\mathbb{D}}), A(\bar{\mathbb{D}}))$, $\mathrm{Iso}_{\mathbb{C}}(\mathrm{Lip}(X_1), \mathrm{Lip}(X_2))$, $\mathrm{Iso}_{\mathbb{C}}(C^1[0,1], C^1[0,1])$ and $\mathrm{Iso}_{\mathbb{C}}(\mathrm{S}^\infty(\mathbb{D}), \mathrm{S}^\infty(\mathbb{D}))$ are iso-reflexive.