On the rationality problem for norm one tori

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Contents

CHAPTER 1

Introduction

Let k be a base field. We consider the rationality problem for algebraic tori over k, which is a k-form of the multiplicative group $\mathbb{G}_m^n$. Rationality problem is significant but difficult question in algebraic geometry; for given an algebraic variety, determine whether it is rational over k, i.e., it is birational to the projective space $\mathbb{P}^n$. We consider this problem in the case of algebraic tori over k, especially norm one tori.

Rationality problem for algebraic tori can be applied to number theoretical problems. One of them is Noether's problem. The motivation considering Noether's problem is to solve inverse Galois problem. Let G be a finite group and $k(x_g \mid g \in G)$ be the rational function field on which G acts naturally by $g \cdot x_h = x_{gh}$ for any $g, h \in G$. Under this situation, Noether's problem asks whether the fixed field $k(x_g \mid g \in G)^G$ is rational over k or not. This problem is related to the existence of G-generic Galois extension over k, which gives affirmative answer to inverse Galois problem (see Swan [**Swa83**]). When G is abelian, Lenstra [**Len74**] solved Noether's problem by using the rationality of algebraic tori (see also Masda[**Mas55**]). The rationality (stably rationality or retract rationality) of algebraic tori over k is related to Hasse norm principle for a field extension K/k if base field k is a number field (see Voskresenskii [**Vos98**, Section 11.6]), and the cryptography based on the discrete logarithm problem in the group of rational points of an algebraic torus if k is a finite field (see Rubin and Silverberg [**RS03**]).

In this thesis, we consider stably and retract rationality of algebraic tori. Tori of dimension n over k correspond bijectively to the elements of the set $H^1(\mathcal{G}, \mathrm{GL}_n(\mathbb{Z}))$ where $\mathcal{G} = \mathrm{Gal}(k_{sep}/k)$ is the absolute Galois group of k and k_{sep} is a separable closure of k. The torus T of dimension n over k is determined uniquely by the integral continuous representation $h : \mathcal{G} \to \mathrm{GL}_n(\mathbb{Z})$ up to conjugacy, and the group $h(\mathcal{G})$ is a finite subgroup of $\mathrm{GL}_n(\mathbb{Z})$. Moreover, stably and retract rationality of tori T over k is determined by conjugacy class of $\mathrm{GL}_n(\mathbb{Z})$. By Jordon's theorem, the number of conjugacy classes of finite subgroup of $\mathrm{GL}_n(\mathbb{Z})$ is finite. Hence, when we consider stably and retract rationality of algebraic tori, we should consider finitely many algebraic tori in each dimension:

n	1	2	3	4	5	6
the number of conjugacy classes of finite subgroup of $\mathrm{GL}_n(\mathbb{Z})$	2	13	73	710	6079	85308

The rationality problem for norm one tori $R^{(1)}_{K/k}(\mathbb{G}_m)$, which is the special case of algebraic tori, is investigated by many mathematicians, e.g., Endo and Miyata [**EM75**], [**End11**], J.-L. Colliot-Thélène and J.-J. Sansuc [**CTS77**], [**CTS87**], W. Hürlimann [**Hür84**], L. Le Bruyn [**LeB95**], A. Cortella and B. Kunyavskii [**CK00**], N. Lemire and M. Lorenz [**LL00**], M. Florence [**Flo**], Hoshi and Yamasaki [**HY17**], [**HY**]. Let K/k be a separable field extension of degree n and L/k be the Galois closure of K/k with Galois group G. When K/k is a Galois extension, the stably and retract rationality of norm one tori $R^{(1)}_{K/k}(\mathbb{G}_m)$ is well understand by Endo and Miyata, Saltman and Colliot-Thélène and Sansuc. However, when K/k is a non-Galois extension, the rationality of norm one tori is only known in very special cases.

Hoshi and Yamasaki [**HY**] classified stably/retract norm one tori when $n \leq 10$ and $n = p$ where p is a prime number using computer algebra system GAP [**GAP**] except for the following cases: (i) $G = 9T27 \simeq \mathrm{PSL}_2(\mathbb{F}_8)$; (ii) $G = 10T11 \simeq A_5 \times C_2$; (iii) $G = \mathrm{PSL}_2(\mathbb{F}_{2^e})$ where $p = 2^e + 1 \geq 17$ is a Fermat prime.

We improve the results of the classification of stably/retract rational norm one tori in [**HY**]. In order to prove the stable rationality of norm one tori, Hoshi and Yamasaki [**HY**] construct an isomorphism of flabby class and some permutation lattice by GAP in the previous method.

On the other hand, in our method, we improve the algorithm of construction of flabby resolution and construct flabby class several times by using improved algorithm. In this way, we prove stable rationality of the torus for the case $G = 10T11 \simeq A_5 \times C_2$, which is unsolvable case of $n = 10$ in [**HY**] (see Chapter 2, Section 2). Furthermore, we give the classification of stably/retract rational norm one tori $R^{(1)}_{K/k}(\mathbb{G}_m)$ of dimension $n-1$ in the following cases:

- $n = 2^e$ $(e \geq 1)$,
- $n = 10, 12, 14, 15$,
- $n = q + 1$ where $q = l^e \equiv 1 \pmod 4$ is an odd prime power and $\mathrm{PSL}_2(\mathbb{F}_q) \leq G \leq \mathrm{P\Gamma L}_2(\mathbb{F}_q) \simeq \mathrm{PGL}_2(\mathbb{F}_q) \rtimes C_e$,
- $G \leq S_{2p}$ is primitive, p is a prime number,
- Mathieu groups $G = M_n \leq S_n$ where $n = 11, 12, 22, 23, 24$.

More precisely, our main results can be stated as follows:

Main Theorem 1 (Theorem 3.1). *Let K/k be a separable field extension of degree n and L/k be the Galois closure of K/k. Let $G = \mathrm{Gal}(L/k)$ be a transitive subgroup of S_n where $n = 2^e$ $(e \geq 1)$ and $H = \mathrm{Gal}(L/K)$ with $[G : H] = n$. Then $R^{(1)}_{K/k}(\mathbb{G}_m)$ is stably k-rational if and only if $G \simeq C_n$. Moreover, if $R^{(1)}_{K/k}(\mathbb{G}_m)$ is not stably k-rational, then it is not retract k-rational.*

Main Theorem 2 (Theorem 3.7). *Let K/k be a separable field extension of degree n and L/k be the Galois closure of K/k. Let $G = \mathrm{Gal}(L/k)$ be a transitive subgroup of S_n and $H = \mathrm{Gal}(L/K)$ with $[G : H] = n$. Then a classification of stably/retract rational norm one tori $T = R^{(1)}_{K/k}(\mathbb{G}_m)$ in dimension $n-1$ for $n = 10, 12, 14, 15$ is given as follows:*

(1) *The case* $10Tm$ $(1 \leq m \leq 45)$.

(i) T *is stably* k*-rational for* $10T1 \simeq C_{10}$, $10T2 \simeq D_5$, $10T3 \simeq D_{10}$, $10T11 \simeq A_5 \times C_2$;

(ii) T *is not stably but retract* k*-rational for* $10T4 \simeq F_{20}$, $10T5 \simeq F_{20} \times C_2$, $10T12 \simeq S_5$, $10T22 \simeq S_5 \times C_2$;

(iii) T *is not retract* k*-rational for* $10Tm$ *with* $6 \leq m \leq 45$ *and* $m \neq 11, 12, 22$.

(2) *The case* $12Tm$ $(1 \leq m \leq 301)$.

(i) T *is stably* k*-rational for* $12T1 \simeq C_{12}$, $12T5 \simeq C_3 \rtimes C_4$, $12T11 \simeq C_4 \times S_3$;

(ii) T *is not retract* k*-rational for* $12Tm$ *with* $1 \leq m \leq 301$ *and* $m \neq 1, 5, 11$.

(3) *The case* $14Tm$ $(1 \leq m \leq 63)$.

(i) T *is stably* k*-rational for* $14T1 \simeq C_{14}$, $14T2 \simeq D_7$, $14T3 \simeq D_{14}$;

(ii) T *is not stably* k*-rational but retract* k*-rational for* $14T4 \simeq F_{42}$, $14T5 \simeq F_{21} \times C_2$, $14T7 \simeq F_{42} \times C_2$, $14T16 \simeq \mathrm{PSL}_3(\mathbb{F}_2) \rtimes C_2$, $14T19 \simeq \mathrm{PSL}_3(\mathbb{F}_2) \times C_2$, $14T46 \simeq S_7$, $14T47 \simeq A_7 \times C_2$, $14T49 \simeq S_7 \times C_2$;

(iii) T *is not retract* k*-rational for* $14Tm$ *with* $6 \leq m \leq 63$ *and* $m \neq 7, 16, 19, 46, 47, 49$.

(4) *The case* $15Tm$ $(1 \leq m \leq 104)$.

(i) T *is stably* k*-rational for* $15T1 \simeq C_{15}$, $15T2 \simeq D_{15}$, $15T3 \simeq D_5 \times C_3$, $15T4 \simeq S_3 \times C_5$, $15T5 \simeq A_5$, $15T7 \simeq D_5 \times S_3$, $15T16 \simeq A_5 \times C_3 \simeq \mathrm{GL}_2(\mathbb{F}_4)$, $15T23 \simeq A_5 \times S_3$;

(ii) T *is not stably* k*-rational but retract* k*-rational for* $15T6 \simeq C_{15} \rtimes C_4$, $15T8 \simeq F_{20} \times C_3$, $15T10 \simeq S_5$, $15T11 \simeq F_{20} \times S_3$, $15T22 \simeq (A_5 \times C_3) \rtimes C_2 \simeq \mathrm{GL}_2(\mathbb{F}_4) \rtimes C_2$, $15T24 \simeq S_5 \times C_3$, $15T29 \simeq S_5 \times S_3$;

(iii) T *is not retract* k*-rational for* $15Tm$ *with* $9 \leq m \leq 104$ *and* $m \neq 10, 11, 16, 22, 23, 24, 29$.

Main Theorem 3 (Theorem 3.12). *Let K/k be a separable field extension of degree n and L/k be the Galois closure of K/k. Let $G = \mathrm{Gal}(L/k)$ be a transitive subgroup of S_n and $H = \mathrm{Gal}(L/K)$ with $[G:H] = n$. Assume that $n = q+1$ where $q = l^e \equiv 1 \pmod 4$ is an odd prime power and $\mathrm{PSL}_2(\mathbb{F}_q) \leq G \leq \mathrm{P\Gamma L}_2(\mathbb{F}_q) \simeq \mathrm{PGL}_2(\mathbb{F}_q) \rtimes C_e$. Then $R^{(1)}_{K/k}(\mathbb{G}_m)$ is not retract k-rational.*

Main Theorem 4 (Theorem 3.13). *Let p be a prime number, K/k be a separable field extension of degree $2p$ and L/k be the Galois closure of K/k. Assume that $G = \mathrm{Gal}(L/k)$ is a primitive subgroup of S_{2p} and $H = \mathrm{Gal}(L/K)$ with $[G:H] = 2p$. Then $R^{(1)}_{K/k}(\mathbb{G}_m)$ is not retract k-rational. More precisely, $R^{(1)}_{K/k}(\mathbb{G}_m)$ is not retract k-rational for the following primitive groups $G \leq S_{2p}$:*
(i) *$G = S_{2p}$ or $G = A_{2p} \leq S_{2p}$;*
(ii) *$G = S_5 \leq S_{10}$ or $G = A_5 \leq S_{10}$;*
(iii) *$G = M_{22} \leq S_{22}$ or $G = \mathrm{Aut}(M_{22}) \simeq M_{22} \rtimes C_2 \leq S_{22}$ where M_{22} is the Mathieu group of degree 22;*
(iv) *$\mathrm{PSL}_2(\mathbb{F}_q) \leq G \leq \mathrm{P\Gamma L}_2(\mathbb{F}_q) \simeq \mathrm{PGL}_2(\mathbb{F}_q) \rtimes C_e$ where $2p = q+1$ and $q = l^e$ is an odd prime power.*

Main Theorem 5 (Theorem 3.14). *Let K/k be a separable field extension of degree n and L/k be the Galois closure of K/k. Let $G = \mathrm{Gal}(L/k)$ be a transitive subgroup of S_n and $H = \mathrm{Gal}(L/K)$ with $[G:H] = n$. Assume that $n = 11$, 12, 22, 23 or 24 and G is isomorphic to the Mathieu group M_n of degree n. Then $R^{(1)}_{K/k}(\mathbb{G}_m)$ is not retract k-rational.*

We organize this thesis as follows. In Chapter 2, we will review the definitions and basic properties of G-lattices and algebraic tori. We also recall some important results in the rationality problem for algebraic tori. In Chapter 3, using the results in Chapter 2, we give the proof of the main theorems.

Notation: Let S_n (resp. A_n, D_n, C_n) be the symmetric (resp. the alternating, the dihedral, the cyclic) group of degree n of order $n!$ (resp. $n!/2$, $2n$, n). Let $F_{pm} \simeq C_p \rtimes C_m \leq S_p$ be the Frobenius group of order pm where $m \mid p-1$. Let nTm be the m-th transitive subgroup of S_n (see Butler and McKay [**BM83**] for $n \leq 11$, Royle [**Roy87**] for $n = 12$, Butler [**But93**] for $n = 14, 15$ and [**GAP**]).

CHAPTER 2

Preliminalies

In this chapter, we will review the definitions and basic properties of G-lattices and algebraic tori. G-lattices play an important role in the stable rationality of algebraic tori. Then, we recall some important results in the rationality problem for algebraic tori and prepare for proving the main theorems in Chapter 3.

1. G-lattices and algebraic tori

In order to classify the stably/retract rational algebraic tori, the flabby class plays an important role. In this section, we recall some basic facts of the theory of flabby G-lattices (see J.-L. Colliot-Thélène and J.-J. Sansuc [**CTS77**], Swan [**Swa83**] [**Swa10**], Voskresenskii [**Vos98**, Chapter 2], Lorenz [**Lor05**, Chapters 1 and 2]) and of algebraic tori (see Ono [**Ono61**], Swan [**Swa83**], Voskresenskii [**Vos98**]).

Definition 2.1. A finitely generated $\mathbb{Z}[G]$-module M is called *G-lattice* if it is $\mathbb{Z}$-free as an abelian group.

Definition 2.2. Let M be a G-lattice.
(1) M is called a *permutation* G-lattice if M has a $\mathbb{Z}$-basis permuted by G.
(2) M is called a *stably permutation* G-lattice if $M \oplus P \simeq P'$ for some permutation G-lattices P and P'.
(3) M is called *invertible* (or *permutation projective*) if it is a direct summand of a permutation G-lattice, i.e. $P \simeq M \oplus M'$ for some permutation G-lattice P and a G-lattice M'.
(4) M is called *flabby* (or *flasque*) if $\widehat{H}^{-1}(H, M) = 0$ for any subgroup H of G where $\widehat{H}$ is the Tate cohomology (see [**Bro82**, Chapter IV]).
(5) M is called *coflabby* (or *coflasque*) if $H^1(H, M) = 0$ for any subgroup H of G.

Remark 2.3. When G-lattice P is a permutation lattice, $P \simeq \oplus_{1\leq i\leq m}\mathbb{Z}[G/H_i]$ for some subgroups $H_1, \ldots, H_m$ of G.

For G-lattice M, we see that "permutation" $\Rightarrow$ "stably permutation" $\Rightarrow$ "invertible" $\Rightarrow$ "flabby and coflabby".

Definition 2.4. Let M be a G-lattice. The G-lattice $M^\circ = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ with G-action

$$(g \cdot f)(m) = f(g^{-1} \cdot m) \quad (f \in M^\circ, m \in M, g \in G)$$

is called *the dual G-lattice* of M.

Note that $(M^\circ)^\circ \simeq M$.

Let M_1 and M_2 be G-lattices. M_1 and M_2 are said to *similar* if there exist permutation G-lattices P_1 and P_2 such that $M_1 \oplus P_1 \simeq M_2 \oplus P_2$. The *similar class* of M is denoted by $[M]$. We can define addition on the set of similarity classes as $[M_1] + [M_2] = [M_1 \oplus M_2]$. Then the set of similarity classes becomes a commutative monoid with $0 = [P]$ as the identity where P is permutation.

Remark 2.5. In terms of similarity classes, we can rephrase the definition of stably permutation and invertible G-lattice;

- M is stably permutation $\iff$ $[M] = 0$;
- M is invertible $\iff$ $[M]$ is invertible element of the above monoid.

Theorem 2.6 (Lenstra [**Len74**, Propositions 1.1 and 1.2]). *Let N be an invertible G-lattice.*
(1) *N is flabby and coflabby.*
(2) *If C is a coflabby G-lattice, then any short exact sequence $0 \to C \to M \to N \to 0$ splits.*

In order to classify the stably/retract rational algebraic tori, flabby class plays crucial role (see Theorems 2.18 and 2.19).

Theorem 2.7 (Endo and Miyata [**EM75**, Lemma 1.1]). *For any G-lattice M, there exists a short exact sequence of G-lattices $0 \to M \to P \to F \to 0$ where P is permutation and F is flabby.*

Definition 2.8. The exact sequence $0 \to M \to P \to F \to 0$ as in Theorem 2.7 is called a *flabby resolution* of the G-lattice M. $\rho_G(M) = [F]$ is called *the flabby class* of M, denoted by $[M]^{fl} = [F]$.

Note that $[M]^{fl}$ is well-defined: if $[M] = [M']$, $[M]^{fl} = [F]$ and $[M']^{fl} = [F']$ then $F \oplus P_1 \simeq F' \oplus P_2$ for some permutation G-lattices P_1 and P_2, and therefore $[F] = [F']$. We say that $[M]^{fl}$ is *invertible* if $[M]^{fl} = [E]$ for some invertible G-lattice E.

The following results are some useful facts about flabby classes.

Lemma 2.9 (Swan [**Swa10**, Lemma 3.1]). *Let* $0 \to M_1 \to M_2 \to M_3 \to 0$ *be a short exact sequence of* G*-lattices with* M_3 *invertible. Then the flabby class* $[M_2]^{fl} = [M_1]^{fl} + [M_3]^{fl}$. *In particular, if* $[M_1]^{fl}$ *is invertible, then* $-[M_1]^{fl} = [[M_1]^{fl}]^{fl}$.

Theorem 2.10 (see Lorentz [**Lor05**, Lemma 2.7.1]). *Let* M_1, M_2 *be* G*-lattices. Then the following conditions are equivalent:*
(i) $[M_1]^{fl} = [M_2]^{fl}$;
(ii) *there exist exact sequences of* G*-lattices*

$$0 \to M_1 \to E \to P_1 \to 0,$$

$$0 \to M_2 \to E \to P_2 \to 0,$$

where P *and* Q *are permutation lattices.*

We have the implications as following:

$$\begin{array}{ccccccc} \text{permutation} & \Rightarrow & \text{stably permutation} & \Rightarrow & \text{invertible} & \Rightarrow & \text{flabby and coflabby} \\ & & \Downarrow & & \Downarrow & & \\ & & [M]^{fl} = 0 & \Rightarrow & [M]^{fl} \text{ is invertible.} & & \end{array}$$

Note that the above implications in each step cannot be reversed (see, for example, [**HY17**, Section 1]).

Next we recall the definitions and basic properties of algebraic tori.

Definition 2.11. An algebraic group T over k is called *algebraic* k*-torus* of dimension n if $T \times_k k_{sep} \simeq \mathbb{G}_{m,k_{sep}}^n$ where k_{sep} is a separable closure of k.

Let X be a quasi-projective k-variety. A k-variety Y is called k*-form* of X if Y is isomorphic to X after extending to the separable closure k_{sep} of k. Isomorphism classes of k-form of X correspond bijectively to the elements of the set $H^1(\mathcal{G}, \mathrm{GL}_n(\mathbb{Z}))$ where $\mathcal{G} = \mathrm{Gal}(k_{sep}/k)$ is the absolute Galois group of k. In this contexts, algebraic k-torus of dimension n is a k-form of a split k-torus $\mathbb{G}_m^n$.

Since $\mathrm{Aut}(\mathbb{G}_m^n) = \mathrm{GL}_n(\mathbb{Z})$, tori of dimension n over k correspond bijectively to the elements of the set $H^1(\mathcal{G}, \mathrm{GL}_n(\mathbb{Z}))$. Then the k-torus T of dimension n is determined uniquely by the integral continuous representation $h : \mathcal{G} \to \mathrm{GL}_n(\mathbb{Z})$ up to conjugacy. Since h is continuous, $\mathrm{GL}_n(\mathbb{Z})$ has the discrete topology and G is compact, the group $h(\mathcal{G})$ is a finite subgroup of $\mathrm{GL}_n(\mathbb{Z})$. Denote $H = \mathrm{Ker}\ (h)$. Let $L = k_{sep}^H$ be the fixed field and $G = \mathcal{G}/H \simeq h(\mathcal{G})$. One can see that L is the intersection of all splitting fields of T in k_{sep}, thus L is the minimal splitting field of T which is a Galois extension of k with Galois group G (see [**Vos98**, page 27, Section 3.4]). Therefore everything can be reduced to the case that splitting field L/k of T is finite Galois extension. Note that the class of stable equivalence of T only depends on the *splitting group* $h(\mathcal{G}) \subset \mathrm{GL}_n(\mathbb{Z})$ but not on the splitting filed L/k (see [**Vos98**, page 57, Section 4.9]).

If T is an algebraic k-torus which splits by finite Galois extension L/k with Galois group G, then the character group $X(T) = \mathrm{Hom}(T, \mathbb{G}_m)$ of T is regarded as a G-lattice. Conversely, for a given G-lattice M, $T = \mathrm{Spec}(L[M]^G)$ is an algebraic torus and splits over L such that $X(T)$ is isomorphic to M where $L([M])^G$ is multiplicative invariant. Hence there is the duality between the category of G-lattices and the category of algebraic k-tori which split over L (see [**Ono61**, Section 1.2], [**Swa83**, Section 12], [**Vos98**, page 27, Example 6]).

Let L be a finite Galois extension of a field k and $G = \mathrm{Gal}(L/k)$ be the Galois group of the extension L/k. Let $M = \bigoplus_{1\leq i\leq n} \mathbb{Z}\cdot u_i$ be a G-lattice with a $\mathbb{Z}$-basis $\{u_1, \ldots, u_n\}$. Let G act on the rational function field $L(x_1, \ldots, x_n)$ over L with n variables $x_1, \ldots, x_n$ by

$$\sigma(x_i) = \prod_{j=1}^{n} x_j^{a_{i,j}}, \quad 1 \leq i \leq n$$

for any $\sigma \in G$, when $\sigma(u_i) = \sum_{j=1}^{n} a_{i,j} u_j$, $a_{i,j} \in \mathbb{Z}$. The field $L(x_1, \ldots, x_n)$ with this action of G will be denoted by $L(M)$.

By the duality explained above, the function field of the algebraic k-torus T may be identified with the invariant field $L(M)^G$ of $L(M)$ under the action of G. Note that the field $L(M)^G$ is always k-unirational (see [**Vos98**, page 40, Example 21]). Therefore we are interested in the stably and retract rationality of algebraic tori.

Next we recall the definition of norm one tori which are main target in this thesis.

Definition 2.12. Let K/k be a separable field extension of degree n and L/k be the Galois closure of K/k. The *norm one torus* $R^{(1)}_{K/k}(\mathbb{G}_{m,K})$ is the kernel of the norm map $R_{K/k}(\mathbb{G}_{m,K}) \xrightarrow{N_{K/k}} \mathbb{G}_m$, where $R_{K/k}$ is a Weil restriction.

The norm one torus $R^{(1)}_{K/k}(\mathbb{G}_m)$ has the *Chevalley module* $J_{G/H}$ as its character module and the field $L(J_{G/H})^G$ as its function field where $J_{G/H} = (I_{G/H})^\circ = \mathrm{Hom}_{\mathbb{Z}}(I_{G/H}, \mathbb{Z})$ is the dual lattice of $I_{G/H} = \mathrm{Ker}\ \varepsilon$ and $\varepsilon : \mathbb{Z}[G/H] \to \mathbb{Z}$ is the augmentation map. We have the exact sequence $0 \to \mathbb{Z} \to \mathbb{Z}[G/H] \to J_{G/H} \to 0$ and $\mathrm{rank}_{\mathbb{Z}}(J_{G/H}) = n - 1$.

2. Classification of stably/retract rational algebraic tori

In this section, we present some important results for the rationality problem for algebraic tori. First, we recall the definitions of some notions of rationality in terms of function fields (see Lorenz [**Lor05**, Chapters 9]).

Definition 2.13. Let K/k be a finitely generated extension of fields.
(1) K is *rational over* k (or k*-rational* for short) if $K \simeq k(x_1, \ldots, x_n)$ for some algebraically independent elements $x_1, \ldots, x_n$ over k.
(2) K is *stably* k*-rational* if $K(y_1, \ldots, y_m)$ is rational for some $y_1, \ldots, y_m$ such that $y_1, \ldots, y_m$ are algebraically independent over K.
(3) When k is an infinite field, K is *retract* k*-rational* if there exists a k-algebra R contained in K such that (i) K is the quotient field of R, and (ii) there exist a non-zero polynomial $f \in k[x_i, \ldots, x_n]$ and k-algebra homomorphisms $\varphi : R \to k[x_1, \ldots x_n]$ and $\psi : k[x_1, \ldots, x_n] \to R$ satisfying $\varphi \circ \psi = 1_R$.
(4) K is k*-unirational* if $k \subset K \subset k(x_1, \ldots, x_n)$ for some m.

We see that "k-rational" $\Rightarrow$ "stably k-rational" $\Rightarrow$ "retract k-rational" $\Rightarrow$ "k-unirational".

Definition 2.14. Let K/k and K'/k be finitely generated extensions of fields. K and K' are called *stably* k*-isomorphic* if $K(y_1, \ldots, y_m) \simeq K'(z_1, \ldots, z_n)$ over k for some algebraically independent elements $y_1, \ldots, y_m$ over K and $z_1, \ldots, z_n$ over K'.

The 1-dimensional algebraic k-tori, i.e. the trivial torus $\mathbb{G}_m$ and the norm one torus $R^{(1)}_{K/k}(\mathbb{G}_m)$, are rational over k. The rationality problem for dimension two were proved by Voskresenskii and a classification of rational algebraic k-tori dimension 3 is given by Kunyavskii:

Theorem 2.15 (Voskresenskii [**Vos67**]). *All the* 2*-dimensional algebraic* k*-tori are* k*-rational.*

Theorem 2.16 (Kunyavskii [**Kun90**, Theorem 1]). *All the* 3*-dimensional algebraic* k*-tori are* k*-rational except for 15 cases in* [**Kun90**, Theorem 1]*. The remaining* 15 *cases are not* k*-retract rational.*

The following result shows that the flabby class of G-lattice corresponding to algebraic torus becomes the stably birational invariant.

Theorem 2.17 (Voskresenskii [**Vos74**, Theorem 2]). *Let* L/k *be a finite Galois extension with Galois group* $G = \mathrm{Gal}(L/k)$*. Let* M *and* M' *be* G*-lattices. Then the following*

conditions are equivalent:
(i) $[M]^{fl} = [M']^{fl}$*;*
(ii) $L(M)^G$ *and* $L(M')^G$ *are stably* k*-isomorphic.*

The following two results give us necessary and sufficient conditions for stable and retract rationality of algebraic tori in terms of the flabby classes.

THEOREM 2.18 (Endo and Miyata [**EM73**, Theorem 1.6]). *Let* L/k *be a finite Galois extension with Galois group* $G = \mathrm{Gal}(L/k)$*. Let* M *be* G*-lattice. Then the following conditions are equivalent:*
(i) $[M]^{fl} = 0$*;*
(ii) $L(M)^G$ *is stably* k*-rational.*

THEOREM 2.19 (Saltman [**Sal84**, Theorem 3.14]). *Let* L/k *be a finite Galois extension with Galois group* $G = \mathrm{Gal}(L/k)$*. Let* M *be* G*-lattice. Then the following conditions are equivalent:*
(i) $[M]^{fl}$ *is invertible;*
(ii) $L(M)^G$ *is retract* k*-rational.*

Using above results, we may determine whether algebraic k-tori are stably/retract rational by the flabby class $[M]^{fl}$. A classification of stably/retract rational algebraic k-tori in dimensions 4 and 5 is given by Hoshi and Yamasaki [**HY17**]. Before presenting the results, we define the G-lattice M_G associated to finite subgroup $G \leq \mathrm{GL}_n(\mathbb{Z})$.

DEFINITION 2.20. Let G be a finite subgroup of $\mathrm{GL}_n(\mathbb{Z})$. *The* G*-lattice* M_G *with* $\mathrm{rank}_{\mathbb{Z}}(M_G) = n$ is defined to be the G-lattice with a $\mathbb{Z}$-basis $\{u_1, \ldots, u_n\}$ on which G acts by $\sigma(u_i) = \sum_{j=1}^{n} a_{i,j} u_j$ for any $\sigma = [a_{i,j}] \in G$.

Let G be a finite subgroup of $\mathrm{GL}_n(\mathbb{Z})$ and L/k be a Galois extension with Galois group $\mathrm{Gal}(L/k) \simeq G$. Let G act on the rational function field $L(x_1, \ldots, x_n)$ over L with n variables $x_1, \ldots, x_n$ by

$$\sigma(x_i) = \prod_{j=1}^{n} x_j^{a_{i,j}}, \quad 1 \leq i \leq n \tag{1}$$

for any $\sigma = (a_{i,j}) \in G$. The field $L(x_1, \ldots, x_n)$ with this action of G is same as $L(M_G)$ where M_G is the corresponding G-lattice as in Definition 2.20.

The following lemma is useful.

Lemma 2.21 (see [**CTS77**, Remarque R2, page 180], [**HY17**, Lemma 2.17]). *Let G be a finite subgroup of $\mathrm{GL}_n(\mathbb{Z})$ and M_G be the corresponding G-lattice as in Definition 2.20. Let $H \leq G$ and $\rho_H(M_H)$ be the flabby class of M_H as an H-lattice.*
(i) *If $\rho_G(M_G) = 0$, then $\rho_H(M_H) = 0$.*
(ii) *If $\rho_G(M_G)$ is invertible, then $\rho_H(M_H)$ is invertible.*

Theorem 2.22 (Hoshi and Yamasaki [**HY17**, Theorem 1.9]). *Let L/k be a Galois extension and $G \simeq \mathrm{Gal}(L/k)$ be a finite subgroup of $\mathrm{GL}(4,\mathbb{Z})$ which acts on $L(x_1,x_2,x_3,x_4)$ via* (1)*.*
(i) *$L(x_1,x_2,x_3,x_4)^G$ is stably k-rational if and only if G is conjugate to one of the* 487 *groups which are not in* Tables 2*,* 3 *and* 4*.*
(ii) *$L(x_1,x_2,x_3,x_4)^G$ is not stably but retract k-rational if and only if G is conjugate to one of the* 7 *groups which are given as in* [**HY17**, Table 2]*.*
(iii) *$L(x_1,x_2,x_3,x_4)^G$ is not retract k-rational if and only if G is conjugate to one of the* 216 *groups which are given as in* [**HY17**, Tables 3 and 4]*.*

Theorem 2.23 (Hoshi and Yamasaki [**HY17**, Theorem 1.12]). *Let L/k be a Galois extension and $G \simeq \mathrm{Gal}(L/k)$ be a finite subgroup of $\mathrm{GL}(5,\mathbb{Z})$ which acts on $L(x_1,x_2,x_3,x_4,x_5)$ via* (1)*.*
(i) *$L(x_1,x_2,x_3,x_4,x_5)^G$ is stably k-rational if and only if G is conjugate to one of the* 3051 *groups which are not in* [**HY17**, Tables 11,12,13,14 and 15]*.*
(ii) *$L(x_1,x_2,x_3,x_4,x_5)^G$ is not stably but retract k-rational if and only if G is conjugate to one of the* 25 *groups which are given as in* [**HY17**, Table 11]*.*
(iii) *$L(x_1,x_2,x_3,x_4,x_5)^G$ is not retract k-rational if and only if G is conjugate to one of the* 3003 *groups which are given as in* [**HY17**, Tables 12,13,14 and 15]*.*

Let K/k be a separable field extension of degree n and L/k be the Galois closure of K/k. When K/k is Galois extension, the classification of stably/retract rationality of the norm one torus $R^{(1)}_{K/k}(\mathbb{G}_{m,K})$ is known by Endo and Miyata, Saltman, Colliot-Thélène and Sansuc.

Theorem 2.24 (Endo and Miyata [**EM75**, Theorem 1.5], Saltman [**Sal84**, Theorem 3.14]). *Let K/k be a finite Galois field extension and $G = \mathrm{Gal}(K/k)$. Then the following conditions are equivalent:*
(i) *$R^{(1)}_{K/k}(\mathbb{G}_m)$ is retract k-rational;*
(ii) *all the Sylow subgroups of G are cyclic.*

THEOREM 2.25 (Endo and Miyata [**EM75**, Theorem 2.3], Colliot-Thélène and Sansuc [**CTS77**, Proposition 3]). *Let K/k be a finite Galois field extension and $G = \mathrm{Gal}(K/k)$. Then the following conditions are equivalent:*
(i) $R^{(1)}_{K/k}(\mathbb{G}_m)$ *is stably k-rational;*
(ii) *all the Sylow subgroups of G are cyclic and $H^4(G,\mathbb{Z}) \simeq \widehat{H}^0(G,\mathbb{Z})$ where $\widehat{H}$ is the Tate cohomology;*
(iii) $G = C_m$ *or* $G = C_n \times \langle \sigma, \tau \mid \sigma^k = \tau^{2^d} = 1, \tau\sigma\tau^{-1} = \sigma^{-1}\rangle$ *where $d \geq 1, k \geq 3$, n, k: odd, and* $\gcd\{n,k\} = 1$*;*
(iv) $G = \langle s, t \mid s^m = t^{2^d} = 1, tst^{-1} = s^r, m : odd,\ r^2 \equiv 1 \pmod{m}\rangle$.

When K/k is non-Galois extension, the rationality of norm one tori $R^{(1)}_{K/k}(\mathbb{G}_{m,K})$ is only known in very special cases:

THEOREM 2.26 (Endo [**End11**, Theorem 2.1]). *Let K/k be a finite non-Galois, separable field extension and L/k be the Galois closure of K/k. Assume that the Galois group of L/k is nilpotent. Then the norm one torus $R^{(1)}_{K/k}(\mathbb{G}_m)$ is not retract k-rational.*

THEOREM 2.27 (Endo [**End11**, Theorem 3.1]). *Let K/k be a finite non-Galois, separable field extension and L/k be the Galois closure of K/k. Let $G = \mathrm{Gal}(L/k)$ and $H = \mathrm{Gal}(L/K) \leq G$. Assume that all the Sylow subgroups of G are cyclic. Then the norm one torus $R^{(1)}_{K/k}(\mathbb{G}_m)$ is retract k-rational, and the following conditions are equivalent:*
(i) $R^{(1)}_{K/k}(\mathbb{G}_m)$ *is stably k-rational;*
(ii) $G = D_n$ *with n odd ($n \geq 3$) or $G = C_m \times D_n$ where m, n are odd, $m, n \geq 3$,* $\gcd\{m,n\} = 1$*, and $H \leq D_n$ is of order 2;*
(iii) $H = C_2$ *and $G \simeq C_r \rtimes H$, $r \geq 3$ odd, where H acts non-trivially on C_r.*

THEOREM 2.28 (Colliot-Thélène and Sansuc [**CTS87**, Proposition 9.1], Le Bruyn [**LeB95**, Theorem 3.1], [**CK00**, Proposition 0.2], [**LL00**], Endo [**End11**, Theorem 4.1], see also [**End11**, Remark 4.2 and Theorem 4.3]). *Let K/k be a non-Galois separable field extension of degree n and L/k be the Galois closure of K/k. Assume that* $\mathrm{Gal}(L/k) = S_n$*, $n \geq 3$, and* $\mathrm{Gal}(L/K) = S_{n-1}$ *is the stabilizer of one of the letters in S_n.*
(i) $R^{(1)}_{K/k}(\mathbb{G}_m)$ *is retract k-rational if and only if n is a prime number;*
(ii) $R^{(1)}_{K/k}(\mathbb{G}_m)$ *is (stably) k-rational if and only if $n = 3$.*

THEOREM 2.29 (Endo [**End11**, Theorem 4.4], Hoshi and Yamasaki [**HY17**, Corollary 1.11]). *Let K/k be a non-Galois separable field extension of degree n and L/k be the*

Galois closure of K/k. Assume that $\mathrm{Gal}(L/k) = A_n$, $n \geq 4$, *and* $\mathrm{Gal}(L/K) = A_{n-1}$ *is the stabilizer of one of the letters in* A_n.
(i) $R^{(1)}_{K/k}(\mathbb{G}_m)$ *is retract k-rational if and only if n is a prime number.*
(ii) $R^{(1)}_{K/k}(\mathbb{G}_m)$ *is stably k-rational if and only if* $n = 5$.

Hoshi and Yamasaki [**HY17**], [**HY**] classified stably/retract norm one tori when $n \leq 10$ and $n = p$ where p is a prime number using computer algebra system GAP [**GAP**] except for some cases:

THEOREM 2.30 (Hoshi and Yamasaki [**HY17**, Theorem 1.10, Theorem 1.14, Theorem 8.5]). *Let K/k be a separable field extension of degree n and L/k be the Galois closure of K/k. Let $G = \mathrm{Gal}(L/k)$ be a transitive subgroup of S_n and $H = \mathrm{Gal}(L/K)$ with $[G : H] = n$. Then a classification of stably/retract rational norm one tori $T = R^{(1)}_{K/k}(\mathbb{G}_m)$ in dimension $n-1$ for $n = 5, 6, 7, 11$ is given as follows:*
(1) *The case* $5Tm$ $(1 \leq m \leq 5)$.
(i) *T is stably k-rational for* $5T1 \simeq C_5$, $5T2 \simeq D_5$ *and* $5T4 \simeq A_5$;
(ii) *T is not stably but retract k-rational for* $5T3 \simeq F_{20}$ *and* $5T5 \simeq S_5$.
(2) *The case* $6Tm$ $(1 \leq m \leq 16)$.
(i) *T is stably k-rational for* $6T1 \simeq C_6$, $6T2 \simeq S_3$ *and* $6T3 \simeq D_6$;
(ii) *T is not retract k-rational for $6Tm$ with $4 \leq m \leq 16$ which is isomorphic to* A_4, $C_3 \times S_3$, $C_2 \times A_4$, S_4, S_4, S_3^2, $C_3^2 \rtimes C_4$, $C_2 \times S_4$, A_5, $S_3^2 \rtimes C_2$ S_5, A_6, S_6 *respectively.*
(3) *The case* $7Tm$ $(1 \leq m \leq 7)$.
(i) *T is stably k-rational for* $7T1 \simeq C_7$ *and* $7T2 \simeq D_7$;
(ii) *T is not stably but retract k-rational for* $7T3 \simeq F_{21}$, $7T4 \simeq F_{42}$, $7T5 \simeq \mathrm{PSL}_3(\mathbb{F}_2) \simeq \mathrm{PSL}_2(\mathbb{F}_7)$, $7T6 \simeq A_7$ *and* $7T7 \simeq S_7$.
(4) *The case* $11Tm$ $(1 \leq m \leq 8)$.
(i) *T is stably k-rational for* $11T1 \simeq C_{11}$ *and* $11T2 \simeq D_{11}$;
(ii) *T is not stably but retract k-rational for* $11T3 \simeq F_{55}$, $11T4 \simeq F_{110}$, $11T5 \simeq \mathrm{PSL}_2(\mathbb{F}_{11})$, $11T6 \simeq M_{11}$, $11T7 \simeq A_{11}$ *and* $11T8 \simeq S_{11}$ *where M_{11} is the Mathieu group of degree* 11.

THEOREM 2.31 (Hoshi and Yamasaki [**HY**, Theorem 1.9]). *Let $p \geq 3$ be a prime number, K/k be a separable field extension of degree p and L/k be the Galois closure of K/k. Let $G = \mathrm{Gal}(L/k)$ be a transitive subgroup of S_p and $H = \mathrm{Gal}(L/K)$ with $[G : H] = p$. Then norm one tori $T = R^{(1)}_{K/k}(\mathbb{G}_m)$ of dimension $p-1$ are retract k-rational and a stably rational classification of T is given as follows:*

(1) *T is stably k-rational for $G \simeq C_p \leq S_p$ and $G \simeq D_p \leq S_p$;*

(2) *T is not stably k-rational for $G \simeq C_p \rtimes C_m \leq S_p$ with $3 \leq m \mid p-1$;*

(3) *T is not stably k-rational for $G \simeq S_p$ where $p \geq 5$;*

(4) *T is stably k-rational for $G \simeq A_5 \leq S_5$ and T is not stably k-rational for $G \simeq A_p \leq S_p$ where $p \geq 7$;*

(5) *T is not stably k-rational for $G \simeq \mathrm{PSL}_2(\mathbb{F}_{11}) \leq S_{11}$;*

(6) *T is not stably k-rational for $G \simeq M_{11} \leq S_{11}$ and $G \simeq M_{23} \leq S_{23}$;*

(7) *T is not stably k-rational for $\mathrm{PSL}_d(\mathbb{F}_q) \leq G \leq \mathrm{P\Gamma L}_d(\mathbb{F}_q) \simeq \mathrm{PGL}_d(\mathbb{F}_q) \rtimes C_e$ where $d \geq 3$, $p = \frac{q^d-1}{q-1}$ and $q = l^e$ is a prime power;*

(8) *T is not stably k-rational for $\mathrm{PSL}_2(\mathbb{F}_{2^e}) < G \leq \mathrm{P\Gamma L}_2(\mathbb{F}_{2^e}) \simeq \mathrm{PSL}_2(\mathbb{F}_{2^e}) \rtimes C_e$ where $p = 2^e + 1$ is a Fermat prime.*

THEOREM 2.32 (Hoshi and Yamasaki [**HY**, Theorem 1.11]). *Let K/k be a separable field extension of degree n and L/k be the Galois closure of K/k. Let $G = \mathrm{Gal}(L/k)$ be a transitive subgroup of S_n and $H = \mathrm{Gal}(L/K)$ with $[G : H] = n$. Then a classification of stably/retract rational norm one tori $T = R^{(1)}_{K/k}(\mathbb{G}_m)$ in dimension $n-1$ for $n = 8, 9, 10$ is given as follows:*

(1) *The case $8Tm$ $(1 \leq m \leq 50)$.*

(i) *T is stably k-rational for $8T1 \simeq C_8$;*

(ii) *T is not retract k-rational for $8Tm$ with $2 \leq m \leq 50$.*

(2) *The case $9Tm$ $(1 \leq m \leq 34)$.*

(i) *T is stably k-rational for $9T1 \simeq C_9$ and $9T3 \simeq D_9$;*

(ii) *T is retract k-rational for $9T27 \simeq \mathrm{PSL}_2(\mathbb{F}_8)$;*

(iii) *T is not retract k-rational for $9Tm$ with $2 \leq m \leq 34$ and $m \neq 3, 27$.*

(3) *The case $10Tm$ $(1 \leq m \leq 45)$.*

(i) *T is stably k-rational for $10T1 \simeq C_{10}$, $10T2 \simeq D_5$ and $10T3 \simeq D_{10}$;*

(ii) *T is retract k-rational for $10T11 \simeq A_5 \times C_2$;*

(iii) *T is not stably but retract k-rational for $10T4 \simeq F_{20}$, $10T5 \simeq F_{20} \times C_2$, $10T12 \simeq S_5$ and $10T22 \simeq S_5 \times C_2$;*

(iv) *T is not retract k-rational for $10Tm$ with $6 \leq m \leq 45$ and $m \neq 11, 12, 22$.*

CHAPTER 3

Rationality problem for norm one tori in small dimensions

In this chapter, we give the classification of stably/retract rational norm one tori $R^{(1)}_{K/k}(\mathbb{G}_m)$ of dimension $n-1$ in the following cases:

- $n = 2^e$ $(e \geq 1)$,
- $n = 10, 12, 14, 15$,
- $n = q+1$ where $q = l^e \equiv 1 \pmod 4$ is an odd prime power and $\mathrm{PSL}_2(\mathbb{F}_q) \leq G \leq \mathrm{P\Gamma L}_2(\mathbb{F}_q) \simeq \mathrm{PGL}_2(\mathbb{F}_q) \rtimes C_e$,
- $G \leq S_{2p}$ is primitive, p is a prime number,
- Mathieu groups $G = M_n \leq S_n$ where $n = 11, 12, 22, 23, 24$.

1. Case $G \leq S_n$ $(n = 2^e)$

In this section, we give the classification of stably/retract rational norm one tori $R^{(1)}_{K/k}(\mathbb{G}_m)$ in dimension $n-1$ for $n = 2^e$ $(e \geq 1)$.

THEOREM 3.1. *Let K/k be a separable field extension of degree n and L/k be the Galois closure of K/k. Let $G = \mathrm{Gal}(L/k)$ be a transitive subgroup of S_n where $n = 2^e$ $(e \geq 1)$ and $H = \mathrm{Gal}(L/K)$ with $[G : H] = n$. Then $R^{(1)}_{K/k}(\mathbb{G}_m)$ is stably k-rational if and only if $G \simeq C_n$. Moreover, if $R^{(1)}_{K/k}(\mathbb{G}_m)$ is not stably k-rational, then it is not retract k-rational.*

Before proving this, we show the following two theorems.

THEOREM 3.2. *Let $n = p^e$ be a prime power and G be a transitive subgroup of S_n. Let $G_p = \mathrm{Syl}_p(G)$ be a p-Sylow subgroup of G. Then G_p is a transitive subgroup of S_n.*

PROOF. Let H be the stabilizer of one of the letters in G and H_p be a p-Sylow subgroup of H with $H_p \leq G_p$. Because $[G : H] = n$ and p does not divide both $[H : H_p]$ and $[G : G_p]$, we have $[G_p : H_p] = n = p^e$. Hence $H_p = G_p \cap H$ becomes the stabilizer of one of the letters in G_p and $G_p \leq S_n$ is transitive. □

THEOREM 3.3. *Let $n = 2^e$ be a power of 2 and G be a transitive subgroup of S_n. Let $G_2 = \mathrm{Syl}_2(G)$ be a 2-Sylow subgroup of G. If $G_2 \simeq C_n$, then $G \simeq C_n$.*

Proof. Let H be the stabilizer of one of the letters in G. We should show that $H=1$ because $[G:H]=n$. We will prove $H=1$ by induction in e. When $e=1$, the assertion holds. For e, we assume that $G_2=\langle\sigma\rangle\simeq C_n$ where $n=2^e$. Without loss of generality, we may assume that $\sigma=(1\cdots n)\in S_n$.

There exist $(n-1)!$ elements of order n in S_n which are conjugate in S_n. Let $Z_{S_n}(G_2)$ be the centralizer of G_2 in S_n and $N_{S_n}(G_2)$ be the normalizer of G_2 in S_n. Then we see that $Z_{S_n}(G_2)=G_2\simeq C_n$ and $N_{S_n}(G_2)=C_n\rtimes \mathrm{Aut}(C_n)\simeq \mathbb{Z}/2^e\mathbb{Z}\rtimes(\mathbb{Z}/2^e\mathbb{Z})^\times$. We also have $G_2=Z_G(G_2)\leq N_G(G_2)\leq G$. Because $N_G(G_2)$ is also a 2-group, we obtain that $Z_G(G_2)=N_G(G_2)=G_2$.

Let $A=\{x\in G\mid \mathrm{ord}(x)=n\}$ be the set of elements of order n in G and $A_2=\{x\in G_2\mid \mathrm{ord}(x)=n\}=\{\sigma^i\mid i\colon\mathrm{odd}\}$ be the set of elements of order n in G_2. If $g\in G_2$, then $gag^{-1}=a$ for any $a\in A_2$. If $g\in G\setminus G_2$, then $gA_2g^{-1}\cap A_2=\emptyset$ because $N_G(G_2)=G_2$. Note that $g_1A_2g_1^{-1}=g_2A_2g_2^{-1}$ if and only if $g_2^{-1}g_2\in G_2$. Hence we have $|A|=|A_2|\cdot[G:G_2]=2^{e-1}\cdot|H|=|G|/2$. This implies that $A=\{x\in G\mid \mathrm{sgn}(x)=-1\}$.

We claim that if $h(j)=k$ $(h\in H)$, then $j\equiv k \pmod 2$. Suppose not. Then there exists $\sigma^{j-k}\in A_2$ such that $\sigma^{j-k}h(j)=j$. But this is impossible because $\mathrm{sgn}(\sigma^{j-k}h)=-1$ and hence $\mathrm{ord}(\sigma^{j-k}h)=n$. This claim implies that $\langle\sigma^2,H\rangle$ acts on $2\mathbb{Z}/n\mathbb{Z}=\{2,4,\ldots,n\}$.

On the other hand, $\langle\sigma^2,H\rangle\leq G\cap A_n$ because $\mathrm{sgn}(\sigma^2)=\mathrm{sgn}(h)=1$ $(h\in H)$. We also see $\langle\sigma^2,H\rangle=G\cap A_n$ because $[\langle\sigma^2,H\rangle:H]=n/2$.

Remember that $|H|=[G:G_2]$ is odd. The restriction $G\cap A_n|_{2\mathbb{Z}/n\mathbb{Z}}$ of $G\cap A_n$ into $2\mathbb{Z}/n\mathbb{Z}$ seems to be a transitive subgroup of $S_{2\mathbb{Z}/n\mathbb{Z}}=S_{\{2,4,\ldots,n\}}$ whose 2-Sylow subgroup is $\langle\sigma^2\rangle|_{2\mathbb{Z}/n\mathbb{Z}}$. By the assumption of induction, we have $H|_{2\mathbb{Z}/n\mathbb{Z}}=1$. Similarly, we get $H|_{1+2\mathbb{Z}/n\mathbb{Z}}=1$. Therefore, we conclude that $H=1$. □

Proof of Theorem 3.1. Take a transitive subgroup $G=\mathrm{Gal}(L/k)\leq S_n$ $(n=2^e)$ and $H=\mathrm{Gal}(L/K)$ with $[G:H]=n$. By Theorem 3.2, the 2-Sylow subgroup $G_2=\mathrm{Syl}_2(G)$ of G is a transitive subgroup of S_n.

($\Rightarrow$) Assume that $G\not\simeq C_n$. By Theorem 3.3, we have $G_2\not\simeq C_n$. Hence $[J_{G_2/H_2}]^{fl}$ is not invertible by Endo and Miyata [**EM75**, Theorem 1.5] and Endo [**End11**, Theorem 2.1] where H_2 is the 2-Sylow subgroup of H. Because G_2 is transitive in S_n, it follows from Lemma 2.21 (ii) that $[J_{G/H}]^{fl}$ is not invertible. Hence $R^{(1)}_{K/k}(\mathbb{G}_m)$ is not retract k-rational.

($\Leftarrow$) By Endo and Miyata [**EM75**, Theorem 2.3], if $G\simeq C_n$, then $R^{(1)}_{K/k}(\mathbb{G}_m)$ is stably k-rational. □

EXAMPLE 3.4 (The case $nTm \leq S_n$ where $n = 2^e$). (1) When $n = 4$, there exist 5 transitive subgroups $4Tm \leq S_4$ ($1 \leq m \leq 5$): $4T1 \simeq C_4$, $4T2 \simeq C_2 \times C_2$, $4T3 \simeq D_4$, $4T4 \simeq A_4$, $4T5 \simeq S_4$.

(2) When $n = 8$, there exist 50 transitive subgroups of $8Tm \leq S_8$ ($1 \leq m \leq 50$). There exist 5 groups $G = 8Tm$ ($1 \leq m \leq 5$) with $|G| = 8$ (see Butler and McKay [**BM83**], [**GAP**]): $8T1 \simeq C_8$, $8T2 \simeq C_4 \times C_2$, $8T3 \simeq (C_2)^3$, $8T4 \simeq D_4$, $8T5 \simeq Q_8$.

(3) When $n = 16$, there exist 1954 transitive subgroups of $16Tm \leq S_{16}$ ($1 \leq m \leq 1954$). There exist 14 groups $G = 16Tm$ ($1 \leq m \leq 14$) with $|G| = 16$ (see Example 3.5): $16T1 \simeq C_{16}$, $16T2 \simeq C_4 \times (C_2)^2$, $16T3 \simeq (C_2)^4$, $16T5 \simeq C_4 \times C_4$, $16T5 \simeq C_8 \times C_2$, $16T6 \simeq M_{16}$, $16T7 \simeq Q_8 \times C_2$, $16T8 \simeq C_4 \rtimes C_4$, $16T9 \simeq D_4 \times C_2$, $16T10 \simeq (C_4 \times C_2) \rtimes C_2$, $16T11 \simeq (C_4 \times C_2) \rtimes C_2$, $16T12 \simeq QD_8$, $16T13 \simeq D_8$, $16T14 \simeq Q_{16}$.

(4) When $n = 32$, there exist 2801324 transitive subgroups of $32Tm \leq S_{32}$ ($1 \leq m \leq 2801324$) (see Cannon and Holt [**CH08**]).

EXAMPLE 3.5 (Computations for $16Tm \leq S_{16}$). For $G = 16Tm \leq S_{16}$, Theorem 3.2 and Theorem 3.3 can be checked by GAP as follows:

```
gap> NrTransitiveGroups(16); # the number of transitive subgroups G=16Tm <= S16
1954
gap> Sy162:=List([1..1954],x->SylowSubgroup(TransitiveGroup(16,x),2));;
gap> Filtered([1..1954],x->IsTransitive(Sy162[x])=false);
# all 2-Syllow subgroups of 16Tm are transitive
[  ]
gap> Filtered([1..1954],x->IsCyclic(Sy162[x])=true);
# all 2-Syllow subgroups of 16Tm are cyclic except for m=1
[ 1 ]
gap> Filtered([1..1954],x->Size(TransitiveGroup(16,x))=16); # 16Tm with |16Tm|=16
[ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14 ]
gap> List([1..14],x->StructureDescription(TransitiveGroup(16,x)));
[ "C16", "C4 x C2 x C2", "C2 x C2 x C2 x C2", "C4 x C4", "C8 x C2", "C8 : C2",
"C2 x Q8", "C4 : C4", "C2 x D8", "(C4 x C2) : C2", "(C4 x C2) : C2", "QD16",
 "D16", "Q16" ]
```

2. Case $G \leq S_n$ ($n = 10, 12, 14, 15$)

In this section, we give the classification of stably/retract rational norm one tori $R^{(1)}_{K/k}(\mathbb{G}_m)$ in dimension $n-1$ for $n = 10, 12, 14, 15$. Note that there exist 45 (resp. 301, 63, 104) transitive groups $10Tm$ (resp. $12Tm$, $14Tm$, $15Tm$) of degree 10 (resp. 12, 14, 15). The case $n=10$ in Theorem 3.7 (1) was solved by [**HY**, Theorem 1.11] except for $G=10T11\simeq A_5\times C_2$ (see Theorem 2.32).

Let K/k be a separable field extension of degree n and L/k be the Galois closure of K/k. Let $G=\mathrm{Gal}(L/k)$ be a transitive subgroup of S_n and $H=\mathrm{Gal}(L/K)$ with $[G:H]=n$. We may assume that H is the stabilizer of one of the letters in G, i.e. $L=k(\theta_1,\ldots,\theta_n)$ and $K=L^H=k(\theta_i)$ where $1\leq i\leq n$.

Let nTm be the m-th transitive subgroup of S_n There exist 2 (resp. 5, 5, 16, 7, 50, 34, 45, 8) transitive subgroups of S_3 (resp. S_4, S_5, S_6, S_7, S_8, S_9, S_{10}, S_{11}) (see Butler and McKay [**BM83**] for $n\leq 11$, Royle [**Roy87**] for $n=12$, Butler [**But93**] for $n=14,15$ and [**GAP**]).

The following GAP algorithm may certify whether $F=[J_{G/H}]^{fl}$ is invertible (resp. zero) (see also Hoshi and Yamasaki [**HY17**, Chapter 5]). Some related programs are available from

`https://www.math.kyoto-u.ac.jp/~yamasaki/Algorithm/RatProbNorm1Tori/`.

Algorithm 3.6 (see Hoshi and Yamasaki [**HY17**, Chapter 5 and Chapter 8]).

(0) Construction of the Chevalley module $J_{G/H}$ (see [**HY17**, Chapter 8]):

`Norm1TorusJ`(n,m) returns $J_{G/H}$ for $G=nTm\leq S_n$ and H is the stabilizer of one of the letters in G.

(1) Whether $F=[J_{G/H}]^{fl}$ is invertible:

`IsInvertibleF(Norm1TorusJ`(n,m)`)` returns true (resp. false) if $[J_{G/H}]^{fl}$ is invertible (resp. not invertible) for $G=nTm\leq S_n$ and H is the stabilizer of one of the letters in G (see [**HY17**, Section 5.2]).

(2) Possibility for $F=0$ where $F=[J_{G/H}]^{fl}$:

`PossibilityOfStablyPermutationF(Norm1TorusJ`(m,n)`)` returns a basis $\mathcal{L}=\{l_1,\ldots,l_s\}$ of possible solutions space $\{(a_1,\ldots,a_r,b_1)\}$ $(a_i,b_1\in\mathbb{Z})$ (see also [**HY17**, Section 5.4]) to

$$\bigoplus_{i=1}^{r}\mathbb{Z}[G/H_i]^{\oplus a_i}\ \simeq\ F^{\oplus(-b_1)}$$

for $G=mTn\leq S_n$, H is the stabilizer of one of the letters in G and $F=[J_{G/H}]^{fl}$. In particular, if all the b_1's are even, then we can conclude that $F=[J_{G/H}]^{fl}\neq 0$.

(3) Verification of $F=0$ where $F=[J_{G/H}]^{fl}$:

`FlabbyResolutionLowRankFromGroup((Norm1TorusJ`(n,m)`,TransitiveGroup`(n,m)`)`
`.actionF` returns a suitable flabby class $F=[J_{G/H}]^{fl}$ of $J_{G/H}$ with low rank for $G=$

$nTm \leq S_n$ and H is the stabilizer of one of the letters in G by using the backtracking techniques. Repeating the algorithm, by defining $[J_{G/H}]^{fl^n} := [[J_{G/H}]^{fl^{n-1}}]^{fl}$ inductively, $[J_{G/H}]^{fl} = 0$ is provided if we may find some n with $[J_{G/H}]^{fl^n} = 0$ (this method is slightly improved to the `flfl` algorithm, see [**HY17**, Section 5.3]).

THEOREM 3.7. *Let K/k be a separable field extension of degree n and L/k be the Galois closure of K/k. Let $G = \mathrm{Gal}(L/k)$ be a transitive subgroup of S_n and $H = \mathrm{Gal}(L/K)$ with $[G : H] = n$. Then a classification of stably/retract rational norm one tori $T = R^{(1)}_{K/k}(\mathbb{G}_m)$ in dimension $n-1$ for $n = 10, 12, 14, 15$ is given as follows:*
(1) *The case* $10Tm$ $(1 \leq m \leq 45)$.
(i) *T is stably k-rational for $10T1 \simeq C_{10}$, $10T2 \simeq D_5$, $10T3 \simeq D_{10}$, $10T11 \simeq A_5 \times C_2$;*
(ii) *T is not stably but retract k-rational for $10T4 \simeq F_{20}$, $10T5 \simeq F_{20} \times C_2$, $10T12 \simeq S_5$, $10T22 \simeq S_5 \times C_2$;*
(iii) *T is not retract k-rational for $10Tm$ with $6 \leq m \leq 45$ and $m \neq 11, 12, 22$.*
(2) *The case* $12Tm$ $(1 \leq m \leq 301)$.
(i) *T is stably k-rational for $12T1 \simeq C_{12}$, $12T5 \simeq C_3 \rtimes C_4$, $12T11 \simeq C_4 \times S_3$;*
(ii) *T is not retract k-rational for $12Tm$ with $1 \leq m \leq 301$ and $m \neq 1, 5, 11$.*
(3) *The case* $14Tm$ $(1 \leq m \leq 63)$.
(i) *T is stably k-rational for $14T1 \simeq C_{14}$, $14T2 \simeq D_7$, $14T3 \simeq D_{14}$;*
(ii) *T is not stably k-rational but retract k-rational for $14T4 \simeq F_{42}$, $14T5 \simeq F_{21} \times C_2$, $14T7 \simeq F_{42} \times C_2$, $14T16 \simeq \mathrm{PSL}_3(\mathbb{F}_2) \rtimes C_2$, $14T19 \simeq \mathrm{PSL}_3(\mathbb{F}_2) \times C_2$, $14T46 \simeq S_7$, $14T47 \simeq A_7 \times C_2$, $14T49 \simeq S_7 \times C_2$;*
(iii) *T is not retract k-rational for $14Tm$ with $6 \leq m \leq 63$ and $m \neq 7, 16, 19, 46, 47, 49$.*
(4) *The case* $15Tm$ $(1 \leq m \leq 104)$.
(i) *T is stably k-rational for $15T1 \simeq C_{15}$, $15T2 \simeq D_{15}$, $15T3 \simeq D_5 \times C_3$, $15T4 \simeq S_3 \times C_5$, $15T5 \simeq A_5$, $15T7 \simeq D_5 \times S_3$, $15T16 \simeq A_5 \times C_3 \simeq \mathrm{GL}_2(\mathbb{F}_4)$, $15T23 \simeq A_5 \times S_3$;*
(ii) *T is not stably k-rational but retract k-rational for $15T6 \simeq C_{15} \rtimes C_4$, $15T8 \simeq F_{20} \times C_3$, $15T10 \simeq S_5$, $15T11 \simeq F_{20} \times S_3$, $15T22 \simeq (A_5 \times C_3) \rtimes C_2 \simeq \mathrm{GL}_2(\mathbb{F}_4) \rtimes C_2$, $15T24 \simeq S_5 \times C_3$, $15T29 \simeq S_5 \times S_3$;*
(iii) *T is not retract k-rational for $15Tm$ with $9 \leq m \leq 104$ and $m \neq 10, 11, 16, 22, 23, 24, 29$.*

PROOF. We may assume that H is the stabilizer of one of the letters in G (see the second paragraph of Section 2).

(1) The case $10Tm$ $(1 \leq m \leq 45)$.

By [**HY**, Theorem 1.11], we should show that T is stably k-rational for $10T11 \simeq A_5 \times C_2$. For $10T11$, by Algorithm 3.6 (3), we may take $F = [J_{G/H}]^{fl}$ with $\mathrm{rank}_{\mathbb{Z}}(F) = 31$, $F' = [F]^{fl}$ with $\mathrm{rank}_{\mathbb{Z}}(F') = 13$ and $F'' = [F']^{fl}$ with $F'' = [\mathbb{Z}] = 0$. This implies that $F = 0$ and hence T is stably k-rational (see Example 3.8).

(2) The case $12Tm$ $(1 \leq m \leq 301)$.

(2-1) The case where K/k is Galois: $1 \leq m \leq 5$. For $12T1 \simeq C_{15}$, $12T2 \simeq C_6 \times C_2$, $12T3 \simeq D_6$, $12T4 \simeq A_4$, $12T5 \simeq C_3 \rtimes C_4$, K/k is a Galois extension. By Endo and Miyata [**EM75**, Theorem 2.3], T is stably k-rational for $12T1$, $12T5$. By Endo and Miyata [**EM75**, Theorem 1.5], T is not retract k-rational for $12T2$, $12T3$, $12T4$.

(2-2) The case where K/k is not Galois: $6 \leq m \leq 301$.

Case 1: $m = 11$. For $12T11 \simeq C_4 \times S_3$, by Algorithm 3.6 (3), we may take $F = [J_{G/H}]^{fl}$ with $\mathrm{rank}_{\mathbb{Z}}(F) = 17$, $F' = [F]^{fl}$ with $\mathrm{rank}_{\mathbb{Z}}(F') = 4$ and F' is permutation. This implies that $F = 0$ and hence T is stably k-rational (see Example 3.9). (We note that $12T1 \leq 12T5 \leq 12T11$.)

Case 2: $m \neq 11$. By using the command

```
List([1..301],x->Filtered([1..x],y->IsSubgroup(TransitiveGroup(12,x),
TransitiveGroup(12,y))))
```

in GAP [**GAP**] (see also Example 3.10 for the case where $n = 14$), we obtain the inclusions $12Tm \leq 12Tm'$ among the groups $G = 12Tm$ with minimal groups $12Tm$ where $m \in I_{12} := \{2, 3, 4, 7, 8, 9, 12, 15, 16, 17, 19, 29, 30, 31, 32, 33, 34, 36, 40, 41, 46, 47, 57, 58, 59, 60, 61, 63, 64, 65, 66, 68, 69, 70, 73, 74, 75, 76, 89, 91, 93, 96, 99, 100, 102, 105, 107, 160, 162, 166, 171, 172, 173, 179, 181, 182, 183, 207, 212, 216, 246, 254, 272, 278, 295\}$.

By using the command

```
Filtered(List(ConjugacyClassesSubgroups(TransitiveGroup(12,m)),Representative),
x->Length(Orbits(x,[1..12]))=1),
```

we also see the following inclusions for $12Tm$ with $m \in I_0 := \{207, 212, 216, 254, 272, 278, 295\}$ (see Example 3.9, we may reduce these cases which take more computational time and resources):

$12T166 \leq 12T207, 12T254$,

$12T46 \leq 12T212, 12T216, 12T272$,

$12T17 \leq 12T278$,

$12T2 \leq 12T295$.

By the inclusion of $G = 12Tm$ above and Lemma 2.21 (ii), it is enough to check that $[J_{G/H}]^{fl}$ is not invertible for $I_{12} \setminus I_0$. By Algorithm 3.6 (1), we obtain that $[J_{G/H}]^{fl}$ is not invertible and hence, by Theorem 2.19, T is not retract k-rational for $m \in I_{12} \setminus I_0$ (see Example 3.9).

(3) The case $14Tm$ ($1 \leq m \leq 63$).

(3-1) The case where K/k is Galois: $m = 1, 2$. For $14T1 \simeq C_{14}$ and $14T2 \simeq D_7$, K/k is a Galois extension. By Endo and Miyata [**EM75**, Theorem 2.3], T is stably k-rational for $14T1$ and $14T2$.

(3-2) The case where K/k is not Galois: $3 \leq m \leq 63$.

Case 1: $m = 3$. For $14T3 \simeq D_{14}$, by Algorithm 3.6 (1), we obtain that $[J_{G/H}]^{fl}$ is invertible and hence T is retract k-rational by Theorem 2.19. By Algorithm 3.6 (3), we may take $F = [J_{G/H}]^{fl}$ with $\mathrm{rank}_{\mathbb{Z}}(F) = 17$ and $F' = [F]^{fl} = \mathbb{Z}^2$ which is permutation. This implies that $F = 0$ and hence T is stably k-rational by Theorem 2.18 (i) (see Example 3.10).

Case 2: $m = 4, 5, 7, 16, 19, 46, 47, 49$. By Algorithm 3.6 (1), we see that $[J_{G/H}]^{fl}$ is invertible and hence T is retract k-rational by Theorem 2.19 for $m = 4, 5, 7, 16, 19, 46, 47, 49$. For $m = 4, 5, 16$, by Algorithm 3.6 (2), we see that $[J_{G/H}]^{fl} \neq 0$ and hence T is not stably k-rational (see Example 3.10). By Lemma 2.21 (i) and the inclusions $14T4 \leq 14T7, 14T46$ and $14T5 \leq 14T19 \leq 14T47 \leq 14T49$, we have $[J_{G/H}]^{fl} \neq 0$ and hence T is also not stably k-rational for $m = 7, 19, 46, 47, 49$.

Case 3: $6 \leq m \leq 63$ and $m \neq 7, 16, 19, 46, 47, 49$.

By using the command

```
List([1..63],x->Filtered([1..x],y->IsSubgroup(TransitiveGroup(14,x),
TransitiveGroup(14,y))))
```

in GAP [**GAP**] (see Example 3.10), we get the inclusions $14Tm \leq 14Tm'$ among the groups $G = 14Tm$ with minimal groups $14Tm$ where $m \in I_{14} := \{6, 8, 10, 12, 26, 30\}$.

By the inclusion of $G = 14Tm$ above and Lemma 2.21 (ii), it is enough to show that $[J_{G/H}]^{fl}$ is not invertible for $m \in I_{14}$. By Algorithm 3.6 (1), we see that $[J_{G/H}]^{fl}$ is not invertible and hence T is not retract k-rational for $m \in I_{14}$ (see Example 3.10).

(4) The case $15Tm$ ($1 \leq m \leq 104$).

(4-1) The case where K/k is Galois: $m = 1$. For $15T1 \simeq C_{15}$, K/k is a Galois extension. It follows from Endo and Miyata [**EM75**, Theorem 2.3] that T is stably k-rational for $15T1$.

(4-2) The case where K/k is not Galois: $2 \leq m \leq 104$.

Case 1: $m = 2, 3, 4$. For $15T2 \simeq D_{15}, 15T3 \simeq D_5 \times C_3, 15T4 \simeq S_3 \times C_5$, it follows from Endo [**End11**, Theorem 3.1] that T is stably k-rational for $15T2, 15T3, 15T4$.

Case 2: $m = 5, 7, 10, 16, 23$. By Algorithm 3.6 (1), we see that $[J_{G/H}]^{fl}$ is invertible and hence T is retract k-rational for $m = 5, 7, 16, 23$.

For $15T5 \simeq A_5$, by Algorithm 3.6 (3), we get $F = [J_{G/H}]^{fl}$ with $\mathrm{rank}_{\mathbb{Z}}(F) = 21$ and $F' = [F]^{fl} = \mathbb{Z}$. This implies that $F = 0$ and hence T is stably k-rational (see Example 3.11).

For $15T7 \simeq D_5 \times S_3, 15T16 \simeq A_5 \times C_3, 15T23 \simeq A_5 \times S_3$, it is enough to prove that $[J_{G/H}]^{fl} = 0$ for $G = 15T23$ because $15T7 \leq 15T23$, $15T16 \leq 15T23$ and Lemma 2.21 (i). By Algorithm 3.6 (3), we obtain that $F = [J_{G/H}]^{fl}$ with $\mathrm{rank}_{\mathbb{Z}}(F) = 27$, $F' = [F]^{fl}$ with $\mathrm{rank}_{\mathbb{Z}}(F') = 8$ and $F'' = [F']^{fl}$ with $F'' = \mathbb{Z}$. This implies that $F = 0$ and hence T is stably k-rational (see Example 3.11).

For $15T10 \simeq S_5$, by Algorithm 3.6 (2), we obtain that $[J_{G/H}]^{fl} \neq 0$ and hence T is not stably k-rational (see Example 3.11).

Case 3: $m = 6, 8, 11, 22, 24, 29$. For $15T6 \simeq C_{15} \rtimes C_4$, $15T8 \simeq F_{20} \times C_3$, it follows from Endo [**End11**, Theorem 3.1] that $[J_{G/H}]^{fl}$ is invertible and $[J_{G/H}]^{fl} \neq 0$. Hence T is not stably but retract k-rational.

For $m = 11, 22, 24, 29$, by Algorithm 3.6 (1), we see that $[J_{G/H}]^{fl}$ is invertible and hence T is retract k-rational. By Lemma 2.21 (i) and the inclusions $15T6 \leq 15T11, 15T22, 15T29$ and $15T8 \leq 15T24$, we obtain that $[J_{G/H}]^{fl} \neq 0$ and hence T is not stably k-rational for $m = 11, 22, 24, 29$.

Case 4: $9 \leq m \leq 104$ and $m \neq 10, 11, 16, 22, 23, 24, 29$.

By using the command

```
List([1..104],x->Filtered([1..x],y->IsSubgroup(TransitiveGroup(15,x),
TransitiveGroup(15,y))))
```

in GAP [**GAP**] (see also Example 3.10 for the case where $n = 14$), we obtain the inclusions $15Tm \leq 15Tm'$ among the groups $G = 15Tm$ with minimal groups $15Tm$ where $m \in I_{15} := \{9, 15, 20, 26\}$.

By the inclusions of groups $G = 15Tm$ above and Lemma 2.21 (ii), it is enough to show that $[J_{G/H}]^{fl}$ is not invertible for $m \in I_{15}$. By Algorithm 3.6 (1), we obtain that $[J_{G/H}]^{fl}$ is not invertible and hence T is not retract k-rational for $m \in I_{15}$ (see Example 3.11). □

We give GAP [**GAP**] computations in the proof of Theorem 3.7 for $n = 10, 12, 14, 15$ in Example 3.8 to Example 3.11 (see [**HY17**, Chapter 5] for the explanation of the functions). Some related programs are available from

`https://www.math.kyoto-u.ac.jp/~yamasaki/Algorithm/RatProbNorm1Tori/.`

Example 3.8 (Computations for $10T11 \leq S_{10}$).

```
gap> Read("FlabbyResolutionFromBase.gap");

gap> J:=Norm1TorusJ(10,11);
<matrix group with 3 generators>
gap> StructureDescription(J);
"C2 x A5"
gap> IsInvertibleF(J); # 10T11 is retract k-rational
true
gap> T:=TransitiveGroup(10,11);
A(5)[x]2
gap> F:=FlabbyResolutionLowRankFromGroup(J,T).actionF;
<matrix group with 3 generators>
gap> Rank(F.1); # F is of rank 31
31
gap> F2:=FlabbyResolutionLowRankFromGroup(F,T).actionF;
<matrix group with 3 generators>
gap> Rank(F2.1); # [F]^fl is of rank 13
13
gap> F3:=FlabbyResolutionLowRankFromGroup(F2,T).actionF;
# 10T11 is stably k-rational because [F]^fl=0
Group([ [ [ 1 ] ], [ [ 1 ] ], [ [ 1 ] ] ])
```

Example 3.9 (Computations for $12Tm \leq S_{12}$).

```
gap> Read("FlabbyResolutionFromBase.gap");

gap> J:=Norm1TorusJ(12,11);
<matrix group with 3 generators>
gap> StructureDescription(J);
```

```
"C4 x S3"
gap> IsInvertibleF(J); # 12T11 is retract k-rational
true
gap> T:=TransitiveGroup(12,11);
S(3)[x]C(4)
gap> F:=FlabbyResolutionLowRankFromGroup(J,T).actionF;
<matrix group with 3 generators>
gap> Rank(F.1); # F is of rank 17
17
gap> F2:=FlabbyResolutionLowRankFromGroup(F,T).actionF;
<matrix group with 3 generators>
gap> Rank(F2.1); # [F]^fl is of rank 4
4
gap> GeneratorsOfGroup(F2);
[ [ [ 1, 0, 0, 0 ], [ 0, 1, 0, 0 ], [ 0, 0, 1, 0 ], [ 0, 0, 0, 1 ] ],
  [ [ 1, 0, 0, 0 ], [ 0, 1, 0, 0 ], [ 0, 0, 1, 0 ], [ 0, 0, 0, 1 ] ],
  [ [ 0, 1, 1, 2 ], [ 0, 1, 0, 0 ], [ 3, -3, -2, -6 ], [ -1, 1, 1, 3 ] ] ]
gap> F3:=FlabbyResolutionLowRankFromGroup(F2,T).actionF;
# 12T11 is stably k-rational because [F]^fl is permutation
[  ]

gap> IsInvertibleF(Norm1TorusJ(12,2)); # 12T2 is not retract k-rational
false
gap> IsInvertibleF(Norm1TorusJ(12,3)); # 12T3 is not retract k-rational
false
gap> IsInvertibleF(Norm1TorusJ(12,4)); # 12T4 is not retract k-rational
false
gap> IsInvertibleF(Norm1TorusJ(12,7)); # 12T7 is not retract k-rational
false
gap> IsInvertibleF(Norm1TorusJ(12,8)); # 12T8 is not retract k-rational
false
gap> IsInvertibleF(Norm1TorusJ(12,9)); # 12T9 is not retract k-rational
false
gap> IsInvertibleF(Norm1TorusJ(12,12)); # 12T12 is not retract k-rational
false
gap> IsInvertibleF(Norm1TorusJ(12,15)); # 12T15 is not retract k-rational
false
```

```
gap> IsInvertibleF(Norm1TorusJ(12,16)); # 12T16 is not retract k-rational
false
gap> IsInvertibleF(Norm1TorusJ(12,17)); # 12T17 is not retract k-rational
false
gap> IsInvertibleF(Norm1TorusJ(12,19)); # 12T19 is not retract k-rational
false
gap> IsInvertibleF(Norm1TorusJ(12,29)); # 12T29 is not retract k-rational
false
gap> IsInvertibleF(Norm1TorusJ(12,30)); # 12T30 is not retract k-rational
false
gap> IsInvertibleF(Norm1TorusJ(12,31)); # 12T31 is not retract k-rational
false
gap> IsInvertibleF(Norm1TorusJ(12,32)); # 12T32 is not retract k-rational
false
gap> IsInvertibleF(Norm1TorusJ(12,33)); # 12T33 is not retract k-rational
false
gap> IsInvertibleF(Norm1TorusJ(12,34)); # 12T34 is not retract k-rational
false
gap> IsInvertibleF(Norm1TorusJ(12,36)); # 12T36 is not retract k-rational
false
gap> IsInvertibleF(Norm1TorusJ(12,40)); # 12T40 is not retract k-rational
false
gap> IsInvertibleF(Norm1TorusJ(12,41)); # 12T41 is not retract k-rational
false
gap> IsInvertibleF(Norm1TorusJ(12,46)); # 12T46 is not retract k-rational
false
gap> IsInvertibleF(Norm1TorusJ(12,47)); # 12T47 is not retract k-rational
false
gap> IsInvertibleF(Norm1TorusJ(12,57)); # 12T57 is not retract k-rational
false
gap> IsInvertibleF(Norm1TorusJ(12,58)); # 12T58 is not retract k-rational
false
gap> IsInvertibleF(Norm1TorusJ(12,59)); # 12T59 is not retract k-rational
false
gap> IsInvertibleF(Norm1TorusJ(12,60)); # 12T60 is not retract k-rational
false
gap> IsInvertibleF(Norm1TorusJ(12,61)); # 12T61 is not retract k-rational
```

```
false
gap> IsInvertibleF(Norm1TorusJ(12,63)); # 12T63 is not retract k-rational
false
gap> IsInvertibleF(Norm1TorusJ(12,64)); # 12T64 is not retract k-rational
false
gap> IsInvertibleF(Norm1TorusJ(12,65)); # 12T65 is not retract k-rational
false
gap> IsInvertibleF(Norm1TorusJ(12,66)); # 12T66 is not retract k-rational
false
gap> IsInvertibleF(Norm1TorusJ(12,68)); # 12T68 is not retract k-rational
false
gap> IsInvertibleF(Norm1TorusJ(12,69)); # 12T69 is not retract k-rational
false
gap> IsInvertibleF(Norm1TorusJ(12,70)); # 12T70 is not retract k-rational
false
gap> IsInvertibleF(Norm1TorusJ(12,73)); # 12T73 is not retract k-rational
false
gap> IsInvertibleF(Norm1TorusJ(12,74)); # 12T74 is not retract k-rational
false
gap> IsInvertibleF(Norm1TorusJ(12,75)); # 12T75 is not retract k-rational
false
gap> IsInvertibleF(Norm1TorusJ(12,76)); # 12T76 is not retract k-rational
false
gap> IsInvertibleF(Norm1TorusJ(12,89)); # 12T89 is not retract k-rational
false
gap> IsInvertibleF(Norm1TorusJ(12,91)); # 12T91 is not retract k-rational
false
gap> IsInvertibleF(Norm1TorusJ(12,93)); # 12T93 is not retract k-rational
false
gap> IsInvertibleF(Norm1TorusJ(12,96)); # 12T96 is not retract k-rational
false
gap> IsInvertibleF(Norm1TorusJ(12,99)); # 12T99 is not retract k-rational
false
gap> IsInvertibleF(Norm1TorusJ(12,100)); # 12T100 is not retract k-rational
false
gap> IsInvertibleF(Norm1TorusJ(12,102)); # 12T102 is not retract k-rational
false
```

```
gap> IsInvertibleF(Norm1TorusJ(12,105)); # 12T105 is not retract k-rational
false
gap> IsInvertibleF(Norm1TorusJ(12,107)); # 12T107 is not retract k-rational
false
gap> IsInvertibleF(Norm1TorusJ(12,160)); # 12T160 is not retract k-rational
false
gap> IsInvertibleF(Norm1TorusJ(12,162)); # 12T162 is not retract k-rational
false
gap> IsInvertibleF(Norm1TorusJ(12,166)); # 12T166 is not retract k-rational
false
gap> IsInvertibleF(Norm1TorusJ(12,171)); # 12T171 is not retract k-rational
false
gap> IsInvertibleF(Norm1TorusJ(12,172)); # 12T172 is not retract k-rational
false
gap> IsInvertibleF(Norm1TorusJ(12,173)); # 12T173 is not retract k-rational
false
gap> IsInvertibleF(Norm1TorusJ(12,179)); # 12T179 is not retract k-rational
false
gap> IsInvertibleF(Norm1TorusJ(12,181)); # 12T181 is not retract k-rational
false
gap> IsInvertibleF(Norm1TorusJ(12,182)); # 12T182 is not retract k-rational
false
gap> IsInvertibleF(Norm1TorusJ(12,183)); # 12T183 is not retract k-rational
false
gap> IsInvertibleF(Norm1TorusJ(12,246)); # 12T246 is not retract k-rational
false

gap> List(Filtered(List(ConjugacyClassesSubgroups(TransitiveGroup(12,207)),
> Representative),x->Length(Orbits(x,[1..12]))=1),Size);
[ 576, 1152 ]
gap> List(Filtered(List(ConjugacyClassesSubgroups(TransitiveGroup(12,212)),
> Representative),x->Length(Orbits(x,[1..12]))=1),Size);
[ 72, 72, 72, 72, 72, 144, 144, 648, 648, 1296 ]
gap> List(Filtered(List(ConjugacyClassesSubgroups(TransitiveGroup(12,216)),
> Representative),x->Length(Orbits(x,[1..12]))=1),Size);
[ 72, 72, 72, 648, 648, 1296 ]
gap> List(Filtered(List(ConjugacyClassesSubgroups(TransitiveGroup(12,254)),
```

```
> Representative),x->Length(Orbits(x,[1..12]))=1),Size);
[ 576, 576, 1152, 1728, 3456 ]
gap> List(Filtered(List(ConjugacyClassesSubgroups(TransitiveGroup(12,272)),
> Representative),x->Length(Orbits(x,[1..12]))=1),Size);
[ 72, 72, 144, 720, 7920 ]
gap> List(Filtered(List(ConjugacyClassesSubgroups(TransitiveGroup(12,278)),
> Representative),x->Length(Orbits(x,[1..12]))=1),Size);
[ 12, 12, 24, 36, 36, 72, 72, 144, 576, 14400 ]
gap> List(Filtered(List(ConjugacyClassesSubgroups(TransitiveGroup(12,295)),
> Representative),x->Length(Orbits(x,[1..12]))=1),Size);
[ 12, 12, 12, 24, 24, 24, 36, 48, 48, 60, 72, 72, 72, 96, 96, 96, 120, 120,
  144, 192, 216, 240, 432, 660, 720, 720, 1440, 7920, 95040 ]
```

Example 3.10 (Computations for $14Tm \leq S_{14}$).

```
gap> Read("FlabbyResolutionFromBase.gap");

gap> List([1..63],x->Filtered([1..x],y->IsSubgroup(TransitiveGroup(14,x),
> TransitiveGroup(14,y))));
[ [ 1 ], [ 2 ], [ 1, 2, 3 ], [ 2, 4 ], [ 1, 5 ], [ 6 ], [ 1, 2, 3, 4, 5, 7 ],
  [ 1, 8 ], [ 1, 6, 9 ], [ 10 ], [ 6, 11 ], [ 12 ], [ 1, 2, 3, 8, 13 ],
  [ 1, 5, 8, 14 ], [ 1, 8, 15 ], [ 16 ], [ 1, 5, 10, 17 ],
  [ 1, 5, 6, 9, 11, 18 ], [ 1, 5, 19 ], [ 1, 2, 3, 8, 12, 13, 20 ], [ 6, 21 ],
  [ 12, 22 ], [ 12, 23 ], [ 1, 2, 3, 4, 5, 7, 8, 13, 14, 24 ],
  [ 1, 2, 3, 8, 13, 15, 25 ], [ 26 ], [ 2, 6, 21, 27 ], [ 6, 21, 28 ],
  [ 1, 6, 9, 21, 29 ], [ 30 ], [ 1, 2, 3, 8, 12, 13, 15, 20, 22, 25, 31 ],
  [ 1, 2, 3, 4, 5, 7, 8, 12, 13, 14, 20, 23, 24, 32 ], [ 6, 11, 33 ],
  [ 6, 10, 11, 34 ], [ 6, 11, 21, 35 ], [ 12, 22, 23, 36 ],
  [ 1, 2, 3, 4, 5, 7, 8, 13, 14, 15, 24, 25, 37 ],
  [ 1, 2, 3, 6, 9, 21, 27, 28, 29, 38 ], [ 30, 39 ],
  [ 2, 4, 6, 11, 21, 27, 35, 40 ], [ 6, 11, 21, 28, 35, 41 ],
  [ 1, 5, 6, 9, 11, 18, 33, 42 ], [ 1, 5, 6, 9, 10, 11, 17, 18, 19, 34, 43 ],
  [ 1, 5, 6, 9, 11, 18, 21, 29, 35, 44 ],
  [ 1, 2, 3, 4, 5, 7, 8, 12, 13, 14, 15, 20, 22, 23, 24, 25, 31, 32, 36, 37, 45
     ], [ 2, 4, 46 ], [ 1, 5, 19, 47 ],
  [ 1, 2, 3, 4, 5, 6, 7, 9, 11, 18, 21, 27, 28, 29, 35, 38, 40, 41, 44, 48 ],
```

```
  [ 1, 2, 3, 4, 5, 7, 19, 46, 47, 49 ], [ 6, 10, 11, 21, 33, 34, 35, 50 ],
  [ 1, 5, 6, 9, 10, 11, 17, 18, 19, 21, 29, 33, 34, 35, 42, 43, 44, 50, 51 ],
  [ 1, 5, 8, 14, 15, 16, 19, 52 ], [ 6, 10, 11, 21, 33, 34, 35, 50, 53 ],
  [ 2, 4, 6, 10, 11, 21, 27, 33, 34, 35, 40, 46, 50, 53, 54 ],
  [ 6, 10, 11, 21, 28, 33, 34, 35, 41, 50, 53, 55 ],
  [ 1, 5, 6, 9, 10, 11, 17, 18, 19, 21, 29, 33, 34, 35, 42, 43, 44, 47, 50, 51,
      53, 56 ],
  [ 1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 17, 18, 19, 21, 27, 28, 29, 33, 34, 35, 38,
      40, 41, 42, 43, 44, 46, 47, 48, 49, 50, 51, 53, 54, 55, 56, 57 ],
  [ 1, 5, 8, 14, 15, 16, 19, 26, 47, 52, 58 ], [ 12, 22, 23, 36, 59 ],
  [ 1, 2, 3, 4, 5, 7, 8, 13, 14, 15, 16, 19, 24, 25, 26, 37, 46, 47, 49, 52,
      58, 60 ],
  [ 1, 2, 3, 4, 5, 7, 8, 12, 13, 14, 15, 16, 19, 20, 22, 23, 24, 25, 26, 31,
      32, 36, 37, 45, 46, 47, 49, 52, 58, 59, 60, 61 ],
  [ 6, 10, 11, 12, 21, 22, 23, 28, 30, 33, 34, 35, 36, 41, 50, 53, 55, 59, 62 ],
  [ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21,
      22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39,
      40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57,
      58, 59, 60, 61, 62, 63 ] ]

gap> IsInvertibleF(Norm1TorusJ(14,4)); # 14T4 is retract k-rational
true
gap> IsInvertibleF(Norm1TorusJ(14,5)); # 14T5 is retract k-rational
true
gap> IsInvertibleF(Norm1TorusJ(14,7)); # 14T7 is retract k-rational
true
gap> IsInvertibleF(Norm1TorusJ(14,16)); # 14T16 is retract k-rational
true
gap> IsInvertibleF(Norm1TorusJ(14,19)); # 14T19 is retract k-rational
true
gap> IsInvertibleF(Norm1TorusJ(14,46)); # 14T46 is retract k-rational
true
gap> IsInvertibleF(Norm1TorusJ(14,47)); # 14T47 is retract k-rational
true
gap> IsInvertibleF(Norm1TorusJ(14,49)); # 14T49 is retract k-rational
true
```

```
gap> PossibilityOfStablyPermutationF(Norm1TorusJ(14,4));
# 14T4 is not stably k-rational by Algorithm 4.1 (2)
[ [ 1, 1, 1, 1, 1, -1, -1, 1, -2 ] ]
gap> PossibilityOfStablyPermutationF(Norm1TorusJ(14,5));
# 14T5 is not stably k-rational by Algorithm 4.1 (2)
[ [ 2, -1, 0, 3, 0, 1, 0, -1, -2 ] ]
gap> PossibilityOfStablyPermutationF(Norm1TorusJ(14,16));
# 14T16 in not stably k-rational by Algorithm 4.1 (2)
[ [ 1, 0, 0, 0, 0, 0, -2, -1, -3, -4, -2, 0, 0, -3, 0, 2, 2, 0, 1, 2, 2, 1, -4, 4 ],
  [ 0, 1, 0, 0, 0, 0, -1, 0, -1, -1, 0, 0, 0, -1, 0, 1, 0, 1, 0, 1, 1, 0, -1, 0 ],
  [ 0, 0, 1, 0, 0, 0, -1, -1, -1, -2, -1, 0, 0, -1, 0, 1, 1, -1, 1, 1, 0, 0, -1, 2 ],
  [ 0, 0, 0, 1, 0, 0, -1, 0, -1, 0, 0, 0, 0, 0, 0, 1, 0, 1, -1, -1, 1, 1, -1, 0 ],
  [ 0, 0, 0, 0, 1, 0, 1, -1, 0, -4, -2, 0, 0, -3, -1, -1, 2, -3, 2, 3, -1, 0, -1, 4 ] ]

gap> J:=Norm1TorusJ(14,3);
<matrix group with 2 generators>
gap> StructureDescription(J);
"D28"
gap> IsInvertibleF(J); # 14T3 is retract k-rational
true
gap> T:=TransitiveGroup(14,3);
D(7)[x]2
gap> F:=FlabbyResolutionLowRankFromGroup(J,T).actionF;
<matrix group with 2 generators>
gap> Rank(F.1); # F is of rank 17
17
gap> F2:=FlabbyResolutionLowRankFromGroup(F,T).actionF;
# 14T3 is stably k-rational because [F]^fl=0
Group([ [ [ 1, 0 ], [ 0, 1 ] ], [ [ 1, 0 ], [ 0, 1 ] ] ])

gap> IsInvertibleF(Norm1TorusJ(14,6)); # 14T6 is not retract k-rational
false
gap> IsInvertibleF(Norm1TorusJ(14,8)); # 14T8 is not retract k-rational
false
gap> IsInvertibleF(Norm1TorusJ(14,10)); # 14T10 is not retract k-rational
false
gap> IsInvertibleF(Norm1TorusJ(14,12)); # 14T12 is not retract k-rational
```

```
false
gap> IsInvertibleF(Norm1TorusJ(14,26)); # 14T26 is not retract k-rational
false
gap> IsInvertibleF(Norm1TorusJ(14,30)); # 14T30 is not retract k-rational
false
```

EXAMPLE 3.11 (Computations for $15Tm \leq S_{15}$).

```
gap> Read("FlabbyResolutionFromBase.gap");

gap> J:=Norm1TorusJ(15,5);
<matrix group with 2 generators>
gap> StructureDescription(J);
"A5"
gap> IsInvertibleF(J); # 15T5 is retract k-rational
true
gap> T:=TransitiveGroup(15,5);
A_5(15)
gap> F:=FlabbyResolutionLowRankFromGroup(J,T).actionF;
<matrix group with 2 generators>
gap> Rank(F.1); # F is of rank 21
21
gap> F2:=FlabbyResolutionLowRankFromGroup(F,T).actionF;
# 15T5 is stably k-rational because [F]^fl=0
Group([ [ [ 1 ] ], [ [ 1 ] ] ])

gap> J:=Norm1TorusJ(15,23);
<matrix group with 3 generators>
gap> StructureDescription(J);
"A5 x S3"
gap> IsInvertibleF(J); # 15T23 is retract k-rational
true
gap> T:=TransitiveGroup(15,23);
A(5)[x]S(3)
gap> F:=FlabbyResolutionLowRankFromGroup(J,T).actionF;
<matrix group with 3 generators>
```

```
gap> Rank(F.1); # F is of rank 27
27
gap> F2:=FlabbyResolutionLowRankFromGroup(F,T).actionF;
<matrix group with 3 generators>
gap> Rank(F2.1); # [F]^fl is of rank 8
8
gap> F3:=FlabbyResolutionLowRankFromGroup(F2,T).actionF;
# 15T23 is stably k-rational because [[F]^fl]^fl=0
Group([ [ [ 1 ] ], [ [ 1 ] ], [ [ 1 ] ] ])

gap> IsInvertibleF(Norm1TorusJ(15,10)); # 15T10 is retract k-rational
true
gap> PossibilityOfStablyPermutationF(Norm1TorusJ(15,10));
# 15T10 is not stably k-rational by Algorithm 4.1 (2)
[ [ 1, 0, 0, 0, 0, 0, 8, 1, -2, 5, -3, 2, 2, 5, 0, -8, -10, -3, 8, -2 ],
  [ 0, 1, 0, 0, 0, -1, -1, 0, 0, -1, 0, 0, 0, 0, 1, 1, 1, 0, -1, 0 ],
  [ 0, 0, 1, 0, 0, 0, -2, 0, 0, -1, 1, 0, -1, -1, 0, 2, 2, 1, -2, 0 ],
  [ 0, 0, 0, 1, 0, 0, 12, 1, -3, 7, -5, 2, 3, 6, 0, -12, -14, -4, 12, -2 ],
  [ 0, 0, 0, 0, 1, 2, -2, 0, 1, -2, 2, -2, -1, -2, -2, 2, 4, 1, -2, 0 ] ]

gap> IsInvertibleF(Norm1TorusJ(15,11)); # 15T11 is retract k-rational
true
gap> IsInvertibleF(Norm1TorusJ(15,22)); # 15T22 is retract k-rational
true
gap> IsInvertibleF(Norm1TorusJ(15,24)); # 15T24 is retract k-rational
true
gap> IsInvertibleF(Norm1TorusJ(15,29)); # 15T29 is retract k-rational
true

gap> IsInvertibleF(Norm1TorusJ(15,9)); # 15T9 is not retract k-rational
false
gap> IsInvertibleF(Norm1TorusJ(15,15)); # 15T15 is not retract k-rational
false
gap> IsInvertibleF(Norm1TorusJ(15,20)); # 15T20 is not retract k-rational
false
gap> IsInvertibleF(Norm1TorusJ(15,26)); # 15T26 is not retract k-rational
false
```

3. Case $\mathrm{PSL}_2(\mathbb{F}_q) \leq G \leq \mathrm{P\Gamma L}_2(\mathbb{F}_q)$

In this section, we give the following result for $\mathrm{PSL}_2(\mathbb{F}_q) \leq G \leq \mathrm{P\Gamma L}_2(\mathbb{F}_q)$:

THEOREM 3.12. *Let K/k be a separable field extension of degree n and L/k be the Galois closure of K/k. Let $G = \mathrm{Gal}(L/k)$ be a transitive subgroup of S_n and $H = \mathrm{Gal}(L/K)$ with $[G:H] = n$. Assume that $n = q+1$ where $q = l^e \equiv 1 \pmod 4$ is an odd prime power and $\mathrm{PSL}_2(\mathbb{F}_q) \leq G \leq \mathrm{P\Gamma L}_2(\mathbb{F}_q) \simeq \mathrm{PGL}_2(\mathbb{F}_q) \rtimes C_e$. Then $R^{(1)}_{K/k}(\mathbb{G}_m)$ is not retract k-rational.*

PROOF. We may assume that H is the stabilizer of one of the letters in G (see the second paragraph of Section 2).

Step 1. It is enough to show that $F = [J_{G/H}]^{fl}$ is not invertible for $G = \mathrm{PSL}_2(\mathbb{F}_q)$ because $\mathrm{PSL}_2(\mathbb{F}_q) \leq G \leq \mathrm{P\Gamma L}_2(\mathbb{F}_q)$ and Lemma 2.21 (ii). The group $G = \mathrm{PSL}_2(\mathbb{F}_q)$ acts on $\mathbb{P}^1(\mathbb{F}_q) = \mathbb{F}_q \cup \{\infty\}$ via linear fractional transformation. Let $\mathbb{F}_q^\times = \langle u \rangle$. Then $\mathbb{P}^1(\mathbb{F}_q) = \mathbb{F}_q^\times \cup \{0\} \cup \{\infty\}$ and $\mathbb{F}_q^\times = \{1, -1, \sqrt{-1}, -\sqrt{-1}, u^i, -u^i, u^{-i}, -u^{-i} \mid 1 \leq i \leq \frac{q-5}{4}\}$ because $q \equiv 1 \pmod 4$.

Step 2. Take a subgroup $V_4 = \langle \sigma, \tau \rangle \simeq C_2 \times C_2 \leq G = \mathrm{PSL}_2(\mathbb{F}_q)$ as

$$\sigma = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}, \ \tau = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The action of $V_4 = \langle \sigma, \tau \rangle$ on $\mathbb{P}^1(\mathbb{F}_q)$ is given as $\sigma : x \mapsto -x$ and $\tau : x \mapsto -1/x$. This action induces the action of V_4 on $J_{G/H}$ given by

$$\sigma : e_1 \leftrightarrow e_{-1},\ e_{\sqrt{-1}} \leftrightarrow e_{-\sqrt{-1}},\ e_{u^i} \leftrightarrow e_{-u^i},\ e_{-u^{-i}} \leftrightarrow e_{u^{-i}},\ e_0 \mapsto e_0,\ e_\infty \mapsto e_\infty,$$

$$\tau : e_1 \leftrightarrow e_{-1},\ e_{\pm\sqrt{-1}} \mapsto e_{\pm\sqrt{-1}},\ e_{u^i} \leftrightarrow e_{-u^{-i}},\ e_{-u^i} \leftrightarrow e_{u^{-i}},\ e_0 \leftrightarrow e_\infty,$$

$$\sigma\tau : e_{\pm 1} \mapsto e_{\pm 1},\ e_{\sqrt{-1}} \leftrightarrow e_{-\sqrt{-1}},\ e_{u^i} \leftrightarrow e_{u^{-i}},\ e_{-u^i} \leftrightarrow e_{-u^{-i}},\ e_0 \leftrightarrow e_\infty$$

where $B = \{e_1, e_{-1}, e_{\sqrt{-1}}, e_{-\sqrt{-1}}, e_{u^i}, e_{-u^i}, e_{u^{-i}}, e_{-u^{-i}}, e_0 \mid 1 \leq i \leq \frac{q-5}{4}\}$ is a $\mathbb{Z}$-basis of $J_{G/H}$ and

$$e_\infty := -\sum_{j \in \mathbb{F}_q} e_j.$$

By Lemma 2.21 (ii), we should show that $[M]^{fl}$ is not invertible where $M = J_{G/H}|_{V_4}$ is a V_4-lattice with $\mathrm{rank}_{\mathbb{Z}}(M) = q = n-1$.

Step 3. We will construct a coflabby resolution $0 \to F^\circ \to P^\circ \to M^\circ \to 0$ where P° is permutation V_4-lattice and F° is coflabby V_4-lattice with $\mathrm{rank}_{\mathbb{Z}}(F^\circ) = 5$.

Step 3-1. The actions of σ and τ on M are represented as matrices

$$
\left(\begin{array}{cccc|cccc|ccc|c}
0 & 1 & & & & & & & & & & \\
1 & 0 & & & & & & & & & & \\
 & & 0 & 1 & & & & & & & & \\
 & & 1 & 0 & & & & & & & & \\ \hline
 & & & & 0 & 1 & 0 & 0 & & & & \\
 & & & & 1 & 0 & 0 & 0 & & & & \\
 & & & & 0 & 0 & 0 & 1 & & & & \\
 & & & & 0 & 0 & 1 & 0 & & & & \\ \hline
 & & & & & & & & \ddots & & & \\
 & & & & & & & & & \ddots & & \\
 & & & & & & & & & & \ddots & \\ \hline
 & & & & & & & & & & & 1
\end{array}\right),
$$

$$
\left(\begin{array}{cccc|cccc|ccc|c}
0 & 1 & & & & & & & & & & \\
1 & 0 & & & & & & & & & & \\
 & & 1 & 0 & & & & & & & & \\
 & & 0 & 1 & & & & & & & & \\ \hline
 & & & & 0 & 0 & 0 & 1 & & & & \\
 & & & & 0 & 0 & 1 & 0 & & & & \\
 & & & & 0 & 1 & 0 & 0 & & & & \\
 & & & & 1 & 0 & 0 & 0 & & & & \\ \hline
 & & & & & & & & \ddots & & & \\
 & & & & & & & & & \ddots & & \\
 & & & & & & & & & & \ddots & \\ \hline
-1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 & \cdots & -1 & -1
\end{array}\right).
$$

Let $B^* = \{e_1^*, e_{-1}^*, e_{\sqrt{-1}}^*, e_{-\sqrt{-1}}^*, e_{u^i}^*, e_{-u^i}^*, e_{u^{-i}}^*, e_{-u^{-i}}^*, e_0^* \mid 1 \leq i \leq \frac{q-5}{4}\}$ be the dual basis of B. By the definition, B^* is a $\mathbb{Z}$-basis of the G-lattice $I_{G/H} = (J_{G/H})^\circ$. The action of $V_4 = \langle \sigma, \tau \rangle$ on M° is given by

$$\sigma : e_1^* \leftrightarrow e_{-1}^*,\ e_{\sqrt{-1}}^* \leftrightarrow e_{-\sqrt{-1}}^*,\ e_{u^i}^* \leftrightarrow e_{-u^i}^*,\ e_{-u^{-i}}^* \leftrightarrow e_{u^{-i}}^*,\ e_0^* \mapsto e_0^*,$$

$$\tau : e_1^* \leftrightarrow e_{-1}^* - e_0^*,\ e_{\pm\sqrt{-1}}^* \leftrightarrow e_{\pm\sqrt{-1}}^* - e_0^*,\ e_{u^i}^* \leftrightarrow e_{-u^{-i}}^* - e_0^*,\ e_{-u^i}^* \leftrightarrow e_{u^{-i}}^* - e_0^*,\ e_0^* \mapsto -e_0^*,$$

$$\sigma\tau : e_{\pm 1}^* \leftrightarrow e_{\pm 1}^* - e_0^*,\ e_{\pm\sqrt{-1}}^* \leftrightarrow e_{\mp\sqrt{-1}}^* - e_0^*,\ e_{u^i}^* \leftrightarrow e_{u^{-i}}^* - e_0^*,\ e_{-u^i}^* \leftrightarrow e_{-u^{-i}}^* - e_0^*,\ e_0^* \mapsto -e_0^*$$

(this action corresponds to the transposed matrices of the above matrices).

We define the permutation V_4-lattice P° of $\mathrm{rank}_{\mathbb{Z}}(P^\circ) = q+5 = n+4$ with $\mathbb{Z}$-basis

$$v_1 := v(e_1^*),\ v_2 := v(e_{-1}^*),\ v_3 := v(e_1^* - e_0^*), v_4 := v(e_{-1}^* - e_0^*),\ v_5 := v(e_1^* + e_{-1}^* - e_0^*),$$
$$v_6 := v(e_{\sqrt{-1}}^*), v_7 := v(e_{-\sqrt{-1}}^*), v_8 := v(e_{\sqrt{-1}}^* - e_0^*),\ v_9 := v(e_{-\sqrt{-1}}^* - e_0^*),$$
$$v_{10} := v(e_{\sqrt{-1}}^* + e_{-\sqrt{-1}}^* - e_0^*),$$
$$v_{i,1} := v(e_{u^i}^*), v_{i,2} := v(e_{-u^i}^*), v_{i,3} := v(e_{u^{-i}}^* - e_0^*), v_{i,4} := v(e_{-u^{-i}}^* - e_0^*) \quad (1 \leq i \leq \tfrac{q-5}{4})$$

where V_4 acts on P° by $g(v(m^*)) = v(g(m^*))$ $(m^* \in M^\circ, g \in V_4)$:

$$\sigma : v_1 \leftrightarrow v_2,\ v_3 \leftrightarrow v_4,\ v_5 \mapsto v_5,\ v_6 \leftrightarrow v_7,\ v_8 \leftrightarrow v_9,\ v_{10} \mapsto v_{10},\ v_{i,1} \leftrightarrow v_{i,2},\ v_{i,3} \leftrightarrow v_{i,4},$$
$$\tau : v_1 \leftrightarrow v_4,\ v_2 \leftrightarrow v_3,\ v_5 \mapsto v_5,\ v_6 \leftrightarrow v_9,\ v_7 \leftrightarrow v_8,\ v_{10} \mapsto v_{10},\ v_{i,1} \leftrightarrow v_{i,4},\ v_{i,2} \leftrightarrow v_{i,3},$$
$$\sigma\tau : v_1 \leftrightarrow v_3,\ v_2 \leftrightarrow v_4,\ v_5 \mapsto v_5,\ v_6 \leftrightarrow v_8,\ v_7 \leftrightarrow v_9,\ v_{10} \mapsto v_{10},\ v_{i,1} \leftrightarrow v_{i,3},\ v_{i,2} \leftrightarrow v_{i,4}.$$

Step 3-2. We define a V_4-homomorphism $f : P^\circ \to M^\circ$, $v(m^*) \mapsto m^*$ $(m^* \in M^\circ)$. Then f is surjective. We define a V_4-lattice F° as $F^\circ = \mathrm{Ker}(f)$. Then we obtain an exact sequence $0 \to F^\circ \to P^\circ \to M^\circ \to 0$ with $\mathrm{rank}_{\mathbb{Z}}(F^\circ) = 5$.

Step 3-3. We will check that F° is coflabby. In order to prove this assertion, we should check that $\widetilde{f} = f|_{H^0(W,P^\circ)} : H^0(W, P^\circ) \to H^0(W, M^\circ)$ is surjective (hence $H^1(W, F^\circ) = 0$) for any $W \leq V_4$ where $H^0(W, P^\circ) = \widehat{Z}^0(W, P^\circ) = (P^\circ)^W$ (see also [**HY17**, Chapter 2]).

Step 3-3-1. $W = V_4 = \langle \sigma, \tau \rangle$. By the orbit decomposition of the action of V_4 on P°,

$$\{v_1 + v_2 + v_3 + v_4, v_5, v_6 + v_7 + v_8 + v_9, v_{10}, v_{i,1} + v_{i,2} + v_{i,3} + v_{i,4} \mid 1 \leq i \leq \tfrac{q-5}{4}\}$$

is a $\mathbb{Z}$-basis of $(P^\circ)^{V_4}$. We also see that

$$\{e_1^* + e_{-1}^* - e_0^*, e_{\sqrt{-1}}^* + e_{-\sqrt{-1}}^* - e_0^*, e_{u^i}^* + e_{-u^i}^* + e_{u^{-i}}^* + e_{-u^{-i}}^* - 2e_0^* \mid 1 \leq i \leq \tfrac{q-5}{4}\}$$

is a $\mathbb{Z}$-basis of $(M^\circ)^{V_4}$. Hence $\widetilde{f}$ is surjective because

$$\begin{aligned}\widetilde{f} : v_1 + v_2 + v_3 + v_4 &\mapsto 2(e_1^* + e_{-1}^* - e_0^*),\ v_5 \mapsto e_1^* + e_{-1}^* - e_0^*,\\ v_6 + v_7 + v_8 + v_9 &\mapsto 2(e_{\sqrt{-1}}^* + e_{-\sqrt{-1}}^* - e_0^*),\ v_{10} \mapsto e_{\sqrt{-1}}^* + e_{-\sqrt{-1}}^* - e_0^*,\\ v_{i,1} + v_{i,2} + v_{i,3} + v_{i,4} &\mapsto e_{u^i}^* + e_{-u^i}^* + e_{u^{-i}}^* + e_{-u^{-i}}^* - 2e_0^* \quad (1 \leq i \leq \tfrac{q-5}{4}).\end{aligned}$$

Step 3-3-2. $W = \langle \sigma \rangle$. The set

$$\{v_1 + v_2, v_3 + v_4, v_5, v_6 + v_7, v_8 + v_9, v_{10}, v_{i,1} + v_{i,2}, v_{i,3} + v_{i,4} \mid 1 \leq i \leq \tfrac{q-5}{4}\}$$

becomes a $\mathbb{Z}$-basis of $(P^\circ)^{\langle\sigma\rangle}$ and

$$\{e_1^* + e_{-1}^*, e_{\sqrt{-1}}^* + e_{-\sqrt{-1}}^*, e_{u^i}^* + e_{-u^i}^*, e_{u^{-i}}^* + e_{-u^{-i}}^*, e_0^* \mid 1 \leq i \leq \tfrac{q-5}{4}\}$$

is a $\mathbb{Z}$-basis of $(M^\circ)^{\langle\sigma\rangle}$. Hence $\widetilde{f}$ is surjective because

$$\widetilde{f}: v_1+v_2\mapsto e_1^*+e_{-1}^*, v_5\mapsto e_1^*+e_{-1}^*-e_0^*,\ v_6+v_7\mapsto e_{\sqrt{-1}}^*+e_{-\sqrt{-1}}^*,$$
$$v_{i,1}+v_{i,2}\mapsto e_{u^i}^*+e_{-u^i}^*,\ v_{i,3}+v_{i,4}\mapsto e_{u^{-i}}^*+e_{-u^{-i}}^*-2e_0^*\quad (1\le i\le \tfrac{q-5}{4}).$$

Step 3-3-3. $W=\langle\tau\rangle$. The set

$$\{v_1+v_4, v_2+v_3, v_5, v_6+v_8, v_7+v_9, v_{10}, v_{i,1}+v_{i,4}, v_{i,2}+v_{i,3}\mid 1\le i\le \tfrac{q-5}{4}\}$$

becomes a $\mathbb{Z}$-basis of $(P^\circ)^{\langle\tau\rangle}$ and

$$\{e_1^*+e_{-1}^*-e_0^*, e_{\sqrt{-1}}^*+e_{-\sqrt{-1}}^*-e_0^*, 2e_{-\sqrt{-1}}^*-e_0^*, e_{u^i}^*+e_{-u^{-i}}^*-e_0^*, e_{u^{-i}}^*+e_{-u^i}^*-e_0^*\mid 1\le i\le \tfrac{q-5}{4}\}$$

is a $\mathbb{Z}$-basis of $(M^\circ)^{\langle\tau\rangle}$. Hence $\widetilde{f}$ is surjective because

$$\widetilde{f}: v_5\mapsto e_1^*+e_{-1}^*-e_0^*,\ v_7+v_9\mapsto 2e_{-\sqrt{-1}}^*-e_0^*,\ v_{10}\mapsto e_{\sqrt{-1}}^*+e_{-\sqrt{-1}}^*-e_0^*,$$
$$v_{i,1}+v_{i,4}\mapsto e_{u^i}^*+e_{-u^{-i}}^*-e_0^*,\ v_{i,2}+v_{i,3}\mapsto e_{-u^i}^*+e_{u^{-i}}^*-e_0^*\quad (1\le i\le \tfrac{q-5}{4}).$$

Step 3-3-4. $W=\langle\sigma\tau\rangle$. The set

$$\{v_1+v_3, v_2+v_4, v_5, v_6+v_9, v_7+v_8, v_{10}, v_{i,1}+v_{i,3}, v_{i,2}+v_{i,4}\mid 1\le i\le \tfrac{q-5}{4}\}$$

becomes a $\mathbb{Z}$-basis of $(P^\circ)^{\langle\sigma\tau\rangle}$ and

$$\{e_1^*+e_{-1}^*-e_0^*, 2e_{-1}^*-e_0^*, e_{\sqrt{-1}}^*+e_{-\sqrt{-1}}^*-e_0^*, e_{u^i}^*+e_{u^{-i}}^*-e_0^*, e_{-u^i}^*+e_{-u^{-i}}^*-e_0^*\mid 1\le i\le \tfrac{q-5}{4}\}$$

is a $\mathbb{Z}$-basis of $(M^\circ)^{\langle\sigma\tau\rangle}$. Hence $\widetilde{f}$ is surjective because

$$\widetilde{f}: v_5\mapsto e_1^*+e_{-1}^*-e_0^*,\ v_2+v_4\mapsto 2e_{-1}^*-e_0^*,\ v_{10}\mapsto e_{\sqrt{-1}}^*+e_{-\sqrt{-1}}^*-e_0^*,$$
$$v_{i,1}+v_{i,3}\mapsto e_{u^i}^*+e_{u^{-i}}^*-e_0^*,\ v_{i,2}+v_{i,4}\mapsto e_{-u^i}^*+e_{-u^{-i}}^*-e_0^*\quad (1\le i\le \tfrac{q-5}{4}).$$

Step 4. We will prove that F is not invertible. By Step 3, we have an exact sequence $0\to F^\circ\to P^\circ\to M^\circ\to 0$ where P° is permutation V_4-lattice and F° is coflabby V_4-lattice with $\mathrm{rank}_{\mathbb{Z}}(F^\circ)=5$.

The set $\{w_1,w_2,w_3,w_4,w_5\}$ becomes a $\mathbb{Z}$-basis of F° where

$$w_1=v_1+v_4-v_5,\ w_2=v_2-v_4+v_8+v_9-v_{10},\ w_3=v_3+v_4-v_5-v_8-v_9+v_{10},$$
$$w_4=v_6+v_9-v_{10},\ w_5=v_7+v_8-v_{10}.$$

The actions of σ and τ on F° are given by

$$\sigma: w_1\mapsto w_2+w_3,\ w_2\mapsto w_1-w_3,\ w_3\mapsto w_3,\ w_4\leftrightarrow w_5,$$
$$\tau: w_1\mapsto w_1,\ w_2\mapsto -w_1+w_3+w_4+w_5,\ w_3\mapsto w_1+w_2-w_4-w_5,\ w_4\leftrightarrow w_5$$

and they are represented as matrices

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

By taking the dual, we get the flabby resolution $0 \to M \to P \to F \to 0$ of M and the actions of σ and τ on F are represented as the following matrices (transposed matrices of the above):

$$S = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 \end{pmatrix}.$$

In order to obtain $H^1(V_4, F)$, we should evaluate the elementary divisors of

$$(S - I \mid T - I) = \left(\begin{array}{ccccc|ccccc} -1 & 1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 & -1 & 1 & -1 \end{array}\right)$$

where I is the 5×5 identity matrix. Multiply the regular matrix

$$Q = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \end{pmatrix}$$

from the left, we have

$$Q\,(S - I \mid T - I) = \left(\begin{array}{ccccc|ccccc} 1 & -1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

Hence we conclude that $H^1(V_4, F) = \mathbb{Z}/2\mathbb{Z}$. This implies that F is not invertible. $\square$

4. Case $G \leq S_{2p}$, primitive

As a consequence of Theorem 3.12, we give a classification of stably/retract rational norm one tori $R^{(1)}_{K/k}(\mathbb{G}_m)$ in dimension $n-1$ where $n = 2p$, p is a prime number and $G = \mathrm{Gal}(L/k) \leq S_{2p}$ is primitive.

THEOREM 3.13. *Let p be a prime number, K/k be a separable field extension of degree $2p$ and L/k be the Galois closure of K/k. Assume that $G = \mathrm{Gal}(L/k)$ is a primitive subgroup of S_{2p} and $H = \mathrm{Gal}(L/K)$ with $[G : H] = 2p$. Then $R^{(1)}_{K/k}(\mathbb{G}_m)$ is not retract k-rational.*

More precisely, $R^{(1)}_{K/k}(\mathbb{G}_m)$ is not retract k-rational for the following primitive groups $G \leq S_{2p}$:

(i) *$G = S_{2p}$ or $G = A_{2p} \leq S_{2p}$;*

(ii) *$G = S_5 \leq S_{10}$ or $G = A_5 \leq S_{10}$;*

(iii) *$G = M_{22} \leq S_{22}$ or $G = \mathrm{Aut}(M_{22}) \simeq M_{22} \rtimes C_2 \leq S_{22}$ where M_{22} is the Mathieu group of degree 22;*

(iv) *$\mathrm{PSL}_2(\mathbb{F}_q) \leq G \leq \mathrm{P\Gamma L}_2(\mathbb{F}_q) \simeq \mathrm{PGL}_2(\mathbb{F}_q) \rtimes C_e$ where $2p = q+1$ and $q = l^e$ is an odd prime power.*

PROOF. We may assume that H is the stabilizer of one of the letters in G

(i) follows from Cortella and Kunyavskii [**CK00**, Proposition 0.2] and Endo [**End11**, Theorem 5.2].

(ii) follows from Theorem 3.7 (1) because $S_5 \simeq 10T13$ and $A_5 \simeq 10T7$.

For (iii), it is enough to show that $F = [J_{G/H}]^{fl}$ is not invertible for $G = M_{22} \leq S_{22}$. We see that there exists $G' \leq G$ such that $[J_{G/H}|_{G'}]^{fl}$ is not invertible. Indeed, we can find such G' which is isomorphic to $(C_2)^3$, Q_8, D_4 or $C_4 \times C_2$ (see Example 3.15). Hence it follows from Lemma 2.21 (ii) that F is not invertible. This implies that T is not retract k-rational by Theorem 2.19.

For (iv), we may assume that $p \geq 3$ (if $p = 2$, then $q = 3$ and $\mathrm{PSL}_2(\mathbb{F}_3) \simeq A_4$, $\mathrm{PGL}_2(\mathbb{F}_3) \simeq S_4$, see (i)). Then $q = 2p-1 \equiv 1 \pmod 4$ because p is odd. Hence the assertion follows from Theorem 3.12 as a special case where $n = 2p$ and $q = l^e$. □

5. **Case** $G = M_n \leq S_n$ $(n = 11, 12, 22, 23, 24)$

Finally, we give the following result for the five Mathieu groups $M_n \leq S_n$ where $n = 11, 12, 22, 23, 24$:

Theorem 3.14. *Let K/k be a separable field extension of degree n and L/k be the Galois closure of K/k. Let $G = \mathrm{Gal}(L/k)$ be a transitive subgroup of S_n and $H = \mathrm{Gal}(L/K)$ with $[G : H] = n$. Assume that $n = 11$, 12, 22, 23 or 24 and G is isomorphic to the Mathieu group M_n of degree n. Then $R^{(1)}_{K/k}(\mathbb{G}_m)$ is not retract k-rational.*

Proof. The assertion for $n = 11$ and $n = 23$ follows from [**HY**, Theorem 1.9 (6)]. The assertion for $n = 12$ and $n = 22$ follows from Theorem 3.7 (2)–(ii) and Theorem 3.13 (iii) respectively.

Let $G = M_{24}$ be the Mathieu group of degree 24. Then there exists $G' \leq G \leq S_{24}$ which is transitive and isomorphic to S_4 (see Example 3.15). Then $[J_{G'}]^{fl}$ is not invertible by Endo and Miyata [**EM75**, Theorem 1.5]. It follows from Lemma 2.21 (ii) that $[J_{G/H}]^{fl}$ is not invertible and hence $R^{(1)}_{K/k}(\mathbb{G}_m)$ is not retract k-rational by Theorem 2.19. □

Example 3.15 (Computations for $22T38 \simeq M_{22} \leq S_{22}$ and $M_{24} \leq S_{24}$).

```
gap> Read("FlabbyResolutionFromBase.gap");

gap> JM22:=Norm1TorusJ(22,38);
<matrix group with 2 generators>
gap> StructureDescription(JM22);
"M22"
gap> M22:=TransitiveGroup(22,38);
gap> M22s:=List(ConjugacyClassesSubgroups2(M22),Representative);;
gap> JM22s:=ConjugacyClassesSubgroups2FromGroup(JM22,M22);;
gap> JM22s8:=Filtered(JM22s,x->Size(x)=8);;
gap> Length(JM22s8);
12
gap> JM22s8false:=Filtered(JM22s8,x->IsInvertibleF(x)=false);;
# [J_{G/H}|G']^fl is not invertible
gap> List(JM22s8false,StructureDescription);
[ "C2 x C2 x C2", "Q8", "D8", "C4 x C2" ]
```

```
gap> M24:=PrimitiveGroup(24,1);
M(24)
gap> M24s:=Filtered(List(ConjugacyClassesSubgroups2(M24),Representative),
> x->Length(Orbits(x,[1..24]))=1 and Size(x)=24);;
gap> M24s4:=Filtered(M24s,x->IdGroup(x)=[24,12]);;
gap> List(M24s4,StructureDescription);
[ "S4", "S4", "S4" ]
```

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   Rationality problem for norm one tori in small dimensions,
   Mathematics of Computation **89** (2020) 923–940.