Propagation behavior of spreading geodesic circles in geodesically convex Finsler surfaces

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Abstract

The geodesics are widely applied to studies of the geometrical structure and topological structure of manifolds. There exists a close link between the behavior of geodesics and curvature of manifolds. In general, a universal covering space has been used to study the behavior of geodesics in manifolds. In this way, the geodesic flows of compact Riemannian manifolds with negative curvature have been studied and contributed to the development of the dynamical systems. Moreover, H. Busemann and F. P. Pedersen have studied geodesics in a G-space whose universal covering spaces is straight, i.e., all geodesics are minimal. Their studies are applied to studies of geodesics in a 2-torus. N. Innami has studied the asymptotic behavior of geodesic circles in a 2-torus of revolution. N. Innami and T. Okura have proved for a Riemannian 2-torus T^2: ε-density of geodesic circles with sufficiently large radii.

In this paper, we study the asymptotic behavior of geodesic circles in an orientable finitely connected and geodesically convex Finsler surface M with genus $g \geq 1$. We have a generalization of their study if all geodesics in M are reversible, by using an intrinsic distance function and the Busemann function on its special covering space. In particular, this paper shows the global behavior of geodesics without assumptions on curvature and geodesically completeness of the surface. Furthermore, the absence of those assumptions is different from other previously studies of geodesics. Additionally, most of the proofs do not need the reversibility assumption on geodesics.

Acknowledgments

I would like to express my deepest gratitude to Professor Nobuhiro Innami who gives me helpful comments and suggestions. I had studied under him since an undergraduate student at Niigata University. His useful advices and enormous support are invaluable during the course of my study and, are the reason that I started the study of the geodesics in Finsler manifolds. Without his encouragement, guidance and persistent help, this paper would not have materialized.

Chapter 1

Introduction

In a connected smooth Finsler manifold M, a *geodesic* is by definition a solution of the Euler-Lagrange equation of the length function of piecewise smooth curves. The definitions of a length of curves and an intrinsic distance induced by a Finsler metric on a Finsler manifold are seen in Appendix. A smooth shortest curve is a geodesic when the boundary of M does not exist. Then it is called a *minimal geodesic*. A *geodesic sphere* $S_M(p,t)$ *with center* p *and radius* t is by definition the set of all terminal points of geodesics emanating from p with length t. A *distance sphere* $S^d_M(p,t)$ *with center* p *and radius* t is by definition the set of all terminal points of minimal geodesics emanating from p with length t. Obviously, $S^d_M(p,t) \subset S_M(p,t)$ and, generally, $S^d_M(p,t) \neq S_M(p,t)$. For a unit tangent vector $u \in S_pM$ let γ_u be a geodesic such that $\gamma_u(0) = p$ and $\dot{\gamma_u}(0) = u$. We define a function $t(u)$ on $u \in S_pM$ as

$$t(u) := \sup\{t > 0 \mid d(p, \gamma_u(t)) = L(\gamma_u(t))\}$$

for $u \in S_pM$. If $t(u) > 0$, then a cut point along γ_u of the point p is defined by $\gamma_u(t(u)) = \exp_p(t(u)u)$ where T_pM is the tangent vector space of M at p and $\exp_p : T_pM \to M$ is the exponential map at p. The set of all cut points of $p \in M$ is called the *cut locus* of $p \in M$. If there exist no cut point of $p \in M$, then $S^d_M(p,t) = S_M(p,t)$.

Light has the nature behaving like both a particle and a wave. Mathematically, the geodesics describe the trajectories as the behavior of the particles, and the geodesic spheres describe the behavior of wavefronts which spread according to Huygens' principle. The indicatrix of a Finsler metric describes the shape of an infinitesimal wavefront (cf. (1.17) in [34]).

We recall Huygens' principle in order to understand how geodesic spheres behave in Finsler manifolds as wavefronts. Let $\phi_p(t)$ be a wavefront from a point source p at time t. For every $q \in \phi_p(t)$, $\phi_q(s)$ is called a *wavelet* of the wavefront for a sufficiently small $s > 0$. For every $q \in \phi_p(t)$, we consider the wavelet at time s, $\phi_q(s)$. Then Huygens' principle states that the wavefront of p at time $t+s$, $\phi_p(t+s)$ is the envelope of all wavefronts $\phi_q(s)$ for $q \in \phi_p(t)$ (cf. [11]).

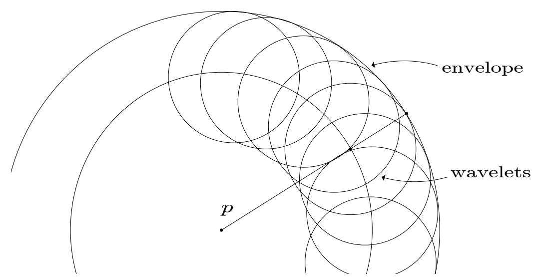

Figure 1.1: The propagation of wavefronts.

The triangle inequality of the intrinsic distance $d(\cdot,\cdot)$ induced by the Finsler metric is almost equivalent to Huygens' principle when we consider the distance spheres as wavefronts. Namely, for any point p_1 in a minimal geodesic from p to q, $S^d_M(p_1, d(p_1, q))$ is inscribed in $S^d_M(p, d(p, q))$ at the point q if and only if for any point q_1 in $S^d_M(p_1, d(p_1, q))$ we have $d(p, q_1) \leq d(p, p_1) + d(p_1, q_1)$ and equality holding $\Leftrightarrow q_1 = q$. Furthermore if there exist geodesics from p to q, then for a point $p_2 \in S^d_M(p, d(p, p_1))$ in another minimal geodesic from p to q, $S^d_M(p_2, d(p_1, q))$ is inscribed in $S^d_M(p, d(p, q))$ at the point q. However, if $p_3 \in S^d_M(p, d(p, p_1))$ is a cut point of p, then for any point q_2 in $S^d_M(p_3, d(p_1, q))$ we have $d(p, q_2) < d(p, p_3) + d(p_3, q_2) = d(p, q)$ and thus $S^d_M(p_3, d(p_1, q))$ does not inscribe in $S^d_M(p, d(p, q))$ (see Figure 1.2). Hence, the distance spheres $S^d_M(p, t)$ do not satisfy Huygens' principle if it contains a cut point of p.

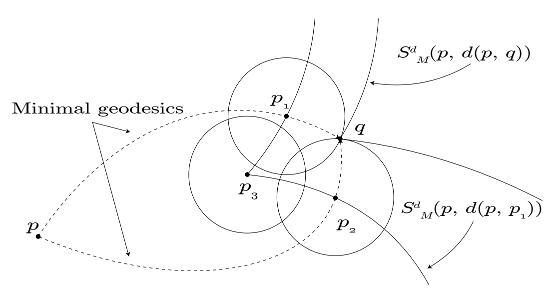

Figure 1.2: The remark of the geodesic circle.

On the other hand, if we impose some condition on Finsler manifolds, then we have Huygens' principle for the distance spheres. In fact, in forward geodesically complete, simply connected Finsler manifolds of non-positive flag curvature without boundary, the distance spheres with center at any point satisfy Huygens' principle, since all geodesics are minimal and Gauss's lemma holds, i.e., geodesics emanating from p intersect the geodesic spheres with center p orthogonally (cf. [4]). Furthermore, as more general settings, we assume the cut locus of p does not exist. Then we have Huygens' principle for the distance spheres with center at any point. Let M be an n-dimensional smooth Finsler manifold and $U \subset M$ a smooth domain. Let A be a compact subset of M and $\rho : M \to \mathbb{R}$ with $\rho(q) := d(A, q)$. We assume that $\rho(U) = [s, r]$ where $0 < s < r$. If $\rho^{-1}(s)$ is the wavefront at time $t = 0$ and there are no cut point in $\rho^{-1}\big([s, r]\big)$. Then, for each $t \in [s, r]$, $\rho^{-1}(t)$ is the wavefront at time $t - s$ and Huygens' principle is satisfied by all the wavefronts $\{\rho^{-1}(t)\}_{t\in[s,r]}$ (cf. [11]). These facts suggest that the geometry of geodesics in a Finsler manifold is useful to study the global behavior of wavefronts.

N. Innami and T. Okura have studied the behavior of geodesics and the asymptotic behavior of geodesic circles in a 2-torus equipped with a Riemannian metric and have proved ε-density of geodesic circles with sufficiently large radii (cf. [23]). The following facts have played a crucial role in the proof:

1. The Jordan curve theorem holds true in the universal covering space N of T^2, since N is topologically a plane $\mathbb{R}^2$.
2. The covering transformation group of a torus is isomorphic to $\mathbb{Z}^2$ where $\mathbb{Z}$ is the set of all integers.

In this paper, we obtain the same properties for surfaces with Finsler metrics and with genus ≥ 1 by making a special covering space in which the Jordan curve theorem is true and on which $\mathbb{Z}^2$ acts, and work in them.

We mention a difference between a shortest curve and a minimal geodesic and prepare definitions to state a generalization of their study. The length of a minimal geodesic from p to q equals the distance from p to q. However, a shortest curve may not be a geodesic if, for example, an interior point of it touches the boundary of M and it is not smooth at the point of contact. We say that M is *geodesically convex* if there exists a minimal geodesic from p to q in M for any points $p, q \in M$. All forward geodesically complete Finsler manifolds without boundary are geodesically convex because of the Hopf-Rinow theorem (cf. [34]).

We say that a geodesic $c : [a,b] \to M$ is *reversible* if the reverse curve $c^{-1} : [a,b] \to M$, $c^{-1}(t) = c(a+b-t)$, is a geodesic as a point set. If F is reversible, i.e., $F(x,y) = F(x,-y)$ for all $y \in T_xM$, then all geodesics are reversible. It is well known that all geodesics are reversible in (M,F) if $F := \alpha+\beta$ is a *Randers metric* where α is the norm induced by a Riemannian metric and β is a closed 1-form with $\|\beta\|_\alpha < 1$ (cf. [14], [20], [34]). Furthermore, all geodesics are reversible in (M,F) if and only if $F := \alpha+\beta+\frac{\beta^2}{\alpha}$ is a *first approximate Matsumoto metric* where α and β are the same as above such that $F_0 := \alpha + \frac{\beta^2}{\alpha}$ is a Finsler metric (cf. [29], [31]). Additionally, $F := \frac{\alpha^2}{\alpha-\beta}$ is called a *Matsumoto metric* (cf. [30], [33]) which is a Finsler metric if $\|\beta\|_\alpha < \frac{1}{2}$ (cf. [5]).

We say that a surface M is *finitely connected* if there exist a compact surface S, with or without boundary, and finitely many points $p_1, \cdots, p_k \in S$ such that M is homeomorphic to $S \setminus \{p_1, \cdots, p_k\}$ (cf. [35], p.41).

The following theorem shows the asymptotic behavior of geodesics without assumptions on curvature and geodesically completeness.

Theorem 1.0.1. *Let (M,F) be an orientable finitely connected and geodesically convex smooth Finsler surface with genus $g \geq 1$. Assume that all geodesics are reversible. Then, for any number $\varepsilon > 0$ and any points $p, q \in M$ there exists a number $R > 0$ such that the geodesic circle $S_M(p,t)$ with center p and radius t meets the ε-ball $B(q,\varepsilon)$ with center q for any $t > R$.*

Here $S_M(p,t) \cap B(q,\varepsilon)$ consists of many subarcs of $S_M(p,t)$ (see Figure 1.3), although we do not count the number of those subarcs (see Corollary 1.0.5 below). In the process of the proof of Theorem 1.0.1, we know the movement of subarcs of $S_M(p,t)$ in $B(q,\varepsilon)$: There exists a geodesic circle $S_M(p,t_0)$ passing through q while a subarc of $S_M(p,t)$ gets into $B(q,\varepsilon)$ and leaves.

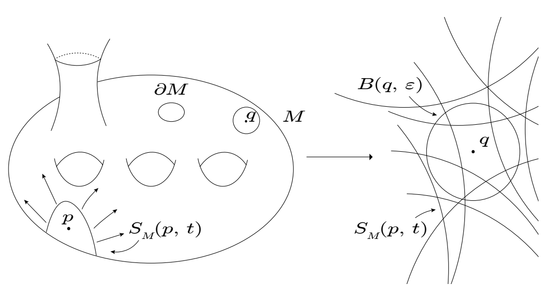

Figure 1.3: The propagation of geodesic circles.

Remark 1.0.2. If a Finsler surface (M, F) is not orientable, then M has an orientable double covering surface $\pi : M_1 \to M$. If a Finsler metric F_1 of M_1 is the pullback of F by π, i.e., $F_1 = \pi^* F$, then the image of any geodesic in (M_1, F_1) by π is a geodesic in (M, F) with same length. Hence, (M_1, F_1) satisfies the assumption of Theorem 1.0.1. Assume that the genus of M_1 is greater than or equal to one. Choosing $p_1, q_1 \in M_1$ such that $\pi(p_1) = p$ and $\pi(q_1) = q$ and applying Theorem 1.0.1 to p_1 and q_1 in M_1, we have the same conclusion for p and q through the projection π.

The following corollary is a direct consequence but the situation may often arise when (M, F) is complete.

Corollary 1.0.3. *Let (M, F) be an orientable Finsler surface such that all geodesics are reversible and $p, q \in M$. Assume that there exists a surface M_1 embedded in M containing p and q and with genus $g \geq 1$ such that M_1 is finitely connected and geodesically convex. Then, for any $\varepsilon > 0$ there exists a number $R > 0$ satisfying the same property in Theorem 1.0.1.*

Let $E : \Omega_{p,q} \to \mathbb{R}$ be the *energy function* on the *path space* $\Omega_{p,q}$ from p to q. The critical points of E are geodesics from p to q in M (cf. [27], [34]).

Corollary 1.0.4. *Let (M, F) be as in Theorem 1.0.1. For any number $\varepsilon > 0$ and any points $p, q \in M$ there exists a number $R > 0$ such that the set of critical values of E is ε-dense in $[R, \infty)$.*

To clarify the role of the geodesic reversibility assumption, we study geodesics and geodesic circles under more general settings. *Hereafter, let (M, F) be an orientable finitely connected and geodesically convex smooth Finsler surface with genus $g \geq 1$.* From the assumption on the genus of M, there exist g simple closed curves $c_1, \cdots, c_g$ in M such that they are disjoint and $M \setminus \cup_{i=1}^{g} c_i$ is connected. *We assume that $g-1$ curves of them, say $c_1, \cdots, c_{g-1}$, are reversible geodesics.* This assumption is automatically satisfied if all geodesics are reversible in the Finsler surface. We study geodesics in $M \setminus \cup_{i=1}^{g-1} c_i$ and the asymptotic behavior of geodesic circles. Namely, we develop geometry of geodesics in M, using no geodesic which intersects $c_1, \cdots, c_{g-1}$. The set $M \setminus \cup_{i=1}^{g-1} c_i$ and its covering space N are geodesically convex when $c_1, \cdots, c_{g-1}$ are reversible geodesics. In fact, if the distance $d_1(p, q)$ from a point p to a point q in $M \setminus \cup_{i=1}^{g-1} c_i$ is defined as the infimum of the lengths of all piecewise smooth curves from p to q in $M \setminus \cup_{i=1}^{g-1} c_i$, then $M \setminus \cup_{i=1}^{g-1} c_i$ is geodesically convex with respect to d_1. The distances of M and $M \setminus \cup_{i=1}^{g-1} c_i$ induced by F are different but a geodesic in $M \setminus \cup_{i=1}^{g-1} c_i$ remains a geodesic in M. We make use of those geodesics which do not intersect $c_1, \cdots, c_{g-1}$ in M to obtain the properties mentioned in Theorem 1.0.1.

The following corollary is a rough estimate of the number of critical points and sufficiently large critical values of E. In the following corollary we use a phrase *'a pencil of geodesics'* which is a set of geodesics converging to or narrowly diverging from a point.

Corollary 1.0.5. *Let (M, F) be as mentioned above. Let n be any positive integer. For any number $\varepsilon > 0$ and any points $p, q \in M \setminus \cup_{i=1}^{g-1} c_i$ there exists a number $R > 0$ such that at least n pencils of geodesics emanating from p with length t intersect the ε-ball with center q for any $t > R$ where the sequences of the lifts of these n pencils of geodesics into N converge to rays with different slopes as $t \to \infty$.*

Here, the covering space N of $M \setminus \cup_{i=1}^{g-1} c_i$ is defined in Section 2.4 and the notion of slopes for rays is defined in Section 4.2. We work in the covering space N where the covering transformation group Φ is isomorphic to $\mathbb{Z}^2$. Such a covering space can be constructed because $M \setminus \cup_{i=1}^{g-1} c_i$ is considered to be a subset in a 2-torus. Hence we find and use many analogous results on the behavior of geodesics on 2-tori. Working in N, we prove Theorem 2.4.4 which is sufficient for Theorem 1.0.1.

The geodesics on 2-tori of revolution embedded in the Euclid space $\mathbb{E}^3$ have been studied by G. A. Bliss [7] and B. F. Kimball [25]. Recently, J. Gravesen, S. Markvorsen, R. Sinclair and M. Tanaka [13] have studied the cut locus in a 2-torus of revolution. N. Innami [18] has studied geodesics in a 2-torus having poles. H. M. Morse [28] and G. A. Hedlund [15] studied the geodesics on arbitrary Riemannian tori whose lifts into the universal covering space are straight lines. H. Busemann and F. P. Pedersen [9] have determined how the straight lines behave in the universal covering planes of 2-tori with one-parameter groups of motions. Their methods are unified by V. Bangert [3] with those of J. N. Mather [26] and S. Aubry and P. Y. Le Daeron [2] to study a monotone twist map of the annulus and the discrete Frenkel-Kontrova model (cf. [24]). The method of finding straight lines by displacement functions can be applied in more general situations. Indeed, in [3], we can see the complete classification of straight lines in the universal covering plane of an arbitrary 2-torus, as an application. Recently, J. P. Schröder [32] has generalized those results for non-symmetric distance cases. We modify the methods in [9] to have analogous results for studying the asymptotic behavior of geodesic circles. In the light of the classification of straight lines, we can study the limit circles which are the level sets of Busemann functions.

Let $G^t : SX \to SX$ be the geodesic flow of a unit tangent bundle SX of a complete Finsler manifold X without boundary. It follows from Poincaré's recurrence theorem that for almost all $y \in SX$ there exists a sequence of numbers t_n such that $t_n \to \infty$ and $G^{t_n}(y) \to y$ as $n \to \infty$ if the volume of X is finite. We can estimate the averages of the return time for almost all $y \in SX$ by using Birkhoff ergodic theorem (cf. [1]). In comparison with these results, Theorem 1.0.1 states that some terminal points of geodesics emanating from p and with length $t > R$ always exist near q. An event occurs at a point p, its influence spreads according to Huygens' principle, and after the time R, at the point q, it is affected every time less than ε.

We say that G^t is *topologically mixing* if for any two open sets U and V of the unit tangent bundle SX there exists a number $R > 0$ such that $G^t(U) \cap V \neq \emptyset$ for all t with $|t| > R$. P. Eberlein [12] has proved that the geodesic flow G^t is topologically mixing on SX if the Riemannian manifold X is a compact visibility manifold of non-positive curvature. We are interested in existence of wavefronts more than the directions of trajectories, so it is important to study the asymptotic behavior of geodesic spheres related to the property of topological mixing in the underlying manifold, since the geodesic circles spread according to Huygens' principle.

We say that the geodesic flow G^t is *topologically sub-mixing* if for any open sets U and V of X there exists a number $R > 0$ such that $\rho(G^t(S_qX)) = \exp_q(tS_qX)$ intersect V for some point $q \in U$ and for all $t > R$, i.e., $G^t(\rho^{-1}(U)) \cap \rho^{-1}(V) \neq \emptyset$, where $\rho : SX \to X$ is the natural projection and $\exp_q : T_qX \to X$ is the exponential map at q. Here we note that $S_X(q,t) = \exp_q(tS_qX)$ is the geodesic sphere with center q and radius t. The geodesic flow of a flat n-torus, $n \geq 2$, is topologically sub-mixing, but not mixing. W. Sierpinski (in 1906) (cf. [16]) has estimated the asymptotic difference between the area πt^2 of the circle $S(t)$ with radius t and the number $N(t)$ of lattice points contained in $S(t)$ in the Euclidean plane, proving that $|\pi t^2 - N(t)| \leq O(t^{2/3})$, which means that $N(t+\varepsilon) - N(t) = \pi(t+\varepsilon)^2 - \pi t^2 + O(t^{2/3}) = 2\pi\varepsilon t + O(t^{2/3}) \to \infty$ as $t \to \infty$. We find the similar estimate for a flat n-torus T^n in [10] where the error term is $O(t^\alpha)$, $0 \leq \alpha < n-1$. These properties prove the topological sub-mixing property of T^n. In [19], N. Innami have investigated the asymptotic behavior of geodesic circles in a 2-torus of revolution and have proved that the geodesic flow of a 2-torus of revolution is topologically sub-mixing. In [23], N. Innami and T. Okura have proved the geodesic flow of any 2-torus is topologically sub-mixing. Theorem 1.0.1 states that the sub-mixing property of geodesic flow is true for much wider class of surfaces.

Chapter 2

A surface cut along simple closed geodesics

2.1 Cutting and opening M along simple closed geodesics

We recall that M is an orientable finitely connected and geodesically convex smooth Finsler surface with genus $g \geq 1$. From the assumption on the genus of M, there exist g simple closed curves $c_1, \cdots, c_g$ in M such that they are disjoint and $M \setminus \cup_{i=1}^{g} c_i$ is connected. *We assumed that $g-1$ curves of them, say $c_1, \cdots, c_{g-1}$, are reversible geodesics.* This assumption is automatically satisfied if all geodesics are reversible in the Finsler surface.

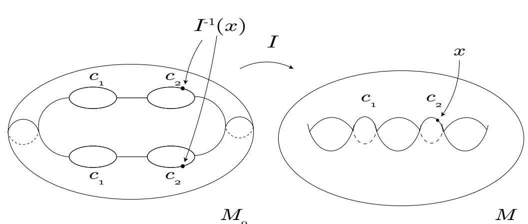

Figure 2.1: The case of M is compact with genus $g = 3$.

We construct an orientable finitely connected Finsler surface (M_0, F_0) with boundary and with genus one (see Figure 2.1) satisfying the following properties : There exists a map $I : M_0 \to M$ such that

1. the interior $\mathrm{Int}(M_0)$ of M_0 is isometric to $M \setminus \cup_{i=1}^{g-1} c_i$, i.e., the restriction $I : \mathrm{Int}(M_0) \to M \setminus \cup_{i=1}^{g-1} c_i$ is a diffeomorphism such that $F_0(x, y) = F(I(x), dI_x(y))$ for all $x \in \mathrm{Int}(M_0)$ and $y \in T_x M_0$,

2. for any point $x \in \cup_{i=1}^{g-1} c_i$, $I^{-1}(x)$ consists of exactly two points in the boundary ∂M_0 of M_0. When those two points are identified, the quotient space M_0/I is naturally thought as M by the quotient map $M_0/I \to M$.

The boundary ∂M_0 of M_0 consists of $I^{-1}(\cup_{i=1}^{g-1} c_i)$ and $I^{-1}(\partial M)$ (the original boundary of M) as surfaces. Hence, ∂M_0 has at least $2(g-1)$ connected components. The set $\{p_1, \cdots, p_k\}$ is contained in the boundary ∂M of $M = S \setminus \{p_1, \cdots, p_k\}$ as a topological sub-space in a certain compact surface S, but we think no point in ∂M_0 is sent to $p_1, \cdots, p_k$ by I.

The finitely connected Finsler surface M_0 with genus one defined as above is geodesically convex, since M is geodesically convex and all c_i, $i = 1, \cdots, g-1$, are reversible geodesics. The boundary of M_0 is not empty if $g > 1$.

Notice that c is a geodesic in the *interior* $\mathrm{Int}(M_0)$ of M_0 if and only if $I(c)$ is a geodesic in $M \setminus \cup_{i=1}^{g-1} c_i$. It should be noted that there exist no geodesic touching ∂M_0 any place other than its endpoints because all curves c_i and reverse curves ${c_i}^{-1}$ are geodesics for $i = 1, \cdots, g-1$ and the geodesic is uniquely determined from the initial condition. Namely, any geodesic c whose end points are in $\mathrm{Int}(M_0)$ satisfies that $I(c) \subset M \setminus \cup_{i=1}^{g-1} c_i$.

We define a distance $d_0(p, q)$ for $p, q \in M_0$ as usual; $d_0(p, q)$ is the infimum of the lengths of piecewise smooth curves from p to q in M_0. Then a shortest curve from p to q in $\mathrm{Int}(M_0)$ is a minimal geodesic c in $\mathrm{Int}(M_0)$, i.e., $d_0(p, q)$ is the length of c.

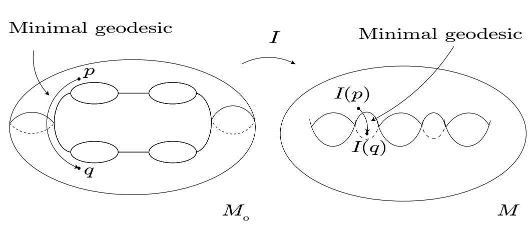

Figure 2.2: Minimal geodesics in M_0 and M.

It is of course that $d_0(p,q) \geq d(I(p), I(q))$ for any $p, q \in M_0$ where $d(\cdot,\cdot)$ is the distance on M induced by F. In fact, if a minimal geodesic γ from $I(p)$ to $I(q)$ in M crosses c_i for some $i = 1, \cdots, g-1$, then the minimal geodesics from p to q in M_0 are longer than γ (see Figure 2.2).

2.2 Another construction of a geodesically convex surface M_0 with genus one

Assume in this section that all geodesics in M are reversible. Since $M \setminus \cup_{i=1}^{g-1} c_i$ is a geodesically convex set, we have $g-1$ simple closed geodesics d_i, $i = 1, \cdots, g-1$, such that

1. $c_i \cap d_i$ is a single point for each $i = 1, \cdots, g-1$,
2. $c_i \cup d_i$, $i = 1, \cdots, g-1$, are mutually disjoint.

Then there exists an open neighborhood U_i of $c_i \cup d_i$ such that $U_i \setminus c_i \cup d_i$ is homeomorphic to an open cylinder $S^1 \times (0,1)$ for each $i = 1, \cdots, g-1$. As was seen before, we make an orientable finitely connected and geodesically convex Finsler surface (M_0, F_0), $M_0 = M \setminus \cup_{i=1}^{g-1} c_i \cup d_i$, with boundary and with genus 1 such that there exists a map $I : M_0 \to M$ satisfying the same property as above. In this construction, the boundary ∂M_0 is the union of $I^{-1}(\cup_{i=1}^{g-1} c_i \cup d_i)$ and $I^{-1}(\partial M)$. Each connected component of $I^{-1}(\cup_{i=1}^{g-1} c_i \cup d_i)$ is a broken geodesic, but not a (smooth) geodesic.

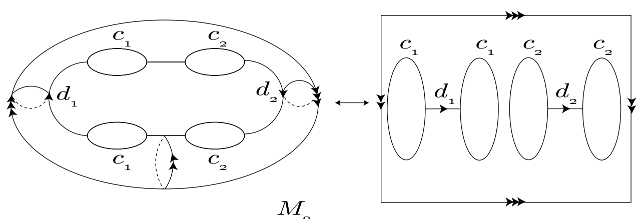

Figure 2.3: Another geodesically convex surface M_0.

2.3 The geodesic circles in M_0

Let M_1 be a finitely connected and geodesically convex Finsler surface. For a point $p \in M_1$ and a unit vector $v \in S_pM_1$ let $\gamma_v : (-a(v), b(v)) \to M_1$ be a unit speed geodesic such that $\dot{\gamma}_v(0) = v$, possibly $a(v) = \infty, b(v) = \infty$. Here the interval $(-a(v), b(v))$ is supposed to be maximal, i.e., there exists no proper extension of γ_v in M_1. If M_1 is complete and without boundary, then $a(v) = \infty$ and $b(v) = \infty$ for all $v \in SM_1$. For a number $t \in (-\infty, \infty)$ let $SM_1(t)$ be the set of all $v \in SM_1$ such that $t \in (-a(v), b(v))$. Then a map $G^t : SM_1(t) \to SM_1$ is defined by

$$G^t(v) := \dot{\gamma}_v(t)$$

for any $v \in SM_1(t)$. Let $\pi : SM_1 \to M_1$ be a natural projection of the unit tangent bundle of M_1. Then $S_{M_1}(p,t) := \pi(G^t(S_pM_1 \cap SM_1(t))) = \{\gamma_v(t) \,|\, v \in S_pM_1(t)\}$ is called a *geodesic circle* with center p and radius t in M_1 for any point p and any $t > 0$. If M_1 is complete and without boundary, then $SM_1(t) = SM_1$ for all $t \in (-\infty, \infty)$ and G^t is called the *geodesic flow* on SM_1 (cf. [1]). *Hereafter let M_0 denotes a surface constructed in Section 2.1 from M.*

Remark 2.3.1. We emphasize that

$$I(S_{M_0}(p,t)) \subset S_M(I(p), t)$$

for all $t > 0$ and any $p \in M_0$. On the other hand, this inclusion relation is not true for the distance spheres. That is

$$I(S^d_{M_0}(p,t)) \not\subset S^d_M(I(p), t),$$

in general. Here $S^d_{M_0}(p,t) := \{q \in M_0 \,|\, d_{M_0}(p,q) = t\}$.

2.4 A covering surface and its transformation group

We make a surface S_0 from S ($\supset M$) in the same way as M_0 from M. Then we think $M_0 = S_0 \setminus \{p_1, \cdots, p_k\}$. Let k' be the number of the connected components of the boundary ∂M of M.

Remark 2.4.1. Recall that the genus of M_0 is one, ∂M_0 has $2(g-1) + k'$ connected components and k points $p_1, \cdots, p_k$ are removed.

If $2(g-1)+k'$ disks K_i, $i=1,\cdots,2(g-1)+k'$, are glued along the boundary ∂M_0 and k points p_j, $j=1,\cdots,k$, are plugged up at the original location in S_0, then this operation turns S_0 into a 2-torus topologically. Hence its universal covering surface is topologically a plane $\mathbb{R}^2$ and the covering transformation group Φ is isomorphic to $\mathbb{Z}^2$. Moreover, Φ is properly discontinuous, i.e., every point $p \in \mathbb{R}^2$ has a neighborhood U_p such that the intersection $\tau(U_p) \cap U_p$ with its translate under the group action via some element $\tau \in \Phi$ is non-empty only for $id. \in \Phi$. Namely, $\tau(U_p) \cap U_p \neq \emptyset \Rightarrow \tau = id.$. We define a surface N by

$$N := \mathbb{R}^2 \setminus \Phi(\cup_{i=1}^{2(g-1)+k'} \mathrm{Int}(\widetilde{K}_i) \cup \{\tilde{p}_1, \cdots, \tilde{p}_k\})$$

where $\widetilde{K}_i$ (resp., $\tilde{p}_j$) is a lift of K_i (resp., p_j) into $\mathbb{R}^2$ for each $i=1,\cdots,2(g-1)+k'$ (resp., $j=1,\cdots,k$). Then N is a covering surface of M_0 with a natural covering map $\pi : N \to M_0$.

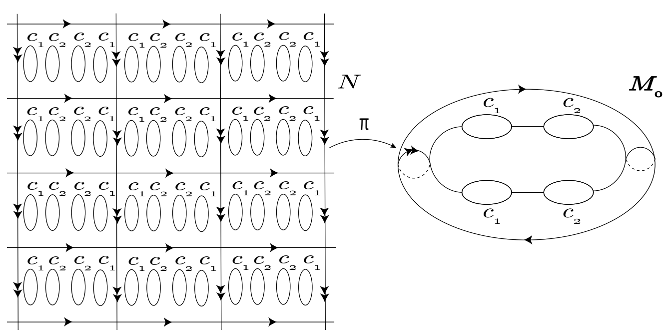

Figure 2.4: The covering space of M_0 with genus 1.

Lemma 2.4.2 (The Jordan curve theorem)**.** *Let C be a simple closed curve in* $\mathrm{Int}(N)$*. Then $N \setminus C$ consists of two connected components.*

Proof. Since $C \subset N \subset \mathbb{R}^2$, the Jordan curve theorem for $\mathbb{R}^2$ implies that $\mathbb{R}^2 \setminus C$ consists of two connected components X_1 and X_2. Then $X_1 \cap N$ and $X_2 \cap N$ are the connected components of $N \setminus C$. □

If we define a Finsler metric $\widetilde{F}$ on N by

$$\widetilde{F}(x,y) := F_0(\pi(x), d\pi(y))$$

for any $x \in N$ and any $y \in T_xN$, then Φ acts on N as an isometry group isomorphic to $\mathbb{Z}^2$ such that $M_0 = N/\Phi$ (see Figure 2.5). From the definitions of geodesic circles and distance circles, we have the following lemma.

Lemma 2.4.3. *Let* (M,F) *be as mentioned above. Then there exist an isometric surface* $I : \mathrm{Int}(M_0) \to M \setminus \cup_{i=1}^{g-1} c_i$ *and its covering surface* $\pi : N \to M_0$ *such that* $M_0 = N/\Phi$ *where* Φ *is a covering transformation group isomorphic to* $\mathbb{Z}^2$*, satisfying that*

$$\begin{aligned} I(\pi(S_N^d(p,t))) &\subset I(S_{M_0}(\pi(p),t)) = I(\pi(S_N(p,t))) \\ &\subset S_M(I(\pi(p)),t) = \rho(G^t(S_{I(\pi(p))}M \cap SM(t))) \end{aligned}$$

for any $p \in N$ *and any* $t > 0$*.*

From Lemma 2.4.3, it suffices to prove Theorem 2.4.4 in order to obtain Theorem 1.0.1.

Theorem 2.4.4. *Let* N *be a covering surface of* M_0 *constructed as above. Let* $p, q \in N$*. Given* $\varepsilon > 0$ *there exists a number* $R > 0$ *such that* $S_N^d(p,t) \cap \Phi(B(q,\varepsilon)) \neq \emptyset$*, equivalently* $\Phi(S_N^d(p,t)) \cap \Phi(B(q,\varepsilon)) \neq \emptyset$ *for all* $t > R$*.*

Thanks to Lemmas 2.4.2 and 2.4.3, the process of the proof for Theorem 2.4.4 is the same as in [23], although N is not homeomorphic to a plane and the distance is not symmetric. However, from the next section up to *Proof of Theorem 1.0.1* in Chapter 5, we progress the study parallel to ones in [9] and [23]. It makes this paper self-contained. The arguments here include some improvements.

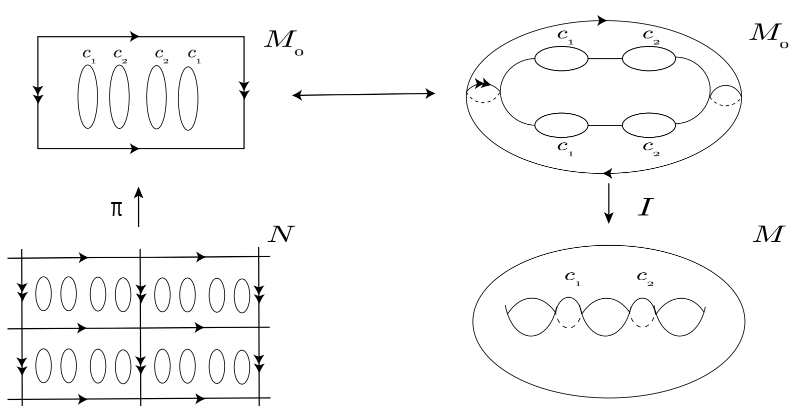

Figure 2.5: The special covering space of M.

Chapter 3

Axial straight lines

3.1 The displacement function on N

Let N be a surface constructed in Section 2.4. Then N is topologically a plane removed many open disks and points, and geometrically a geodesically convex Finsler surface on which the isometry group Φ isomorphic to $\mathbb{Z}^2$ acts so that $M_0 = N/\Phi$. Therefore, N has many properties which a plane has: A simple closed curve and a simple curve diverging both directions as a curve in $\mathbb{R}^2$ divides N into two connected components, etc.. Let $d(\cdot,\cdot)$ be the distance on N induced by $\widetilde{F} = \pi^* F_0$. Since we do not assume that F is reversible, we have $d(p,q) \neq d(q,p)$ for points $p, q \in N$, in general. For $\tau \in \Phi$, let $d_\tau : N \to \mathbb{R}$ denote the *displacement function* of τ which is defined by

$$d_\tau(p) := d(p, \tau(p))$$

for all $p \in N$. We say that a minimal geodesic T is a *straight line* in N if T is unbounded in both directions as a curve in $\mathbb{R}^2$ where N lies.

Note that if M_0 contains two unbounded tubes, then there exists a minimal geodesic T defined on $\mathbb{R}$ in N which is not a straight line. There exists a parametrization $\gamma : (-\infty, \infty) \to N$ of T such that $d(\gamma(s), \gamma(t)) = t - s$ for any $s, t \in (-\infty, \infty)$ with $s < t$. The unboundedness of T in $\mathbb{R}^2$ implies that any half part of T does not stay in a *fundamental domain* in N over $S_0 = M_0 \cup \{p_1, \cdots, p_k\}$ and is not contained in any tube of N. Moreover, the reverse curve T^{-1} of T may not be a geodesic. We have seen in [20], [21] and [23] what phenomenon happens on geodesics in Finsler 2-tori, in comparison with the case of a Riemannian surface.

Proposition 3.1.1 (cf. [9]). *Let $\tau \in \Phi$, $\tau \neq id.$. Then d_τ takes a positive minimum. If $p \in N$ is a minimum point of d_τ, then there exists a unique straight line $\gamma : (-\infty, \infty) \to N$ such that $\gamma(0) = p$ and $\tau(\gamma(t)) = \gamma(t + c)$ for all $t \in (-\infty, \infty)$ where $c = \min d_\tau > 0$.*

We call a straight line γ as in Proposition 3.1.1 an *axis* of τ.

Remark 3.1.2. It follows that $\tau^{-1}(\gamma(t)) = \gamma(t - c)$ for all $t \in (-\infty, \infty)$ for an axis γ of τ. However, the reverse curve $\gamma^-(t) = \gamma(-t)$ for $t \in (-\infty, \infty)$ is neither axis of τ^{-1} nor geodesic, in general (cf. [20]).

Proposition 3.1.1 is certified by the following Lemmas 3.1.3 to 3.1.5.

Lemma 3.1.3. *For all $\tau \in \Phi$, $\tau \neq id.$, the displacement function d_τ take the positive minimum on N. The set of all minimum points of d_τ is contained in $\mathrm{Int}(N)$ and invariant under Φ.*

Proof. Since Φ is abelian, we have $d_\tau(\sigma(q)) = d_\tau(q)$ for all $q \in N$ and all $\sigma \in \Phi$. Hence, the set of all minimum points of d_τ is invariant under Φ.

Since Φ is a covering transformation group and $\tau \in \Phi$, we have $m := \inf\{d_\tau(q) \mid q \in N\} > 0$. We prove that the set of minimum points of d_τ is not empty, and if $d_\tau(q) = \min d_\tau$ for a point $q \in N$, then $q \in \mathrm{Int}(N)$. Let $q_j \in N$ be a sequence of points in a fundamental domain $\widetilde{M}_0$ for M_0 such that $d_\tau(q_j)$ converges to m as $j \to \infty$. We suppose for indirect proof that q_j converges to a point $q \in \partial N$ or $q = \tilde{p}_i \in \mathbb{R}^2$ for some $i = 1, \cdots, k$ where $\pi(\tilde{p}_i) = p_i \in S_0$. In case $q = \tilde{p}_i$, M_0 is bounded around p_i with respect to the distance d_0, since q_j and $\tau(q_j)$ belong to different fundamental domain. This is not the case when M is geodesically complete. Then the minimal geodesics $T(q_j, \tau(q_j))$ from q_j to $\tau(q_j)$ in N satisfy $T(q_j, \tau(q_j)) \setminus \{q_j, \tau(q_j)\} \subset \mathrm{Int}(N)$, since N is geodesically convex and any connected component of ∂N can not contain both q_j and $\tau(q_j)$. In particular, the midpoint $r_j \in T(q_j, \tau(q_j))$ is contained in $\mathrm{Int}(N)$. We assume that r_j converges to a point r as well. Then we have $r \in \mathrm{Int}(N)$ because r is a interior point of a minimal geodesic. Furthermore, $T(q, \tau(q)) \cup \tau(T(q, \tau(q)))$ is the union of minimal geodesics broken at $\tau(q)$. Since N is geodesically convex and $r \in \mathrm{Int}(N)$, a minimal geodesic $T(r, \tau(r))$ is contained in $\mathrm{Int}(N)$. Hence we have that

$$d_\tau(r) < d(r, \tau(q)) + d(\tau(q), \tau(r)) = d(r, \tau(q)) + d(q, r) = \lim_{j \to \infty} d_\tau(q_j) = m,$$

a contradiction. Therefore, we have $q \in \mathrm{Int}(N)$. □

Lemma 3.1.4. *Let $\tau \in \Phi$, $\tau \neq id.$. If $p \in N$ is a minimum point of d_τ, then*
$$T_\tau(p) := \bigcup_{n=-\infty}^{\infty} T(\tau^n(p), \tau^{n+1}(p))$$
is a unique τ-invariant and simple geodesic through p in N.

Proof. We first prove that $T_\tau(p)$ is a geodesic in N. Let $q \in T(p, \tau(p))$ be a point between p and $\tau(p)$, i.e., $q \in T(p, \tau(p)) \setminus \{p, \tau(p)\}$. We then have

$$\begin{aligned} d(p, \tau(p)) &\leq d(q, \tau(q)) \\ &\leq d(q, \tau(p)) + d(\tau(p), \tau(q)) \\ &= d(p, q) + d(q, \tau(p)) \\ &= d(p, \tau(p)). \end{aligned}$$

Therefore, we have

$$d(p, \tau(p)) = d(q, \tau(q)) = d(q, \tau(p)) + d(\tau(p), \tau(q)),$$

meaning that $T(p, \tau(p))$ and $T(\tau(p), \tau^2(p))$ is smoothly joined at $\tau(p)$ to make a geodesic segment $T(p, \tau(p)) \cup T(\tau(p), \tau^2(p))$ in N. In particular, we note that there exists a unique minimal geodesic segment $T(p, \tau(p))$ from p to $\tau(p)$, because τ preserves the orientation of N. In fact, if there exist two minimal geodesics T_1 and T_2 from p to $\tau(p)$, then both $T_1 \cup \tau(T_1)$ and $T_2 \cup \tau(T_2)$ are smooth geodesics having the same end points p and $\tau^2(p)$ and crossing at $\tau(p)$. However, two simple closed curves $T_1 \cup {T_2}^{-1}$ and $\tau(T_1 \cup {T_2}^{-1}) = \tau(T_1) \cup \tau(T_2)^{-1}$ have different orientations, a contradiction. From the uniqueness of the minimal geodesic from p to $\tau(p)$, the joined geodesics $T_\tau(p)$ is a unique τ-invariant geodesic passing through p.

Since $\{\tau^n(p) \,|\, n \in \mathbb{Z}\}$ is unbounded, $T_\tau(p)$ is not a closed geodesic in N. We next prove that $T_\tau(p)$ is simple. Suppose for indirect proof that $\tau^n(T(p, \tau(p))) \cap \tau^m(T(p, \tau(p))) \neq \emptyset$ for some integers n and m, $n \neq m$. Since $T_\tau(p)$ is not a closed geodesic, $\tau^n(T(p, \tau(p))) \cap \tau^m(T(p, \tau(p)))$ consists of a single point q. However, it is impossible because $\tau^n(T(p, \tau(p)))$ and $\tau^m(T(p, \tau(p)))$ contains a sub-segment of $T(q, \tau(q))$ in common. $\square$

The straightness of $T_\tau(p)$ in N can be proved by the same way as H. Busemann and F. P. Pedersen [9]. We then use Lemma 2.4.2 (The Jordan curve theorem) for N.

Lemma 3.1.5. *Let $\tau \in \Phi$, $\tau \neq id.$. If $p \in N$ is a minimum point of d_τ, then $T_\tau(p)$ is a straight line in N invariant under τ.*

Proof. Suppose for indirect proof that $T_\tau(p)$ is not minimal in N. There exists a minimum integer k such that $T_\tau(p)^k := \cup_{n=0}^{k-1} T(\tau^n(p), \tau^{n+1}(p))$ is not a minimal geodesic segment in N. We then have $k \geq 2$ and

$$d(p, \tau^k(p)) < k \min d_\tau.$$

Since $T_\tau(p)^k$ is not minimal, a minimal geodesic $T(p, \tau^k(p))$ from p to $\tau^k(p)$ is different from $T_\tau(p)^k$. In fact, we have

$$T(p, \tau^k(p)) \cap T_\tau(p)^k = \{p, \tau^k(p)\},$$

because both $T(p, \tau^{k-1}(p))$ and $T(\tau^{k-1}(p), \tau^k(p))$ are minimal. Since τ is an orientation preserving isometry of N and $T_\tau(p)$ is invariant under τ, we see from Lemma 2.4.2 that $T(p, \tau^k(p))$ intersects $\tau(T(p, \tau^k(p))) = T(\tau(p), \tau^{k+1}(p))$ at one point q (see Figure 3.1). Furthermore, we have

$$\tau(q) \in T(\tau(p), \tau^{k+1}(p)) \cap T(\tau^2(p), \tau^{k+2}(p)).$$

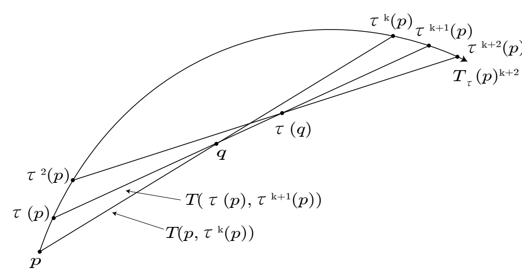

Figure 3.1: $T_\tau(p)^k$ in N.

Since

$$\begin{aligned} k \min d_\tau &> d(p, \tau^k(p)) \\ &= d(\tau(p), \tau^{k+1}(p)) \\ &= d(\tau(p), q) + d(q, \tau(q)) + d(\tau(q), \tau^{k+1}(p)) \\ &= d(\tau(p), q) + d(q, \tau(q)) + d(q, \tau^k(p)) \\ &\geq d(\tau(p), \tau^k(p)) + d(q, \tau(q)) \\ &= (k-1) \min d_\tau + d(q, \tau(q)), \end{aligned}$$

we have $\min d_\tau > d(q, \tau(q))$, a contradiction. $\square$

3.2 Isometries leaving an Axis invariant

Any point in $T_\tau(p)$ is a minimum point of d_τ. Hence, the parametrization $\gamma : (-\infty, \infty) \to N$ of $T_\tau(p)$ satisfies the property $\tau(\gamma(t)) = \gamma(t + c)$ as in Proposition 3.1.1. There are some phenomena which do not happen in the case of reversible geodesics.

Remark 3.2.1. The following are true.

1. Let $\tau \in \Phi$, $\tau \neq id.$. If $p, q \in N$ are minimum points of d_τ, then either $T_\tau(p) = T_\tau(q)$ or $T_\tau(p) \cap T_\tau(q) = \emptyset$ is true. Furthermore, $\tau^{-1}(T_\tau(p)) = T_\tau(p)$, but the reverse curve $T_\tau(p)^{-1}$ may be neither axis of τ^{-1} nor straight line (cf. [20]).

2. Let $\tau, \sigma \in \Phi$, $\tau, \sigma \neq id.$, $\tau \neq \sigma$. Assume that an axis $\gamma : (-\infty, \infty) \to N$ of τ intersects an axis $\alpha : (-\infty, \infty) \to N$ of σ at $p = \gamma(0) = \alpha(0)$. Then $\gamma((0, \infty)) \cap \alpha((0, \infty)) = \emptyset$ and $\gamma((-\infty, 0)) \cap \alpha((-\infty, 0)) = \emptyset$ are true. However, $\gamma((0, \infty)) \cap \alpha((-\infty, 0)) \neq \emptyset$ and $\gamma((-\infty, 0)) \cap \alpha((0, \infty)) \neq \emptyset$ may happen (cf. [20]).

A straight line $\gamma : (-\infty, \infty) \to N$ divides N into two connected components. We call them the *right side* $E(\gamma)$ and the *left side* $W(\gamma)$ of γ.

In conjunction with Proposition 3.1.1, we have the following Proposition 3.2.2, using the same argument in [8].

Proposition 3.2.2. *Let $\gamma : (-\infty, \infty) \to N$ be a straight line in N. If γ is positively invariant under $\tau \in \Phi$, i.e., $\tau(\gamma(t)) = \gamma(t + c)$ for some $c > 0$, then $c = \min d_\tau$ and γ is an axis of τ. Hence all points $p \in \gamma((-\infty, \infty))$ are minimum points of d_τ and $\gamma((-\infty, \infty)) = T_\tau(p)$. Moreover, there exists $\tau_0 \in \Phi$ such that, if $\tau \in \Phi$ leaves γ invariant, then $\tau = {\tau_0}^k$ for some $k \in \mathbb{Z}$. If $\tau_0 = \varphi^m \circ \psi^n$, then m and n are relatively prime where φ and ψ are the generators of Φ.*

Proof. Let $p = \gamma(t)$ for a number $t \in (-\infty, \infty)$ and $q \in N$. From the assumption, we then have $c = d_\tau(p)$ and

$$\begin{aligned} nd(p, \tau(p)) &= d(p, \tau^n(p)) \\ &\leq d(p, q) + \sum_{k=1}^{n} d(\tau^{k-1}(q), \tau^k(q)) + d(\tau^n(q), \tau^n(p)) \\ &= d(p, q) + nd(q, \tau(q)) + d(q, p). \end{aligned}$$

Hence, we have

$$d(p, \tau(p)) \leq d(q, \tau(q)) + \frac{d(p,q) + d(q,p)}{n}.$$

As $n \to \infty$, we conclude that $c = d_\tau(p) \leq d_\tau(q)$, meaning that p is a minimum point of d_τ.

Let $\Phi_1 := \{\tau \in \Phi \,|\, \tau(\gamma(t)) = \gamma(t + \min d_\tau) \text{ for all } t \in (-\infty, \infty)\ \}$ and $c := \inf\{\min d_\tau \,|\, \tau \in \Phi_1, \tau \neq id.\}$. Since Φ is properly discontinuous, there exists $\tau_0 \in \Phi_1$ such that $\min d_{\tau_0} = c > 0$. Let $\tau \in \Phi_1$ and $d := \min d_\tau$. If $d = c$, then $\tau = \tau_0$. Let $d = kc + e$ for some $k \in \mathbb{Z}$ with $k \geq 0$ and some number e with $0 \leq e < c$. We prove $e = 0$. In fact, $\tau_1 = {\tau_0}^{-k} \circ \tau$ satisfies that $\tau_1 \in \Phi_1$ and $e = \min d_{\tau_1}$, contradicting the choice of c if $e \neq 0$ (see Figure 3.2).

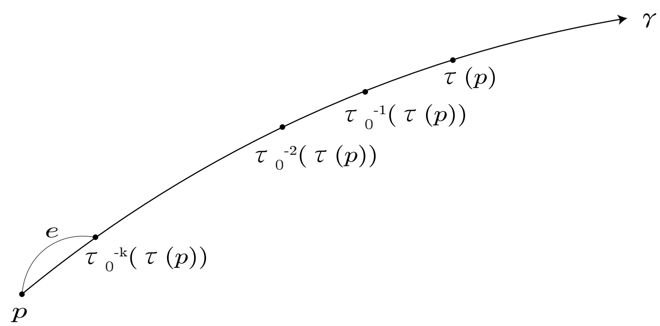

Figure 3.2: An axis of τ.

Since $e = 0$, we have $\tau_1 = id.$. Hence $\tau = {\tau_0}^k$ with $k > 0$. This implies that if γ is positively invariant under $\tau \in \Phi$, we then have $\tau = {\tau_0}^k$ for some $k > 0$. In case there exists a number $c > 0$ such that $\tau(\gamma(t)) = \gamma(t - c)$ for all $t \in (-\infty, \infty)$, we have $\tau^{-1} = {\tau_0}^k$ for some $k > 0$, since $\tau^{-1}(t) = \gamma(t + c)$ for all $t \in (-\infty, \infty)$. Then $\tau = \tau_0^{-k}$.

Suppose for indirect proof that m and n is not relatively prime, i.e., $m = km_1$ and $n = kn_1$ for some integers $k > 1$, m_1 and n_1. Let $\tau_1 = \varphi^{m_1} \circ \psi^{n_1}$. From the choice of c, we have $\tau_1(\gamma(-\infty, \infty)) \cap \gamma((-\infty, \infty)) = \emptyset$ because both $\tau_1 \circ \gamma$ and γ are axes of τ_0 (see Remark 3.2.1). Since τ_1 preserves the orientation of N, if $\tau_1 \circ \gamma$ is contained in $E(\gamma)$ (resp., $W(\gamma)$), then ${\tau_1}^k \circ \gamma$ is also contained in $E(\gamma)$ (resp., $W(\gamma)$). This contradicts that ${\tau_1}^k \circ \gamma((-\infty, \infty)) = \tau_0 \circ \gamma((-\infty, \infty)) = \gamma(-\infty, \infty)$. $\square$

Chapter 4

Straight lines and slopes

4.1 Busemann functions and limit circles

Let $\gamma : (-\infty, \infty) \to N$ be a straight line. We define the Busemann function $B_\gamma : N \to \mathbb{R}$ of γ by

$$B_\gamma(p) := \lim_{t \to -\infty} d(\gamma(t), p) + t$$

for all $p \in N$. It follows that

$$-d(p, q) \leq B_\gamma(p) - B_\gamma(q) \leq d(q, p)$$

for all $p, q \in N$. Hence, B_γ is differentiable on a full measure set in N. The structure of the level sets of a Busemann function has been studied in [23] and [36]. We say that a ray $\alpha : (-\infty, 0] \to N$ is a *co-ray* to $\gamma^- : (-\infty, 0] \to N$, $\gamma^-(t) = \gamma(t)$, ending at $p = \alpha(0)$ if there exist a sequence of numbers $t_j \to -\infty$ and a sequence of points $p_j \in N$ such that a sequence of minimal geodesics $\alpha_j : [-d(\gamma(t_j), p_j), 0] \to N$ converges to α as $j \to \infty$ where $\alpha_j(-d(\gamma(t_j), p_j)) = \gamma(t_j)$ and $p_j = \alpha_j(0)$. From [8], we see that a curve $\alpha : (-\infty, 0] \to N$ is a co-ray to γ^- ending at $\alpha(0)$ if and only if $B_\gamma(\alpha(t)) = t + B_\gamma(\alpha(0))$ for all $t \leq 0$. We call the end point of a maximal co-ray to γ^- a *co-point* to γ^-. Let $C(\gamma^-)$ denote the set of all co-points to γ^-. Then B_γ is of class C^1 on $N \setminus C(\gamma^-)$ and the gradient vector of B_γ at $p \notin C(\gamma^-)$ is $\dot{\alpha}(0)$ where $\alpha : (-\infty, 0] \to N$ is a unique co-ray to γ^- ending at $p = \alpha(0)$ (cf. [17]). We say that a straight line $\alpha : (-\infty, \infty) \to N$ is an *asymptote* to γ^- if $B_\gamma(\alpha(t)) = t + B_\gamma(\alpha(0))$ for all $t \in (-\infty, \infty)$. In addition, if a restriction $\alpha : [a, \infty) \to N$ is a co-ray to γ, i.e., there exists a sequence of minimal geodesics α_j from $p_j = \alpha_j(a)$ to $\gamma(t_j) = \alpha_j(d(p_j, \gamma(t_j))$

such that α_j converges to α and $t_j \to \infty$ as $j \to \infty$, we call α a *parallel* to γ. The Busemann functions on the universal covering spaces of Finsler 2-tori are studied in [21] and [23].

For a function f on N, let $[f = a] := \{p \in N \mid f(p) = a\}$, $[f \leq a] := \{p \in N \mid f(p) \leq a\}$ and so on. When γ is a straight line, it follows from (22.14) in [8], p.133, that $[B_\gamma = a] = \lim_{t\to-\infty} S^d_N(\gamma(t), a - t)$ for all $a \in \mathbb{R}$. We call $[B_\gamma = a]$ a *limit circle* with central ray γ^- (see Figure 4.1).

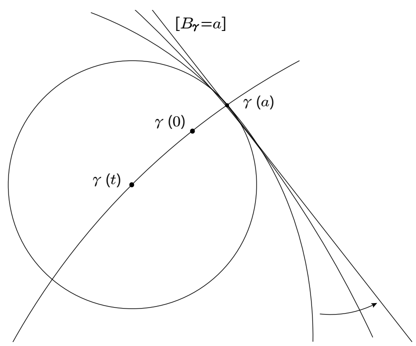

Figure 4.1: The limit circle with central ray γ^-.

Lemma 4.1.1 (cf. Theorem (32.4) in [8])**.** *Let $\tau \in \Phi$, $\tau \neq id.$. Then all axes of τ are parallels to each other.*

Proof. Let $c = \min d_\tau$ and let γ and α be two axes of τ. We prove that $\alpha|_{(-\infty,s]}$ is a co-ray to γ^- for any $s \in (-\infty, \infty)$. Since

$$\begin{aligned}
B_\gamma(\alpha(s-c)) &= B_\gamma(\tau^{-1}(\alpha(s))) \\
&= \lim_{t\to-\infty} d(\gamma(t), \tau^{-1}(\alpha(s))) + t \\
&= \lim_{t\to-\infty} d(\tau^{-1}(\gamma(t+c)), \tau^{-1}(\alpha(s))) + t \\
&= \lim_{t\to-\infty} d(\gamma(t+c), \alpha(s)) + t \\
&= B_\gamma(\alpha(s)) - c,
\end{aligned}$$

$\alpha(s-c)$ is a foot of $\alpha(s)$ on $[B_\gamma = B_\gamma(\alpha(s)) - c]$. From (22.17) and (22.18) in [8], we conclude that α is an asymptote to γ^-. The similar argument proves that $\alpha|_{[s,\infty)}$ is a co-ray to γ. □

4.2 Fundamental domains over M_0 and Slopes of straight lines

Assume that Φ is generated by two motions $\{\varphi, \psi\}$. Let $\mu : (-\infty, \infty) \to N$ be an axis of φ. Then $\psi \circ \mu$ is an axis of φ also. *We may assume that* $\psi \circ \mu \in W(\mu)$. We take a simple curve $c : [0,1] \to N$ in the strip bounded by $\mu((-\infty,\infty))$ and $\psi \circ \mu((-\infty,\infty))$ such that $c(0) \in \mu((-\infty,\infty))$ and $c(1) = \psi(c(0))$. Let $\nu : (-\infty,\infty) \to N$ be a parametrization of a curve $\cup_{i=-\infty}^{\infty} \psi^i(c([0,1]))$ such that $\nu(t) = \psi^i(c(s))$ if $t = i + s$, $0 \le s < 1$, for some integer i. We use this ν instead of any axis of ψ because of the fact (2) in Remark 3.2.1. The domain bounded by μ, $\psi \circ \mu$, ν and $\varphi \circ \nu$ is denoted by $N(0,0)$. Obviously, $N(0,0)$ covers M_0, i.e., $\pi(N(0,0)) = M_0$. If we set $N(i,j) = \varphi^i \circ \psi^j(N(0,0))$, then $N = \cup_{(i,j)\in\mathbb{Z}^2} N(i,j)$.

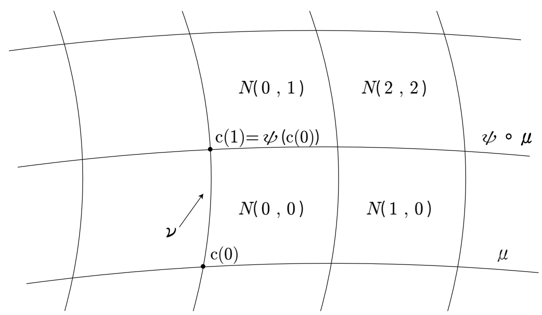

Figure 4.2: Domains $N(i,j)$.

Using this notation, if $\gamma : (-\infty,\infty) \to N$ is a straight line and $\gamma(t) \in N(i(t), j(t))$ for $t \in (-\infty,\infty)$, we then have $|i(t)| \to \infty$ or $|j(t)| \to \infty$ as $t \to \pm\infty$. *Hereafter, we use the word "ray" in the following sense: a minimal geodesic* $\gamma : [0,\infty) \to N$ *(resp.,* $(-\infty, 0] \to N$*) such that* $\gamma(t) \in N(i(t), j(t))$ *for all* t *is a* ray *if* $|i(t)|$ *or* $|j(t)|$ *goes to* ∞ *as* $t \to \infty$ *(resp.,* $-\infty$*).* The half parts of axes and their co-rays are rays.

Let $\gamma : (-\infty, 0] \to N$ be a ray. We define the *slope* $A(\gamma)$ of γ by

$$A(\gamma) := \liminf_{t\to-\infty} \left\{ \frac{j(t)}{i(t)} \;\middle|\; \gamma(t) \in N(i(t), j(t)) \right\}.$$

We prove that "lim inf" is replaced by "lim" in Lemma 4.2.2.

Lemma 4.2.1. *If* $\gamma : (-\infty,\infty) \to N$ *is an axis of* $\tau = \varphi^m \circ \psi^n \in \Phi$, $\tau \neq id.$, *we then have* $A(\gamma) = n/m$ *if* $m \neq 0$ *and* $A(\gamma) = \infty$ *if* $m = 0$.

Proof. Assume that $p = \gamma(0) \in N(m_0, n_0)$ is a minimum point of d_τ and $c = \min d_\tau$. Let L be the maximum of those numbers $|m_0 - i|$ and $|n_0 - j|$ where $N(i, j)$ intersects a c-ball with center p with respect to $d(\cdot, p)$. If $t = kc + r$ for some integer k and some number r with $0 \leq r < c$, then $\gamma(t) \in N(m_0 + km + m_1, n_0 + kn + n_1)$ for some m_1 and n_1 with $0 \leq |m_1| < L$ and $0 \leq |n_1| < L$. Hence we have

$$A(\gamma) = \lim_{t \to -\infty} \frac{n_0 + kn + n_1}{m_0 + km + m_1} = \frac{n}{m}.$$

□

All axes of τ and τ^{-1} have the same slope.

Lemma 4.2.2. *Let $\gamma : (-\infty, 0] \to N$ be a ray. We then have*

$$A(\gamma) = \lim_{t \to -\infty} \left\{ \frac{j}{i} \,\middle|\, \gamma(t) \in N(i, j) \right\}.$$

Furthermore, for a straight line $\gamma : (-\infty, \infty) \to N$, we have

$$A(\gamma) = \lim_{t \to \pm\infty} \left\{ \frac{j}{i} \,\middle|\, \gamma(t) \in N(i, j) \right\}.$$

Proof. Suppose for indirect proof that there exists a rational number n/m such that

$$\liminf_{t \to -\infty} \left\{ \frac{j}{i} \,\middle|\, \gamma(t) \in N(i, j) \right\} < \frac{n}{m} < \limsup_{t \to -\infty} \left\{ \frac{j}{i} \,\middle|\, \gamma(t) \in N(i, j) \right\}.$$

Then there exists an axis of $\tau = \varphi^m \circ \psi^n$ such that it intersects γ many times. Since the axis and ray γ are minimal, this is impossible, proving this lemma.

The second statement is proved in the same way. □

Under a slightly different definition of slopes or rotation numbers, we see the complete structure of all sets of all straight lines with slope $h \in \mathbb{R}$ when N is the universal covering plane of a 2-torus with Riemannian or reversible Finsler metric (cf. [3], [32]).

Instead of classifying the straight lines in N, we pay our attention to a restricted set of straight lines with slope $h \in \mathbb{R}$: Let X_h denote a set of straight lines for $h \in \mathbb{R}$:

1. If $h = n/m$ is a rational number, then X_h is the set of all axes of $\tau = \varphi^m \circ \psi^n$ in N for some $(m, n) \in \mathbb{Z}^2$ with $m > 0$.
2. If h is an irrational number, then X_h is the set of all straight lines α such that there exists a sequence of axes in X_ℓ converging to α as $\ell \to h - 0$ where ℓ are rational numbers.

For two straight lines γ and α, we write $\gamma > \alpha$ when α is contained in $E(\gamma)$. The relation "$>$" is a partial order on the set of all straight lines in N. Because all straight lines in X_h are mutually disjoint, the following lemma is obvious.

Lemma 4.2.3. *All geodesics in X_h are straight lines with slope $h \in \mathbb{R}$ and X_h is Φ-invariant, i.e., $\tau \circ \gamma \in X_h$ for any $\gamma \in X_h$ and any $\tau \in \Phi$. The set X_h is a totally ordered set. If $\alpha, \gamma \in X_h$ such that $\alpha < \gamma$, then α is an asymptote to γ^-.*

4.3 Level sets of Busemann functions

Let $\gamma : (-\infty, \infty) \to N$ be a straight line. Note that the boundary of $[B_\gamma > a]$ possibly contains sub-arcs of the boundary of N, and that $[B_\gamma = a]$ may be divided by a removed point if M is not complete.

Lemma 4.3.1. *For all $a \in \mathbb{R}$ there exists the unique connected component of $[B_\gamma > a]$ whose boundary is unbounded in N.*

Proof. Since $\gamma([a + 1, \infty)) \subset [B_\gamma > a]$, there exists at least one unbounded connected component W_1 of $[B_\gamma > a]$. Because of the topological structure of N and Theorem 2.4.2 (The Jordan curve theorem), the boundary of W_1 is unbounded. Suppose for indirect proof that there exists another unbounded connected component W_2 of $[B_\gamma > a]$ such that the boundary of W_2 is unbounded. Then we have a compact set K in N such that $N \setminus W_1 \cup W_2 \cup K$ has at least two unbounded connected components one of which contains $\gamma((-\infty, a - 1])$. If p_k is a boundary point of W_2 contained in another unbounded connected component of $N \setminus W_1 \cup W_2 \cup K$ such that $B_\gamma(p_k) = a$, then we have a co-ray $\alpha_k : (-\infty, a_k] \to N$ from $p_k = \alpha_k(a_k)$ to γ^- such that $\alpha_k(0) \in K$. As p_k goes to ∞, choosing a subsequence of α_k converging a straight line α, we have an asymptote α to γ^-. However this is impossible because $B_\gamma(\alpha(t))$ is bounded above by a. $\square$

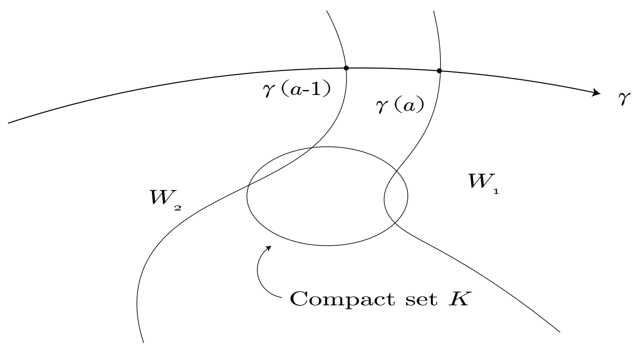

Figure 4.3: Two connected components W_1, W_2 and compact set K.

Let $Y_\gamma(a)$ denote the boundary of the unbounded connected component of $[B_\gamma > a]$ containing $\gamma([a+1,\infty])$ for each $a \in \mathbb{R}$ in N. Obviously, $[B_\gamma = a]_0 := Y_\gamma(a) \setminus \partial N \subset [B_\gamma = a]$. Furthermore, $Y_\gamma(a)$ divides N into two connected components N^+ and N^- such that $\gamma((a,\infty)) \subset N^+$ and $\gamma((-\infty,a)) \subset N^-$. If $p \in N^+$, then $p \in [B_\gamma > a]$. If $p \in [B_\gamma < a]$, then $p \in N^-$. In general, it follows that $[B_\gamma > a] \cap N^- \neq \emptyset$. The parameterized curve $Y_\gamma(a)(t)$, $t \in \mathbb{R}$, is assumed to cross the co-rays to γ^- from left to right.

Let $\gamma \in X_h$ and let $X_h(\gamma)$ denote a subset of X_h consisting of all straight lines contained in $E(\gamma)$. Then all straight lines $\alpha \in X_h(\gamma)$ are asymptotes to γ^- because of the definition of X_h (see Lemma 4.2.3). We use a parametrization of $\alpha \in X_h$ such that $B_\gamma(\alpha(t)) = t$ for all $t \in (-\infty,\infty)$ if $\alpha < \gamma$ and $B_\alpha(\gamma(t)) = t$ for all $t \in (-\infty,\infty)$ if $\alpha > \gamma$.

Lemma 4.3.2. *If $\alpha \in X_h(\gamma)$, then $B_\alpha = B_\gamma$ on $E(\alpha)$.*

Proof. If β is a co-ray from $p \in E(\alpha)$ to γ^-, then β is a co-ray to α^- as well, since α is an asymptote to γ^-. Conversely, a co-ray β to α^- in $E(\alpha)$ is a co-ray to γ^-. Hence, $B_\gamma - B_\alpha$ is constant on $E(\alpha)$ because the distribution of co-rays of γ^- and α^- in $E(\alpha)$ are identified. In particular, the gradient vectors of B_γ and B_α are equal almost everywhere (see Section 4.1). We have $B_\gamma(p) - B_\alpha(p) = B_\gamma(\alpha(0)) - B_\alpha(\alpha(0)) = 0$. □

From this lemma we can define a function $B_h : N \to \mathbb{R}$ by $B_h(p) = B_\alpha(p)$ for all $p \in N$ where α is a straight line in X_h such that $p \in E(\alpha)$.

Lemma 4.3.3. *Let $h,k \in \mathbb{R}$ with $h \neq k$. If $Y_h(a)(t_0) = Y_k(b)(t_1) =: p$, then $Y_h(a)((t_0,\infty)) \cap Y_k(b)((t_1,\infty)) \setminus \partial N = \emptyset$.*

Proof. We may assume that $h < k$. Suppose for indirect proof that there exist numbers $s_0 > t_0$ and $s_1 > t_1$ such that $Y_h(a)((t_0, s_0)) \cap [B_h = a] \cap Y_k(b)((t_1, s_1)) \cap [B_k = b] = \emptyset$ and $Y_h(a)(s_0) = Y_k(b)(s_1) =: q \in [B_h = a] \cap [B_k = b]$.

Let $\alpha : (-\infty, \infty) \to N$ (resp., $\beta : (-\infty, \infty) \to N$) be a straight line in X_h (resp., X_k) such that $p, q \in E(\alpha)$ (resp., $p, q \in E(\beta)$). We may assume that the sequences of minimal geodesics $T(\alpha(t), p)$, $T(\beta(t), p)$, $T(\alpha(t), q)$ and $T(\beta(t), q)$ converge to α_1, β_1, α_2 and β_2, respectively. Then α_1 and β_1 (resp., α_2 and β_2) are co-rays from p to α^- and β^-, respectively, (resp., from q to α^- and β^-, respectively). Since $h < k$, the co-ray β_1 intersects the co-ray α_2 at some point $r \in N$. (see Figure 4.4).

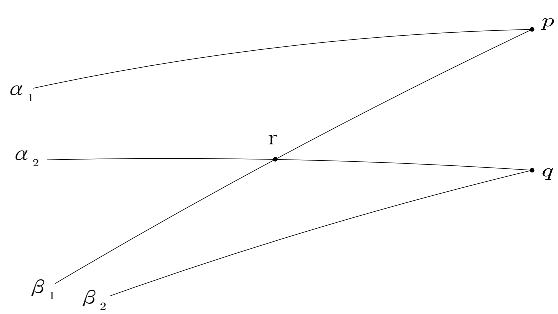

Figure 4.4: α_1, β_1, α_2 and β_2.

This means that

$$\liminf_{t \to -\infty} \left(d(\alpha(t), q) + d(\beta(t), p) - d(\alpha(t), p) - d(\beta(t), q) \right) > 0,$$

since there exists a number $\delta > 0$ such that

$$\begin{aligned} d(\alpha(t), q) + d(\beta(t), p) &= d(\alpha(t), r_t) + d(r_t, q) + d(\beta(t), r_t) + d(r_t, p) \\ &> d(\alpha(t), p) + d(\beta(t), q) + \delta \end{aligned}$$

for any $t < 0$ with sufficiently large $|t|$ and $r_t \to r$ as $t \to -\infty$ where $r_t = T(\alpha(t), q) \cap T(\beta(t), p)$. This contradicts the following equality.

$$\begin{aligned} 0 &= (B_h(q) - B_h(p)) - (B_k(q) - B_k(p)) \\ &= \lim_{t \to -\infty} \left(d(\alpha(t), q) + d(\beta(t), p) - d(\alpha(t), p) - d(\beta(t), q) \right). \end{aligned}$$

□

Lemma 4.3.4. *Let $\tau \in \Phi$. Then the function $f_h(\tau) = B_h \circ \tau - B_h$ is constant on N. Moreover, $f_h : \Phi \to \mathbb{R}$ is a homomorphism, i.e., $f_h(\tau \circ \sigma) = f_h(\tau) + f_h(\sigma)$ for all $\tau, \sigma \in \Phi$. In particular, if $\tau = \varphi^m \circ \psi^n \in \Phi$, we then have $f_h(\tau) = m f_h(\varphi) + n f_h(\psi)$.*

Proof. For any points $p, q \in N$, let $\gamma \in X_h$ be a straight line such that p and q are in the right side of γ and $\tau^{-1} \circ \gamma$, i.e., $p, q \in E(\gamma) \cap E(\tau^{-1} \circ \gamma)$. We then have

$$\begin{aligned} B_h(\tau(p)) - B_h(\tau(q)) &= \lim_{t \to -\infty} d(\gamma(t), \tau(p)) - d(\gamma(t), \tau(q)) \\ &= \lim_{t \to -\infty} d(\tau^{-1} \circ \gamma(t), p) - d(\tau^{-1} \circ \gamma(t), q) \\ &= B_h(p) - B_h(q). \end{aligned}$$

From this we conclude that $f_h(\tau)$ is constant on N.

Since

$$\begin{aligned} f_h(\tau \circ \sigma)(p) &= B_h(\tau(\sigma(p))) - B_h(p) \\ &= (B_h(\tau(\sigma(p))) - B_h(\sigma(p))) + (B_h(\sigma(p)) - B_h(p)) \\ &= f_h(\tau)(\sigma(p)) + f_h(\sigma)(p) \end{aligned}$$

for all $p \in N$, we have $f_h(\tau \circ \sigma) = f_h(\tau) + f_h(\sigma)$. ☐

Let $\Phi_0(h) = \mathrm{Ker}(f_h) = \{\tau \,|\, f_h(\tau) = 0\}$ for each slope $h \in \mathbb{R}$. If $\tau \in \Phi_0(h)$, then $\tau(Y_h(a)) = Y_h(a)$ for all $a \in \mathbb{R}$. There exists $\tau_0 \in \Phi_0(h)$ such that $\tau = {\tau_0}^k$ for any $\tau \in \Phi_0(h)$ and some $k \in \mathbb{Z}$, as was seen in the proof of Proposition 3.2.2. In fact, if $c(t)$, $t \in (-\infty, \infty)$, is a parametrization of $Y_h(0)$ such that $c(0) = \gamma(0)$ and $c((0, \infty))$ is in the right side of γ in N and $t(\tau)$ are the numbers such that $\tau(c(0)) = c(t(\tau))$ for any $\tau \in \Phi_0(h)$, then τ_0 or ${\tau_0}^{-1}$ satisfies that $t(\tau_0) = \min\{t(\tau) > 0 \,|\, \tau \in \Phi_0(h) \setminus \{id.\}\}$.

Lemma 4.3.5. *Let $\Phi_0(h)$ be generated by $\tau_0 = \varphi^{m_0} \circ \psi^{n_0} \neq id.$. Then m_0 and n_0 are relatively prime and $f_h(\psi)/f_h(\varphi) = -m_0/n_0$ if $f_h(\varphi) \neq 0$, and $f_h(\varphi)/f_h(\psi) = -n_0/m_0$ if $f_h(\psi) \neq 0$.*

Proof. Suppose for indirect proof that $m_0 = k m_1$ and $n_0 = k n_1$ for some $k \in \mathbb{Z}$ with $k \neq 1$. Hence, if $\tau_1 = \varphi^{m_1} \circ \psi^{n_1}$, then $\tau_1(\gamma(0)) \notin Y_h(0)$, implying that $f_h(\tau_1) \neq 0$. Then we get a contradiction; $0 = f_h(\tau_0) = k f_h(\tau_1) \neq 0$.

The second part of the theorem follows from $0 = f_h(\tau_0) = m_0 f_h(\varphi) + n_0 f_h(\psi)$. ☐

Lemma 4.3.6. *Let $\tau \in \Phi_0(h)$. If a straight line $\gamma \in \cup_{k \in \mathbb{R}} X_k$ is not any axis of τ, then $\gamma((-\infty, \infty))$ intersects $Y_h(a)$ for all $a \in \mathbb{R}$.*

Proof. We first assume that $\gamma \in X_k$ for some rational number $k \in \mathbb{R}$ such that it is an axis of $\tau_1 \in \Phi$ with $f_h(\tau_1) \neq 0$. Then $|f_h(\tau_1{}^n)| = |nf_h(\tau_1)|$ goes to ∞ as $n \to \pm\infty$. This implies that $|B_h(\gamma(t))|$ goes to ∞ as $t \to \pm\infty$.

If the slope k of γ is irrational, then there exist a sequence of rational numbers k_j with $k_j < k$ converging to k and a sequence of axes γ_j with slopes k_j converging to γ. Since all axes γ_j intersect $Y_h(a)$, γ intersects $Y_h(a)$ for all $a \in \mathbb{R}$. □

Let $\ell_h = \inf\{f_h(\tau) \,|\, \tau \in \Phi \setminus \Phi_0(h) \text{ such that } f_h(\tau) > 0\}$. Since

$$\ell_h = \inf\{|mf_h(\varphi) + nf_h(\psi)| \mid (m,n) \in \mathbb{Z}^2 \text{ such that } f_h(\varphi^m \circ \psi^n) \neq 0\},$$

if $f_h(\psi)/f_h(\varphi)$ or $f_h(\varphi)/f_h(\psi)$ is an irrational number, we then have $\ell_h = 0$ (cf. [1], [24]). Assume that $f_h(\psi)/f_h(\varphi) = i/j$ where i and j are relatively prime integers. Then we have

$$f_h(\varphi^m \circ \psi^n) = \frac{mj + ni}{j} f_h(\varphi).$$

Since i and j are relatively prime integers, there exist integers m and n such that $mi + nj = 1$. Therefore, we see that

$$\ell_h = \min\left\{\left|\frac{f_h(\varphi)}{j}\right|, \left|\frac{f_h(\psi)}{i}\right|\right\}.$$

Note that $|f_h(\varphi)| \leq \min d_\varphi$ and $|f_h(\psi)| \leq \min d_\psi$. If one of the denominators i and j in the above estimate of ℓ_h is greater than $Q := \max\{\min d_\varphi, \min d_\psi\}/\varepsilon$ for a number $\varepsilon > 0$, we then have $\ell_h < \varepsilon$.

Lemma 4.3.7. *For any $\varepsilon > 0$ the number of slopes $h \in \mathbb{R}$ such that $\ell_h > \varepsilon$ is finite.*

Proof. Assume that $\ell_h > \varepsilon$. Then there exists a $\tau_1 \in \Phi$ such that $f_h(\tau_1) = \ell_h$. Here we write $\tau_1 = \varphi^{m_1} \circ \psi^{n_1}$. Since $f_h(\Phi)$ is a subgroup generated by ℓ_h, there exists an integer k_1 such that $f_h(\varphi) = k_1 f_h(\tau_1)$. Hence, we then have $(k_1 m_1 - 1) f_h(\varphi) + k_1 n_1 f_h(\psi) = 0$. We assume that $k_1 m_1 - 1 = k m_0$ and $k_1 n_1 = k n_0$ for some integer k where the integers m_0 and n_0 are relatively prime. Set $\tau_0 = \varphi^{m_0} \circ \psi^{n_0}$. Then τ_0 is a generator of $\Phi_0(h)$. It follows from the argument just before Lemma 4.3.7 and Lemma 4.3.5 that both $|m_0|$ and $|n_0|$ are less than Q. Thus we have at most finitely many $\tau_0 = \varphi^{m_0} \circ \psi^{n_0}$ such that $f_h(\tau_0) = 0$ even if there exist infinitely many $\tau_1 \in \Phi$ such that $f_h(\tau_1) = \ell_h$. Furthermore, how to choose m_0 and n_0 depends only on Q which does not depend on the slope h. From Lemma 4.3.3, there exists at most one slope $h \in \mathbb{R}$ such that $f_h(\tau_0) = 0$ for each τ_0. This implies that the number of the slopes h with $\ell_h > \varepsilon$ is finite. □

Chapter 5

The asymptotic behavior of geodesic circles in M

5.1 A domain consisting of slices covering M_0

Let $h \in \mathbb{R}$ be a slope and $\gamma : (-\infty, \infty) \to N$ a straight line in N such that $\gamma \in X_h$. Take an isometry $\tau \in \Phi$ such that $\tau \circ \gamma \neq \gamma$. Let $\square(i, j; u, v)$ denote the rectangle bounded by $Y_\gamma(-i f_h(\tau))$, $Y_\gamma(-j f_h(\tau))$, $\tau^u \circ \gamma$ and $\tau^v \circ \gamma$.

Lemma 5.1.1. *Under the notation above, we have*

$$\tau^s(\square(i, j; u, v)) = \square(i - s, j - s; u + s, v + s).$$

Proof. This lemma follows from the fact that $\tau^s \circ Y_\gamma(a) = Y_\gamma(a + s f_h(\tau))$ and $\tau^s \circ \tau^u = \tau^{s+u}$. $\square$

Let $\Phi(\tau)$ denote the infinite cyclic subgroup of Φ generated by τ. Then $N_1 = N/\Phi(\tau)$ is topologically a cylinder with disks and points removed. If $\rho_1 : N \to N_1$ is the quotient map, then $\rho_1 \circ \gamma$ may not be a minimal geodesic in N_1. By the way, $\rho_1(Y_\gamma(0))$ is a curve like a helix contained in N_1 with pitch $|f_h(\tau)|$ if $|f_h(\tau)| \neq 0$. In particular, we note that $\rho_1(Y_\gamma(0))$ is not a level set of the Busemann function $B_{\rho_1 \circ \gamma}$ in N_1 even if $\rho_1 \circ \gamma$ is a straight line in N_1.

We may assume that $\min\{B_h(x) \,|\, x \in N(0, 0)\} = 0$ (see Section 4.2 for the definition of $N(i, j)$). Let $b > \max\{a \in \mathbb{R} \,|\, Y_\gamma(a) \cap N(0, 0) \neq \emptyset\}$. Hence, $N(0, 0)$ is contained in the strip bounded by $Y_\gamma(0)$ and $Y_\gamma(b)$. It does not imply that $B_h(x) \leq b$ for all $x \in N(0, 0)$, although $0 \leq B_h(x)$ are true for all $x \in N(0, 0)$. Furthermore, when $f_h(\tau) < 0$, we may assume that the domain bounded by γ, $Y_\gamma(0)$, $\tau \circ \gamma$ and $Y_\gamma(b)$ contains $N(0, 0)$, i.e., $N(0, 0) \subset E(\gamma)$

and $N(0,0) \subset W(\tau \circ \gamma)$. If $b > |f_h(\tau)|$, we have an integer k such that $k|f_h(\tau)| \geq b$, i.e., $N(0,0) \subset \Box(0,k;0,1)$. In particular, $M_0 = \pi(\Box(0,k;0,1))$ where $\pi : N \to M_0$ is the covering map.

Lemma 5.1.2. *Assume that $f_h(\tau) < 0$ and $b > |f_h(\tau)|$. Let k be an integer such that $k|f_h(\tau)| > b$. We then have $\Box(0,k;0,1) \subset \cup_{i=0}^{k-1} \Box(i,i+1;-i,k-i)$ and $\pi(\Box(i,i+1;-i,k-i)) = M_0$ for each $i = 0, \cdots, k-1$.*

Proof. The first part of the statement follows from the definition.

We prove the second part. Since $\tau^{-i}(\Box(0,1;i,i+1)) = \Box(i,i+1;0,1)$, we have

$$\begin{aligned} \Box(0,k;0,1) &= \cup_{i=0}^{k-1} \Box(i,i+1;0,1) \\ &= \cup_{i=0}^{k-1} \tau^{-i}(\Box(0,1;i,i+1)). \end{aligned}$$

Therefore, we have

$$\begin{aligned} \pi((\Box(0,1;0,k)) &= \pi(\cup_{i=0}^{k-1} \Box(0,1;i,i+1) \\ &= \pi(\cup_{i=0}^{k-1} \tau^{-i}(\Box(0,1;i,i+1))) \\ &= \pi(\Box(0,k;0,1)) = M_0. \end{aligned}$$

Since $\Box(i,i+1;-i,k-i) = \tau^{-i}(\Box(0,1;0,k))$, we have

$$\pi(\Box(i,i+1;-i,k-i)) = \pi(\Box(0,1;0,k)) = M_0.$$

□

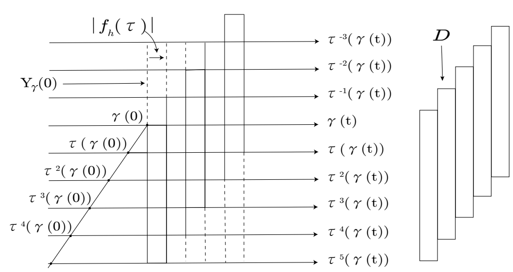

Figure 5.1: The domain covered M_0.

5.2 The asymptotic behavior of distance circles

We start preparing the notations which are used in the proof of Theorem 2.4.4. For any $\varepsilon > 0$ we choose a slope $h \in \mathbb{R}$ and a straight line $\gamma \in X_h$ such that $N(0,0)$ lies between $Y_\gamma(0)$ and $Y_\gamma(b)$, $\ell_h < \varepsilon$, and then choose an isometry $\tau \in \Phi$ such that $-\varepsilon < f_h(\tau) < 0$. The integer k_1 is defined to satisfy $k_1|f_h(\tau)| \geq b$.

Let $\gamma_1 \in X_h$ be a straight line and let α be a ψ-invariant curve such that α intersects γ_1 and $\psi \circ \gamma_1$ exactly once, respectively. Then the domains $Q(j)$ bounded by $\varphi^j \circ \alpha$, $\varphi^{j+1} \circ \alpha$, γ_1 and $\psi \circ \gamma_1$ cover M_0 for all integers $j \in \mathbb{Z}$. Therefore, for any point $p \in N$, there exists a sequence of points $p_j \in Q(j)$ such that $\pi(p_j) = \pi(p)$, i.e., $\tau_j(p) = p_j$ for some $\tau_j \in \Phi$. Since $-\infty < h < \infty$ and the perimeters of $\psi^i(Q(j))$ equal for all $i, j \in \mathbb{Z}$, there exists a number K_1 such that $d(p_j, p_{j+1}) < K_1$ for all $j \in \mathbb{Z}$ (as was seen in the proof of Lemma 7.1 in [23], p.356). Let L be a number such that $L > \max\{b, K_1\}$ and k, k_2 integers such that $k|f_h(\tau)| > L$, $k = k_1 + k_2$. We change the parameterization of γ such that $\tilde{\gamma}(s) = \gamma(s + (k_2 - 1)f_h(\tau))$.

After those preparations, using $\tilde{\gamma}$, we construct a domain

$$D := \cup_{i=0}^{k-1} \Box(i, i+1; -i + k_2, k - i)$$

each of whose slices covers M_0, i.e., $\pi(\Box(i, i+1; -i + k_2, k - i)) = M_0$ for each $i = 0, \cdots, k-1$. We may assume that $\gamma_1 \in X_h$ satisfies $D \subset E(\gamma_1)$.

Lemma 5.2.1 (cf. [23], Assertion 7.2)**.** *There exists an integer j_1 such that*

$$d(p_j, \gamma_1(0)) < d(p_{j+1}, \gamma_1(L))$$

for all integers $j < j_1$.

Proof. The sequences of minimal geodesics $T(p_j, \gamma_1(0))$ and $T(p_{j+1}, \gamma_1(L))$ converge to sub-rays of γ_1 as $j \to -\infty$, so there exists a sequence of points $r_{j+1} \in T(p_{j+1}, \gamma_1(L))$ converging to $\gamma_1(0)$ as $j \to -\infty$. Therefore, there exists an integer j_1 such that

$$\begin{aligned}
& d(p_{j+1}, \gamma_1(L)) - d(p_j, \gamma_1(0)) \\
= \;& d(p_{j+1}, r_{j+1}) + d(r_{j+1}, \gamma_1(L)) - d(p_j, \gamma_1(0)) \\
> \;& -(d(p_j, p_{j+1}) + d(r_{j+1}, \gamma_1(0))) + d(r_{j+1}, \gamma_1(L)) \\
> \;& -K_1 + L + d(r_{j+1}, \gamma_1(0)) - d(\gamma_1(0), r_{j+1}) > 0
\end{aligned}$$

for all $j < j_1$, since $d(p_j, p_{j+1}) < K_1 < L$ and $d(r_{j+1}, \gamma_1(0)) \to 0$, $d(\gamma_1(0), r_{j+1}) \to 0$ as $j \to -\infty$. $\Box$

Let $a_j = d(p_j, \gamma_1(0))$ and $b_j = d(p_j, \gamma_1(L))$. Then, for any $t \in [a_j, b_j]$, there exists a point $x_t \in \gamma_1([0, L])$ such that $d(p_j, x_t) = t$. Since $d(p_j, \gamma_1(0)) \to \infty$ as $j \to -\infty$, there exists an integer j_0 with $j_0 < j_1$ such that $a_j < b_j$ and $d(p_{j_0}, \gamma_1(0)) \leq d(p_j, \gamma_1(0))$ for all integers $j < j_0$. Hence, when $R_1 := a_{j_0}$, we have $R_1 = \min\{a_j \mid j \leq j_0\}$.

Lemma 5.2.2 (cf. [23], Assertion 7.3)**.** *For any $t > R_1$, there exist a point $x_t \in \gamma_1([0, L])$ and an integer $j < j_0$ such that $d(p_j, x_t) = t$.*

Proof. Let $K_j = \cup_{i=j}^{j_0} [a_i, b_i]$ for $j < j_0$. We prove that K_j is connected for all $j \leq j_0$. Suppose for indirect proof that K_{i_0} is connected but not K_{i_0-1}. From the definition of R_1, we have $K_{i_0} = [R_1, b_{j_2}]$ for some $j_2 < j_0$. Since K_{i_0-1} is not connected and $R_1 \leq a_{i_0-1}$, we have $b_{j_2} < a_{i_0-1}$. On the other hand, we have $b_{i_0} > a_{i_0-1}$ because of Lemma 5.2.1. Since $b_{i_0} \leq b_{j_2}$, we have $a_{i_0-1} \leq b_{j_2}$, a contradiction. Since $d(p_j, \gamma_1(0)) \to \infty$ as $j \to -\infty$, we have $\cup_{i>-\infty}^{j_0} [a_i, b_i] = [R_1, \infty)$.

For any $t > R_1$, if we choose an integer j such that $t \in [a_j, b_j]$, then there exists a point $x_t \in \gamma_1([0, L])$ such that $d(p_j, x_t) = t$. $\square$

Lemma 5.2.3 (cf. [23], Lemma 6.1)**.** *Let $\varepsilon > 0$, γ_1, L, D, $p \in N$ and $p_j \in \Phi(p)$ be as above. Then there exists an integer $j_0 = j_0(D, \varepsilon) > 0$ such that*

$$B_h{}^{-1}(B_h(x)) \cap D \subset B(S_N^d(p_j, d(p_j, x)), \varepsilon)$$

for all points $x \in \gamma_1([0, L])$ and all integers $j < j_0$. In particular, for any point $q \in B_h{}^{-1}(B_h(x)) \cap D$, we have $B(q, \varepsilon) \cap S(p_j, d(p_j, x)) \neq \emptyset$.

Proof. Since $g(z, t) = d(\gamma_1(t), z) + t$ is monotone increasing for $t < 0$ and converges to $B_h(z)$ uniformly on any compact set contained in D as $t \to -\infty$, there exists a number $T < 0$ such that $0 \leq g(z, t) - B_h(z) < \varepsilon/3$ for all $z \in D$ and $t < T$.

If $q \in B_h{}^{-1}(B_h(x)) \cap D$ for a point $x \in \gamma_1([0, L])$, we then have

$$0 \leq d(\gamma_1(t), q) - d(\gamma_1(t), x) < \varepsilon/3 \tag{5.1}$$

for any number $t < T$, because

$$\begin{aligned} 0 &\leq d(\gamma_1(t), q) - d(\gamma_1(t), x) \\ &= (d(\gamma_1(t), q) + t) - (d(\gamma_1(t), x) + t) \\ &= g(q, t) - B_h(x) \\ &= g(q, t) - B_h(q) < \frac{\varepsilon}{3}. \end{aligned}$$

Set $A = ({B_h}^{-1}(B_h(x)) \cap D) \setminus B(x, \varepsilon/2)$. Since γ_1 is an asymptote to $(\psi \circ \gamma_1)^-$, there exists a positive integer $j_0 = j_0(D, \varepsilon)$ such that, for all integers $j < j_0$, a minimal geodesic segment $T(p_j, x)$ from p_j to the point $x \in \gamma_1([0, L])$ (resp., any point $q \in A$) passes through $B(\gamma_1(T + 1), \varepsilon/3)$ (resp., intersects γ_1 at $\gamma_1(t_j)$ with some $t_j < T$).

If $p' \in T(p_j, x)$ satisfies $\max\{d(p', \gamma_1(T+1)), d(\gamma_1(T+1), p')\} < \varepsilon/3$, we then have, from (5.1) (see Figure 5.2),

$$\begin{aligned} 0 &< d(\gamma_1(t_j), q) - d(\gamma_1(t_j), x) \\ &= d(p_j, q) - d(p_j, \gamma_1(t_j)) - d(\gamma_1(t_j), x) \\ &\leq d(p_j, q) - d(p_j, x) \\ &\leq d(p', q) + d(p_j, p') - d(p_j, x) \\ &= d(p', q) - d(p', x) \\ &< (d(\gamma_1(T+1), q) + \varepsilon/3) - (d(\gamma_1(T+1), x) - \varepsilon/3) \\ &< \varepsilon \end{aligned}$$

for all $q \in A$. Therefore, we have

$$d(p_j, x) < d(p_j, q) < d(p_j, x) + \varepsilon.$$

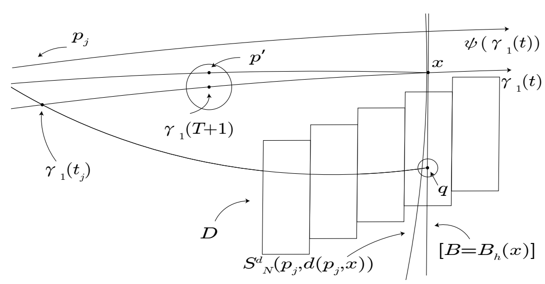

Figure 5.2: The asymptotic behavior of the geodesic from p_j.

If $y_j(q)$ is a point at which $T(p_j, q)$ and $S^d_N(p_j, d(p_j, x))$ intersect, we then have $q \in B(y_j(q), \varepsilon)$ and, therefore, $q \in B(S^d_N(p_j, d(p_j, x)), \varepsilon)$.

For $q \in ({B_h}^{-1}(B_h(x)) \cap D) \cap B(x, \varepsilon/2)$, we have $q \in B(S^d_N(p_j, d(p_j, x)), \varepsilon)$, since $x \in S^d_N(p_j, d(p_j, x))$ and $d(x, q) < \varepsilon/2$. □

Lemma 5.2.3 states that we can find a distance sphere $S^d_N(p_j, d(p_j, x))$ meeting the ε-ball $B(q, \varepsilon)$ for any point $q \in D$ with $B_h(q) = B_h(x)$. From Lemma 5.2.2, any point $q \in (\cup_{x \in \gamma_1([0,L])} Y_{\gamma_1}(B_h(x)) \setminus \partial N) \cap D$ satisfies this condition. We have to treat another case, $q \notin Y_{\gamma_1}(a) \setminus \partial N$ for any $a \in \mathbb{R}$, in order to complete the proof of Theorem 2.4.4.

Proof of Theorem 1.0.1 and 2.4.4. We prove Theorem 2.4.4 which is sufficient for Theorem 1.0.1. Let p, q and ε be as in Theorem 2.4.4. If $q \in (Y_{\gamma_1}(0) \smallsetminus \partial N) \cap D$ for a suitable parametrization of γ_1, then it follows from Lemmas 5.2.1, 5.2.2 and 5.2.3 that for any $t > R_1$ there exist sequences of points $p_j \in \Phi(p)$ and $q_j \in D \cap \Phi(q)$ such that $S^d_N(p_j, t) \cap B(q_j, \varepsilon) \neq \emptyset$.

In case $q \notin Y_{\gamma_1}(a) \smallsetminus \partial N$ for any $a \in \mathbb{R}$, we find a point $q_1 \in D \cap \Phi(q)$ in a strip bounded by $Y_{\gamma_1}(0)$ and $Y_{\gamma_1}(|f_h(\tau)|)$. Assume that a sequence of minimal geodesics from p_j to q_1 converges to a co-ray $\alpha : (-\infty, 0] \to N$ from q_1 to $\gamma_1{}^-$ and $r_1 = \alpha(-d(Y_{\gamma_1}(0), q_1))$. Then the sequence of intersection points $r_j = T(p_j, q_1) \cap Y_{\gamma_1}(0)$ converges to $r_1 \in Y_{\gamma_1}(0)$ as $j \to -\infty$. This implies that for any $t > R_1 + d(r_1, q_1)$ we have $S^d_N(p_j, t) \cap B(q_j, \varepsilon) \neq \emptyset$ for some $p_j \in \Phi(p)$ and $q_j \in D \cap \Phi(q_1)$. □

Remark 5.2.4. In the above argument, if $p_j = \tau_j(p)$ and $q_j = \tau'_j(q)$ for $\tau_j, \tau'_j \in \Phi$, we then have $S^d_N(p_j, t) \cap B(\tau_j{}^{-1} \circ \tau'_j(q), \varepsilon) = S^d_N(p_j, t) \cap B(q_j, \varepsilon) \neq \emptyset$.

For any $\varepsilon > 0$ and any points $p, q \in M$, let $\widetilde{p}$ (resp., $\widetilde{q}_k \in \Phi(D)$) be the lifts of p (resp., q). Then it follows from the above consequence that the geodesic circle with center $\widetilde{p}$ meets the union of $B(\widetilde{q}_k, \varepsilon)$'s for any $t > R$ on N. Combining with Lemma 2.4.3, we can see the asymptotic behavior of the distance circles emanating from $\widetilde{p}$ in N (see Figure 5.3).

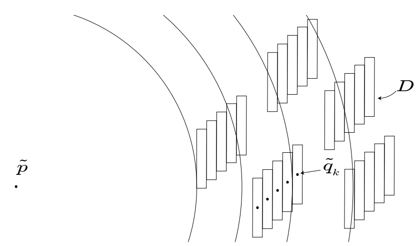

Figure 5.3: The geodesic circle with center $\widetilde{p}$ in N.

Proof of Corollary 1.0.5. We work in M_0 instead of M. Let $n > 0$ be an integer and $\varepsilon > 0$. Let $\tilde{p} \in N$ be chosen so that $\pi(\tilde{p}) = p$. From Lemma 4.3.7, there exists slopes h_i, $i = 1, \cdots n$, such that $h_i \neq h_k$ for $i \neq k$ and $\ell_{h_i} < \varepsilon$ for all i. As was seen in the proof of Theorem 1.0.1, for each slope h_i we can find domains D_i and numbers R_i satisfying the following; for any $t > R_i$ there exist sequences of points $p_{ij} \in N$ and $q_{ij} \in D_i$ such that $\pi(p_{ij}) = p$, $\pi(q_{ij}) = q$ and $S_{M_0}(p_{ij}, t) \cap B(q_{ij}, \varepsilon) \neq \emptyset$. Let $\tau_{ij} \in \Phi$ be such that $\tau_{ij}(p_{ij}) = \tilde{p}$. The sequence of minimal geodesics $T(\tilde{p}, \tau_{ij}(q_{ij}))$ from $\tilde{p}$ to $\tau_{ij}(q_{ij})$ converges to a ray with slope h_i as $j \to -\infty$ for each $i = 1, \cdots, n$. □

Chapter 6

Appendix : Finsler manifolds and geodesics

We define a Finsler metric and induce the Euler-Lagrange equation of piecewise smooth curves and define a geodesic in a Finsler manifold. Let M be an n-dimensional manifold and TM its tangent bundle. Let $(x^1, x^2, \cdots, x^n, y^1, y^2, \cdots, y^n)$ be a local coordinate system in TM where $(x^1, x^2 \cdots, x^n)$ is a local coordinate system in M and y^i's are coefficients of $y = y^i \frac{\partial}{\partial x^i}|_x \in T_xM$. A continuous function $F : TM \to [0, \infty)$ is *Finsler metric* of M if F satisfies the following properties:

1. *Regularity* : F is smooth on $TM \setminus \{0\}$.

2. *Positive homogeneity* : $F(x, \lambda y) = \lambda F(x, y)$ for all $\lambda > 0$.

3. *Strong convexity* : The Hessian matrix

$$(g_{ij}) := \left(\left[\frac{1}{2} F^2(x, y) \right]_{y^i y^j} \right)$$

is positive-definite at every point in $TM \setminus \{0\}$. Here (g_{ij}) is a symmetric $n \times n$ matrix and is called a *fundamental tensor*.

The pair (M, F) is called a *Finsler manifold*. We give some example of Finsler metrics.

Example 6.0.1. Let $x \in M$ and $g_x(\cdot, \cdot)$ a Riemannian metric on T_xM. We consider a norm induced by the Riemannian metric. For $y \in T_xM$,

$$F(x, y) := \sqrt{g_x(y, y)}.$$

F is a Finsler metric on M and reversible. It is said to be *Riemannian*.

The Finsler metric F may be not reversible and satisfies the triangle inequality
$$F(x, u + v) \leq F(x, u) + F(x, v)$$
for $u, v \in T_xM$. We can express the strong convexity in index free from. Namely, for $y \in T_xM \setminus \{0\}$, $u, v \in T_xM$,
$$\mathbf{g}_y(u, v) = \frac{1}{2}\frac{\partial^2}{\partial s \partial t}[F^2(y + su + tv)]\bigg|_{s=t=0}.$$
The strong convexity gives an inner product on Finsler manifolds.

For a piecewise smooth curve $c : [a, b] \to M$, we define the length of the curve c by
$$L(c) := \int_a^b F(c(t), \dot{c}(t))dt.$$
This length function L induces a function $d : M \times M \to \mathbb{R}$ as
$$d(p, q) := \inf_c L(c)$$
where the infimum is taken over all piecewise smooth curves $c : [a, b] \to M$ with $c(a) = p$, $c(b) = q$. The function $d(\cdot, \cdot)$ is called an *intrinsic distance* induced by the Finsler metric F. The intrinsic distance $d(\cdot, \cdot)$ satisfies the triangle inequality
$$d(p, q) \leq d(p, r) + d(r, q)$$
for any $p, q, r \in M$. If a geodesic is reversible, then the intrinsic distance may not be symmetric. However, the Finsler metric F is reversible if and only if the intrinsic distance is symmetric.

We begin to define a geodesic in Finsler manifolds. Let $\lambda > 0$ and $c : [a, b] \to M$ a constant speed piecewise smooth curve with $F(c, c') = \lambda$. By definition, there is a partition of $[a, b]$, $a = t_0 < t_1 < \cdots < t_{k-1} < t_k = b$ such that c is smooth in each $[t_{i-1}, t_i]$. Fix this partition, then we consider a piecewise smooth map $H : (-\varepsilon, \varepsilon) \times [a, b] \to M$ with the following properties:

1. H is continuous on $(-\varepsilon, \varepsilon) \times [a, b]$,
2. H is smooth in each $(-\varepsilon, \varepsilon) \times [t_{i-1}, t_i], i = 1, \cdots, k$,
3. $c(t) = H(0, t)$ for $a \leq t \leq b$.

Set $c_u(t) := H(u, t)$ for each $u \in (-\varepsilon, \varepsilon)$. $H(u, t)$ is called a *variation of a piecewise smooth curve* c and a length of the variation curve c_u is given by
$$L(u) = \int_a^b F\bigg(H(u, t), \frac{\partial H}{\partial t}(u, t)\bigg)dt$$

If c is an extremal of the length function L, then $\dot{L}(0) = 0$. We then have the following equation which is called the *Euler-Lagrange equation*:

$$\frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial y^i} = 0, \ i = 1, \cdots, n.$$

Set $G^i(y) := \frac{1}{4} g^{il}(x,y)\{[F^2(x,y)]_{x^k y^l} y^k - [F^2(x,y)]_{x^l}\}$ where (g^{ij}) is the inverse matrix of (g_{ij}). Then the *geodesic curvature* of c at $c(t)$ is defined by

$$\kappa(t) := \frac{1}{F(c,\dot{c})^2}\{\ddot{c}^i + 2G^i(\dot{c})\}\frac{\partial}{\partial x^i}\bigg|_{c(t)}.$$

If c has minimal length, then $\dot{L}(0) = 0$ for any variation H of c fixing endpoints (see Figure 6.1). If the curve c is smooth, then $\kappa(t) = 0$. Thus a constant speed smooth shortest curve satisfies $\kappa(t) = 0$. Therefore, we define a geodesic in (M, F) as follows. A smooth curve c is a *geodesic* in (M, F) if c has constant speed and its geodesic curvature $\kappa = 0$.

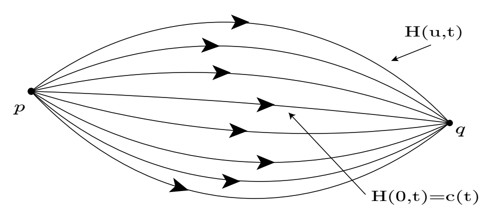

Figure 6.1: The variation H of c fixing endpoints.

Example 6.0.2. Let F be a Finsler metric. We decompose F into the symmetric part A and the skew-symmetric part B by setting for $y \in T_xM \setminus \{0\}$,

$$F(x,y) = A(x,y) + B(x,y)$$

where $A(x,y) := \frac{1}{2}(F(x,y) + F(x,-y))$, $B(x,y) := \frac{1}{2}(F(x,y) - F(x,-y))$. Then, all geodesics in (M,A) are reversible. Let $t = t(s)$ a reversed change of parameter for $s \in \mathbb{R}$, i.e., $t'(s) < 0$. As the relation between geodesics γ in (M, F) and geodesics in (M, A), we see that if $\alpha(s) = \gamma(t(s))$, then γ is reversible in (M, F) if and only if it is a geodesic in (M, A). Furthermore, we set $G := 2A$. Let d_G be an intrinsic distance induced by G and (M, F) a complete Finsler manifold. Then, $m = d_G$ if and only if a metric space (M, m) is *Menger convex*, i.e.,if for any $p, q \in X$ with $p \neq q$, there exists a point $r \in X$ such that $r \neq p$, $r \neq q$ and $d(p,r) + d(r,q) = d(p,q)$ (cf. [20]).

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