Closed polynomials and kernels of higher derivations over an integral domain

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Contents

Standard notation

Item	**Meaning**
$\mathbb{Z}$	the ring of integers
$\mathbb{Q}$	the field of rational numbers
$\mathbb{R}$	the field of real numbers
$\mathbb{C}$	the field of complex numbers
$\mathbb{N}$	the set of non-negative integers (including 0)
$\mathbb{F}_q$	the finite field of order q
$\mathbb{A}^n_k$	the affine space of dimension n over a field k
$\mathbb{G}_a$	the additive group of a field k
$\mathbb{G}_m$	the multiplicative group of units of a field k
$S \subset T$ or $S \subseteq T$	S is a subset of T
$S \subsetneq T$	S is a proper subset of T
$\#S$	the number of elements in S

Introduction

In the study of Affine Algebraic Geometry, one of the our aims is to understand affine spaces over a given ground field k. For example, when an affine algebraic k-variety X is given, we need to understand whether X is isomorphic to $\mathbb{A}_k^n$ or not, as algebraic varieties. Although this question sounds like quite simple, it contains a lot of mysteries. In order to consider this question, we often use theories of $\mathbb{G}_m$-actions and $\mathbb{G}_a$-actions on X. For a field k, we denote $k^{[n]}$ by the polynomial ring in n variables over k.

First of all, we recall that theories of $\mathbb{G}_m$-actions on X. It is well known that to give a $\mathbb{G}_m$-action on $X = \operatorname{Spec} B$ is equivalent to give a $\mathbb{Z}$-grading on B. We say a $\mathbb{G}_m$-action is **elliptic** if the corresponding $\mathbb{Z}$-grading is positive (see *Definition 1.3*).

When X is a smooth affine variety over $\mathbb{C}$, then we can regard X as a topological manifold. We say X is an **exotic structure on** $\mathbb{A}_{\mathbb{C}}^n$ if it is diffeomorphic to $\mathbb{A}_{\mathbb{R}}^{2n}$ and $X \not\cong_{\mathbb{C}} \mathbb{A}_{\mathbb{C}}^n$. The existence of an elliptic $\mathbb{G}_m$-action on X is a strong form of contractibility as below: In this case, the $\mathbb{G}_m$-action has a unique (attractive) fixed point $x_0 \in X$. If the action is given by ${}^{\lambda}x$ ($\lambda \in \mathbb{C}^*$, $x \in X$), then since all the weights of the action are positive integers, restriction to the real interval $t \in (0, 1]$ yields:

$$\lim_{t \to +0} ({}^{t}x) = x_0, \quad \forall x \in X.$$

So the requisite contracting homotopy is given by $F : X \times [0, 1] \to X$, where

$$F(x,t) = \begin{cases} {}^{t}x & (t \neq 0), \\ x_0 & (t = 0). \end{cases}$$

A well-known theorem of Ramanujam [**64**] says that a smooth affine surface over $\mathbb{C}$ which is contractible and simply connected at infinity is isomorphic to $\mathbb{A}_{\mathbb{C}}^2$. This can be used to show that any smooth affine surface over $\mathbb{C}$ with an elliptic $\mathbb{G}_m$-action is isomorphic to $\mathbb{A}_{\mathbb{C}}^2$; see [**17**]. In the same paper, Ramanujam showed that any smooth contractible affine variety over $\mathbb{C}$ of dimension $n \geq 3$ is diffeomorphic to $\mathbb{A}_{\mathbb{R}}^{2n}$, and is therefore either isomorphic to $\mathbb{A}_{\mathbb{C}}^n$ or an exotic structure on $\mathbb{A}_{\mathbb{C}}^n$.

A well-known example of this phenomenon is the Koras-Russell threefold $X = \operatorname{Spec} B$, that is, the coordinate ring is defined by

$$B = \mathbb{C}[x, y, z, t]/(x + x^2y + z^2 + t^3).$$

It is known that B is a rational UFD and it has a $\mathbb{Z}$-grading defined by $\deg(x, y, z, t) = (6, -6, 3, 2)$. Moreover, X is smooth and contractible, and $X \ncong_{\mathbb{C}} \mathbb{A}^3_{\mathbb{C}}$. So X is an exotic structure on $\mathbb{A}^3_{\mathbb{C}}$. In [**19**, Theorem 6.2], Freudenburg and the author showed that X does not have the stronger form of contractibility imposed by an elliptic $\mathbb{G}_m$-action, i.e., B does not admit positive $\mathbb{Z}$-gradings.

Similarly, we consider the Asanuma threefolds over a field k of positive characteristic $p > 0$. In [**2**], Asanuma introduced the following family of rational threefolds

$$A_m = k[x, y, z, t]/(x^my + f(z, t)),$$

where $m \geq 1$, $f(z, t) \in k[z, t]$ and $k[z, t]/(f) \cong_k k^{[1]}$ but $k[z, t] \neq k[f]^{[1]}$. Segre [**69**] gives such non-standard line embeddings in $\mathbb{A}^2_k$, for example, defined by

$$f(z, t) = z^{p^e} + t + t^{sp},$$

where $s, e \in \mathbb{Z}_{>0}$ and p^e and sp do not divide each other; see also the Introduction in [**21**]. Asanuma showed that $A_m^{[1]} \cong_k k^{[4]}$ for each $m \geq 1$. From this, it follows that A_m is a rational UFD and that each threefold $\operatorname{Spec} A_m$ is smooth. Furthermore, Gupta in [**22**] and [**23**] showed that $A_m \ncong_k k^{[3]}$ when $m \geq 2$. Freudenburg and the author showed that these also do not admit elliptic $\mathbb{G}_m$-actions, which is a consequence of the following ([**19**, Theorem 6.4]):

Theorem. *For any field k and positive integers n, m, let X be an affine k-variety such that $X \times \mathbb{A}^m_k \cong_k \mathbb{A}^{n+m}_k$. Then $X \cong_k \mathbb{A}^n_k$ if and only if X admits an elliptic $\mathbb{G}_m$-action.*

In the case where $k = \mathbb{C}$, in one direction of the above theorem, for $X = \operatorname{Spec} B$, the condition $X \times \mathbb{A}^m_k \cong_k \mathbb{A}^{n+m}_k$ ensures that X is smooth, affine and contractible, but does not imply that B has a positive $\mathbb{Z}$-grading. In the other direction, if B is an affine rational UFD with a positive $\mathbb{Z}$-grading and $X = \operatorname{Spec} B$ is smooth, then either $X \cong_{\mathbb{C}} \mathbb{A}^n_{\mathbb{C}}$ or X is an exotic structure on $\mathbb{A}^n_{\mathbb{C}}$. In [**19**, Section 7], we conjecture the following characterization of affine space:

Conjecture. *Let X be a factorial rational affine variety of dimension n over an algebraically closed field k. If X is smooth and admits an elliptic $\mathbb{G}_m$-action, then $X \cong_k \mathbb{A}^n_k$.*

By [**19**, Corollary 4.7 and Theorem 5.1], the conjecture is true for $n = 1$ and $n = 2$.

Next, recall that theories of $\mathbb{G}_a$-actions on X. It is well known that to give a $\mathbb{G}_a$-action on $X = \operatorname{Spec} B$ is equivalent to give a locally finite iterative higher derivation on B, in particular, when the characteristic of k is zero, it is equivalent to give a locally nilpotent derivation on B. Indeed, let $\varphi : \mathbb{G}_a \to \operatorname{Aut} X$ be a $\mathbb{G}_a$-action on X. Then, for $\lambda \in k$ and $f \in B$,

$$\varphi(\lambda)(f) = \sum_{\ell=0}^{\infty} D_\ell(f)\lambda^\ell,$$

where $D = \{D_\ell\}_{\ell=0}^{\infty}$ is a locally finite iterative higher derivation on B (see *Chapter 1.3* for details). This mapping $\varphi \mapsto D$ gives a one-to-one correspondence between the $\mathbb{G}_a$-actions on X and the locally finite iterative higher derivations on B.

Here, we recall some characterizations of affine spaces in the terms of $\mathbb{G}_a$-actions. Denote $X_n = \operatorname{Spec} B_n$ by an affine variety over k of dimension n, then the following characterizations of affine spaces are well known:

- $X_1 \cong_k \mathbb{A}^1_k \iff X_1$ admits a non-trivial $\mathbb{G}_a$-action.
- $X_2 \cong_k \mathbb{A}^2_k \iff X_2$ admits a non-trivial $\mathbb{G}_a$-action and B_2 is a UFD with $B_2^* = k^*$.

If B is a k-algebra, the **Makar-Limanov invariant** $\mathrm{ML}(B)$ of B is the intersection of all invariant rings of $\mathbb{G}_a$-actions on B, and the **Derksen invariant** $\mathcal{D}(B)$ of B is the subring generated by invariants of non-trivial $\mathbb{G}_a$-actions. See [**18**] for details.

If $B \cong_k k^{[n]}$, then $\mathrm{ML}(B) = k$ and $\mathcal{D}(B) = B$. Hence, by calculating that invariants, we can understand whether $X = \operatorname{Spec} B$ is isomorphic to $\mathbb{A}^n_k$ or not. For example, Asanuma threefolds A_m are considered by Gupta in [**22**] and [**23**], showing that, when $m \geq 2$, $\mathrm{ML}(A_m) = k[x]$ and $\mathcal{D}(A_m) = k[x, z, t]$. So $A_m \ncong_k k^{[3]}$ when $m \geq 2$. These give counterexamples for the cancellation problem for affine spaces in positive characteristic. It is an open problem whether $A_1 \cong_k k^{[3]}$.

By the way, in order to understand affine spaces $\mathbb{A}^n_k$ over a given ground field k, it is important to consider the automorphism group $\operatorname{Aut} \mathbb{A}^n_k$ as algebraic varieties. In fact, since $\mathbb{A}^n_k = \operatorname{Spec} k^{[n]}$ admits a lot of $\mathbb{G}_m$-actions and $\mathbb{G}_a$-actions, namely, $\operatorname{Aut} \mathbb{A}^n_k$ contains many kinds of $\mathbb{G}_m$ and $\mathbb{G}_a$ as subgroups, we see that $\operatorname{Aut} \mathbb{A}^n_k$ is a huge group. For this reason, it is difficult to understand $\operatorname{Aut} \mathbb{A}^n_k$, especially when $n \geq 3$. Although it is known that the structure theorem for $\operatorname{Aut} \mathbb{A}^2_k$, several open problems are still left even if $n = 2$. For example the Jacobian conjecture is well known.

A k-automorphism on $B \cong_k k^{[n]}$ is given by the n-tuple of polynomials $f_1, \dots, f_n \in B$. In this thesis, we discuss about such polynomial f_i and the k-algebra $k[f_i]$. We say that each f_i is a **variable** (or **coordinate**). In *Proposition 2.1*, we show that if $f \in B$ is a variable, then the k-subalgebra $k[f]$ is integrally closed in B. We say that such a polynomial is a **closed polynomial** (see *Chapter 2*). Also, a pair of polynomials (f, g) is said to be a **closed-pair** if $k[f, g] \cong_k k^{[2]}$ and $k[f, g]$ is algebraically closed in B (see *Chapter 5*).

When the characteristic of k is zero, there are some constructions of closed polynomials and closed-pairs as below. If d is a non-zero derivation on $k^{[2]}$ such that $\ker d \neq k$, then $\ker d = k[f]$ and f is a closed polynomial (see *Theorem 3.1*). If d is a non-zero locally nilpotent derivation on $k^{[3]}$, then $\ker d = k[f, g]$ and (f, g) is a closed-pair (see Miyanishi [**51**]). Furthermore, for $f, g \in k^{[n]}$, if $k[f, g] \cong_k k^{[2]}$ and is a retract of $k^{[n]}$, then (f, g) is a closed-pair (see *Chapter 5*). More preciously, the author showed in [**56**] that, if A is a retract of $k^{[n]}$ of transcendence degree 2 over k, then $A = k[f, g]$ and (f, g) is a closed-pair.

In this thesis, we give several characterizations and criteria of integrally closed subalgebras of the polynomial ring over an integral domain. In *Chapter 1*, we recall several definitions of degree functions, $\mathbb{Z}$-gradings, derivations and higher derivations. These are important methods and techniques which are used in this thesis. By using these techniques, in *Theorem 1.14*, we give some characterizations of algebraically closed subalgebras of a given integral domain.

In *Chapter 2*, we study closed polynomials and some other concepts of polynomials (factorially closed polynomials, univariate polynomials and variables). In particular, these classes of polynomials have geometric meanings as below: Let k be an algebraically closed field and let $f \in B \backslash k$, where $B \cong_k k^{[n]}$. Then we can consider the morphism

$$\Phi_f : \mathbb{A}_k^n \cong_k \operatorname{Spec} B \to \operatorname{Spec} k[f] \cong_k \mathbb{A}_k^1$$

defined by the natural inclusion $k[f] \to B$. Using this morphism Φ_f, we show that a polynomial $f \in B \backslash k$ is a closed polynomial (resp. factorially closed polynomial) if and only if a general fiber (resp. every fiber) of Φ_f is irreducible and reduced (see *Theorems 2.3* and *2.29*). Similarly, in the case where $n = 2$, we give a characterization of univariate polynomials in *Theorem 2.28*.

By the way, in *Theorem 2.4*, we give characterizations of closed polynomials over an integral domain. These characterizations are conclusions of *Theorem 1.14*. In *Theorem 2.11*, we give some criteria of closed polynomials, using techniques of $\mathbb{Z}$-gradings on the polynomial ring. The key strategy is to find an appropriate $\mathbb{Z}$-grading. Since the polynomial ring has many kinds of $\mathbb{Z}$-gradings, we can find such a $\mathbb{Z}$-grading. Furthermore, in *Chapter 2.3*, by using the criteria, we give classifications of closed polynomials in special cases (the monomials, the polynomials with the Jacobian condition and the polynomials whose degree is prime).

In *Chapter 3*, we study closed polynomials, derivations and higher derivations in the polynomial ring $R[x, y] \cong_R R^{[2]}$ over an integral domain R. In this chapter, we consider the following three cases:

- R is an arbitrary integral domain of characteristic zero.
- R is a UFD of characteristic zero.
- R is a field of positive characteristic.

In fact, the second case is studied by several mathematicians. See e.g., Nowicki [**58**], Nowicki and Nagata [**62**], Ayad [**4**], Arzhantsev and Petravchuk [**3**], Kato and Kojima [**32**], etc. In *Proposition 3.11*, we show relations between derivations and closed polynomials in $R[x, y]$, under some equivalence relations. From the result, we can understand the kernel of a derivation on $R[x, y]$ by using the theory of closed polynomials. This observation will be used in *Chapter 4*. The most important tool is what is called the Jacobian derivation. This derivation makes sense if the characteristic of R is zero. When R is a field of positive characteristic, in *Theorem 3.16*, we introduce higher derivations which look like the Jacobian derivation. By using the higher derivation, we give characterizations of variables.

In *Chapter 4*, as an application of results on closed polynomials in *Chapters 2* and *3*, we study kernels of monomial derivations on the polynomial ring in two variables over a UFD. *Theorem 4.2* gives a classification of kernels of monomial derivations. Furthermore, in *Theorem 4.5*, by using the argument in [**63**, Section 5] and *Theorem 4.2*, we determine the non-zero monomial derivations d on $k[x, y]$ such that the quotient field of the kernel of d is not equal to the kernel of d in $k(x, y)$.

Finally in *Chapter 5*, we explain definitions and some properties of retracts and closed-pairs, and give several examples. In particular, we consider the following question:

Question. Let k be a field of characteristic zero and let d be a non-zero k-derivation on $k^{[3]}$. Is the kernel generated by at most two polynomials?

In *Chapter 5.2*, we give partial affirmative answers and examples for this question.

CHAPTER 1

Preliminaries

Let R be an integral domain of characteristic $p \geq 0$. For a positive integer $n \geq 1$, we denote $R^{[n]}$ by the polynomial ring in n variables over R, $Q(R)$ by the field of fractions of R and R^* by the group of units in R. For an R-subalgebra A of B, we denote $\mathrm{tr.deg}_R\, A := \mathrm{tr.deg}_{Q(R)}\, Q(A)$.

Through in this chapter, assume that B is an integral domain containing R. B is referred to as an R-domain. Let A be a subring of B. We say an element $f \in B$ is **algebraic** over A if there exists a non-zero polynomial $P \in B^{[1]} \setminus \{0\}$ such that $P(f) = 0$, especially say **integral** over A if we can choose such a polynomial $P \in B^{[1]}$ to be monic. We say that A is **algebraically closed** (resp. **integrally closed**) in B if there are no algebraic (resp. integral) elements in B other than A. Also, say A is **factorially closed** in B if for non-zero elements $f, g \in B$, the condition $fg \in A$ implies $f \in A$ and $g \in A$. Here, we consider the following three conditions.

(a) A is integrally closed in B.

(b) A is algebraically closed in B.

(c) A is factorially closed in B.

It is easy to show that the implications "(c) $\Longrightarrow$ (b)" and "(b) $\Longrightarrow$ (a)" hold true. When A is factorially closed in B, we have that $A^* = B^*$ and every irreducible element of A is also irreducible in B. Furthermore, the following result holds.

Proposition 1.1. *Let A be a subring of B. Then the following two conditions are equivalent*:

(i) *A is integrally closed in B and $Q(A) \cap B = A$.*

(ii) *A is algebraically closed in B.*

Proof. **(i) $\Longrightarrow$ (ii)** If $b \in B$ is algebraic over A, then there exist $m \geq 1$ and $a_0, \ldots, a_m \in A$ such that $a_0 b^m + a_1 b^{m-1} + \cdots + a_{m-1} b + a_m = 0$. By multiplying ${a_0}^{m-1}$ on the both sides of this equation, we see that $a_0 b$ is integral over A, so $a_0 b \in A$. Hence $b \in Q(A) \cap B = A$.

(ii) $\Longrightarrow$ (i) It is enough to show that $Q(A) \cap B = A$. Let $b \in Q(A) \cap B$. Then b is algebraic over A, hence $b \in B$. $\square$

Proposition 1.2. (cf. [**28**, Lemma 3.2]) *Suppose that B is a UFD. Let A be a subring of B such that $A^* = B^*$. Then the following two conditions are equivalent*:

(i) *Every irreducible element of A is also irreducible in B.*

(ii) *A is factorially closed in B.*

Proof. It is clear that the implication "(ii) $\Longrightarrow$ (i)" holds true.

(i) $\Longrightarrow$ (ii) Since $A^* = B^*$, A satisfies the ascending chain condition for principal ideals of A. Hence, every element of A has an irreducible decomposition. Taking non-zero elements $f, g \in B$ such that $fg \in A$. Considering an irreducible decomposition of fg in A, we have

$$fg = u{a_1}^{e_1} \cdots {a_r}^{e_r}$$

for some $u \in A^*$, distinct irreducible element $a_i \in A$ and $e_i \geq 1$. Since B is a UFD and each a_i is irreducible in B, the above equation gives the irreducible decomposition of fg in B. Without loss of generality, we may assume that $f = u{a_1}^{e_1-d_1} \cdots {a_r}^{e_r-d_r}$ and $g = {a_1}^{d_1} \cdots {a_r}^{d_r}$ for some $0 \leq d_i \leq e_i$. Therefore $f \in A$ and $g \in A$. $\square$

1. Degree functions and $\mathbb{Z}$-gradings

Let R be an integral domain of characteristic $p \geq 0$ and let B be an R-domain. In this section, we discus about degree functions and $\mathbb{Z}$-gradings on B. For more detail, refer to [**19**, Section 3].

A **degree function** on B is a map $\deg : B \to \mathbb{Z} \cup \{-\infty\}$ such that, for $f, g \in B$, the following three conditions are satisfied.

(a) $\deg f = -\infty$ if and only if $f = 0$,

(b) $\deg(fg) = \deg f + \deg g$,

(c) $\deg(f + g) \leq \max\{\deg f, \deg g\}$.

Here, it is understood that $(-\infty) + (-\infty) = -\infty$ and $(-\infty) + \deg f = -\infty$ for any $f \in B$. It is easy to show that the equality holds in condition (c) if $\deg f \neq \deg g$. The induced filtration is

$$B = \bigcup_{d \in \mathbb{Z}} \mathcal{F}_d,$$

where the set $\mathcal{F}_d = \{f \in B \mid \deg f \leq d\}$ is the associated **degree modules**. We say deg is a **degree function over** R if $\deg(R^*) = 0$.

Definition 1.3. Let $\deg : B \to \mathbb{Z} \cup \{-\infty\}$ be a degree function over R on B.

(a) deg is **non-negative** if $\mathcal{F}_d = \{0\}$ for $d < 0$.

(b) deg is **positive** if it is non-negative and $\mathcal{F}_0 = R$.

When B admits a degree function over R, we have the following properties for the associated degree modules.

Proposition 1.4. *Let* $\deg : B \to \mathbb{Z} \cup \{-\infty\}$ *be a degree function over R on B. Then the following assertions hold true.*

(a) *$\mathcal{F}_0$ is a subring of B which is integrally closed in B.*

(b) *$\mathcal{F}_d$ is an $\mathcal{F}_0$-module for each $d \in \mathbb{Z}$.*

(c) *$\mathcal{F}_d$ is an ideal of $\mathcal{F}_0$ for each $d \leq 0$.*

(d) *If* deg *is non-negative, then $\mathcal{F}_0$ is factorially closed in B and $B^* \subset \mathcal{F}_0$. In particular, if* deg *is positive, then R is factorially closed in B.*

Proof. **(a)** Let $f, g \in \mathcal{F}_0$. Then $\deg fg = \deg f + \deg g \leq 0$, hence $fg \in \mathcal{F}_0$. Therefore $\mathcal{F}_0$ is a subring of B. Extend deg to $Q(B)$ and let $\mathcal{R} = \{\xi \in Q(B) \mid \deg \xi \leq 0\}$, namely, $\deg(f/g) := \deg f - \deg g$ for $f \in B$ and $g \in B \setminus \{0\}$. Then $\mathcal{R}$ is a valuation ring of $Q(B)$ with the valuation deg, and $\mathcal{F}_0 = \mathcal{R} \cap B$. This implies that $\mathcal{F}_0$ is integrally closed in B.

(b) Let $a \in \mathcal{F}_0$ and $f \in \mathcal{F}_d$. Then $\deg af = \deg a + \deg f \leq \deg f \leq d$. Hence $af \in \mathcal{F}_d$.

(c) Let $d \leq 0$. Then $\mathcal{F}_d \subset \mathcal{F}_0$. By (b), $\mathcal{F}_d$ is an ideal of $\mathcal{F}_0$.

(d) Let $f, g \in B \setminus \{0\}$ with $fg \in \mathcal{F}_0$. Since deg is non-negative, $\deg f \geq 0$ and $\deg g \geq 0$. Hence we have

$$0 \leq \deg f + \deg g = \deg fg \leq 0.$$

Therefore $\deg f = \deg g = 0$, which implies that $f \in \mathcal{F}_0$ and $g \in \mathcal{F}_0$. $\square$

Let $\mathfrak{g}$ be a $\mathbb{Z}$-grading of B over R, that is, there exist subgroups B_d of $(B, +)$ such that

(a) $B = \bigoplus_{d \in \mathbb{Z}} B_d$,

(b) $B_d B_e \subset B_{d+e}$ for all $d, e \in \mathbb{Z}$,

(c) $R \subset B_0$.

Definition 1.5. Let $\mathfrak{g}$ be a $\mathbb{Z}$-grading of B over R.

(a) $\mathfrak{g}$ is **non-negative** if $B_d = \{0\}$ for $d < 0$.

(b) $\mathfrak{g}$ is **positive** if it is non-negative and $B_0 = R$.

A $\mathbb{Z}$-grading $\mathfrak{g}$ gives a degree function $\deg_{\mathfrak{g}} : B \to \mathbb{Z} \cup \{-\infty\}$ on B over R by the natural way. It is clear that $\mathfrak{g}$ is non-negative (resp. positive) if and only if $\deg_{\mathfrak{g}}$ is non-negative (resp. positive).

For example, we shall consider the case where B is the polynomial ring $R[x_1, \ldots, x_n] \cong_R R^{[n]}$. For $w_1, \ldots, w_n \in \mathbb{Z}$, define the $\mathbb{Z}$-grading $\mathfrak{g}_{\mathbf{w}}$ by $\deg_{\mathfrak{g}_{\mathbf{w}}} x_i = w_i$ for each i. Then $\mathfrak{g}_{\mathbf{w}}$ is non-negative (resp. positive) if and only if w_i is a non-negative (resp. positive) integer for each i. This implies that every polynomial ring over R has a positive $\mathbb{Z}$-grading, especially has a positive degree function. Hence we have the following.

Proposition 1.6. *R is factorially closed in $R^{[n]}$ for any $n \geq 1$. In particular, R is also factorially closed in $R^{[[n]]}$, where $R^{[[n]]}$ is the formal power series ring in n variables over R.*

Proof. By the above discussion, $R^{[n]}$ admits a positive $\mathbb{Z}$-grading $\mathfrak{g}$ over R. Then $\mathcal{F}_0 = B_0 = R$. By *Proposition 1.4* (d), R is factorially closed in $R^{[n]}$.

In the case where $B = R^{[[n]]}$, we consider $\mathrm{ord} : B \to \mathbb{N} \cup \{-\infty\}$, where ord is the usual order function on B. By using the similar argument of the proof of *Proposition 1.4* (d), we see that R is factorially closed in $R^{[[n]]}$. $\square$

2. Properties of derivations

In this section, we explain fundamental properties of derivations over an integral domain R of characteristic zero. Let B be an R-domain. An R-**derivation** $d : B \to B$ is a map such that, for $r \in R$ and $f, g \in B$,

(a) $d(f + g) = d(f) + d(g)$,

(b) $d(rf) = rd(f)$,

(c) $d(fg) = gd(f) + fd(g)$.

For an R-derivation d on B, we denote B^d by the kernel of d. An R-derivation d is **locally nilpotent** if for any $f \in B$, there exists $m \geq 1$ such that $d^m(f) = 0$. Denote $\mathrm{Der}_R B$ by the set of all R-derivation on B, $\mathrm{LND}_R B$ by the set of all locally nilpotent derivations on B. It is well known that $\mathrm{Der}_R B$ has a structure of B-modules. Remark also, although $\mathrm{LND}_R B$ is a subset of $\mathrm{Der}_R B$, it is not necessarily closed under the sum.

For a locally nilpotent derivation d on B, we define the R-automorphism $\exp(td)$ on $B[t] \cong_B B^{[1]}$ as below. We extend d to an R-derivation $\tilde{d}$ on

$B[t]$ by the following formula:

$$\tilde{d}\left(\sum_{\ell=0}^{m} f_\ell t^\ell\right) := \sum_{\ell=0}^{m} d(f_\ell)t^\ell$$

for $f_\ell \in B$. It is clear that $\tilde{d}$ is also locally nilpotent on $B[t]$. Then for $\xi \in B[t]$, we define $\exp(td)$ by

$$\exp(td)(\xi) := \sum_{\ell=0}^{\infty} \frac{1}{\ell!}\tilde{d}^\ell(\xi)t^\ell,$$

whenever R contains $\mathbb{Q}$. Since $\tilde{d}$ is locally nilpotent, the above definition makes sense. Moreover, it is easy to show that $\exp(td) \circ \exp(-td) = \mathrm{id}_{B[t]}$ and $\exp(-td) \circ \exp(td) = \mathrm{id}_{B[t]}$. Therefore $\exp(td)$ gives an R-automorphism on $B[t]$.

Properties of derivations. We denote some fundamental notations and recall properties for derivations (P.1)–(P.8) as below. Let d be an R-derivation on B.

- **(P.1)** B^d is an algebraically closed subring of B.
- **(P.2)** If d is locally nilpotent, then B^d is factorially closed in B.
- **(P.3)** An element $s \in B$ satisfying $d(s) = 1$ is called a **slice** of d.
- **(P.4)** d is **irreducible** if the only principal ideal of B containing the image of d is B itself (or equivalently, if d is not of the form fd' with f a non-unit element of B and $d' \in \mathrm{Der}_R B$).
- **(P.5)** If B is UFD, then there exist an irreducible R-derivation $d_0 \in \mathrm{Der}_R B$ and $f \in B \setminus \{0\}$ such that $d = fd_0$, and d_0 is unique up to multiplication by units of B.
- **(P.6)** We say that $f \in B \setminus \{0\}$ is an **integral element** if $d(f)$ belongs to the ideal generated by f in B. In the case where B is the polynomial ring over R, it is called also a **Darboux polynomial**.
- **(P.7)** An element t of B is called an **integral factor** or an **eigenvalue** if there exists an integral element $f \in B \setminus \{0\}$ such that $d(f) = tf$.
- **(P.8)** We denote $\chi_d(B)$ by the set of all integral factors in B. Then $\chi_d(B)$ is an abelian monoid under the addition of B.

We prove only the assertion (P.8) (see [**18**] or [**60**] for more details).

Proof. **(P.8)** It is clear that $0 \in \chi_d(B)$. Let $s, t \in \chi_d(B)$. Then there exist $f, g \in B \setminus \{0\}$ such that $d(f) = sf$ and $d(g) = tg$. Hence we have

$$d(fg) = d(f)g + fd(g) = fgs + fgt = (s+t)fg.$$

This implies that $s + t \in \chi_d(B)$. $\square$

Lemma 1.7. (cf. [**52**, Lemma 1.5 (1)]) *Suppose that B is a UFD containing R. Let d be an R-derivation on B. Then the following two conditions are equivalent*:

(i) *B^d is factorially closed in B.*

(ii) *$\chi_d(B)$ contains no abelian subgroups other than* (0).

Proof. **(i) $\Longrightarrow$ (ii)** Suppose that B^d is factorially closed in B. If $\chi_d(B)$ contains an abelian subgroup other than (0), then there exists an integral factor $t \in \chi_d(B) \setminus \{0\}$ such that $-t \in \chi_d(B)$. Then there exist $f, g \in B \setminus \{0\}$ such that $d(f) = tf$ and $d(g) = -tg$. We have

$$d(fg) = d(f)g + fd(g) = tfg - tfg = 0,$$

hence $fg \in B^d$. But $d(f) \neq 0$ and $d(g) \neq 0$, this is a contradiction.

(ii) $\Longrightarrow$ (i) Let $f, g \in B \setminus \{0\}$ with $fg \in B^d$. Then $0 = d(fg) = d(f)g + fd(g)$. Without loss of generality, we may assume that f and g have no common factors. Hence we have $f \mid d(f)$ and $g \mid d(g)$. Then there exists $t \in B$ such that $d(f) = tf$ and $d(g) = -tg$, which implies that $t, -t \in \chi_d(B)$. Since $\chi_d(B)$ has no abelian subgroups other than (0), we have $t = 0$. So $d(f) = d(g) = 0$ and hence B^d is factorially closed in B. $\square$

The following theorem is one of the most important result in the study of locally nilpotent derivations, and is called **Slice Theorem**. In 1968, Rentschler [**65**] announced the result. Following [**14**], we denote the theorem as a generalized form as below.

Theorem 1.8. **(Slice Theorem)** *Let R be an integral domain containing $\mathbb{Q}$ and let B be an R-domain. Let d be a locally nilpotent derivation on B. If d has a slice $s \in B$, then B is the polynomial ring in s over B^d, that is, $B = B^d[s] \cong_{B^d} (B^d)^{[1]}$.*

As the end of this section, we consider the case where B is the polynomial ring in n variables over R. Set variables $x_1, \dots, x_n$ of B, namely, $B = R[x_1, \dots, x_n] \cong_R R^{[n]}$. For an R-derivation d, we define the **divergence** of d by

$$\operatorname{div}(d) := \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(d(x_i)).$$

Proposition 1.9. *For a locally nilpotent derivation d on B,* $\operatorname{div}(d) = 0$.

Proof. Set $K = Q(R)$ and $B_K = K \otimes_R B$. We use the same symbol d as the extension of d to B_K. Clearly $d \in \operatorname{Der}_K B_K$ is also locally nilpotent. We consider the K-automorphism $\exp(td)$ on $B_K[t] \cong {B_K}^{[1]}$. Without loss of generality, we may assume $\det J(\exp(td)) = 1$, where we denote $J(\exp(td))$ by the Jacobian matrix with respect to $\exp(td)$, that is, set $f_i = \exp(td)(x_i)$ and $g = \exp(td)(t)$, then

$$J(\exp(td)) = \frac{\partial(f_1, \ldots, f_n, g)}{\partial(x_1, \ldots, x_n, t)}.$$

Then $g = t$ and for $1 \le i \le n$,

$$f_i = x_i + d(x_i)t + \sum_{\ell \ge 2} \frac{1}{\ell!} d^\ell(x_i) t^\ell.$$

Therefore the coefficient of t in $\det J(\exp(td))$ is equal to $\operatorname{div}(d)$, which implies that $\operatorname{div}(d) = 0$. □

For $f_1, \ldots, f_{n-1} \in B$, let $F = (f_1, \ldots, f_{n-1})$ and we define the **Jacobian derivation** Δ_F with respect to $f_1, \ldots, f_{n-1}$ by, for any $g \in B$,

$$\Delta_F(g) = \det \frac{\partial(f_1, \ldots, f_{n-1}, g)}{\partial(x_1, \ldots, x_n)}.$$

Proposition 1.10. *For $f_1, \ldots, f_{n-1} \in B$, let $F = (f_1, \ldots, f_{n-1})$. Then the following assertions hold true.*

(a) $\Delta_F = 0$ *if and only if $f_1, \ldots, f_{n-1}$ are algebraically dependent over R.*

(b) *If $\Delta_F \neq 0$, then B^{Δ_F} is the algebraic closure of $R[f_1, \ldots, f_{n-1}]$ in B.*

Proof. (Following [**42**]) **(a)** Suppose that $f_1, \ldots, f_{n-1}$ are algebraically dependent over R. Let $A = R[f_2, \ldots, f_{n-1}]$. Then f_1 is algebraic over A. Choose a non-zero polynomial $0 \neq P(t) \in A[t] \cong_A A^{[1]}$ of minimal degree such that $P(f_1) = 0$. Then

$$0 = \Delta_{(P(f_1), f_2, \ldots, f_{n-1})} = P'(f_1)\Delta_{(f_1, \ldots, f_{n-1})} = P'(f_1)\Delta_F,$$

where $P'(t) = dP(t)/dt$. By the minimality of the degree of $P(t)$, we have $P'(f_1) \neq 0$, hence $\Delta_F = 0$.

Conversely, we suppose that $f_1, \ldots, f_{n-1}$ are algebraically independent over R. Here, we choose $f_n \in B$ which is transcendence over $R[f_1, \ldots, f_{n-1}]$ and set $A = R[f_1, \ldots, f_n]$. Then B is algebraic over A, hence there exist $0 \neq P_i(y_1, \ldots, y_{n+1}) \in R[y_1, \ldots, y_{n+1}] \cong_R R^{[n+1]}$ such that $P_i(f_1, \ldots, f_n, x_i) = 0$ for $1 \le i \le n$. If $\partial P_i/\partial y_{n+1} = 0$, then

$P_i \in R[y_1, \ldots, y_n]$. This implies that $f_1, \ldots, f_n$ are algebraically dependent over R. This contradicts the choice of f_n. Hence $\partial P_i/\partial y_{n+1} \neq 0$ for each i. Thus we may assume that $\deg_{y_{n+1}} P_i$ is minimal.

Define the homomorphism of R-algebras $\varphi_i : R[y_1, \ldots, y_{n+1}] \to B$ by $\varphi_i(y_j) = f_j$ for $1 \leq j \leq n$ and $\varphi_i(y_{n+1}) = x_i$. Then, for $1 \leq i, j \leq n$,

$$0 = \frac{\partial}{\partial x_j}(\varphi_i(P_i)) = \sum_{\ell=1}^{n} \varphi_i\left(\frac{\partial P_i}{\partial y_\ell}\right)\frac{\partial f_\ell}{\partial x_j} + \varphi_i\left(\frac{\partial P_i}{\partial y_{n+1}}\right)\frac{\partial x_i}{\partial x_j}.$$

Let $M = (\varphi_i(\partial P_i/\partial y_j))_{1\leq i,j\leq n}$ and $N = -\sum_{i=1}^{n} \varphi_i(\partial P_i/\partial y_{n+1})\mathbf{e}_i$, wehre $\mathbf{e}_i$ is the standard basis for $R^{\oplus n}$. Then $J(F, f_n) \times M = N$, hence

$$\det(J(F, f_n)) \det M = \det N = (-1)^n \prod_{i=1}^{n} \varphi_i\left(\frac{\partial P_i}{\partial y_{n+1}}\right).$$

By the minimality of the degree of P_i with respect to y_{n+1}, we have $\varphi_i(\partial P_i/\partial y_{n+1}) \neq 0$. Thus $\det(J(F, f_n)) \neq 0$, which implies that $\Delta_F(f_n) \neq 0$, namely, $\Delta_F \neq 0$.

(b) Suppose that $\Delta_F \neq 0$. Then $\mathrm{tr.deg}_R\,(B^{\Delta_F}) \leq n-1$. Furthermore, the assertion (a) implies that $f_1, \ldots, f_{n-1}$ are algebraically independent over R. Since $R^{[n-1]} \cong_R R[f_1, \ldots, f_{n-1}] \subset B^{\Delta_F}$, we have $\mathrm{tr.deg}_R\,(B^{\Delta_F}) = n-1$. Therefore B^{Δ_F} is algebraic over $R[f_1, \ldots, f_{n-1}]$. By (P.1), B^{Δ_F} is algebraically closed in B, hence it is the algebraic closure of $R[f_1, \ldots, f_{n-1}]$ in B. $\square$

3. Properties of higher derivations

In this chapter, we explain fundamental properties of higher derivations over an integral domain R of characteristic $p \geq 0$. Let B be an R-domain. Let $D = \{D_\ell\}_{\ell=0}^{\infty}$ be a family of R-linear maps $D_\ell : B \to B$ for $\ell \geq 0$. We say that D is a **higher R-derivation** on B if, for $f, g \in B$ and $\ell \geq 0$,

(a) $D_0 = \mathrm{id}_B$,

(b) $D_\ell(fg) = \displaystyle\sum_{i+j=\ell} D_i(f)D_j(g)$.

Note that for a higher R-derivation $D = \{D_\ell\}_{\ell=0}^{\infty}$, it is easy to show that D_1 is an R-derivation on B. We say also, D is a **rational higher R-derivation** if every D_ℓ is a homomorphism of R-modules from B to $Q(B)$.

For a higher R-derivation $D = \{D_\ell\}_{\ell=0}^{\infty}$ on B, we define the map $\varphi_D : B \to B[[t]]$, where $B[[t]]$ is the formal power series ring in one variable over B, by

$$\varphi_D(f) = \sum_{i=0}^{\infty} D_i(f)t^i$$

for $f \in B$ (if D is rational, then we consider $\varphi_D : B \to Q(B)[[t]]$). The above condition (b) implies that φ_D is a homomorphism of R-algebras, condition (a) implies that $\varphi_D(f)|_{t=0} = f$. We call the mapping φ_D the *homomorphism associated to* D. We denote B^D by the intersections of the kernel of D_ℓ for $\ell \geq 1$, that is,

$$B^D = \bigcap_{\ell \geq 1} \ker D_\ell.$$

We say that D is **trivial** if $B^D = B$. For a (rational) higher R-derivation $D = \{D_\ell\}_{\ell=0}^{\infty}$ on B, we have a unique higher $Q(R)$-derivation $\overline{D} = \{\overline{D}_\ell\}_{\ell=0}^{\infty}$ on $Q(B)$ such that $\overline{D}_\ell|_B = D_\ell$ for any $\ell \geq 0$. We call $\overline{D}$ the *extension of* D *to* $Q(B)$. For more details on the construction of $\overline{D}$, we refer to [**34**, Section 1]. It is clear that $Q(B^D) \subset Q(B)^{\overline{D}}$.

A higher R-derivation $D = \{D_\ell\}_{\ell=0}^{\infty}$ on B is **locally finite** if D satisfies

(c) for any $f \in B$, there exists a positive integer $N_f \geq 1$ such that $D_\ell(f) = 0$ for any $\ell \geq N_f$,

and is **iterative** if D satisfies

(d) $D_i \circ D_j = \binom{i+j}{j} D_{i+j}$ for any $i, j \geq 0$.

When $D = \{D_\ell\}_{\ell=0}^{\infty}$ satisfies the above conditions (a), (b), (c) and (d), we say D is a **locally finite iterative higher R-derivation**.

Properties of higher derivations. We denote some fundamental notations and recall properties for higher derivations (P.9)–(P.11) as below. Let $D = \{D_\ell\}_{\ell=0}^{\infty}$ be a higher R-derivation on B.

(P.9) B^D is an algebraically closed subring of B.

(P.10) If D is locally finite, then B^D is factorially closed in B.

(P.11) If $p = 0$ and D is locally finite and iterative, then for $\ell \geq 1$, $\ell! D_\ell = D_1$ and D_1 is a locally nilpotent derivation on B.

Here, we prove (P.9) and (P.10).

Proof. Let $D = \{D_\ell\}_{\ell=0}^{\infty}$ be a higher R-derivations of B and let φ_D : $B \to B[[t]]$ be the homomorphism associated to D. We note also that B^D is the ring of invariant for φ_D, namely, $B^D = \{b \in B \mid \varphi_D(b) = b\}$.

(P.9) Let $f \in B$ such that it is algebraic over B^D. Then there exists $m \geq 1$ and $b_0, \ldots, b_m \in B^D$ such that

$$b_0 f^m + b_1 f^{m-1} + \cdots + b_{m-1} f + b_m = 0.$$

Applying φ_D for the above equation, we have

$$b_0 \varphi_D(f)^m + b_1 \varphi_D(f)^{m-1} + \cdots + b_{m-1}\varphi_D(f) + b_m = 0.$$

Then $\varphi_D(f) \in B[[t]]$ is algebraic over B. By *Proposition 1.6*, we have $\varphi_D(f) \in B$, which implies that $\varphi_D(f) = f$. Thus $f \in B^D$.

(P.10) Since $D = \{D_\ell\}_{\ell=0}^{\infty}$ is locally finite, the image of φ_D is contained in $B[t]$. Here, we define the degree function with respect to D

$$\deg_D : B \to \mathbb{Z} \cup \{-\infty\}$$

by $\deg_D f := \deg_t(\varphi_D(f))$ for $f \in B$. It is easy to show that $\deg_D$ gives a non-negative degree function on B and $\mathcal{F}_0 = B^D$. By *Proposition 1.4* (d), B^D is factorially closed in B. □

4. Algebraically closed subalgebras

This section is based on [**37**, Section 2] and [**55**, Section 3]. Let R be an integral domain of characteristic $p \geq 0$ and let B be an R-domain. The aim of this section is to prove *Theorem 1.14* which gives some characterizations of algebraically closed R-subalgebras of B.

We denote the set of R-subalgebras of B whose transcendence degree r over R by $\mathfrak{S}(r, B)$, that is,

$$\mathfrak{S}(r, B) := \{A \mid A \text{ is an } R\text{-subalgebra of } B \text{ of } \operatorname{tr.deg}_R A = r\}.$$

Here, this set $\mathfrak{S}(r, B)$ is ordered by partial inclusion. First of all, we prove three lemmas (*Lemmas 1.11*, *1.12* and *1.13*) needed later. From now on, we set $K := Q(R)$, $A_K := K \otimes_R A$ and $B_K := K \otimes_R B$.

Lemma 1.11. *Let A be an R-subalgebra of B such that $Q(A) \cap B = A$. Then the following two conditions are equivalent*:

(i) *A is integrally* (*resp. algebraically*) *closed in B.*

(ii) *A_K is integrally* (*resp. algebraically*) *closed in B_K.*

Proof. **(i) $\Longrightarrow$ (ii)** Suppose that A is integrally closed in B. If $\beta \in B_K$ is integral over A_K, then there exist $m \geq 1$ and $\alpha_1, \ldots, \alpha_m \in A_K$ such that

$$\beta^m + \alpha_1 \beta^{m-1} + \cdots + \alpha_{m-1}\beta + \alpha_m = 0.$$

We choose elements $a, b \in R \setminus \{0\}$ with $a\alpha_i \in A$ and $b\beta \in B$ for any $1 \leq i \leq m$ and set $c := ab \in R$. Then we have

$$(c\beta)^m + c\alpha_1(c\beta)^{m-1} + \cdots + c^{m-1}\alpha_{m-1}(c\beta) + c^m\alpha_m = 0.$$

Since $c^i\alpha_i \in A$ and $c\beta \in B$, $c\beta$ is integral over A, and so $c\beta \in A$. Hence $\beta \in A_K$.

(ii) $\Longrightarrow$ **(i)** Suppose that A_K is integrally closed in B_K. If $b \in B$ is integral over A, then it is also integral over A_K as an element of B_K. Hence $b \in A_K \cap B \subset Q(A) \cap B = A$. Therefore A is integrally closed in B. □

Lemma 1.12. *Suppose that the characteristic of R equals zero and B is finitely generated over R. For an R-subalgebra A of B, the following two conditions are equivalent*:

(i) *There exists an R-derivation d on B such that $B^d = A$.*

(ii) *$Q(A) \cap B = A$ and there exists a K-derivation δ of B_K such that $(B_K)^\delta = A_K$.*

Proof. We may assume that B is generated by $b_1, \ldots, b_r \in B$ over R.

(i) $\Longrightarrow$ **(ii)** Let d be an R-derivation on B such that $B^d = A$. By (P.1), A is algebraically closed in B. By *Proposition 1.1*, we have $Q(A) \cap B = A$. We denote d_K by the natural extension of d to B_K. It is clear that A_K is contained in $(B_K)^{d_K}$. On the other hand, if $\beta \in (B_K)^{d_K} \subset B_K$, then $r\beta \in B$ for some $r \in R \setminus \{0\}$ and

$$0 = rd_K(\beta) = d_K(r\beta) = d(r\beta).$$

Therefore $r\beta \in B^d = A$, hence $\beta \in A_K$.

(ii) $\Longrightarrow$ **(i)** We suppose that $Q(A) \cap B = A$ and there exists a K-derivation δ on B_K with $(B_K)^\delta = A_K$. We can take a non-zero element $r \in R \setminus \{0\}$ such that $r\delta(b_i) \in A$ for $1 \leq i \leq r$. Here, we define the R-derivation d on B by $d := r\delta$. By the construction of d, we have $B^d = A$. □

Lemma 1.13. *Suppose that $B \cong_R R^{[n]}$ is the polynomial ring in n variables over R. Let A be an R-subalgebra of B. Then the following conditions are equivalent*:

(i) *A is algebraically closed in B and $Q(B)$ is separably generated over $Q(A)$.*

(ii) *There exists a rational higher R-derivation D on B such that $A = B^D$ and $Q(B)^{\overline{D}} = Q(A)$, where $\overline{D}$ denotes the extension of D to $Q(B)$.*

Proof. **(ii)** $\Longrightarrow$ **(i)** By (P.1), A is algebraically closed in B. Since $\overline{D}$ is a higher K-derivation on $Q(B)$ with $Q(A) = Q(B)^{\overline{D}}$, it follows from [**26**, Theorem (2.3)] that $Q(B)/Q(A)$ is a regular field extension. In particular, the field extension $Q(B)/Q(A)$ is separable.

(i) $\Longrightarrow$ **(ii)** Since B_K is normal and A is algebraically closed in B, we know that $Q(A)$ is algebraically closed in $Q(B)$. So $Q(A)/Q(B)$ is a regular field extension. It follows from [**73**, Theorem 1] that there exists a higher K-derivation $\widetilde{D} = \{\widetilde{D}_\ell\}_{\ell=0}^{\infty}$ on $Q(B)$ such that $Q(B)^{\widetilde{D}} = Q(A)$. Set $D_\ell = \widetilde{D}_\ell|_B$ for each non-negative integer ℓ and set $D = \{\widetilde{D}_\ell\}_{\ell=0}^{\infty}$. Then D is a rational higher R-derivation on B and $A \subset B^D$. Since

$$B^D \subset Q(B^D) \cap B \subset Q(B)^{\widetilde{D}} \cap B = Q(A) \cap B = A,$$

we have $A = B^D$.

We claim that $\overline{D} = \widetilde{D}$. Indeed, let $\varphi_{\widetilde{D}} : Q(B) \to Q(B)[[t]]$ be the homomorphism associated to $\widetilde{D}$. For $f/g \in Q(B)$ with $f, g \in B$ and $g \neq 0$, we have

$$\varphi_{\widetilde{D}}\left(\frac{f}{g}\right) = \frac{\varphi_{\widetilde{D}}(f)}{\varphi_{\widetilde{D}}(g)}$$

because $\varphi_{\widetilde{D}}$ is a homomorphism of K-algebras. Since $\varphi_{\widetilde{D}}(f) = \varphi_D(f)$ and $\varphi_{\widetilde{D}}(g) = \varphi_D(g)$, we know that $\varphi_{\widetilde{D}}(f/g) = \varphi_{\overline{D}}(f/g)$ by the definition of the extension $\overline{D}$ of D to $Q(B)$. So $\varphi_{\widetilde{D}} = \varphi_{\overline{D}}$ and hence $\widetilde{D} = \overline{D}$. Therefore $Q(A) = Q(B)^{\widetilde{D}} = Q(B)^{\overline{D}}$. $\square$

The following is the main result in this section.

Theorem 1.14. *Let R be an integral domain of characteristic $p \geq 0$ and let B be an R-domain of transcendence degree n over R. For $1 \leq r \leq n$ and $A \in \mathfrak{S}(r, B)$, the following conditions are equivalent*:

(i) *A is algebraically closed in B.*

(ii) *A is integrally closed in B and $Q(A) \cap B = A$.*

(iii) *A is a maximal element of $\mathfrak{S}(r, B)$.*

If the characteristic of R equals zero and B is finitely generated over R, then the condition (i) *is equivalent to the following condition* (iv):

(iv) *There exists an R-derivation d of B such that $B^d = A$.*

If $B \cong_R R^{[n]}$ and $Q(B)$ is separably generated over $Q(A)$, then the condition (i) *is equivalent to the following condition* (v):

(v) *There exists a rational higher R-derivation D on B such that $B^D = A$ and $Q(B)^{\overline{D}} = Q(A)$, where $\overline{D}$ denotes the extension of D to $Q(B)$.*

Proof. The part "(i) $\Longleftrightarrow$ (ii)" follows from *Proposition 1.1*. When $B \cong_R R^{[n]}$ and $Q(B)$ is separably generated over $Q(A)$, the part "(i) $\Longleftrightarrow$ (v)" follows from *Lemma 1.13*.

(i) $\Longrightarrow$ (iii) Let $\widetilde{A}$ be any element of $\mathfrak{S}(r, B)$ containing A. Since A and $\widetilde{A}$ have the same transcendence degree over R, the ring extension $\widetilde{A}$ over A is algebraic. Hence $\widetilde{A} = A$, which implies A is a maximal element of $\mathfrak{S}(r, B)$.

(iii) $\Longrightarrow$ (i) We denote $\overline{A}$ by the algebraic closure of A in B. Then $\overline{A}$ is an element of $\mathfrak{S}(r, B)$. Since $A \subset \overline{A}$ and A is a maximal element of $\mathfrak{S}(r, B)$, we have $A = \overline{A}$.

Next, in order to prove the equivalence of (ii) and (iv), we assume further that the characteristic of R equals zero and B is finitely generated over R.

(ii) $\Longrightarrow$ (iv) By *Lemma 1.11*, A_K is integrally closed in B_K. Since K is a field, it follows from [**60**, Theorem 5.4] that there exists a K-derivation δ of B_K such that $(B_K)^\delta = A_K$. By *Lemma 1.12* and the assumption on A, we have an R-derivation d of B satisfying $B^d = A$.

(iv) $\Longrightarrow$ (ii) Suppose that there exists an R-derivation d on B with $B^d = A$. By *Lemma 1.12*, $Q(A) \cap B = A$ and there exists a K-derivation δ of B_K such that $(B_K)^\delta = A_K$. Then A_K is integrally closed in B_K. By *Lemma 1.11*, A is integrally closed in B. □

CHAPTER 2

Closed polynomials in polynomial rings

In this chapter, we assume that $B \cong_R R^{[n]}$ is the polynomial ring in n variables over an integral domain R of characteristic $p \geq 0$. Set a system of variables of B by $x_1, \ldots, x_n$, that is, $B = R[x_1, \ldots, x_n]$. A polynomial $f \in B$ is a **closed polynomial** over R if $f \notin R$ and the ring $R[f]$ is integrally closed in B. A polynomial $f \in B$ is called a **variable** (or **coordinate**) over R if there exist $g_2, \ldots, g_n \in B$ such that $R[f, g_2, \ldots, g_n] = B$, or equivalently, there exists an R-automorphism φ of B such that $\varphi(x_1) = f$.

Proposition 2.1. *If $f \in B$ is a variable over R, then it is a closed polynomial over R.*

Proof. It is clear that f is non-constant. Take $g_2, \ldots, g_n \in B$ such that $R[f, g_2, \ldots, g_n] = B$. Define the $\mathbb{Z}$-grading $\mathfrak{g}$ on B by $\deg_{\mathfrak{g}} f = 0$ and $\deg_{\mathfrak{g}} g_i = 1$. Then $\deg_{\mathfrak{g}}$ gives a positive degree function on B over $R[f]$. By *Proposition 1.4* (d), $R[f]$ is factorially closed in B. Therefore, f is a closed polynomial over R. □

We denote the set $\{R[f] \mid f \in B \setminus R\}$ of R-subalgebras of B by $S(1, B)$. It is clear that the set is partially ordered by inclusion, and $S(1, B) \subsetneq \mathfrak{S}(1, B)$. From now on, we set $K := Q(R)$, $A_K := K \otimes_R A$ and $B_K := K \otimes_R B \cong_K K^{[n]}$ for an R-subalgebra A of B.

Lemma 2.2. *Let $S \in \mathfrak{S}(1, B_K)$. If S is maximal in $\mathfrak{S}(1, B_K)$, then $S \in S(1, B_K)$. In particular, the set of maximal element of $S(1, B_K)$ and the set of maximal element of $\mathfrak{S}(1, B_K)$ are coincide.*

Proof. Let $S \in \mathfrak{S}(1, B_K)$ be a maximal element of $\mathfrak{S}(1, B_K)$. By *Theorem 1.14*, S is algebraically closed in B_K, and has transcendence degree 1 over K. Since B_K is a normal domain, S is a Dedekind K-subalgebra of the polynomial ring $B_K \cong_K K^{[n]}$. It follows from [**75**, Theorem 8] that there exists $f \in B_K \setminus K$ such that $S = K[f]$. Hence $S \in S(1, B_K)$ and it is maximal as an element of $S(1, B_K)$.

Conversely, let $K[f] \in S(1, B_K)$ be a maximal element of $S(1, B_K)$. Let $S = \overline{K[f]}$ be the algebraic closure of $K[f]$ in B_K. Then $S \in \mathfrak{S}(1, B_K)$. Since S is algebraically closed in B_K, by *Theorem 1.14*, S is a maximal element of $\mathfrak{S}(1, B_K)$. From the above discussion, $S = K[g]$ for some $g \in B_K \setminus K$. Then we have $K[f] \subset S = K[g]$. However, since $S \in S(1, B_K)$, we have $K[f] = S$, which implies that $K[f]$ is a maximal element of $\mathfrak{S}(1, B_K)$. $\square$

1. Characterizations of closed polynomials over domains

Let k be a field and let $k[\mathbf{X}] \cong_k k^{[n]}$ be the polynomial ring in n variables over k. In this case, closed polynomials in $k[\mathbf{X}]$ are studied by several mathematicians. Historically, in 1988, Nowicki and Nagata [**62**] introduced the notation of closed polynomials for understanding derivations and their kernels. Around the same time, in 1989, Stein [**71**] announced the concepts about the total reducibility order of a polynomial. This concept is essentially the same as closed polynomials. However, his approach was differ from [**62**]. After that, in 2007, Arzhantsev and Petravchuk [**3**] improved these results as below.

Theorem 2.3. (cf. [**3**, Theorem 1]) *Let k be a field and let $k[\mathbf{X}] \cong_k k^{[n]}$ be the polynomial ring in n variables over k. Denote $\bar{k}$ by an algebraically closed field containing k. For a non-constant polynomial $f \in k[\mathbf{X}] \setminus k$, the following conditions are equivalent*:

(i) *f is a closed polynomial over k.*

(ii) *$k[f]$ is a maximal element of $S(1, k[\mathbf{X}])$.*

If the characteristic of k equals zero, then the condition (i) *is equivalent to the following condition* (iii):

(iii) *There exists a k-derivation d on $k[\mathbf{X}]$ such that $k[\mathbf{X}]^d = k[f]$.*

If k is a perfect field, then the condition (i) *is equivalent to the following conditions* (iv) *and* (v):

(iv) *$f - \lambda$ is irreducible over $\bar{k}$ for all but finitely many $\lambda \in \bar{k}$.*

(v) *$\#\{\lambda \in \bar{k} \mid f - \lambda$ is reducible $\} < \deg f$, where $\deg$ is the standard degree function on $k[\mathbf{X}]$.*

The condition (iv) in *Theorem 2.3* gives a geometrical meaning for closed polynomials over an algebraically closed field k. Let $f \in k[\mathbf{X}] \setminus k$ and let

$$\Phi_f : \mathbb{A}_k^n \cong_k \operatorname{Spec} k[\mathbf{X}] \to \operatorname{Spec} k[f] \cong_k \mathbb{A}_k^1$$

be the surjective morphism defined by the natural inclusion $k[f] \to k[\mathbf{X}]$. Then the condition (iv) in *Theorem 2.3* implies that f is a closed polynomial if and only if general fibers of Φ_f are irreducible and reduced.

The following is the main result in this section, which is based on [**37**, Sections 3 and 4]. This theorem is a generalization of *Theorem 2.3* in the case where the coefficient ring of the polynomial ring is an arbitrary integral domain.

Theorem 2.4. *Let R be an integral domain and $B \cong_R R^{[n]}$ the polynomial ring in n variables over R. Let $f \in B \setminus R$ be a non-constant polynomial such that $K[f] \cap B = R[f]$. Then the following conditions* (i) *and* (ii) *are equivalent*:

(i) *f is a closed polynomial over R.*

(ii) *$K[f]$ is a maximal element of $S(1, B_K)$.*

Moreover, if the characteristic of R equals zero (*resp. if the field extension $Q(B)/Q(R[f])$ is separable*)*, then the condition* (i) *is equivalent to the following condition* (iii) (*resp.* (iv)):

(iii) *There exists an R-derivation d on B such that $B^d = R[f]$.*

(iv) *There exists a rational higher R-derivation D on B such that $B^D = R[f]$ and $Q(R[f]) = Q(B)^{\overline{D}}$, where $\overline{D}$ denotes the extension of D to $Q(B)$.*

Proof. The assertions follow from *Theorem 1.14* and *Lemma 2.2*. □

We give a sufficient condition for a non-constant polynomial $f \in B \setminus R$ over an integral domain R to be satisfied $K[f] \cap B = R[f]$. Later, in *Lemma 2.18*, we shall give a necessary and sufficient condition in the case where R is a UFD.

Example 2.5. Let R and B be the same as in *Theorem 2.4*. Let $f \in B \setminus R$. If the ideal generated by the coefficients of $f - f(0, \ldots, 0)$ in R equals R, then $K[f] \cap B = R[f]$.

Proof. It follows from [**68**, Lemma 2.6.1] that $K(f) \cap B = R[f]$. Since $R[f] \subset K[f] \cap B \subset K(f) \cap B$, we have $K[f] \cap B = R[f]$. □

We give some remarks and examples on *Theorem 2.4*. Let R, $K = Q(R)$ and B be the same as in *Theorem 2.4*. Let $f \in B \setminus R$ be a non-constant polynomial. We know that the part "(i) $\Longrightarrow$ (ii)" of *Theorem 2.4* remains true without assuming $K[f] \cap B = R[f]$ (see the proof of [**37**, Theorem 3.1]). However, in order to prove the part "(ii) $\Longrightarrow$ (i)" of *Theorem 2.4*, we need the hypothesis $K[f] \cap B = R[f]$. We give an example below.

Example 2.6. Let k be a field and set $k[y] \cong_k k^{[1]}$, $R = k[y^2, y^3]$ and $B = R[x] \cong_R R^{[1]}$. Set $f = y^2x \in B \setminus R$. Then $K[f] = K[x]$, where $K = Q(R)$, and so $K[f]$ is a maximal element of $S(1, B_K)$ and $K[f] \cap B \supsetneq R[f]$. However, for $g = y^3x \in B \setminus R[f]$, we have $g^2 - y^2f^2 = 0$. So g is integral over $R[f]$. Therefore f is a not closed polynomial over R.

Here we prove the following result.

Proposition 2.7. *Let $f \in B \setminus R$ such that $K[f] \cap B = R[f]$. Then the following assertions hold true.*

(a) *If $K[f]$ is a maximal element of $S(1, B_K)$, then $R[f]$ is a maximal element of $S(1, B)$.*

(b) *Suppose that R is a UFD. Then $R[f]$ is a maximal element of $S(1, B)$ if and only if $K[f]$ is a maximal element of $S(1, B_K)$.*

Proof. **(a)** Let $R[g]$ be any element of $S(1, B)$ containing $R[f]$. By the hypothesis, $K[f] = K[g]$. Then $R[f] = K[g] \cap B \supseteq R[g]$ and hence $R[f] = R[g]$. This proves the assertion.

(b) It suffices to prove the "only if" part. It follows from [**32**, Theorem 1.1] that f is a closed polynomial over R. Then it is also closed over K and hence, by *Theorem 2.3*, $K[f]$ is a maximal element of $S(1, B_K)$. □

In *Proposition 2.7*, the converse of the assertion (a) does not hold true in general. We give an example below.

Example 2.8. Let k be a field of characteristic $p > 0$. Set $R = k[z^p, z^{p+1}] \subset k[z] \cong_k k^{[1]}$, $R[x, y] \cong_R R^{[2]}$ and $f = x^p + z^py^p \in R[x, y]$. Then R is not normal and $K := Q(R) = k(z)$. The following assertions hold true.

(a) $R[f] = K[f] \cap R[x, y]$.

(b) $R[f]$ is a maximal element of $S(1, R[x, y])$.

(c) $K[f]$ is not a maximal element of $S(1, K[x, y])$.

(d) f is not a closed polynomial over R.

Proof. **(a)** The ideal generated by the coefficients of $f - f(0, 0)$ (as a polynomial over R) in R equals $(1, z^p) = R$. By *Example 2.5*, we see that $R[f] = K[f] \cap R[x, y]$.

(b) It suffices to show that f is irreducible as a polynomial over R. Let $g, h \in R[x, y] \setminus \{0\}$ be polynomials such that $f = gh$. Suppose that $\deg g$ and $\deg h$ are positive, where we consider the standard degree function on $R[x,y]$ over R. Since $f = (x+zy)^p$ as a polynomial in $k[x,y,z] \cong_k k^{[3]}$, we know that $g = \alpha(x+zy)^{\deg g}$ and $h = \beta(x+zy)^{\deg h}$ for some $\alpha, \beta \in k^*$. However, since $0 < \deg g, \deg h < p$, we have

$$\alpha(x+zy)^{\deg g} \notin R[x,y], \ \ \beta(x+zy)^{\deg h} \notin R[x,y].$$

This is a contradiction. Therefore f is irreducible as a polynomial over R.

(c) The assertion follows from $K[f] \subsetneq K[x+zy]$.

(d) The assertion follows from *Theorem 2.4* (Of course, we can prove the assertion directly). $\square$

The author has not yet given an example of $f \in B \setminus R$ such that $K[f] \cap B = R[f]$, $R[f]$ is a maximal element of $S(1,B)$ and f is not closed in B, where $B = R^{[n]}$ is the polynomial ring in n variables over an integral domain R of characteristic zero.

2. Criteria of closed polynomials

This section is based on [**54**, Sections 3 and 4]. Let $B \cong_R R^{[n]}$ be the polynomial ring in n variables over an integral domain R of characteristic $p \geq 0$. Set a system of variables of B by $x_1, \ldots, x_n$, that is, $B = R[x_1, \ldots, x_n]$ and set $K = Q(R)$. For polynomials $f, g \in B$, we write $f \sim_R g$ if there exists $r \in R \setminus \{0\}$ such that $f = rg$. For $\mathbf{w} = (w_1, \ldots, w_n) \in \mathbb{N}^n$, we consider the $\mathbb{Z}$-grading $\mathfrak{g}_{\mathbf{w}}$ on B with respect to $\mathbf{w}$ by $\deg_{\mathfrak{g}_{\mathbf{w}}} x_i = w_i$ for $1 \leq i \leq n$, and set $\deg_{\mathbf{w}} := \deg_{\mathfrak{g}_{\mathbf{w}}}$. We denote simply deg by the standard degree function on B. Since each w_i is non-negative, $\deg_{\mathbf{w}}$ gives a non-negative degree function on B.

Let $\mathfrak{g}_{\mathbf{w}}$ be a non-negative $\mathbb{Z}$-grading on B for some $\mathbf{w} \in \mathbb{N}^n$. A homogeneous polynomial $f \in B$ for $\mathfrak{g}_{\mathbf{w}}$ is **decomposable** with respect to $\mathfrak{g}_{\mathbf{w}}$ if there exists a homogeneous polynomial $g \in B$ for $\mathfrak{g}_{\mathbf{w}}$ such that $f \sim_R g^m$ for some $m \geq 2$, also say f is **primitive** with respect to $\mathfrak{g}_{\mathbf{w}}$ if it is not decomposable with respect to $\mathfrak{g}_{\mathbf{w}}$.

For a polynomial $f \in B$, we define $\hat{f} \in B_K$ by

$$\hat{f} := \gcd(f_{x_1}, \ldots, f_{x_n}),$$

where f_{x_i} is the partial derivative of f with respect to x_i and we take the greatest common divisor of $f_{x_1}, \ldots, f_{x_n}$ as polynomials in B_K, hence $\hat{f} \in B_K$.

Definition 2.9. Let $f \in B$ and $\mathbf{w} \in \mathbb{N}^n$. Assume that $\deg_{\mathbf{w}} f \geq 2$. Then we denote by $\mathrm{LD}_{\mathbf{w}}(f)$ the smallest positive prime integer dividing $\deg_{\mathbf{w}} f$.

The number $\mathrm{LD}_{\mathbf{w}}(f)$ is the most important concept for *Theorem 2.11.* We give some examples as below.

Example 2.10. For $f = x^9 + x^6y^2 + x^3y^4 \in \mathbb{Z}[x, y] \cong_{\mathbb{Z}} \mathbb{Z}^{[2]}$, we can easily see that:

(a) for $\mathbf{u} = (1, 1)$, $\deg_{\mathbf{u}} f = \deg f = 9$ and $\mathrm{LD}_{\mathbf{u}}(f) = 3$,

(b) for $\mathbf{v} = (0, 1)$, $\deg_{\mathbf{v}} f = 4$ and $\mathrm{LD}_{\mathbf{v}}(f) = 2$,

(c) for $\mathbf{w} = (1, 2)$, $\deg_{\mathbf{w}} f = 11$ and $\mathrm{LD}_{\mathbf{w}}(f) = 11$.

As seeing the above examples, it is clear that if $\deg_{\mathbf{w}} f$ is a prime number, then $\deg_{\mathbf{w}} f = \mathrm{LD}_{\mathbf{w}}(f)$. The aim of this section is to prove the following theorem.

Theorem 2.11. *Let $B \cong_R R^{[n]}$ be the polynomial ring in n variables over an integral domain R. Let $f \in B \setminus R$ be a non-constant polynomial such that $K[f] \cap B = R[f]$. Then the following assertions hold true.*

(a) *Suppose that f is homogeneous for some $\mathbb{Z}$-grading $\mathfrak{g}_{\mathbf{w}}$, $\mathbf{w} \in \mathbb{N}^n$. Then f is a closed polynomial if and only if it is primitive in $B \otimes_R K$with respect to $\mathfrak{g}_{\mathbf{w}}$.*

(b) *Suppose that the characteristic R is zero. If there exists $\mathbf{w} \in \mathbb{N}^n$ such that $\deg_{\mathbf{w}} f = 1$ or, $\deg_{\mathbf{w}} f \geq 2$ and*

$$\deg_{\mathbf{w}} \hat{f} < \frac{\mathrm{LD}_{\mathbf{w}}(f) - 1}{\mathrm{LD}_{\mathbf{w}}(f)} \deg_{\mathbf{w}} f,$$

then f is a closed polynomial.

First of all, we prove three lemmas (*Lemmas 2.12*, *2.13* and *2.14*) needed later.

Lemma 2.12. *Let $f \in B \setminus R$ such that $K[f] \cap B = R[f]$. For $g \in B$ such that $f \sim_R g$, we have $R[f] = R[g]$.*

Proof. Since $f \sim_R g$, there exists $r \in R \setminus 0$ such that $f = rg$, hence $R[f] \subset R[g]$. On the other hand,

$$R[g] \subset K[g] \cap B = K[f] \cap B = R[f].$$

Therefore $R[f] = R[g]$. □

Lemma 2.13. *Let $f \in B \setminus R$. Assume that f is homogeneous for some $\mathfrak{g}_{\mathbf{w}}$ of $\deg_{\mathbf{w}} f > 0$, where $\mathbf{w} \in \mathbb{N}^n$. For $g \in B$ such that $f \in R[g]$ and $g(0, \ldots, 0) = 0$, the following assertions hold true.*

(a) *g is homogeneous for $\mathfrak{g}_{\mathbf{w}}$.*

(b) *$f \sim_R g^m$ for some positive integer $m \geq 1$.*

Proof. Since f is homogeneous for $\mathfrak{g}_{\mathbf{w}}$ of $\deg_{\mathbf{w}} f > 0$, we see that $f(0,\dots,0) = 0$. Hence we can write f as

$$f = u(g) = \sum_{i=1}^{m} u_{m-i} g^i,$$

where $u_i \in R, u_0 \neq 0$ and $m \geq 1$. Set $g_1 = \sum_{i=1}^{m} u_{m-i} g^{i-1}$. Then $f = gg_1$. Since f is homogeneous, g and g_1 are also homogeneous for $\mathfrak{g}_{\mathbf{w}}$. This completes the proof of the part (a).

Next, we look at the constant term of g_1. The term is $g_1(0,\dots,0) = u_{m-1}$, and $\deg_{\mathbf{w}} g_1 = (m-1)\deg_{\mathbf{w}} g \geq 0$. If $m = 1$, then $f = u_0 g$ and hence $f \sim_R g$. If $m \geq 2$, then we have $u_{m-1} = 0$ and $g_1 = gg_2$, where $g_2 = \sum_{i=1}^{m} u_{m-i} g^{i-2}$. By the above argument, g_2 is also homogeneous. Using the same argument repeatedly, we have $f = u_0 g^m$, which implies $f \sim_R g^m$. □

Lemma 2.14. *Let $\mathbf{w} \in \mathbb{N}^n$ and let $f, g \in B \setminus R$ with $f \in R[g]$. Assume that $\deg_{\mathbf{w}} f > 0$ and $f = u(g)$ for a polynomial $u(t) \in R[t] \cong_R R^{[1]}$ of degree $m \geq 1$. Then the following assertions hold true.*

(a) *$\deg_{\mathbf{w}} f = m \deg_{\mathbf{w}} g$, hence m divides $\deg_{\mathbf{w}} f$.*

(b) *If the characteristic of R equals zero, then*

$$\deg_{\mathbf{w}} \hat{f} \geq \frac{m-1}{m} \deg_{\mathbf{w}} f.$$

Proof. **(a)** We write $f = u(g)$ as follows:

$$f = u(g) = u_0 g^m + u_1 g^{m-1} + \cdots + u_{m-1} g + u_m,$$

for $u_0 \in R \setminus \{0\}$ and $u_1, \dots, u_m \in R$. Since $\deg_{\mathbf{w}} f > 0$, $\deg_{\mathbf{w}} g > 0$. This implies that $\deg_{\mathbf{w}} g^i \geq \deg_{\mathbf{w}} g^j$ if $i \geq j$. So,

$$\deg_{\mathbf{w}} f = \deg_{\mathbf{w}}(u(g)) = \deg_{\mathbf{w}}(u_0 g^m) = m \deg_{\mathbf{w}} g.$$

(b) Since $f = u(g)$, $f_{x_i} = u'(g) g_{x_i}$ for $1 \leq i \leq n$, where $u'(t) = du/dt$. This implies that each f_{x_i} is divided by $u'(g)$, so $u'(g)$ divides $\hat{f}$ as a polynomial over K. Since $\deg_{\mathbf{w}}$ is non-negative, this implies that $\deg_{\mathbf{w}} \hat{f} \geq \deg_{\mathbf{w}}(u'(g))$. On the other hand, since the characteristic of R equals zero, $mu_0 \neq 0$. Therefore $\deg_{\mathbf{w}} u'(g) = (m-1)\deg_{\mathbf{w}} g$, so we have

$$\deg_{\mathbf{w}} \hat{f} \geq \deg_{\mathbf{w}}(u'(g)) = (m-1)\deg_{\mathbf{w}} g = \frac{m-1}{m} \deg_{\mathbf{w}} f.$$

□

Now, we start the proof of *Theorem 2.11*.

Proof of Theorem 2.11. **(a)** Suppose that f is homogeneous for some $\mathbb{Z}$-grading $\mathfrak{g}_{\mathbf{w}}$, $\mathbf{w} \in \mathbb{N}^n$. It is clear that, if f is not primitive for $\mathfrak{g}_{\mathbf{w}}$, then f is not a closed polynomial over R.

Conversely, we suppose that f is primitive in $B \otimes_R K$ for $\mathfrak{g}_{\mathbf{w}}$. According to *Theorem 2.4*, it suffices to prove the maximality of the ring $K[f]$ in $S(1, B_K)$. Let $g \in B_K \setminus K$ be any polynomial such that $f \in K[g]$. Without loss of generality we may assume that $g(0, \ldots, 0) = 0$. By *Lemma 2.13* (a) and (b), g is homogeneous for $\mathfrak{g}_{\mathbf{w}}$ and $f \sim_K g^m$ for some positive integer $m \geq 1$. Since f is primitive, we have $m = 1$, hence $f \sim_K g$. By *Lemma 2.12*, we have $K[f] = K[g]$. Therefore, $K[f]$ is a maximal element of $S(1, B_K)$.

(b) By *Theorem 2.4*, it is enough to show the maximality of the ring $K[f]$ in $S(1, B_K)$. Let $g \in B_K \setminus K$ with $f \in K[g]$. Since $f \in K[g]$, there exists $u(t) \in K[t]$ of degree m such that $f = u(g)$. We write $u(t)$ as

$$u(t) = u_0 t^m + u_1 t^{m-1} + \cdots + u_{m-1} t + u_m,$$

for some $u_i \in K$ and $u_0 \neq 0$. By *Lemma 2.14* (a), $\deg_{\mathbf{w}} f = m \deg_{\mathbf{w}}(g)$. It is enough to show that $m = 1$. Indeed, if $m = 1$, then $f = u_0 g + u_1$. This implies $g \in K[f]$, so $K[f] = K[g]$.

If $\deg_{\mathbf{w}} f = 1$, then obviously $m = 1$. On the other hand, we suppose that $\mathbf{w} \in \mathbb{N}^n$ satisfies $\deg_{\mathbf{w}} f \geq 2$ and

$$\deg_{\mathbf{w}} \hat{f} < \frac{\mathrm{LD}_{\mathbf{w}}(f) - 1}{\mathrm{LD}_{\mathbf{w}}(f)} \deg_{\mathbf{w}} f.$$

Since the characteristic of R equals zero, by *Lemma 2.14* (b),

$$\deg_{\mathbf{w}} \hat{f} \geq \frac{m - 1}{m} \deg_{\mathbf{w}} f.$$

By comparing the above two inequalities, we have $\mathrm{LD}_{\mathbf{w}}(f) > m$. By using *Lemma 2.14* (a) again, we see that m divides $\deg_{\mathbf{w}} f$. But the number $\mathrm{LD}_{\mathbf{w}}(f)$ is the smallest positive prime number dividing $\deg_{\mathbf{w}} f$, hence $m = 1$. Therefore f is a closed polynomial over R. $\square$

By using *Theorem 2.11* (b), we have the following. This is a generalization of [**4**, Proposition 14].

Corollary 2.15. *Let $f \in B \setminus R$ such that $K[f] \cap B = R[f]$. Suppose that the characteristic of R is zero. If there exists $1 \leq i < j \leq n$ such that $f_{x_i} \neq 0$, $f_{x_j} \neq 0$ and $\hat{f} \in R[x_i]$, then f is a closed polynomial over R.*

Proof. If $\hat{f} \in R$, then $\deg_{\mathbf{w}} \hat{f} \leq 0$ for any $\mathbf{w} \in \mathbb{N}^n$. Hence the inequality in *Theorem 2.11* (b) is satisfied for any $\mathbf{w} \in \mathbb{N}^n$.

Suppose that $\hat{f} \in R[x_i] \setminus R$. Set $\mathbf{w}_{i,j} = (w_1, \ldots, w_n) \in \mathbb{N}^n$, where $w_i = 0$, $w_j = 2$ and $w_\ell = 1$ for $\ell \neq i, j$. Then $\deg_{\mathbf{w}_{i,j}} \hat{f} = 0$ and $\deg_{\mathbf{w}_{i,j}} \hat{f} \geq 2$. Then the inequality in *Theorem 2.11* (b) is satisfied for $\mathbf{w}_{i,j}$. Therefore f is a closed polynomial over R. □

We give some remarks on *Theorem 2.11* (b). In the case where R is a field, the assumption "$K[f] \cap B = R[f]$" is satisfied for any $f \in B$ automatically. Furthermore, in *Lemma 2.18*, we will give a necessary and sufficient condition for a polynomial f to be satisfied "$K[f] \cap B = R[f]$" when R is a UFD. For this reason, we can confirm that almost all a given polynomial is to be a closed polynomial. However, there are examples of closed polynomials which do not satisfy the assumption on *Theorem 2.11* (b) for any $\mathbf{w} \in \mathbb{N}^n$ as below:

Example 2.16. Let $f = x^6y^4 + x^4y^6 \in \mathbb{Q}[x, y] \cong_{\mathbb{Q}} \mathbb{Q}^{[2]}$. Then f is a closed polynomial over $\mathbb{Q}$, but it does not satisfy the assumption on *Theorem 2.11* (b) for any $\mathbf{w} \in \mathbb{N}^2$.

Proof. It is clear that f is primitive and homogeneous for the standard $\mathbb{Z}$-grading. By *Theorem 2.11* (a), f is a closed polynomial.

On the other hand, for any $\mathbf{w} = (w_1, w_2) \in \mathbb{N}^2$, we have

$$\deg_{\mathbf{w}} f = \max\{6w_1 + 4w_2, 4w_1 + 6w_2\}.$$

Hence $\deg_{\mathbf{w}} f$ is divided by at least 2, which implies that $\deg_{\mathbf{w}} f \geq 2$ and $\mathrm{LD}_{\mathbf{w}}(f) = 2$. We may assume that $\deg_{\mathbf{w}} f = 6w_1 + 4w_2$. Also, we can see easily that $\hat{f} = x^3y^3$, hence $\deg_{\mathbf{w}} \hat{f} = 3(w_1 + w_2)$. Then

$$\frac{\mathrm{LD}_{\mathbf{w}}(f) - 1}{\mathrm{LD}_{\mathbf{w}}(f)} \deg_{\mathbf{w}} f = 3w_1 + 2w_2 \leq 3w_1 + 3w_2 = \deg_{\mathbf{w}} \hat{f}.$$

Therefore f does not satisfy the assumption on *Theorem 2.11* (b) for any $\mathbf{w} \in \mathbb{N}^2$. □

Also, in the case where the characteristic of R is positive, *Theorem 2.11* (b) and *Corollary 2.15* do not hold true in general. We give two examples below.

Example 2.17. Let k be a field of characteristic $p > 0$ and let $k[x, y] \cong_k k^{[2]}$. The following two polynomials are not closed.

(a) $f = x^p + y^p + (x + y)^{p-1}$.

(b) $g = x^p + y^p + x + y$.

Indeed, for (a), we can see easily that $\hat{f} = (x + y)^{p-2}$. Hence $\deg_{\mathbf{w}} f$ satisfies the assumption on *Theorem 2.11* (b) for $\mathbf{w} = (1, 1)$. However, $k[f] \subsetneq k[x + y]$, which implies that f is not a closed polynomial.

Also, for (b), we see that $\hat{g} = 1$. Hence $\deg_{\mathbf{w}} g$ satisfies the assumption on *Corollary 2.15*. However, $k[g] \subsetneq k[x+y]$, which implies that g is not a closed polynomial.

Finally, we give a necessary and sufficient condition for a polynomial f to be satisfied "$K[f] \cap B = R[f]$" when R is a UFD. This is a refinement of the statement in *Example 2.5*.

Lemma 2.18. *Suppose that R is a UFD. For $f \in B \setminus R$, we denote $c(f) \in R$ by the greatest common divisor of the coefficients of f. Then the following two conditions are equivalent*:

(i) $c(f - f(0, \ldots, 0)) \in R^*$.

(ii) $K[f] \cap B = R[f]$.

Proof. Without loss of generality, we may assume that $f(0, \ldots, 0) = 0$. Then $c(f - f(0, \ldots, 0)) = c(f)$. We note also $c(gh) = c(g)c(h)$ for $g, h \in B$.

(i) $\Longrightarrow$ (ii) Suppose that $c(f) \in R^*$. Let $g \in K[f] \cap B$. Then there exist $u_0, u_1, \ldots, u_m \in K$ such that

$$g = u_0 f^m + u_1 f^{m-1} + \cdots + u_{m-1} f + u_m.$$

Since $f(0, \ldots, 0) = 0$ and $g \in B$, we see that $g(0, \ldots, 0) = u_m$ and $u_m \in R$. Now, we choose $r \in R \setminus \{0\}$ with $ru_i \in R$ for $0 \leq i \leq m$. Let $g_1 := r(g - u_m)/f \in B$, namely, $g_1 = \sum_{i=0}^{m-1} ru_{m-1-i} f^i$. Then $c(g_1) = rc(f)^{-1} c(g - u_m) \in rR$. Hence $g_1 \in rB$, especially, $ru_{m-1} = g_1(0, \ldots, 0) \in rR$. This implies $u_{m-1} \in R$. Next, let

$$g_2 := \frac{r(g - u_{m-1} f - u_m)}{f^2} \in B.$$

By the same augment, we have $u_{m-2} \in R$. Using the same augment inductively, we have $u_i \in R$ for $0 \leq i \leq m$, so $g \in R[f]$.

(ii) $\Longrightarrow$ (i) Suppose that $c(f) \notin R^*$. Let $f^* := f/c(f) \in B$. Then $R[f] \subsetneq R[f^*]$ and $K[f] = K[f^*]$. Since $c(f^*) \in R^*$, by the consequence of "(i) $\Longrightarrow$ (ii)", $K[f^*] \cap B = R[f^*]$. Therefore

$$R[f] \subsetneq R[f^*] = K[f^*] \cap B = K[f] \cap B = R[f].$$

This is a contradiction. □

3. Closed polynomials in special cases

This section in based on [**33**, Section 2]. Let $B \cong_R R^{[n]}$ be the polynomial ring in n variables over an integral domain R of characteristic $p \geq 0$. Set a system of variables of B by $x_1, \dots, x_n$, that is, $B = R[x_1, \dots, x_n]$ and set $K = Q(R)$. For $\mathbf{w} = (w_1, \dots, w_n) \in \mathbb{N}^n$, we define $\deg_{\mathbf{w}}$ by $\deg_{\mathbf{w}} x_i = w_i$ and deg by the standard degree function on B.

In this section, we study closed polynomials in the following cases:

- Monomials (*Example 2.19*).
- Polynomials with the Jacobian condition (*Proposition 2.20*).
- Polynomials whose degree is prime (*Theorem 2.22*).

The following example gives a classification of monomials which are closed polynomials.

Example 2.19. Let $ux_1^{m_1} \cdots x_n^{m_n}$ be a monomial of B. Then the following two conditions are equivalent:

(i) $ux_1^{m_1} \cdots x_n^{m_n}$ is a closed polynomial over R and $K[f] \cap B = R[f]$.

(ii) $u \in R^*$ and $\gcd(m_1, \dots, m_n) = 1$.

Proof. Let $f = ux_1^{m_1} \cdots x_n^{m_n}$ and let $d = \gcd(m_1, \dots, m_n)$.

(i) $\Longrightarrow$ (ii) If $u \notin R^*$, then

$$R[f] = K[f] \cap B = R[x_1^{m_1} \cdots x_n^{m_n}] \supsetneq R[f].$$

This is a contradiction. Suppose that $u \in R^*$ and $d \geq 2$. For $1 \leq i \leq n$, let $\ell_i = m_i/d \in \mathbb{N}$. Then $u^{-1}f = (x_1^{\ell_1} \cdots x_n^{\ell_n})^d$, so $x_1^{\ell_1} \cdots x_n^{\ell_n} \notin R[f]$, but it is integral over $R[f]$. This is a contradiction.

(ii) $\Longrightarrow$ (i) Since $u \in R^*$, we have $K[f] \cap B = R[f]$. Furthermore, the condition $\gcd(m_1, \dots, m_n) = 1$ implies that f is primitive in B_K for the standard $\mathbb{Z}$-grading. By *Theorem 2.11* (a), f is a closed polynomial. □

For polynomials $f_1, \dots, f_n \in B$, let $F := (f_1, \dots, f_n)$. We denote $J(F)$ by the Jacobian matrix of F with respect to variables $x_1, \dots, x_n$, namely, $J(F) = (\partial f_i / \partial x_j)_{1 \leq i,\, j \leq n}$.

Proposition 2.20. *Suppose that the characteristic of R is zero. Let $F := (f_1, \dots, f_n)$ for polynomials $f_1, \dots, f_n \in B$. Assume that* $\det J(F) \in R \setminus \{0\}$ *and $K[f_i] \cap B = R[f_i]$ for $1 \leq i \leq n$. Then these polynomials $f_1, \dots, f_n$ are closed polynomials. In particular, for $g \in B \setminus R$ satisfying*

$K[g] \cap B = R[g]$, if $\hat{g} = \gcd(g_{x_1}, \ldots, g_{x_n}) \in R \setminus \{0\}$, then it is a closed polynomial over R.

Proof. Suppose that $\det J(F) \in R \setminus \{0\}$. Then there exist $g_{ij} \in B_K$ such that, for $1 \leq i, j \leq n$,

$$\frac{\partial f_i}{\partial x_j} = g_{ij} \hat{f}_i.$$

Here, we note that $\hat{f}_i$ is a polynomial over K. We have

$$\begin{aligned}\det J(F) &= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \frac{\partial f_1}{\partial x_{\sigma(a)}} \cdots \frac{\partial f_n}{\partial x_{\sigma(n)}} \\ &= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) g_{1\sigma(a)} \hat{f}_1 \cdots g_{n\sigma(n)} \hat{f}_n \\ &= (\hat{f}_1 \cdots \hat{f}_n) \cdot \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) g_{1\sigma(a)} \cdots g_{n\sigma(n)},\end{aligned}$$

where S_n is the symmetric group on n elements. For each permutation $\sigma \in S_n$, $\operatorname{sgn}(\sigma)$ denotes the signature of σ. Since $\det J(F) \in R \setminus \{0\}$, we have $\hat{f}_i \in K^*$. Hence $\deg \hat{f}_i = 0$ for $1 \leq i \leq n$. Therefore each f_i satisfies the assumption on *Theorem 2.11* (b) for $\mathbf{w} = (1, \ldots, 1)$. This implies that f_i is a closed polynomial for $1 \leq i \leq n$. □

In the case where R is a field of characteristic zero, the above proposition gives a relation between the Jacobian conjecture and closed polynomials. Let k be a field of characteristic zero and let $k[\mathbf{X}] = k[x_1, \ldots, x_n] \cong_k k^{[n]}$ be the polynomial ring in n variables over k. For $f_1, \ldots, f_n \in k[\mathbf{X}]$, let $F = (f_1, \ldots, f_n)$. We can regard F as a k-endomorphism on $k[\mathbf{X}]$. Here, we consider the following two conditions (a) and (b).

(a) F is a k-automorphism on $k[\mathbf{X}]$.

(b) $\det J(F) \in k^*$.

It is easy to show that the implication "(a) $\Longrightarrow$ (b)" holds true. The Jacobian conjecture says that the converse implication "(b) $\Longrightarrow$ (a)" holds true, namely, the above two conditions (a) and (b) are equivalent. Let $f \in k[\mathbf{X}]$ be a polynomial. We say that f **satisfies the Jacobian condition** if there exist $f_2, \ldots, f_n \in k[\mathbf{X}]$ such that $F = (f, f_2, \ldots, f_n)$ satisfies the condition (b). We denote $\mathrm{JC}_k(k[\mathbf{X}])$ by the set of polynomials satisfying the Jacobian condition. Then it is well known that the Jacobian conjecture is equivalent to the following assertion (c) (see e.g., [**6**]):

(c) Every $f \in \mathrm{JC}_k(k[\mathbf{X}])$ is a variable.

In fact, *Proposition 2.20* implies that every $f \in \mathrm{JC}_k(k[\mathbf{X}])$ is at least a closed polynomial.

As the end of this section, we consider polynomials of degree prime. We prove the following lemma needed later.

Lemma 2.21. *Suppose that the characteristic of R is zero. For $f \in B \setminus R$, the following conditions are equivalent*:

(i) $\deg \hat{f} = \deg f - 1$.

(ii) *There exist $r_1, \ldots, r_n \in K$ with $(r_1, \ldots, r_n) \neq (0, \ldots, 0)$ such that $f \in K[r_1x_1 + \cdots + r_nx_n]$.*

Proof. **(i) $\Longrightarrow$ (ii)** Let $d = \deg f$. There exist $r_1, \ldots, r_n \in B_K$ such that $f_{x_i} = r_i\hat{f}$ for $1 \leq i \leq n$. We may assume that $f_{x_1} \neq 0$. Then

$$d - 1 = \deg \hat{f} \leq \deg f_{x_1} \leq d - 1,$$

so we have $\deg f_{x_1} = d - 1 = \deg \hat{f}$ and $r_1 \in R \setminus \{0\}$. For $1 \leq i \leq n$ such $f_{x_i} \neq 0$, using the same argument, we have $r_i \in K^*$. On the other hand, for $1 \leq i \leq n$ with $f_{x_i} = 0$, we have $r_i = 0$. So r_i is either a non-zero constant polynomial or 0 for $1 \leq i \leq n$. Set $g := r_1x_1 + \cdots + r_nx_n$. Since $\deg g = 1$, we see easily that g is a closed polynomial in B_K. By *Theorem 2.4*, there exists a K-derivation δ on B_K such that $\ker \delta = K[g]$. Then

$$\begin{aligned}
\delta(f) &= \delta(x_1)f_{x_1} + \cdots + \delta(x_n)f_{x_n} \\
&= \delta(x_1)r_1\hat{f} + \cdots + \delta(x_n)r_n\hat{f} \\
&= \delta(g)\hat{f} \\
&= 0.
\end{aligned}$$

Therefore $f \in \ker \delta = K[g]$.

(ii) $\Longrightarrow$ (i) Let $d = \deg f$ and $g := r_1x_1 + \cdots + r_nx_n$. Since $f \in K[g]$, there exists $u(t) \in K[t]$ of degree d with $f = u(g)$. Then $f_{x_i} = r_iu'(g)$ for $1 \leq i \leq n$, where $u'(t) = du(t)/dt$. Then $\deg u'(g) = d - 1$ and $u'(g)$ divides $\hat{f}$. So we have

$$\deg u'(g) \leq \deg \hat{f} \leq d - 1.$$

Therefore $\deg \hat{f} = d - 1$. □

Using *Lemma 2.21*, we give a necessary and sufficient condition for polynomials of degree prime to be a closed polynomial.

Theorem 2.22. *Let $B \cong_R R^{[n]}$ be the polynomial ring in n variables over an integral domain R of characteristic zero. For a non-constant polynomial $f \in B \setminus R$ of prime degree such that $K[f] \cap B = R[f]$, the following conditions are equivalent*:

(i) *f is a closed polynomial.*

(ii) $\deg \hat{f} < \deg f - 1$.

Proof. **(i)** $\Longrightarrow$ **(ii)** Suppose that $\deg \hat{f} = \deg f - 1$. By *Lemma 2.21*, there exist $r_1, \ldots, r_n \in K$ such that $(r_1, \ldots, r_n) \neq (0, \ldots, 0)$ and $f \in K[g]$, where $g := r_1 x_1 + \cdots + r_n x_n$. Since $\deg f$ is prime, especially $\deg f \geq 2$, we have $K[f] \subsetneq K[g]$. By *Theorem 2.4*, f is not a closed polynomial.

(ii) $\Longrightarrow$ **(i)** Suppose that $\deg \hat{f} < \deg f - 1$. Since $\deg f$ is prime, $\mathrm{LD}_{\mathbf{w}}(f) = \deg f \geq 2$, where $\mathbf{w} = (1, \ldots, 1)$. Then

$$\frac{\mathrm{LD}_{\mathbf{w}}(f) - 1}{\mathrm{LD}_{\mathbf{w}}(f)} \deg f = \frac{\deg f - 1}{\deg f} \deg f = \deg f - 1.$$

Therefore we have

$$\deg \hat{f} < \deg f - 1 = \frac{\mathrm{LD}_{\mathbf{w}}(f) - 1}{\mathrm{LD}_{\mathbf{w}}(f)} \deg f.$$

By *Theorem 2.11* (b), f is a closed polynomial. $\square$

4. Other classes of polynomials

This section is based on [**55**, Section 2] and a part of [**33**, Section 2]. Let $B \cong_R R^{[n]}$ be the polynomial ring in n variables over an integral domain R of characteristic $p \geq 0$. Set $K = Q(R)$ and $B_K = B \otimes_R K \cong K^{[n]}$. For $\mathbf{w} = (w_1, \ldots, w_n) \in \mathbb{N}^n$, we define $\deg_{\mathbf{w}}$ by $\deg_{\mathbf{w}} x_i = w_i$ and deg by the standard degree function on B.

A non-constant polynomial $f \in B \setminus R$ is **univariate** over R if there exists a variable $u \in B$ such that $f \in R[v]$. Also f is said to be a **factorially closed polynomial** over R if $R[f]$ is factorially closed in B. So far, we have the following four classes of polynomials:

$\mathrm{CL}_R(B) :=$ the set of closed polynomials over R in B.

$\mathrm{FCL}_R(B) :=$ the set of factorially closed polynomials over R in B.

$\mathrm{UV}_R(B) :=$ the set of univariate polynomials over R in B.

$\mathrm{VA}_R(B) :=$ the set of variables over R in B.

It is easy to show that $\mathrm{CL}_R(B) \supset \mathrm{FCL}_R(B) \supset \mathrm{VA}_R(B)$ and $\mathrm{UV}_R(B) \supset \mathrm{VA}_R(B)$. Furthermore, it follows from *Theorem 2.4* that $\mathrm{VA}_K(B_K) = \mathrm{UV}_K(B_K) \cap \mathrm{CL}_K(B_K)$. We show some examples as below.

Example 2.23. Let $B = R[x, y, z_1, \ldots, z_{n-2}] \cong_R R^{[n]}$ for $n \geq 2$. Then the following assertions hold true.

(a) If R is not a field, then $ax \in \mathrm{UV}_R(B) \cap \mathrm{CL}_R(B)$ for a prime element $a \in R$. However, ax is not a variable.

(b) $xy \in \mathrm{CL}_R(B)$, but it is not factorially closed.

(c) $x^2 + y^3 \in \mathrm{FCL}_R(B)$, but it is not a variable.

Proof. **(a)** It is clear that ax is univariate and is not a variable. Let $f \in B$ such that f is integral over $R[ax]$. Then there exist $m \geq 1$ and $P_i \in R^{[1]}$ such that

$$f^m + P_1(ax)f^{m-1} + \cdots + P_{m-1}(ax)f + P_m(ax) = 0.$$

For $\lambda \in R$, let $g_\lambda = f(1, y, z_1, \ldots, z_{n-2}) \in R[y, z_1, \ldots, z_{n-2}]$. Applying $x = \lambda$ for the above equation, g_λ is integral over R, hence $g_\lambda \in R$ for any $\lambda \in R$. Since R is an infinite set, this implies that $f \in R[x]$.

If $f \notin R[ax]$, then there exists $\ell \geq 1$ such that $a^\ell f \in R[ax]$ and $a^{\ell-1} f \notin R[ax]$. Multiplying $a^{\ell m}$ on the both side of the equation, we have

$$(a^\ell f)^m = -a(a^{\ell-1}P_1(ax)(a^\ell f)^{m-1} + \cdots + a^{\ell m-1}P_m(ax)),$$

hence $(a^\ell f)^m \in aR[ax]$. Since $aR[ax]$ is a prime ideal of $R[ax]$, we have $a^\ell f \in aR[ax]$. Then we have $a^{\ell-1} f \in R[ax]$, however, this contradicts to the choice of ℓ. Hence $f \in R[ax]$, which implies that ax is a closed polynomial over R.

(b) It is clear that xy is not a factorially closed polynomial. It follows from *Example 2.19* that $xy \in \mathrm{CL}_R(B)$.

(c) It is clear that $x^2 + y^3$ is not a variable. It follows from *Example 4.9* (see *Chapter 4*) that $x^2 + y^3 \in \mathrm{FCL}_R(B)$. □

As seeing the above examples, if $n \geq 2$, then we have

- $\mathrm{CL}_R(B) \supsetneq \mathrm{FCL}_R(B) \supsetneq \mathrm{VA}_R(B)$,
- $\mathrm{UV}_R(B) \supsetneq \mathrm{VA}_R(B)$,
- $\mathrm{VA}_R(B) \neq \mathrm{UV}_R(B) \cap \mathrm{CL}_R(B)$,

for an arbitrary integral domain R which is not a field.

In the rest of this section, we assume that k is a field and $B \cong_k k^{[n]}$ is the polynomial ring in n variables over k.

Lemma 2.24. *Let $f \in B \setminus k$. Then the following two conditions are equivalent*:

(i) *$k[f]$ is algebraically closed in B.*

(ii) *$k[f]$ is integrally closed in B.*

Proof. "(i) $\Longrightarrow$ (ii)" is clear. "(ii) $\Longrightarrow$ (i)" follows from $k(f) \cap k^{[n]} = k[f]$ and *Proposition 1.1*. $\square$

We denote $\overline{k[f]}$ by the algebraic closure of $k[f]$ in B. Then $\overline{k[f]}$ is a Dedekind subring of B containing k. It follows from [**75**, Theorem 8] that there exists ${}^{\mathrm{cl}}f \in B \backslash k$ such that $\overline{k[f]} = k[{}^{\mathrm{cl}}f]$ and $({}^{\mathrm{cl}}f)(0, \ldots, 0) = 0$. Furthermore, ${}^{\mathrm{cl}}f$ is unique up to multiplication by k^*. Then the following holds true.

Lemma 2.25. *With the above notations, the following two conditions are equivalent*:

(i) f *is univariate.*

(ii) ${}^{\mathrm{cl}}f$ *is a variable.*

Proof. The part "(ii) $\Longrightarrow$ (i)" is obvious. We prove the converse implication. Suppose that f is univariate. Then there exists a coordinate $u \in k^{[n]}$ such that $f \in k[u]$. Since $k[{}^{\mathrm{cl}}f] \subset k[u]$ and $k[{}^{\mathrm{cl}}f]$ is algebraically closed in B, $k[{}^{\mathrm{cl}}f] = k[u]$. Hence ${}^{\mathrm{cl}}f$ is a variable. $\square$

In the case where $n = 2$, there are some criteria of a polynomial $f \in k^{[2]}$ to be a variable. The following theorem is proved by Abhyankar and Moh ([**1**]). There are several algebraic proofs of this theorem, see e.g., Richman [**66**], Kang [**31**], Makar-Limanov [**44**], Nowicki [**61**] and Essen [**15**, Theorem 5.4.1].

Theorem 2.26. *Let k be a field of characteristic $p \geq 0$ and let $k[t] \cong_k k^{[1]}$ be the polynomial ring in one variable over k. Let $f, g \in k[t] \setminus \{0\}$ such that $k[f, g] = k[t]$ and p does not divide* $\gcd(\deg f, \deg g)$ *whenever $p > 0$. Then either* $\deg f$ *divides* $\deg g$ *or* $\deg g$ *divides* $\deg f$.

Although the statement of the above theorem is very simple, it has a lot of applications for the study of the polynomial ring in two variables. Among them, the most famous result is the following, and it is called the **Abhyankar-Moh-Suzuki Theorem**.

Theorem 2.27. (cf. Abhyankar-Moh [**1**], Suzuki [**72**]) *Let k be a field of characteristic zero and let $\xi \in k[x, y] \cong_k k^{[2]}$. Then ξ is a variable if and only if $k[x, y]/\xi k[x, y] \cong_k k^{[1]}$.*

Proof. The "only if" part is clear. We prove the "if" part. Suppose that $k[x, y]/\xi k[x, y] \cong_k k^{[1]}$. Let $f, g \in k[t]$ be the image of x, y respectively. We can define the surjective homomorphism of k-algebras

$$\varphi : k[x, y] \to k[t]$$

by $\varphi(x) = f$ and $\varphi(y) = g$. Here, we define $\deg \varphi := \deg f + \deg g$. If $\deg \varphi = 1$, then we may assume that $\deg f = 1$ and $\deg g = 0$, that is, $f = at + b$ and $g = c$ for some $a, c \in k^*$ and $b \in k$. Then

$$\xi k[x, y] = \ker \varphi = (y - c)k[x, y],$$

hence ξ is a variable.

Assume that $\deg \varphi \geq 2$. Since $k[f, g] = k[t]$, by *Theorem 2.27*, we may assume that $\deg g$ is divided by $\deg f$, namely, $\deg g = d \deg f$ for some $d \geq 1$. We can choose $a \in k^*$ such that $\deg g > \deg(g - af^d)$. Here, we define the k-automorphism E on $k[x, y]$ by $E(x) = x$ and $E(y) = y - ax^d$. Then $\deg(\varphi \circ E) < \deg \varphi$ and

$$k[f, g - af^d] = k[f, g] = k[t].$$

By the induction hypothesis, we see that ξ is a variable. □

When k is an algebraically closed field of characteristic zero, the above theorem says that, for $f \in k[x, y]$, f is a variable if and only if every fiber of $\Phi_f : \operatorname{Spec} k^{[2]} \to \operatorname{Spec} k[f]$ is isomorphic to $\mathbb{A}^1_k$.

By using the Abhyankar-Moh-Suzuki Theorem, we give characterizations of univariate polynomials in $k[x, y] \cong_k k^{[2]}$ as below.

Theorem 2.28. *Let k be an algebraically closed field of characteristic zero, let $f \in k^{[2]} \setminus k$ be a non-constant polynomial and let $\Phi_f : \operatorname{Spec} k^{[2]} \to \operatorname{Spec} k[f]$ be the morphism associated to the inclusion $k[f] \to k^{[2]}$. Let m be the degree of field extension $Q(\overline{k[f]})$ over $k(f)$. Then the following three conditions are equivalent:*

(i) *f is univariate.*

(ii) *Δ_f is locally nilpotent.*

(iii) *For any closed point $P \in \operatorname{Spec} k[f]$,*

$$\Phi_f^{-1}(P) = \sum_{i=1}^{r} m_i F_i,$$

where $F_i \cong_k \mathbb{A}^1_k$ for $1 \leq i \leq r$ and $\sum_{i=1}^{r} m_i = m$.

Proof. The part "(i) $\iff$ (ii)" follows from Rentschler's Theorem (see e.g., [**18**, Corollary 4.6]). We show the equivalence of (i) and (iii).

First of all, we give some arguments. By the discussion preceding the statement of *Lemma 2.25*, we have $\overline{k[f]} = k[{}^{\mathrm{cl}}f]$. Now, we consider the morphism $\Psi : \operatorname{Spec} k[{}^{\mathrm{cl}}f] \to \operatorname{Spec} k[f]$ with respect to the inclusion

$k[f] \to k[{}^{\mathrm{cl}}f]$, that is, we consider the following diagram:

$$\begin{tikzcd} \mathbb{A}_k^2 \cong_k \operatorname{Spec} k[x,y] \arrow[r, "\Phi_f"] \arrow[dr, "\Phi_{{}^{\mathrm{cl}}f}"'] & \operatorname{Spec} k[f] \\ & \operatorname{Spec} k[{}^{\mathrm{cl}}f]. \arrow[u, "\Psi"'] \end{tikzcd}$$

Let $m \geq 1$ be the degree of field extension $k({}^{\mathrm{cl}}f)$ over $k(f)$. Then for any closed point $P \in \operatorname{Spec} k[f]$,

$$\Psi^{-1}(P) = \sum_{i=1}^{r} m_i Q_i,$$

where each Q_i is a distinct closed point of $\operatorname{Spec} k[{}^{\mathrm{cl}}f]$ and $\sum_{i=1}^{r} m_i = m$. Therefore

$$\Phi_f^{-1}(P) = \Phi_{{}^{\mathrm{cl}}f}^{-1}\left(\sum_{i=1}^{r} m_i Q_i\right) = \sum_{i=1}^{r} m_i \Phi_{{}^{\mathrm{cl}}f}^{-1}(Q_i).$$

(i) $\Longrightarrow$ (iii) By *Lemma 2.25*, if f is univariate, then ${}^{\mathrm{cl}}f$ is a variable. It follows from *Theorem 2.27* that $\Phi_{{}^{\mathrm{cl}}f}^{-1}(Q_i)$ is isomorphic to $\mathbb{A}_k^1$ for any $1 \leq i \leq r$. Therefore we have the assertion.

(iii) $\Longrightarrow$ (i) It is enough to show that ${}^{\mathrm{cl}}f$ is a variable. Since ${}^{\mathrm{cl}}f$ is a closed polynomial, it follows from *Theorem 2.3* that there exists $Q_0 \in \operatorname{Spec} k[{}^{\mathrm{cl}}f]$ such that $\Phi_{{}^{\mathrm{cl}}f}^{-1}(Q_0)$ is irreducible and reduced. Put $P := \Psi(Q_0) \in \operatorname{Spec} k[f]$. By the assumption, $\Phi_f^{-1}(P) = \sum_{i=1}^{r} m_i F_i$ with $F_i \cong_k \mathbb{A}_k^1$, where $\sum_{i=1}^{r} m_i = m$. Since $\Phi_{{}^{\mathrm{cl}}f}^{-1}(Q_0) \subset \Phi_f^{-1}(P)$ and $\Phi_{{}^{\mathrm{cl}}f}^{-1}(Q_0)$ is irreducible, there exists $1 \leq i \leq r$ such that $\Phi_{{}^{\mathrm{cl}}f}^{-1}(Q_0) \subset F_i \cong_k \mathbb{A}_k^1$. Since the dimension of $\Phi_{{}^{\mathrm{cl}}f}^{-1}(Q_0)$ is one, $\Phi_{{}^{\mathrm{cl}}f}^{-1}(Q_0) \cong_k \mathbb{A}_k^1$. By using *Theorem 2.27* again, we see that ${}^{\mathrm{cl}}f$ is a variable. $\square$

Next, we give characterizations of factorially closed polynomials in $B \cong_k k^{[n]}$, where k is an algebraically closed field of characteristic $p \geq 0$. Let $f \in B \setminus k$ and let

$$\Phi_f : \mathbb{A}_k^n \cong_k \operatorname{Spec} B \to \operatorname{Spec} k[f] \cong_k \mathbb{A}_k^1$$

be the surjective morphism defined by the natural inclusion $k[f] \to B$.

Theorem 2.29. *Let k be an algebraically closed field. For a non-constant polynomial $f \in B \cong_k k^{[n]}$, the following conditions are equivalent:*

(i) *f is a factorially closed polynomial.*

(ii) *$f - \lambda$ is irreducible for any $\lambda \in k$.*

(iii) *Every fiber of Φ_f is irreducible and reduced.*

Proof. The part "(ii) $\iff$ (iii)" is obvious. We show the equivalence of (i) and (ii).

(i) $\Longrightarrow$ **(ii)** Suppose that $k[f]$ is a factorially closed in B. If there exists $\lambda \in k$ such that $f - \lambda$ is reducible, then $f - \lambda = gh$ for some $g, h \in B \setminus k$. Then $gh \in k[f - \lambda] = k[f]$, however, since $\deg g$ and $\deg h$ are less than $\deg(f - \lambda)$, $g \notin k[f]$ and $h \notin k[f]$. This is a contradiction.

(ii) $\Longrightarrow$ **(i)** Let $g, h \in B \setminus \{0\}$ such that $gh \in k[f]$. Since k is an algebraically closed field, there exist $\lambda_1, \ldots, \lambda_s \in k$ and $\varepsilon \in k^*$ such that

$$gh = \varepsilon \prod_{i=1}^{s} (f - \lambda_i).$$

By reordering $\lambda_1, \ldots, \lambda_s \in k$ if necessary, we have $g = \varepsilon_1 \prod_{i=1}^{r} (f - \lambda_i)$ and $h = \varepsilon_2 \prod_{j=r+1}^{s} (f - \lambda_j)$ for $\varepsilon_1, \varepsilon_2 \in k^*$. Then $g, h \in k[f]$, so $k[f]$ is factorially closed in B. $\square$

Later in *Chapter 4.3*, by using the above theorem, we give some examples of factorially closed polynomials.

As the end of this section, we show a relation between factorially closed polynomials and Darboux polynomials (see (P.6)) in the case where $n = 2$.

Proposition 2.30. *Let k be an algebraically closed field of characteristic zero and let $f \in k[x, y] \cong_k k^{[2]}$. If f is a factorially closed polynomial, then Δ_f has no Darboux polynomials any other than elements of the kernel of Δ_f.*

Proof. We define a morphism $\Phi_f : \operatorname{Spec} k[x, y] \to \operatorname{Spec} k[f]$ by the inclusion $k[f] \subset k[x, y]$. By *Theorem 2.29*, every fiber of Φ_f is irreducible and reduced, in particular it gives a fibration. By [**12**, Corollary 2.4], $\gcd(f_x, f_y) = 1$, so Δ_f is irreducible. Moreover $k(x, y)^{\Delta_f}$ contains $k(f)$. Therefore f and Δ_f satisfy the assumptions of [**52**, Lemma 2.4]. By [**52**, Lemma 2.4 (2)], $X_{\Delta_f} = 0$, which means that if g is a Darboux polynomial of Δ_f, then $g \in k[x, y]^{\Delta_f}$. $\square$

CHAPTER 3

Dimension two

In this chapter, we study derivations, higher derivations and closed polynomials in the polynomial ring in two variables over an integral domain. Let R be an integral domain of characteristic $p \geq 0$ and let $R[x, y] \cong_R R^{[2]}$ be the polynomial ring in two variables x, y over R. When R is a UFD, the following result is important.

Theorem 3.1. *Let R be a UFD of characteristic $p \geq 0$. Then the following two assertions hold true.*

(a) *Suppose that $p = 0$. For any non-zero R-derivation d on $R[x, y]$, $R[x, y]^d = R[f]$ for some $f \in R[x, y]$.*

(b) *For any non-trivial higher R-derivation $D = \{D_\ell\}_{\ell=0}^{\infty}$ on $R[x, y]$, $R[x, y]^D = R[f]$ for some $f \in R[x, y]$.*

When R is a field of characteristic zero, *Theorem 3.1* (a) is proved by Nowicki and Nagata [**62**] in 1988. After that, this result is generalized by Berson [**7**] in 1999 or El Kahoui [**13**] in 2004, when R is a UFD of characteristic zero. Furthermore, in 2011 Kojima and Wada [**38**] proved the assertion in *Theorem 3.1* (b) and Wada [**74**] gave a simple proof. The remarkable point of Wada's proof is what he only used Lüroth's theorem and some elementary discussions to prove the assertion. For this reason, his proof makes sense for the assertion (a).

On the other hand, when R is not necessarily a UFD, *Theorem 3.1* dose not hold in general.

Example 3.2. (cf. [**18**, Example 4.4]) Let k be a field of characteristic zero and let $k[t] \cong_k k^{[1]}$. Set $R = k[t^2, t^3]$. Here, we define the R-derivation d on $R[x, y] \cong_R R^{[2]}$ by

$$d = t^2 \frac{\partial}{\partial x} + t^3 \frac{\partial}{\partial y}.$$

Then $R[x, y]^d = R[f^m t^2 \mid m \geq 1]$, where $f = tx - y \notin R[x, y]$. Therefore the kernel is not finitely generated, especially, is not generated by one polynomial.

1. Over an integral domain of characteristic zero

This section is based on [**37**, Section 3] and [**54**, Section 2]. By using *Theorem 2.4*, we generalize some of Nowicki's results in [**58**, Section 2]. The aim of this section is to show *Theorem 3.7.* When R is a UFD of characteristic zero, by *Theorem 3.1* (a), we know already that, for every non-zero R-derivation on $R[x,y]$, its kernel is generated by one element over R. However, as seen in *Example 3.2*, it does not hold over an integral domain which is not necessarily a UFD. When R is an integral domain containing $\mathbb{Q}$, in *Theorem 3.7*, we give a necessary and sufficient condition for kernels of derivations to be generated by one polynomial over R.

Through in this section, we assume that R is an integral domain of characteristic zero, $K = Q(R)$ and $R[x,y] \cong_R R^{[2]}$. First of all, we consider derivations of Jacobian type, that is, it has the form $d = \Delta_f$ for some $f \in R[x,y]$.

Lemma 3.3. *For $f \in R[x,y]$, the following two assertions hold true.*

(a) $R[x,y]^{\Delta_f} = R[x,y]$ *if and only if* $f \in R$.

(b) $R[x,y]^{\Delta_f} = R[f]$ *if and only if f is a closed polynomial over R and* $K[f] \cap R[x,y] = R[f]$.

Proof. **(a)** If $f \in R$, then $f_x = f_y = 0$ and so $R[x,y]^{\Delta_f} = R[x,y]$. Suppose that $f \notin R$. Since the characteristic of R is zero, $f_x \neq 0$ or $f_y \neq 0$, which means that $\Delta_f \neq 0$. Hence $R[x,y]^{\Delta_f} \neq R[x,y]$.

(b) Suppose that $R[x,y]^{\Delta_f} = R[f]$. Since the characteristic of R is zero, by (a) and (P.1), $R[f]$ is algebraically closed in $R[x,y]$. By *Proposition 1.1*, $R[f]$ is integrally closed in $R[x,y]$ and $K(f) \cap R[x,y] = R[f]$. Therefore, f is a closed polynomial over R and $K[f] \cap R[x,y] \subset K(f) \cap R[x,y] = R[f]$. Hence $K[f] \cap R[x,y] = R[f]$.

Conversely, we suppose that f is a closed polynomial over R and $K[f] \cap R[x,y] = R[f]$. By *Proposition 1.1*, $R[f]$ is algebraically closed in $R[x,y]$. It follows from *Theorem 1.14* that $R[f]$ is a maximal element of $\mathfrak{S}(1, R[x,y])$. Since $R[x,y]^{\Delta_f} \in \mathfrak{S}(1, R[x,y])$ and $R[f] \subset R[x,y]^{\Delta_f}$, we have $R[f] = R[x,y]^{\Delta_f}$. $\square$

As seeing in *Theorem 2.29*, we give a characterization of univariate polynomials in the terms of Jacobian derivations. Similarly, we can characterize the closed polynomials in the terms of Jacobian derivations as below.

Proposition 3.4. *Let $f \in R[x,y] \setminus R$ such that $K[f] \cap R[x,y] = R[f]$. Then the following conditions are equivalent*:

(i) *f is a closed polynomial over R in $R[x,y]$.*

(ii) *The ring $K[f]$ is a maximal element of $\mathfrak{S}(1, R[x,y])$.*

(iii) *The ring $K[f]$ is a maximal element of $S(1, R[x,y])$.*

(iv) $R[x,y]^{\Delta_f} = R[f]$.

Proof. The part "(iv) $\Longrightarrow$ (i)" is a fundamental fact of derivations (see (P.1)). The equivalence of (i), (ii) and (iii) follows from *Theorem 2.4*. Furthermore, the part "(i) $\Longrightarrow$ (iv)" follows from *Lemma 3.3* (b). $\square$

When R contains $\mathbb{Q}$, the following lemma is useful to find an element of the kernel of a derivation on $R[x,y]$. Although this result follows from [**60**, Lemma 2.5.3], we write its proof more kindly as below. The remarkable point is what we can construct such an element whenever $\operatorname{div}(d) = 0$. This construction is also used in the next chapter.

Lemma 3.5. *Suppose that R contains $\mathbb{Q}$. Let d be a non-zero R-derivation on $R[x,y]$. Then* $\operatorname{div}(d) = 0$ *if and only if there exists a non-constant polynomial $f \in R[x,y] \setminus R$ such that $d = \Delta_f$. In particular, if one of the above equivalent conditions holds, then $R[x,y]^d \neq R$.*

Proof. If there exists $f \in R[x,y] \setminus R$ such that $d = \Delta_f$, then it is clear that $\operatorname{div}(d) = 0$.

Suppose that $\operatorname{div}(d) = 0$. Set $p := d(x)$ and $q := d(y)$. Then we can write p, q as below:

$$p = \sum_{m,n \geq 0} [m,n]_p x^m y^n, \quad q = \sum_{m,n \geq 0} [m,n]_q x^m y^n,$$

where $[m,n]_p, [m,n]_q \in R$. Then

$$\frac{\partial p}{\partial x} = \sum_{m \geq 1,\, n \geq 0} m[m,n]_p x^{m-1} y^n = \sum_{m,\, n \geq 1} m[m,n-1]_p x^{m-1} y^{n-1},$$

$$\frac{\partial q}{\partial y} = \sum_{m \geq 0,\, n \geq 1} n[m,n]_q x^m y^{n-1} = \sum_{m,\, n \geq 1} n[m-1,n]_q x^{m-1} y^{n-1}.$$

Since $0 = \operatorname{div}(d) = \partial p/\partial x + \partial q/\partial y$, we have that, for $m, n \geq 1$,

$$-n^{-1}[m,n-1]_p = m^{-1}[m-1,n]_q.$$

Here, we define a polynomial $f \in R[x,y]$ by

$$[m,n]_f = \begin{cases} 0 & (m = n = 0), \\ m^{-1}[m-1,0]_q & (m \geq 1,\ n = 0), \\ -n^{-1}[m,n-1]_p & (m \geq 0,\ n \geq 1). \end{cases}$$

Therefore we have,

$$f = \sum_{m \geq 1} \frac{1}{m}[m-1,0]_q x^m + \sum_{m \geq 0,\, n \geq 1} -\frac{1}{n}[m,n-1]_p x^m y^n.$$

Then $f_x = q = d(y)$ and $f_y = -p = -d(x)$, hence we have $d = \Delta_f$. Since $d(f) = 0$, $R[x,y]^d \neq R$. □

We prove the following result.

Lemma 3.6. *For $f \in R[x,y] \setminus R$, the following two conditions are equivalent*:

(i) $R[x,y]^{\Delta_f} = R[g]$ *for some* $g \in R[x,y] \setminus R$.

(ii) *There exists* $g \in R[x,y] \setminus R$ *such that* $f \in R[g]$ *and* g *is a closed polynomial over* R *in* $R[x,y]$ *with* $K[g] \cap R[x,y] = R[g]$.

Proof. **(i) $\Longrightarrow$ (ii)** Let $g \in R[x,y] \setminus R$ such that $R[x,y]^{\Delta_f} = R[g]$. Since $R[g]$ is algebraically closed in $R[x,y]$, we see that g is a closed polynomial over R and $K[g] \cap R[x,y] = R[g]$.

(ii) $\Longrightarrow$ (i) By *Proposition 1.1*, $R[g]$ is algebraically closed in $R[x,y]$. By *Proposition 1.10* (b), $R[x,y]^{\Delta_f}$ is the algebraic closure of $R[f]$ in $R[x,y]$. Since $R[f] \subset R[g]$, we have $R[x,y]^{\Delta_f} = R[g]$. □

For a non-zero R-derivation on $R[x,y]$, by *Lemma 3.5*, if $\operatorname{div}(d) = 0$, then we can find $f \in R[x,y] \setminus R$ such that $d = \Delta_f$. Furthermore, assuming $f(0,0) = 0$, such polynomial is uniquely determined. In this case, we denote the polynomial by P_d, that is, $P_d \in R[x,y] \setminus R$ such that $P_d(0,0) = 0$ and $d = \Delta_{P_d}$.

Theorem 3.7. *Let R be an integral domain containing $\mathbb{Q}$. Let d be a non-zero R-derivation on $R[x,y]$ with* $\operatorname{div}(d) = 0$. *Then the following two conditions are equivalent*:

(i) $R[x,y]^d = R[f]$ *for some* $f \in R[x,y] \setminus R$.

(ii) *There exists* $f \in R[x,y] \setminus R$ *such that* $P_d \in R[f]$ *and* f *is a closed polynomial over* R *in* $R[x,y]$ *with* $K[f] \cap R[x,y] = R[f]$.

Proof. Since $\operatorname{div}(d) = 0$, according to the previous discussion of *Theorem 3.7*, we can write $d = \Delta_{P_d}$. By applying *Lemma 3.6* for Δ_{P_d}, we have the assertion. □

We consider again the derivation $d = t^2\partial_x + t^3\partial_y$ in *Example 3.2.* We see that $\mathrm{div}(d) = 0$ and $P_d = t^3x - t^2y$. Since the standard degree with respect to x and y of P_d is one, there are no polynomials f such that $P_d \in R[f]$ any other than P_d up to multiplication by k^*. However, we have

$$K[P_d] \cap R[x, y] = R[(tx - y)^m t^2 \mid m \geq 1] \supsetneq R[P_d].$$

Therefore d does not satisfy the condition (ii) in *Theorem 3.7.*

2. Over a UFD of characteristic zero

This section is based on a part of [**37**, Section 3] and [**54**, Section 2]. In this section, we study derivations on $R[x, y] \cong_R R^{[2]}$ in the case where R is a UFD of characteristic zero.

Proposition 3.8. *Let $f, g \in R[x, y] \setminus R$. Suppose that $\Delta_f(g) = 0$. Then the following assertions hold true.*

(a) $R[x, y]^{\Delta_f} = R[x, y]^{\Delta_g}$.

(b) *There exist $h \in R[x, y]$ and $u(t), v(t) \in R[t] \cong_R R^{[1]}$ such that $f = u(h)$ and $g = v(h)$.*

Proof. **(a)** By the assumptions on g, we see that

$$R \subsetneq R[x, y]^{\Delta_g} \subsetneq R[x, y].$$

Hence $R[x, y]^{\Delta_g} \in \mathfrak{S}(1, R[x, y])$. Since R is a UFD, by *Theorem 3.1*, there exists $h \in R[x, y] \setminus R$ such that $R[x, y]^{\Delta_g} = R[h]$. Then we can write $g \in R[x, y]^{\Delta_g} = R[h]$ as

$$g = a_0h^m + a_1h^{m-1} + \cdots + a_{m-1}h + a_m,$$

where $a_0, a_1, \ldots, a_m \in R$ and $a_0 \neq 0$. Applying Δ_f for the both side of the above equation, we have

$$(ma_0h^{m-1} + (m - 1)a_1h^{m-2} + \cdots + a_{m-1})\Delta_f(h) = 0.$$

Since $h \notin R$, $ma_0h^{m-1} + \cdots + a_{m-1} \neq 0$, hence $\Delta_f(h) = 0$. Therefore we have $R[x, y]^{\Delta_g} = R[h] \subset R[x, y]^{\Delta_f}$, hence $R[x, y]^{\Delta_f} = R[x, y]^{\Delta_g}$.

(b) By the proof of the assertion (a), we have $R[x, y]^{\Delta_f} = R[x, y]^{\Delta_g} = R[h]$ for some $h \in R[x, y] \setminus R$. Hence f and g are written as polynomials with respect to h over R. $\square$

For $d, \delta \in \mathrm{Der}_R\, R[x, y]$, we write $d \sim \delta$ when $fd = g\delta$ for some $f, g \in R[x, y] \setminus \{0\}$. In the rest of this section, we assume further R contains $\mathbb{Q}$.

Theorem 3.9. *Let R be a UFD containing $\mathbb{Q}$ and let d be a non-zero R-derivation on $R[x,y] \cong_R R^{[2]}$. Then the following two conditions are equivalent*:

(i) $R[x,y]^d \neq R$.

(ii) *There exists an R-derivation δ with* $\operatorname{div}(\delta) = 0$ *such that $d \sim \delta$.*

Proof. **(ii) $\Longrightarrow$ (i)** If $d \sim \delta$, then $R[x,y]^d = R[x,y]^\delta$. Hence it follows from *Lemma 3.5* that $R[x,y]^d \neq R$.

(i) $\Longrightarrow$ (ii) Suppose that $R[x,y]^d \neq R$. Since $R[x,y]^d = R[x,y]^{hd}$ for any non-zero polynomial $h \in R[x,y] \setminus \{0\}$, we may assume that $d(x)$ and $d(y)$ are relatively prime. Let $f \in R[x,y]^d \setminus R$ and set $h = \gcd(f_x, f_y) \neq 0$. Then $f_x = Ph$ and $f_y = Qh$ for some $P, Q \in R[x,y]$ which are relatively prime. We have

$$d(x)P + d(y)Q = 0.$$

Therefore we have $d(x) = -Q$ and $d(y) = P$, hence $d = h\Delta_f$, that is, $d \sim \Delta_f$ and $\operatorname{div}(\Delta_f) = 0$. □

Theorem 3.10. *Let R be a UFD containing $\mathbb{Q}$ and let d and δ be R-derivations of $R[x,y] \cong_R R^{[2]}$ such that $R[x,y]^d \neq R$ and $R[x,y]^\delta \neq R$. Then $R[x,y]^d = R[x,y]^\delta$ if and only if $d \sim \delta$.*

Proof. The "if" part is obvious. We prove the "only if" part.

If $R[x,y]^d = R[x,y]^\delta = R[x,y]$, then $d = \delta = 0$ and hence $d \sim \delta$. Hence we may assume that $d \neq 0$ and $\delta \neq 0$. It follows from *Lemma 3.5* and *Theorem 3.9* that $d \sim \Delta_f$ and $\delta \sim \Delta_g$ for some $f, g \in R[x,y] \setminus R$. Then

$$R[x,y]^{\Delta_f} = R[x,y]^d = R[x,y]^\delta = R[x,y]^{\Delta_g}.$$

By *Proposition 3.8* (b), there exists $h \in R[x,y]$ and $u(t), v(t) \in R[t] \cong_R R^{[1]}$ such that $f = u(h)$ and $g = v(h)$. Then

$$\Delta_f = -f_y\partial_x + f_x\partial_y = u'(h)(-h_y\partial_x + h_x\partial_y) = u'(h)\Delta_h,$$

where $u'(t) = du(t)/dt$. Similarly, we have also $\Delta_g = v'(h)\Delta_h$. Therefore we have $\Delta_f \sim \Delta_g$, which implies that $d \sim \delta$. □

Suppose that R is a UFD containing $\mathbb{Q}$. Let $d \in \operatorname{Der}_R R[x,y] \setminus \{0\}$. Then there exists $f \in R[x,y]$ such that $R[x,y]^d = R[f]$. If $R[f] \neq R$, then f is a closed polynomial. For $f, g \in R[x,y]$, we write $f \equiv g$ if $f = ag + b$ for some $a \in R^*$ and $b \in R$. Clearly, $f \equiv g$ if and only if $R[f] = R[g]$. Here, we define the following:

$\mathfrak{Der}_R R[x,y] :=$ the set of equivalence classes of $\operatorname{Der}_R R[x,y] \setminus \{0\}$,

$\mathfrak{CL}_R R[x,y] :=$ the set of equivalence classes of $\mathrm{CL}_R(R[x,y])$,

$\mathrm{KDer}_R R[x,y] := \{ R[x,y]^d \mid d \in \operatorname{Der}_R R[x,y] \setminus \{0\} \}$.

By *Theorem 3.10*, we have the following result:

Proposition 3.11. *We define* $\alpha : \mathfrak{Der}_R R[x,y] \to \mathfrak{CL}_R R[x,y] \cup \{1\}$ *by, for* $[d] \in \mathfrak{Der}_R R[x,y]$,

$$\alpha([d]) = \begin{cases} f & \text{if } R[x,y]^d = R[f] \neq R, \\ 1 & \text{if } R[x,y]^d = R, \end{cases}$$

where we denote $[d]$ *by the equivalence class of* $d \in \operatorname{Der}_R R[x,y]$. *Then* α *is a well-defined bijective mapping.*

Proof. The above map α factors through $\mathrm{KDer}_R R[x,y]$ as below:

$$\beta : \mathrm{KDer}_R R[x,y] \to \mathfrak{CL}_R R[x,y] \cup \{1\},$$

where β is defined by $\beta(R[x,y]^d) = f$ if $R \neq R[x,y]^d = R[f]$, otherwise $\beta(R[x,y]^d) = 1$. It is clear that β is injective. By *Theorem 2.4*, β is surjective, hence β is bijective. Furthermore, by *Theorem 3.10*, the natural mapping $[d] \mapsto R[x,y]^d$ is bijective. Therefore, α is bijective. □

Suppose that R is a UFD containing $\mathbb{Q}$. For a given $d \in \operatorname{Der}_R R[x,y] \setminus \{0\}$, it is important to determine the generator of $R[x,y]^d$. By *Proposition 3.11*, it is enough to understand closed polynomials up to "$\equiv$". We already have some criteria and partial classifications of closed polynomials (see e.g., *Chapter 2*). By using these ideas, we classify the kernel of a monomial derivation on $R[x,y]$ and $k(x,y)$ in *Chapter 4*.

3. Over a field of positive characteristic

The aim of this section is to define higher derivations of Jacobian type (see *Definition 3.14*) and observe them. By using this kind of higher derivations, we show *Theorem 3.16* which gives a characterization of variables of the polynomial ring in two variables over a field of positive characteristic. This is a generalization of [**14**, Proposition 2.3] in the case where the characteristic of the ground field is positive.

Let k be a field of characteristic $p \geq 0$ and let $B = k[x,y] \cong_k k^{[2]}$. For a given polynomial $f \in B$, we denote $\mathrm{HD}_{k[f]}B$ by the set of higher derivations on B whose kernels contain $k[f]$. When $p = 0$, we know that the set $\mathrm{HD}_{k[f]}B$ has a non-trivial element. Indeed, it is enough to take

$D = \{\ell!^{-1}{\Delta_f}^{\ell}\}_{\ell=0}^{\infty}$. However, when $p > 0$, $\ell!^{-1}{\Delta_f}^{\ell}$ does not make sense for $\ell \geq p$. For this reason, we do not know whether the set $\mathrm{HD}_{k[f]}B$ contains a non-trivial element or not. *Proposition 3.15* gives a partial answer for this question.

First of all, we prepare some notation and results of general commutative ring theory (see [**25**] or [**45**]). Let A be a commutative ring and let B be a commutative A-algebra via a homomorphism $\varphi : A \to B$. We say that B is **smooth** over A if for any A-algebra C with $g : A \to C$, an ideal $N \subset C$ with $N^2 = 0$ and homomorphism of A-algebras $u : B \to C/N$, there exists a homomorphism of A-algebras $v : B \to C$ such that $v \circ \pi = u$, where $\pi : C \to C/N$ is the natural homomorphism. That is, v commutes the following diagram:

$$\begin{tikzcd} B \arrow[r, "u"] \arrow[dr, dashed, "\exists v"] & C/N \\ A \arrow[u, "\varphi"] \arrow[r, "g"'] & C. \arrow[u, "\pi"'] \end{tikzcd}$$

For $\mathfrak{p} \in \operatorname{Spec} A$, we denote the residue field by $\kappa(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$.

Proposition 3.12. *Let $\varphi : A \to B$ be a homomorphism of commutative rings. For $\mathfrak{p} \in \operatorname{Spec} A$, let $\iota_{\mathfrak{p}} : \kappa(\mathfrak{p}) \to B \otimes_A \kappa(\mathfrak{p})$ be the natural homomorphism of A-algebras. If φ is smooth, then $\iota_{\mathfrak{p}}$ is also smooth for any $\mathfrak{p} \in \operatorname{Spec} A$.*

Proof. Let $\mathfrak{p} \in \operatorname{Spec} A$. Let C be a $\kappa(\mathfrak{p})$-algebra with $g : \kappa(\mathfrak{p}) \to C$, let N be an ideal of C with $N^2 = 0$ and let $u : B \otimes_A \kappa(\mathfrak{p}) \to C/N$ be a homomorphism of $\kappa(\mathfrak{p})$-algebras. Then we have the following diagram:

$$\begin{tikzcd} B \arrow[r, "p"] & B \otimes_A \kappa(\mathfrak{p}) \arrow[r, "u"] & C/N \\ A \arrow[u, "\varphi"] \arrow[r, "f"'] & \kappa(\mathfrak{p}) \arrow[u, "\iota_{\mathfrak{p}}"] \arrow[r, "g"'] & C, \arrow[u, "\pi"'] \end{tikzcd}$$

where $p : B \to B \otimes_A \kappa(\mathfrak{p})$ and $f : A \to \kappa(\mathfrak{p})$ are the natural homomorphisms. When we regard C as an A-algebra via $g \circ f$, we can regard $u \circ p : B \to C/N$ as a homomorphism of A-algebras. Since φ is smooth, there exists a homomorphism $v_0 : B \to C$ as A-algebras such that $\pi \circ v_0 = u \circ p$. Here, we define $v : B \otimes_A \kappa(\mathfrak{p}) \to C$ by $v(b \otimes a) = v_0(b)g(a)$ for $b \in B$ and $a \in \kappa(\mathfrak{p})$. Then v is a homomorphism of $\kappa(\mathfrak{p})$-algebras satisfying $v \circ \pi = u$. Therefore $\iota_{\mathfrak{p}}$ is smooth. □

Example 3.13. Let k be a field of characteristic $p \geq 0$ and let $k[x, y] \cong_k k^{[2]}$. Then the following assertions hold true.

(a) The natural inclusion $k \to k[x, y]$ is smooth.

(b) For a variable $f \in k[x,y]$, the natural inclusion $k[f] \to k[x,y]$ is smooth.

(c) For $xy \in k[x,y]$, the natural inclusion $\iota : k[xy] \to k[x,y]$ is not smooth.

Proof. (a) and (b) are obvious. We prove the assertion (c).

Assume to the contrary that $\iota : k[xy] \to k[x,y]$ is smooth. By *Proposition 3.12*, the 0-fiber of ι is also smooth, that is,

$$\iota_0 : k \cong_k \kappa(0) \to k[x,y] \otimes_{k[xy]} \kappa(0) \cong_k k[x,y]/(xy)$$

is smooth. Set $C = k[t]/(t^3)$ and $N = t^2C$, where $k[t] \cong_k k^{[1]}$. Then $N^2 = 0$. Here, we define $u : k[x,y]/(xy) \to C/N$ by $u(x) = u(y) = t$. Since ι_0 is smooth, there exists a homomorphism $v : k[x,y]/(xy) \to C$ of k-algebras such that $\pi \circ v = u$, namely, v commutes the following diagram:

$$\begin{tikzcd} k[x,y]/(xy) \arrow[r, "u"] \arrow[dr, dashed, "\exists v"] & C/N \\ k \arrow[u, "\iota_0"] \arrow[r, "g"'] & C. \arrow[u, "\pi"'] \end{tikzcd}$$

Then $v(x) = t + at^2$ and $v(y) = t + bt^2$ for some $a, b \in k$. However,

$$0 = v(xy) = v(x)v(y) = t^2,$$

which is a contradiction. $\square$

From now on, let k be a field of positive characteristic $p > 0$ and let $B = k[x,y] \cong_k k^{[2]}$ be the polynomial ring in two variables over k.

Definition 3.14. A higher derivation $D = \{D_\ell\}_{\ell=0}^{\infty}$ on B is **of Jacobian type** if there exists $f \in k[x,y]$ such that

(a) $f \in k[x,y]^D$,

(b) For $0 \leq \ell \leq p-1$, $D_\ell = \dfrac{1}{\ell!} D_1^\ell$ and $D_1 = \Delta_f$.

Note that the above condition (a) is equivalent to $\varphi_D(f) = f$, where $\varphi_D : B \to B[[t]] \cong_B B^{[[1]]}$ is the homomorphism associated to D. That is, φ_D is a homomorphism as $k[f]$-algebras.

The following proposition guarantees the existence of higher derivations of Jacobian type in the special case.

Proposition 3.15. *Let $f \in B \setminus k[x^p, y^p]$. If the natural inclusion $k[f] \to B$ is smooth, then there exists a higher derivation D on B of Jacobian type with respect to f.*

Proof. Define $D_0 = \mathrm{id}_B$ and $D_1 = \Delta_f (\neq 0)$. Let $B[t] \cong_B B^{[1]}$. Here, we define a map $\varphi_\ell : B \to B[t]/(t^{\ell+1})$ by, for $g \in B$ and $1 \leq \ell \leq p-1$,

$$\varphi_\ell(g) = \sum_{i=0}^{\ell} \frac{1}{i!} {\Delta_f}^i(g) t^i.$$

Then φ_ℓ is a homomorphism of $k[f]$-algebras such that $\varphi_\ell(g)|_{t=0} = g$ for any $g \in B$.

For $r \geq 0$, let $C_r = B[t]/(t^{p+r})$ and $N_r = t^{p+r-1}C_r$. Then $N_r^2 = 0$ and $C_r/N_r \cong_B B[t]/(t^{p+r-1})$. Since $k[f] \to B$ is smooth, there exists a homomorphism $\varphi_{p+r} : B \to C_r$ of $k[f]$-algebras such that $\pi_r \circ \varphi_{p+r} = \varphi_{p+r-1}$, that is, we have the following diagram:

$$\begin{tikzcd}
B \arrow[r, "\varphi_{p+r-1}"] \arrow[dr, dashed, "\exists \varphi_{p+r}"] & B[t]/(t^{p+r-1}) \arrow[r, "\cong_B"] & C_r/N_r \\
k[f] \arrow[u] \arrow[r] & B[t]/(t^{p+r}) \arrow[r, "="] & C_r. \arrow[u, "\pi_r"']
\end{tikzcd}$$

Moreover $\varphi_{p+r}(g)|_{t=0} = g$ for any $g \in B$. For $0 \leq i \leq r-1$, by using φ_{p+r}, we define a homomorphism of k-modules $D_{p+i} : B \to B$ by the following formula:

$$\varphi_{p+r}(g) = \sum_{\ell=0}^{p-1} \frac{1}{\ell!} {\Delta_f}^\ell(g) t^\ell + \sum_{i=0}^{r-1} D_{p+i}(g) t^{p+i}$$

for $g \in B$. By constructing such homomorphisms inductively, we have a homomorphism of $k[f]$-algebras $\varphi = \varphi_\infty : B \to B[[t]] \cong_B B^{[[1]]}$ such that, for $g \in B$, $\varphi(g)|_{t=0} = g$ and

$$\varphi(g) = \sum_{\ell=0}^{p-1} \frac{1}{\ell!} {\Delta_f}^\ell(g) t^\ell + \sum_{i=0}^{\infty} D_{p+i}(g) t^{p+i}.$$

Set $D_\ell = \ell!^{-1} {\Delta_f}^\ell$ for $0 \leq \ell \leq p-1$ and $D = \{D_\ell\}_{\ell=0}^\infty$. By the construction of each D_ℓ, we see that D is a higher derivation on B of Jacobian type with respect to f. $\square$

The following is the main result in this section which is a generalization of [**14**, Proposition 2.3] in the case where the characteristic of the ground field is positive.

Theorem 3.16. *Let k be a field of characteristic $p > 0$ and let $f \in B \cong_k k^{[2]}$. Then the following conditions are equivalent*:

(i) *f is a variable.*

(ii) *There exists a locally finite iterative higher derivation D on B such that $B^D = k[f]$.*

(iii) *There exists a locally finite iterative higher derivation D on B of Jacobian type with respect to f such that $B^D = k[f]$.*

In order to prove *Theorem 3.16*, we introduce some definitions as below. For a positive integer $\ell \geq 1$, we write $\ell! = p^{e(\ell)}m$, where p does not divide m. Let $\ell \geq 1$ and $f \in B$. For a non-zero derivation $d \in \operatorname{Der}_k B$, we say that $\ell!^{-1}d^\ell$ is **defined** at f if there exists $f_\ell \in B$ such that $d^\ell(f) = p^{e(\ell)}f_\ell$ and define the value by $\ell!^{-1}d^\ell(f) = m^{-1}f_\ell$. When $\ell!^{-1}d^\ell$ is defined at any f and $\ell \geq 1$, we consider the map $\operatorname{Exp}(td) : B \to B[[t]] \cong_B B^{[[1]]}$ defined by

$$\operatorname{Exp}(td)(f) := \sum_{\ell=0}^{\infty} \frac{1}{\ell!} d^\ell(f) t^\ell.$$

By the definition of $\operatorname{Exp}(td)$, it is a homomorphism of k-algebras and $\operatorname{Exp}(td)(f)|_{t=0} = f$ for $f \in B$. In order to check whether the map $\operatorname{Exp}(td)$ is defined or not, it is enough to see that $\ell!^{-1}d^\ell$ is defined at x and y for any $\ell \geq 1$.

Example 3.17. Let $B = \mathbb{F}_5[x, y] \cong_{\mathbb{F}_5} \mathbb{F}_5^{[2]}$. Set $d_1 = y^6\partial_x + \partial_y$ and $d_2 = y\partial_x + x\partial_y$. Then $\operatorname{Exp}(td_1)$ is defined, but $\operatorname{Exp}(td_2)$ is not defined. Indeed, for d_1, it is clear that $\ell!^{-1}{d_1}^\ell$ is defined at y for $\ell \geq 1$. Also, it is defined at x for $\ell \geq 1$ as the following table. Therefore $\operatorname{Exp}(td_1)$ is defined.

ℓ	m	${d_1}^\ell(x)$	$\ell!^{-1}{d_1}^\ell(x)$
1	1	y^6	y^6
2	2	y^5	$3y^5$
3	$3!$	0	0
4	$4!$	0	0
5	$4!$	$5 \cdot 2y^2$	$3y^2$
6	$6 \cdot 4!$	$5 \cdot 4y$	y
7	$7 \cdot 6 \cdot 4!$	$5 \cdot 4$	3
$\ell \geq 8$	$8 \cdot 7 \cdot 6 \cdot 4!$	0	0

On the other hand, for d_2,

$${d_2}^\ell(x) = \begin{cases} y & (\ell \text{ is odd}), \\ x & (\ell \text{ is even}), \end{cases}$$

hence $\ell!^{-1}{d_2}^\ell$ is not defined at x when $\ell \geq 2$.

Remark 3.18. Let $d \in \operatorname{Der}_k B$ be a non-zero derivation. We consider $\operatorname{Aut}_k B$ as a subgroup of $\operatorname{Aut}_k B[[t]]$ by $\sigma(t) = t$ for $\sigma \in \operatorname{Aut}_k B$. If

$\mathrm{Exp}(td)$ can be defined, then $\mathrm{Exp}(t \cdot {}^{\sigma}d)$ can be defined for any $\sigma \in \mathrm{Aut}_k B$, where ${}^{\sigma}d := \sigma^{-1} \circ d \circ \sigma$. In particular, the following holds:

$$\mathrm{Exp}(t \cdot {}^{\sigma}d) = \sigma^{-1} \circ \mathrm{Exp}(td) \circ \sigma.$$

Here, we prove *Theorem 3.16*.

Proof of Theorem 3.16. The part "(iii) $\Longrightarrow$ (ii)" is obvious. The implication "(ii) $\Longrightarrow$ (i)" follows from [**35**, Theorem 1]. So, we prove the part "(i) $\Longrightarrow$ (iii)".

Since f is a variable, $f \notin k[x^p, y^p]$ and there exists $g \in B$ such that $k[f, g] = B$. Define the k-automorphism $\sigma : B \to B$ by $\sigma(x) = f$ and $\sigma(y) = g$. We may assume that $\Delta_f(g) = 1$. Then ${}^{\sigma}\Delta_f = \partial_y$. It is clear that $\mathrm{Exp}(t\partial_y)$ can be defined, hence $\mathrm{Exp}(t \cdot {}^{\sigma}\Delta_f) = \sigma^{-1} \circ \mathrm{Exp}(t\partial_y) \circ \sigma$. This implies that $\ell!^{-1}{\Delta_f}^{\ell}$ is defined at any $h \in B$ and $\ell \geq 1$. Set $D = \{\ell!^{-1}{\Delta_f}^{\ell}\}_{\ell=0}^{\infty}$. Then D is a locally finite iterative higher derivation on Jacobian type with respect to f such that $B^D = k[f]$.

□

CHAPTER 4

Applications and examples

In this chapter, as an application of results on closed polynomials, we study kernels of monomial derivations on the polynomial ring in two variables over a UFD (see *Theorem 4.2*). Also, by using the argument in [**63**, Section 5] and *Theorem 4.2*, we determine the non-zero monomial derivations d on $k[x,y] \cong_k k^{[2]}$ such that the quotient field of the kernel of d is not equal to the kernel of d in $k(x,y)$ (see *Theorem 4.5*). Furthermore, we give observations for important three kinds of polynomials; Vénéreau polynomial, Danielewski surface and Koras-Russell threefold.

1. Kernels of monomial derivations on $R[x,y]$

This section is based on [**33**, Section 3]. Let R be a UFD containing $\mathbb{Q}$ and $K = Q(R)$. In this section, we study the kernels of R-derivations on the polynomial ring $R[x,y]$ in two variables x and y over R. A non-zero R-derivation d on $R[x,y]$ is said to be **monomial** if $d(x)$ and $d(y)$ are monomials, here we assume that a monomial may not be monic. Using results on closed polynomials in *Chapters 2* and *3*, we determine generators of the kernel of monomial derivations on $R[x,y]$.

In the case where R is a UFD, the kernel of a non-zero derivation on $R[x,y]$ is generated by one polynomial (see *Theorem 3.1* (a)) and it is integrally closed in $R[x,y]$. Thus, if $R[x,y]^d \neq R$, then it is generated by a closed polynomial. Therefore, in order to determine a generator of the kernel of a derivation on $R[x,y]$, it is sufficient to find a closed polynomial which is vanished by the derivation (see *Proposition 3.11*). Indeed, the following holds true.

Lemma 4.1. *Let R be an integral domain (not necessarily a UFD) of characteristic zero and let d be a non-zero R-derivation on $R[x,y]$. If there exist a closed polynomial $f \in R[x,y] \setminus R$ such that $d(f) = 0$ and $K[f] \cap R[x,y] = R[f]$, then $R[x,y]^d = R[f]$.*

Proof. Since $R[f]$ is integrally closed in $R[x,y]$ and $K[f] \cap R[x,y] = R[f]$, by *Theorem 1.14*, $R[f]$ is a maximal element of $\mathfrak{S}(1, R[x,y])$. Since

$d(f) = 0$, we have that $R[f] \subset R[x,y]^d$ and $R[x,y]^d \in \mathfrak{S}(1, R[x,y])$. By the maximality of $R[f]$, we have $R[f] = R[x,y]^d$. $\square$

The following is the main result in this section, which gives the classification of kernels of monomial derivations on $R[x,y]$.

Theorem 4.2. *Let d be a non-zero R-derivation on the polynomial ring $R[x,y]$ in two variables over a UFD R containing $\mathbb{Q}$. Assume that $d(x)$ and $d(y)$ are monomial,* $\gcd(d(x), d(y)) = 1$ *and d is none of the following* (a)–(c):

(a) ∂_x *or* ∂_y,

(b) $ay^m\partial_x + bx^n\partial_y$, *where* $m, n \in \mathbb{N}$ *and* $a, b \in R \setminus \{0\}$,

(c) $nx\partial_x - my\partial_y$, *where m and n are positive integers.*

Then $R[x,y]^d = R$.

In order to prove *Theorem 4.2* we show the following two lemmas. By the following lemma, we see that for derivations as in *Theorem 4.2* (a)–(c), their kernels are generated by a closed polynomial.

Lemma 4.3. *For the derivations as in Theorem 4.2* (a), (b) *and* (c), *the following assertions hold true.*

(a) $d_1 := \partial_x$. *Then* $R[x,y]^{d_1} = R[y]$.

(b) $d_2 := ay^m\partial_x + bx^n\partial_y$, *where* $m, n \in \mathbb{N}$ *and* $a, b \in R \setminus \{0\}$ *with* $\gcd(a,b) = 1$. *Then* $R[x,y]^{d_2} = R[b(m+1)x^{n+1} - a(n+1)y^{m+1}]$.

(c) $d_3 := nx\partial_x - my\partial_y$, *where m and n are relatively prime positive integers. Then* $R[x,y]^{d_3} = R[x^m y^n]$.

Proof. **(a)** Obvious.

(b) Since $\operatorname{div}(d_2) = 0$, by *Lemma 3.5*, there exists $f \in R[x,y] \setminus R$ such that $d_2(f) = 0$. By the proof of *Lemma 3.5*, we can write f as

$$f = \frac{1}{n+1} bx^{n+1} - \frac{1}{m+1} ay^{m+1}.$$

By *Lemma 2.18*, we see that $K[f] \cap R[x,y] = R[f]$. Moreover, we can check easily that $\gcd(f_x, f_y) = 1$. By *Proposition 2.20*, f is a closed polynomial, also $(m+1)(n+1)f$ is a closed polynomial. Thus $R[x,y]^{d_2} = R[(m+1)(n+1)f]$.

(c) Let $g = x^{m-1}y^{n-1} \in R[x,y]$. Then $\operatorname{div}(gd_3) = 0$. By *Lemma 3.5*, we can construct a polynomial $h \in R[x,y]$ such that $gd_3(h) = 0$ by $h = -x^m y^n$. By *Lemma 2.18*, we have $K[h] \cap R[x,y] = R[h]$. Since m and n are relatively prime, by *Example 2.19*, h is a closed polynomial. Thus $R[x,y]^{d_3} = R[x,y]^{gd_3} = R[h] = R[x^m y^n]$. $\square$

Next, we show the following lemma. This gives some types of derivations whose kernels have only constant polynomials.

Lemma 4.4. *For $g \in R[x,y] \setminus \{0\}$, let $d = \partial_x + g\partial_y$. If $\deg_y g \geq 1$, then $R[x,y]^d = R$.*

Proof. Let $g = b_0 y^{\deg_y g} +$ (the lower y-degree terms), for $b_0 \in R[x] \setminus \{0\}$. We take any element $h \in R[x,y] \setminus \{0\}$ and put

$$h = a_0 y^s + a_1 y^{s-1} + \cdots + a_{s-1} y + a_s,$$

where $s = \deg_y h (\geq 0)$, $a_0, \ldots, a_s \in R[x]$ and $a_0 \neq 0$. Then

$$d(h) = (d(a_0)y^s + \cdots + d(a_s)) + g(sa_0 y^{s-1} + \cdots + a_{s-1}).$$

Since $\deg_y g \geq 1$, we have $s \leq s - 1 + \deg_y g$.

Now, we suppose that $d(h) = 0$. If $s < s - 1 + \deg_y g$, then by comparing the coefficients of y^s in the equation $d(h) = 0$, we obtain the equality $a_0 b_0 s = 0$, so $s = 0$. Then $0 = d(h) = d(a_0) = \partial_x(a_0)$, hence $h = a_0 \in R$. On the other hand, if $s = s - 1 + \deg_y g$, then we obtain the equality $d(a_0) + a_0 b_0 s = 0$. Since $\deg_x d(a_0) < \deg_x a_0 \leq \deg_x a_0 b_0$, we have $s = 0$. Hence $h = a_0 \in R$. □

Now, we shall prove *Theorem 4.2*.

Proof of Theorem 4.2. From now on, we assume that d is none of (a)–(c) of *Theorem 4.2* and prove that $R[x,y]^d = R$. We denote d_K by the K-derivation on $K[x,y]$ which is the natural extension of d. In order to prove $R[x,y]^d = R$, it is enough to show that $K[x,y]^{d_K} = K$. Therefore it is enough to show that, for the following K-derivation d, the kernel of that is equal to K:

$$d = x^m \partial_x + a y^n \partial_y,$$

where $a \in K^*$, $m, n \in \mathbb{N}$. If $m = 0$ and $n \geq 1$, then d is the form in *Lemma 4.4*. So we know already that the kernel is K. Therefore we may assume that $n \geq m \geq 1$. Let ℓ be the greatest common divisor of $m - 1$ and $n - 1$ as integers, $m' := (m-1)/\ell$ and $n' := (n-1)/\ell$, here we assume $m' = n' = 1$ if $m = n = 1$. We set $\mathbf{w} := (n', m')$ and consider the $\mathbb{Z}$-grading $\mathfrak{g}_{\mathbf{w}}$ on $K[x,y]$, that is, $\deg_{\mathbf{w}}(x) = n'$ and $\deg_{\mathbf{w}}(y) = m'$. Then we can see easily that if $f \in K[x,y]$ is homogeneous for $\mathfrak{g}_{\mathbf{w}}$ then so is $d(f)$.

Let $f \in K[x,y]^d \setminus \{0\}$. In order to prove $K[x,y]^d = K$, we may assume that f is homogeneous for $\mathfrak{g}_{\mathbf{w}}$. Then we have $(\alpha_0, \beta_0) \in \mathbb{N}^2$ such that

$$f = \sum_{\substack{i \geq 0 \\ \beta_0 - in' \geq 0}} c_i x^{\alpha_0 + im'} y^{\beta_0 - in'}, \qquad (*)$$

where $c_i \in K$. Since $f \in K[x,y]^d$, we have

$$\begin{aligned} 0 &= d(f) \\ &= \sum_{\substack{i \geq 0 \\ \beta_0 - in' \geq 0}} c_i \left((\alpha_0 + im')x^{\alpha_i} y^{\beta_0 - in'} + (\beta_0 - in')a x^{\alpha_0 + im'} y^{\beta_i} \right), \qquad (**) \end{aligned}$$

where $\alpha_i = \alpha_0 + im' + m - 1$ and $\beta_i = \beta_0 - jn' + n - 1$. Here we set the following subsets A and B of $\mathbb{N}^2$:

$$A := \{(\alpha_0 + im' + m - 1, \beta_0 - in') \mid i \geq 0, \beta_0 - in' \geq 0\},$$
$$B := \{(\alpha_0 + jm', \beta_0 - jn' + n - 1) \mid j \geq 0, \beta_0 - jn' \geq 0\}.$$

Suppose that $A \cap B = \emptyset$. Then, by taking $i = 0$ in A, we see from $(**)$ that $c_0\alpha_0 = 0$. So, $\alpha_0 = 0$. Similarly, we have $\beta_0 = 0$. Hence $f = c_0 x^{\alpha_0} = c_0 \in K$.

Suppose that $A \cap B \neq \emptyset$. Then there exist $i, j \in \mathbb{N}$ such that

$$\begin{aligned} &\alpha_0 + im' + m - 1 = \alpha_0 + jm', \\ &\beta_0 - in' = \beta_0 - jn' + n - 1, \\ &\beta_0 - in' \geq 0, \\ &\beta_0 - jn' \geq 0. \end{aligned}$$

Then $(j - i)m' = m - 1$ and $(j - i)n' = n - 1$. Here we may assume that $j \geq i$. Then $j - i = \ell$. We consider the cases $n \geq 2$ and $n = 1$ separately.

Case: $n \geq 2$. Then $j > i$. By considering the term $i = 0$ in $(**)$, we have $c_0\beta_0 a = 0$. So $\beta_0 = 0$. Since $f = c_0 x^{\alpha_0} \in K[x,y]^d$, we have $\alpha_0 = 0$. Therefore, $f = c_0 \in K$.

Case: $n = 1$. Then $m' = n' = 1$ and so $i = j$. By $(**)$, we have

$$\begin{aligned} &c_0\alpha_0 + c_0\beta_0 a = 0, \\ &c_1(\alpha_0 + 1) + c_1(\beta_0 - 1)a = 0, \\ &\vdots \\ &c_{\beta_0 - 1}(\alpha_0 + \beta_0 - 1) + c_{\beta_0 - 1} a = 0, \\ &c_{\beta_0}(\alpha_0 + \beta_0) + c_{\beta_0} a = 0. \end{aligned}$$

Since $c_0 \neq 0$, we have $\alpha_0 + \beta_0 a = 0$. If $\beta_0 > 0$, then $a = -\alpha_0/\beta_0 \in \mathbb{Q}_{<0}$. So d is (c) of *Theorem 4.2*. If $\beta_0 = 0$, then $\alpha_0 = 0$ and hence $f \in K$. $\square$

We note here that the condition "R is a UFD" is necessary. Even if d is a monomial derivation, the kernel may not be finitely generated over R in the case where R is not a UFD (see e.g., *Example 3.2*). Indeed, the derivation d in *Example 3.2* is a monomial derivation, however, that kernel is not generated by one polynomial, in particular, it needs infinite generators.

2. Kernels of monomial derivations on $k(x, y)$

This section is based on [**33**, Section 4]. Let $k[x, y] \cong_k k^{[2]}$ be the polynomial ring in two variables over a field k of characteristic zero and $k(x, y)$ its quotient field. For a k-derivation d on $k[x, y]$, we denote the same notation d by the k-derivation on $k(x, y)$ which is the natural extension of the original d, and its kernel is denoted by $k(x, y)^d$. In this section, by using the argument in [**63**, Section 5] and *Theorem 4.2*, we determine the non-zero monomial derivations d on $k[x, y]$ such that $Q(k[x, y]^d) \neq k(x, y)^d$.

Let d be a monomial k-derivation on $k[x, y]$. In order to study $k(x, y)^d$, by switching the role of x and y, we may assume that the following conditions are satisfied:

(i) $d(x)$ is monic.

(ii) $\gcd(d(x), d(y)) = 1$.

(iii) $\deg d(x) \leq \deg d(y)$ provided $d(y) \neq 0$.

The following is the main result in this section.

Theorem 4.5. *Let d be a non-zero monomial k-derivation on the polynomial ring $k[x, y]$ in two variables over a field k of characteristic zero. Assume that d satisfies the above three conditions* (i)–(iii)*, $k[x, y]^d = k$ and $k(x, y)^d \neq k$. Then d is one of the following* (a)–(c)*.*

(a) $d = \partial_x + ax^m y^{n+1} \partial_y$*, where $m \in \mathbb{N}$, $n \in \mathbb{Z}_{>0}$ and $a \in k^*$.*

(b) $d = x^{m+1}\partial_x + ay^{n+1}\partial_y$*, where $m, n \in \mathbb{Z}_{>0}$ with $m \leq n$ and $a \in k^*$.*

(c) $d = x\partial_x + ay\partial_y$*, where a is a positive rational number.*

Let d be a k-derivation on $k[x, y]$. If $k[x, y]^d \neq k$, then $Q(k[x, y]^d) = k(x, y)^d$. See [**77**, Theorem], which is generalized in [**5**] and [**34**]. So *Theorem 4.5* also gives the classification of the monomial k-derivations d on $k[x, y]$ such that $Q(k[x, y]^d) \neq k(x, y)^d$.

In order to prove *Theorem 4.5* we show the following two lemmas.

Lemma 4.6. *Let $d = \partial_x + ax^m y^{n+1}\partial_y$, where $m, n \in \mathbb{N}$ and $a \in k^*$. Then $k(x, y)^d = k$ if and only if $n = 0$.*

Proof. If $n \geq 1$, then $nx^{m+1} + (m + 1)a^{-1}y^{-n} \in k(x, y)^d \setminus k$. We assume that $n = 0$. By *Lemma 4.4*, $k[x, y]^d = k$. Let $f \in k[x, y] \setminus k$ be a non-constant polynomial and put

$$f = a_s y^s + a_{s-1} y^s + \cdots + a_1 y + a_0,$$

where $s = \deg_y f(\geq 0)$, $a_0, \ldots, a_s \in k[x]$ and $a_s \neq 0$. Assume that $g := d(f)/f \in k[x,y]$, namely, f is a Darboux polynomial of d.

Assume further that $a_0 \neq 0$, i.e., $y \not| f$. Since f is non-constant and $d(f) = gf$, $g \neq 0$. We have

$$s \geq \deg_y d(f) = \deg_y g + \deg_y f = \deg_y g + s.$$

This implies $g \in k[x]$. Comparing the constant terms with respect to y in the equation $d(f) = gf$, we have $a_0' = ga_0$, where a_0' is the derivative of a_0 with respect to x, which is a contradiction. Hence $a_0 = 0$.

The argument in the previous paragraph implies that f can be expressed as $f = f_1 y^t$, where $t \in \mathbb{Z}_{>0}$, $f_1 \in k[x,y]$ and $y \not| f_1$. By [**63**, Proposition 2.4], f_1 is also a Darboux polynomial of d and so $f_1 \in k^*$. Therefore, f can be expressed as $f = a_s y^s$, where $a_s \in k^*$. We infer from [**63**, Proposition 2.5] that $k(x,y)^d = k$. □

Lemma 4.7. *Assume that $d = x^{m+1}\partial_x + ay^{n+1}\partial_y$, where $a \in k^*$, $m, n \in \mathbb{N}$ and $m \leq n$, and that $k(x,y)^d \neq k$. Then one of the following conditions* (a) *and* (b) *holds true.*

(a) *$m, n > 0$.*

(b) *$m = n = 0$ and $a \in \mathbb{Q} \setminus \{0\}$.*

Proof. If d satisfies the condition (a) (resp. (b)), then $ma^{-1}y^{-n} - nx^{-m} \in k(x,y)^d \setminus k$ (resp. $x^p y^{-q} \in k(x,y)^d \setminus k$, where p and q are relatively prime integers such that $a = p/q$). We consider the following cases separately.

Case: $m = n = 0$ and $a \notin \mathbb{Q}$. By *Theorem 4.2*, $B^d = k$. Let $f \in R \setminus k$ be a non-constant polynomial and put

$$f = a_s y^s + a_{s-1} y^s + \cdots + a_1 y + a_0,$$

where $s = \deg_y f(\geq 0)$, $a_0, \ldots, a_s \in k[x]$ and $a_s \neq 0$. Assume that f is a Darboux polynomial and set $g = d(f)/f$.

Assume further that $a_0 \neq 0$, i.e., $y \not| f$. Since f is non-constant and $d(f) = gf$, $g \neq 0$. We have $\deg_y d(f) = \deg_y g + s$. Since $\deg_y d(f) \leq s$, $g \in k[x]$ and $xa_0' = ga_0$. So, $n_0 := g = \deg_x a_0 \in \mathbb{Z}_{>0}$ and $a_0 = bx^{n_0}$ for some $b \in k^*$. Assume further that $s > 0$. Comparing the highest terms with respect to y in the equation $d(f) = n_0 f$, we have $xa_s' = (n_0 - sa)a_s$. Then $n_0 - sa = \deg_x a_s$ and so $a \in \mathbb{Q}$. This is a contradiction. Therefore, $s = 0$, i.e., $f = bx^{n_0}$.

Assume next that $a_0 = 0$. We set as $f = f_1 y^t$, where $t \in \mathbb{Z}_{>0}$, $f_1 \in B$ and $y \not| f_1$. Then f_1 is also a Darboux polynomial of d. So the argument in the previous paragraph implies that $f_1 = bx^{\deg_x f_1}$ for some $b \in k^*$.

Therefore, f can be expressed as $bx^i y^j$ for some $i, j \in \mathbb{N}$ and $b \in k^*$. Since $a \notin \mathbb{Q}$, we infer from [**63**, Proposition 2.5] that $k(x, y)^d = k$.

Case: $n = 0$, $m \geq 1$. Set $d_1 = \partial_x + ax^{m+1} y\partial_y$, where m and a are the same as in d. By *Lemma 4.6*, $k(x, y)^{d_1} = k$. Let $\sigma : k(x, y) \to k(x, y)$ be the k-automorphism defined by $\sigma(x) = x^{-1}$ and $\sigma(y) = y-1$. Then $d = -x^{m-1}\sigma d_1 \sigma^{-1}$. Hence $k(x, y)^d = k$.

Case: $m = 0$, $n \geq 1$. By using the same argument as in the previous case, we have $k(x, y)^d = k$.

The proof of *Lemma 4.7* is thus verified. □

Theorem 4.5 is a consequence of *Theorem 4.2*, *Lemmas 4.6* and *4.7*.

3. Examples

In this section, by using *Theorem 2.29*, we give some examples of closed polynomials and factorially closed polynomials. Let R be an integral domain of characteristic $p \geq 0$, $K = Q(R)$ and let $B \cong_R R^{[n]}$.

Lemma 4.8. *Let $f \in B \setminus R$ such that $K[f] \cap B = R[f]$. Then the following assertions hold true.*

(a) *The following two conditions are equivalent*:

  (i) *f is factorially closed over R.*

  (ii) *f is factorially closed over K.*

(b) *If there exist a field L such that L/K is Galois extension and f is factorially closed over L, then f is factorially closed over K.*

Proof. For a field L containing R, denote $B_L = B \otimes_R L \cong_L L^{[n]}$.

(a) (i) $\Longrightarrow$ (ii) Take $g, h \in B_K \setminus \{0\}$ with $gh \in K[f]$. Without loss of generality, we may assume that $g, h \in B$. Since $gh \in K[f]$, there exist $a_0, ..., a_m \in R$ and $b_0, ..., b_m \in R \setminus \{0\}$ such that

$$gh = \frac{a_0}{b_0} f^m + \frac{a_1}{b_1} f^{m-1} + \cdots + \frac{a_{m-1}}{b_{m-1}} f + \frac{a_m}{b_m}.$$

Set $b := \prod_{i=0}^m b_i$ and $c_i := b(a_i/b_i) \in R$ for $0 \leq i \leq m$. Then

$$bgh = c_0 f^m + \cdots + c_{m-1} f + c_m \in R[f].$$

Since f is factorially closed over R, we have $g, h \in R[f] \subset K[f]$.

(ii) $\Longrightarrow$ (i) Take $g, h \in B \setminus \{0\}$ with $gh \in R[f]$. Since $gh \in K[f]$ and f is factorially closed over K, we have that $g, h \in K[f] \cap B = R[f]$.

(b) Let G be the Galois group of L over K. Here, G acts on a polynomial of B_L by acting on its coefficients. Then we can see easily that $L[f]^G = K[f]$, where $L[f]^G$ is the ring of G-invariant of $L[f]$.

Take $g, h \in B_K \setminus \{0\}$ with $gh \in K[f]$. Since $K[f] \subset L[f]$ and f is factorially closed over L, $g \in L[f]$ and $h \in L[f]$. Hence

$$g, h \in L[f] \cap B_K = L[f]^G = K[f].$$

Therefore f is a factorially closed polynomial over K. $\square$

We give an example of factorially closed polynomials below.

Example 4.9. Suppose that the characteristic of R is zero. Let $f \in B$ such that $K[f] \cap B = R[f]$. Assume that $f - \lambda$ is irreducible in $B \otimes_R L$ for any $\lambda \in L$, where L is an algebraic closure of K. Then f is a factorially closed polynomial over R.

Proof. This assertion follows from *Theorem 2.29* and *Lemma 4.8*. $\square$

As the end of this section, we give observations for important three kinds of polynomials; Vénéreau polynomial, Danielewski surface and Koras-Russell threefold over $\mathbb{C}$. For $f \in \mathbb{C}[x_1, \ldots, x_n] \cong_{\mathbb{C}} \mathbb{C}^{[n]}$, we define the **Makar-Limanov invariant** for f by

$$\mathrm{ML}(f) := \bigcap_d (\mathbb{C}[x_1, \ldots, x_n]/(f))^d,$$

where d runs through the non-zero locally nilpotent derivations on the $\mathbb{C}$-algebra $\mathbb{C}[x_1, \ldots, x_n]/(f)$. We can easily see that if f is a variable, then $\mathrm{ML}(f) = \mathbb{C}$. By using this fact, we get the following examples:

Example 4.10. Let $R = \mathbb{C}[x] \cong_{\mathbb{C}} \mathbb{C}^{[1]}$ and $B = R[y, z, t] = \mathbb{C}[x, y, z, t] \cong_{\mathbb{C}} \mathbb{C}^{[4]}$. For $n \geq 1$, let

$$v_n := y + x^n(xz + y(yt + z^2)) \in B.$$

This is called the n-th **Vénéreau polynomial**. If $n \geq 2$, then it is known to be a variable over R, however, we do not know whether v_1 is a variable over R or not (see [**18**, Example 3.18] and [**39**, Corollary 14]). Here, we can show that v_1 is a closed polynomial over R (of course, v_n is a closed polynomial for $n \geq 2$).

Proof. Since $c(v_1 - v_1(0,0,0)) = \gcd(1, x^2, x^1) = 1$, by *Lemma 2.18*, $Q(R)[v_1] \cap B = R[v_1]$. It is easy to show that

$$\widehat{v_1} = \gcd((v_1)_y, (v_1)_z, (v_1)_t) = 1.$$

By *Corollary 2.15*, v_1 is a closed polynomial over R. In other words, $\mathbb{C}[x, v_1]$ is integrally (especially algebraically) closed in $B = \mathbb{C}[x, y, z, t]$. $\square$

Example 4.11. (a) (Danielewski surface). Let $\mathbb{C}[x, y, z] \cong_{\mathbb{C}} \mathbb{C}^{[3]}$. The **Danielewski surface** is the polynomial in $\mathbb{C}[x, y, z]$ defined by

$$f := x^n z - y^2 - y,$$

where $n \geq 2$. Then f is a factorially closed polynomial, but is not a variable.

(b) (Koras-Russell threefold). Let $\mathbb{C}[x, y, z, t] \cong_{\mathbb{C}} \mathbb{C}^{[4]}$. The **Koras-Russell threefold** is the polynomial in $\mathbb{C}[x, y, z, t]$ defined by

$$g := x + x^2 y + z^2 + t^3.$$

Then g is a factorially closed polynomial, but is not a variable.

Proof. **(a)** For $\lambda \in \mathbb{C}$, let $f_\lambda := f - \lambda$. We assume that $f_\lambda = gh$ for some $g, h \in \mathbb{C}[x, y, z] \setminus \{0\}$. Computing the z-degree of $f_\lambda = gh$, we may assume that $\deg_z g = 1$ and $\deg_z h = 0$. Here, we write $g = g_1 z + g_2$ for $g_1, g_2 \in \mathbb{C}[x, y]$. Then we have $x^n = g_1 h$ and $-y^2 - y - \lambda = g_2 h$. Hence $\deg_y h = \deg_x h = 0$, which means $h \in \mathbb{C}^*$. Therefore f_λ is irreducible for any $\lambda \in \mathbb{C}$. By *Theorem 2.29*, f is a factorially closed polynomial. On the other hand, it follows from [**18**, Theorem 9.2] (see also [**43**, Proposition (ii)]) that $\mathrm{ML}(f) = \mathbb{C}[x]$. So, f is not a variable.

(b) For $\lambda \in \mathbb{C}$, let $g_\lambda := g - \lambda$. We assume that $g_\lambda = pq$ for some $p, q \in \mathbb{C}[x, y, z, t] \setminus \{0\}$. Computing the y-degree of $g_\lambda = pq$, we may assume that $\deg_y p = 1$ and $\deg_y q = 0$. Here, we write $p = p_1 y + p_2$ for $p_1, p_2 \in \mathbb{C}[x, z, t]$. Then we have $x^2 = p_1 q$ and $x + z^2 + t^3 - \lambda = p_2 q$. By the first equation, we have $\deg_z q = \deg_t q = 0$ and q is a component of x^2. If x divides q, then this contradicts the second equation. Thus $\deg_x q = 0$, so $q \in \mathbb{C}^*$. By *Theorem 2.29*, g is a factorially closed polynomial. On the other hand, it follows from [**18**, Theorem 9.9] (see also [**41**, Section 1]) that $\mathrm{ML}(g) = \mathbb{C}[x]$. So, g is not a variable. $\square$

CHAPTER 5

Partial theories for higher dimensional cases

Let k be a field and let $B = k[x_1, \ldots, x_n] \cong_k k^{[n]}$ be the polynomial ring in n variables over k. For $1 \leq r \leq n$, let $f_1, \ldots, f_r \in B$ such that they are algebraically independent over k and the ring $k[f_1, \ldots, f_r]$ is algebraically closed in B, and let $F = (f_1, \ldots, f_r)$. Here we consider the dominant morphism

$$\Phi_F : \mathbb{A}_k^n \cong_k \operatorname{Spec} B \to \operatorname{Spec} k[f_1, \ldots, f_r] \cong_k \mathbb{A}_k^r$$

associated by the natural inclusion $k[f_1, \ldots, f_r] \to B$. When k is an algebraically closed field, by the first Bertini theorem (see e.g., [**70**, Theorem 1, p.139]), we see that general fibers of Φ_F are irreducible and reduced. This is a natural generalization of theories of closed polynomials (see e.g., *Theorem 2.3*). As a first step, we will discuss the case where $r = 2$.

Let $f, g \in B$. A pair of polynomials (f, g) is said to be a **closed-pair** if f and g are algebraically independent over k, and the ring $k[f, g]$ is algebraically closed in B. Clearly, if $n = 2$, then a closed-pair (f, g) is a system of variables of $B \cong_k k^{[2]}$, that is, $k[f, g] = B$. For this reason, the concept of closed-pair makes sense for the polynomial ring having at least three variables.

In this chapter, we construct several examples of closed-pairs and give some observations for them. In particular, we consider the following situation:

$$\Phi_{(f,g)} : \mathbb{A}_k^3 \to \operatorname{Spec} k[f, g] \cong_k \mathbb{A}_k^2,$$

wehre $f, g \in B \cong_k k^{[3]}$ and $k[f, g] \cong_k k^{[2]}$.

First of all, we show the following result. This is a characterization of closed-pairs (see also *Proposition 3.4*).

Proposition 5.1. *Let k be a field of characteristic zero, $B \cong_k k^{[3]}$ and let $f, g \in B$. Then the following conditions are equivalent*:

(i) *(f, g) is a closed-pair.*

(ii) $B^{\Delta_{(f,g)}} = k[f, g]$.

Proof. **(i) $\Longrightarrow$ (ii)** Suppose that (f, g) is a closed-pair. By *Proposition 1.10* (a) and (b), we have $\Delta_{(f,g)} \neq 0$ and $B^{\Delta_{(f,g)}} = k[f, g]$.

(ii) $\Longrightarrow$ (i) Similarly, if $B^{\Delta_{(f,g)}} = k[f,g]$, then $\Delta_{(f,g)} \neq 0$. By *Proposition 1.10* (a), f and g are algebraically independent over k. Therefore (f,g) is a closed-pair. □

Remark 5.2. A pair of closed polynomials is not necessarily a closed-pair. For example, in $\mathbb{Q}[x,y,z] \cong_{\mathbb{Q}} \mathbb{Q}^{[3]}$, consider polynomials xy and $x+y$. Then these polynomials are closed polynomials which are algebraically independent over $\mathbb{Q}$. However, $\mathbb{Q}[xy, x+y] \subsetneq \mathbb{Q}[x,y]$, which implies that there exists $f \in \mathbb{Q}[x,y,z] \setminus \mathbb{Q}[xy, x+y]$ such that f is algebraic over $\mathbb{Q}[xy, x+y]$. Thus, $(xy, x+y)$ is not a closed-pair.

1. Retracts of polynomial rings

Let R be an integral domain. Let B be an R-algebra and let A be an R-subalgebra of B. We say that A is a **retract** of B if there exists an ideal I of B such that $B \cong_R A \oplus I$ as R-modules. There are some equivalent conditions of retracts as below.

Lemma 5.3. *The following conditions are equivalent*:

(i) *A is a retract of B.*

(ii) *There exists a homomorphism $\varphi : B \to A$ of R-algebras such that the following sequence of R-modules is exact and split*:
$$0 \to \ker \varphi \to B \xrightarrow{\varphi} A \to 0.$$

(iii) *There exists a homomorphism $\varphi : B \to A$ of R-algebras such that $\varphi|_A = \mathrm{id}_A$.*

Proof. Obvious. □

The homomorphism $\varphi : B \to A$ in (ii) is called a **retraction**. The followings are basic properties of retracts.

Proposition 5.4. (cf. [**11**, Section 1]) *Let A be a retract of B. Then the following assertions hold true.*

(a) *If B is an integral domain, then A is algebraically closed in B.*

(b) *If B is a UFD, then A is also a UFD.*

(c) *If B is regular, then A is also regular.*

Corollary 5.5. *Suppose that the characteristic of R is zero and B is finitely generated R-domain. For every retract A of B, there exists an R-derivation on B such that $A = B^d$.*

Proof. By *Proposition 5.4* (a), A is algebraically closed in B. It follows from *Theorem 1.14* that $A = B^d$ for some R-derivation d. □

In [**11**], Costa asks us the following interesting question.

Question 5.6. Let k be a field and let $B \cong_k k^{[n]}$. Is every retract of B a polynomial ring over k?

If $n \leq 2$, then the above question is affirmative and proved by Costa ([**11**, Theorem 3.5]). On the other hand, it is well known that *Question 5.6* is related to Zariski's cancellation problem as below.

Proposition 5.7. *Let k be a field. If Question 5.6 holds for $n+1$, then Zariski's cancellation problem has an affirmative answer for $\mathbb{A}_k^n$, that is, for an affine variety X over k, $X \times \mathbb{A}_k^1 \cong_k \mathbb{A}_k^{n+1}$ implies $X \cong_k \mathbb{A}_k^n$.*

Proof. Suppose that *Question 5.6* holds for $n+1$. Let X be an affine variety over k such that $X \times \mathbb{A}_k^1 \cong_k \mathbb{A}_k^{n+1}$ and let A be the coordinate ring of X. Then $A^{[1]} \cong_k k^{[n+1]}$ and $\mathrm{tr.deg}_k\, A = n$. Let $B = A[t] \cong_A A^{[1]} \cong_k k^{[n+1]}$. Define the homomorphism of k-algebras

$$\varphi : B = A[t] \to A$$

by $\varphi(t) = 0$ and $\varphi(a) = a$ for $a \in A$. Then φ is a retraction. By the assumption, $A \cong_k k^{[n]}$, which implies that $X \cong_k \mathbb{A}_k^n$. □

It is well known that if $n \leq 2$, then Zariski's cancellation problem holds true for any field (see Fujita [**20**], Miyanishi-Sugie [**53**] and Russell [**67**]). On the other hand, when the characteristic of k is positive, Gupta in [**22**] and [**23**] proved that Zariski's cancellation problem does not hold for $\mathbb{A}_k^n$ if $n \geq 3$. Therefore *Question 5.6* does not hold in the case where k is a field of positive characteristic and $n \geq 4$. So, the remaining cases are:

- the characteristic of k is positive and $n = 3$,
- the characteristic of k is zero and $n \geq 3$.

In [**56**], the author gave an affirmative answer for *Question 5.6* in the case where k is a field of characteristic zero and $n = 3$.

Let k be a field of characteristic zero. We consider retracts of the polynomial ring $B \cong_k k^{[n]}$. Since B is a UFD and regular, by *Proposition 5.4*, every retract of B has the same properties. Furthermore, it follows from [**56**, Theorem 2.5] that if A is a retract of $B \cong_k k^{[n]}$ of transcendence degree 2 over k, then there exist $f, g \in B$ such that $A = k[f, g] \cong_k k^{[2]}$. By *Proposition 5.4* (a), the pair (f, g) is a closed-pair. In this way, we can find closed-pairs.

Example 5.8. Let $B = k[x, y, z] \cong_k k^{[3]}$ be the polynomial ring over a field k of characteristic zero. Let $f = x + x^2 z$ and $g = y + y^2 z$. Then $k[f, g]$ is a retract of B. In particular, $k[f, g] = B^{\Delta_{(f,g)}}$ and (f, g) is a closed-pair.

Proof. Define the homomorphism of k-algebras $\varphi : k[x, y, z] \to k[f, g]$ by $\varphi(x) = f$, $\varphi(y) = g$ and $\varphi(z) = 0$. Then φ is a retraction. Since $\Delta_{(f,g)} \neq 0$, by *Proposition 1.10*, we have that $k[f, g] \cong_k k^{[2]}$ and $k[f, g] = B^{\Delta_{(f,g)}}$. Hence (f, g) is a closed-pair. $\square$

2. Closed-pairs and kernels of derivations on $k[x, y, z]$

Let k be a field of characteristic zero and let $B \cong_k k^{[n]}$ be the polynomial ring in n variables over k. For a derivation $d \in \operatorname{Der}_k B$, we define the **corank** of d by the maximal integer i such that there exists a partial system of variables $\{x_1, \ldots, x_i\}$ of B contained in B^d and denote it by $\operatorname{corank}(d)$. The **rank** of d is defined by

$$\operatorname{rank}(d) := n - \operatorname{corank}(d),$$

in particular, $\operatorname{rank}(d) \in \{0, 1, \ldots, n\}$. By definition, the rank and corank are invariants of d.

We know already that the kernel of a derivation on $k^{[3]}$ is finitely generated over k (see Zariski's Theorem [**76**]). Also, if d is a locally nilpotent derivation on $k^{[3]}$, then the kernel is generated by two polynomials (see Miysnishi [**51**]). Then the following is the natural question.

Question 5.9. Let $d \in \operatorname{Der}_k k^{[3]}$ be a non-zero derivation. Is the kernel generated by at most two polynomials?

In the rest of this section, we consider the above question. From now on, we assume that $B = k[x, y, z] \cong_k k^{[3]}$. Let $d \in \operatorname{Der}_k B \setminus \{0\}$ be a non-zero k-derivation on B.

Lemma 5.10. *If* $\operatorname{tr.deg}_k B^d \leq 1$*, then* $B^d = k[f]$ *for some* $k \in B$*.*

Proof. If $\operatorname{tr.deg}_k B^d = 0$, then $B^d = k$. So $B^d = k[1]$. If $\operatorname{tr.deg}_k B^d = 1$, then B^d is a maximal element of $\mathfrak{S}(1, B)$. By *Lemma 2.2*, there exists $f \in B \setminus k$ such that $B^d = k[f]$. $\square$

Proposition 5.11. *If* $\operatorname{tr.deg}_k B^d = 2$ *and* $\operatorname{rank}(d) \leq 2$*, then there exists a closed-pair* (f, g) *such that* $B^d = k[f, g]$*.*

Proof. Since $\operatorname{rank}(d) \leq 2$, we may assume that $z \in B^d$. Set $R = k[z]$. Then we can regard d as an R-derivation on $R[x, y] \cong_R R^{[2]}$. By *Theorem*

3.1 (a), $B^d = R[x,y]^d = R[f]$ for some $f \in R[x,y]$. Since $\operatorname{tr.deg}_k B^d = 2$, we have $f \notin R$. Therefore $B^d = k[f,z] \cong_k k^{[2]}$, which implies that (f,z) is a closed-pair. □

We give a necessary condition for kernels of derivations to be isomorphic to $k^{[2]}$ as follows (see also *Theorem 3.9*).

Proposition 5.12. (cf. [**42**, Lemma 6]) *Assume that $B^d = k[f,g] \cong_k k^{[2]}$ for some $f,g \in B$. Then $d \sim \Delta_{(f,g)}$, that is, there exist $\xi, \eta \in B$ such that $\eta d = \xi \Delta_{(f,g)}$.*

Proof. By *Proposition 1.10* (b), $B^{\Delta_{(f,g)}}$ is the algebraic closure of $k[f,g] = B^d$ in B. Choose $h \in B$ such that $d(h) \neq 0$. Then h is transcendental over $k[f,g]$. Hence $\Delta_{(f,g)}(h) \neq 0$. Let $\eta = \Delta_{(f,g)}(h)$ and $\xi = d(h)$. Then we have $\eta d = \xi \Delta_{(f,g)}$ on the subalgebra $k[f,g,h] \subset B$. Since the extension $B/k[f,g,h]$ is algebraic, it follows from [**18**, Proposition 1.14] that $\eta d = \xi \Delta_{(f,g)}$ on B. □

By *Lemma 5.10* and *Proposition 5.11*, it is enough to consider the case where $\operatorname{rank}(d) = 3$ and $\operatorname{tr.deg}_k A = 2$. We introduce some kinds of types for derivations as below.

Definition 5.13. For $d \in \operatorname{Der}_k B$, let $A = B^d$. Assume that $\operatorname{rank}(d) = 3$ and $\operatorname{tr.deg}_k A = 2$. Then we say d is

- **of type I** if d is locally nilpotent and A is not a retract,
- **of type II** if d is not locally nilpotent and A is a retract,
- **of type III** if d is not locally nilpotent and A is not a retract.

See also the following table:

type	locally nilpotent	retract	$A = B^d$
I	yes	no	$k^{[2]}$
II	no	yes	$k^{[2]}$
III	no	no	??

In [**40**], Liu and Sun announced the following result.

Proposition 5.14. *Let A be a retract of B. If A is the kernel of a non-zero locally nilpotent derivation on B, then there exist a system of variables $s,t,u \in B$ such that $A = k[u,v]$ and $B = A[w]$.*

Corollary 5.15. *If d is a locally nilpotent derivation of rank 3, then B^d is not a retract. In particular, d is of type I.*

Proof. Assume to the contrary that $A = B^d$ is a retract of B. Then A is a retract and the kernel of the non-zero locally nilpotent derivation $\Delta_{(f,g)}$. By *Proposition 5.14*, there exists a system of variables u, v, w of B such that $A = k[u, v]$ and $B = A[w]$. This is a contradiction. $\square$

According to *Corollary 5.15*, we know that there are no derivations d such that d is locally nilpotent of rank 3 and B^d is a retract. The following is an example of derivations of type I.

Example 5.16. (Derivations of type I) Let $f = xz - y^2$ and $g = zf^2 + 2x^2yf + x^5$. Then $\Delta_{(f,g)}$ is locally nilpotent and $\mathrm{rank}(\Delta_{(f,g)}) = 3$. Furthermore, $A = B^{\Delta_{(f,g)}}$ is not a retract of B. Hence $\Delta_{(f,g)}$ is of type I. In particular, (f, g) is a closed-pair.

Proof. It follows from [**18**, Theorem 5.23] that $\Delta_{(f,g)}$ is locally nilpotent and $\mathrm{rank}(\Delta_{(f,g)}) = 3$. Hence, by *Corollary 5.15*, $\Delta_{(f,g)}$ is of type I. $\square$

Next, we give two examples of derivations of type II.

Example 5.17. (Derivations of type II) Define $d = x\partial_x + y\partial_y - z\partial_z$. Then $A = B^d = k[xz, yz]$, $\mathrm{rank}(d) = 3$, d is not locally nilpotent and A is a retract of B. Hence d is of type II. In particular, (xz, yz) is a closed-pair.

Proof. It is easy to show that d is not locally nilpotent, $k[xz, yz] \cong_k k^{[2]}$ and $k[xz, yz] \subset A$. Define the homomorphism $\varphi : B \to A$ of k-algebras by $\varphi(x) = xz$, $\varphi(y) = yz$ and $\varphi(z) = 1$. Then $\varphi|_A = \mathrm{id}_A$. Therefore A is a retract of B, hence $A = k[xz, yz]$. It is clear that A does not contain variables, that is, $\mathrm{rank}(d) = 3$. $\square$

Example 5.18. (Derivations of type II) Define $d = x\partial_x + z\partial_y + y\partial_z$. Then $A = B^d = k[xy - xz, y^2 - z^2]$, $\mathrm{rank}(d) = 3$, d is not locally nilpotent and A is a retract of B. Hence d is of type II. In particular, $(xy - xz, y^2 - z^2)$ is a closed-pair.

Proof. It is easy to show that d is not locally nilpotent, $k[xy - xz, y^2 - z^2] \cong_k k^{[2]}$ and $k[xy - xz, y^2 - z^2] \subset A$. Define the homomorphism $\varphi : B \to A$ of k-algebras by $\varphi(x) = xy - xz$ and

$$\varphi(y) = \frac{1}{2}(y^2 - z^2 + 1), \ \ \varphi(z) = \frac{1}{2}(y^2 - z^2 - 1).$$

Then $\varphi|_A = \mathrm{id}_A$. Therefore A is a retract of B, hence $A = k[xy - xz, y^2 - z^2]$. It is clear that A does not contain variables, that is, $\mathrm{rank}(d) = 3$. $\square$

As seeing the above table, we know already that the kernel of a derivation of type I or II is isomorphic to $k^{[2]}$. However, the author has not yet given an example of derivations of type III.

Finally, we give another example of closed-pair.

Example 5.19. Let $d = x\partial_y + 2y\partial_z$. Then d is locally nilpotent and $B^d = k[x, xz - y^2]$. Hence B^d is factorially closed in B, in particular, $(x, xz - y^2)$ is a closed-pair. However, B^d is not a retract of B.

Proof. Let $A = B^d$. Assume to the contrary that A is a retract of B. Then there exists a homomorphism $\varphi : B \to A$ such that $\varphi|_A = \mathrm{id}_A$. Then $\varphi(x) = x$ and

$$xz - y^2 = \varphi(xz - y^2) = x\varphi(z) - \varphi(y)^2.$$

Note that $\varphi(y), \varphi(z) \in A$. Set $s = x$ and $t = xz - y^2$. Since $A = k[s, t] \cong_k k^{[2]}$, we can consider the standard degree function on A with respect to s and t. By the above equation, we have $t = \varphi(z)s - \varphi(y)^2$. Comparing the t-degree of the both side, we have a contradiction. $\square$

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