Characterization of Set Relations in Set Optimization and its Application to Set-Valued Alternative Theorems

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Introduction

In recent information society, it is instantly confirmed by seeing performance of computer, developments of network services, or worldwide information sharing that almost everything can be integrated and operated by computers. They sometimes work like humans thinking objects from multiple points of view and improving their performance by themselves. Actually, substitution with computers for humans results in reduction of costs and time. However, the base framework of them consists of computer programmings in whole cases and here one can find a reason to study mathematical optimization.

Characters of vector or set optimization are in a sense, dependent on imposed partial (non-total) order relations. We may be involved in a situation where we choose a superior element from possibly uncomparable components. When we nominate one of those whose personality or expertise completely differ, when we change our attitude toward investment conservatively for low risks or aggressively for high risks but with high returns, or when we make a machine detect physical disorder by comparing pictures of tomographic imaging in practice, we face uneven tradeoff and they cannot be solved by single-valued comparisons. For multi-valued criteria, binary relations such as preorders or partial orders are commonly introduced. Convex cones can stand for partial orders and this fact implies vector and set optimization are strongly linked with convex analysis. Set optimization, that is a direct generalization of the vector one, has developed together with set-valued analysis (for an overview of the history, see [25, 35] and references cited therein).

Set optimization is basically composed of two concepts: set orderings and optimality conditions. Set orderings describe preference of sets as criteria of deciding which set of two is preferred to the other. As a particular concept, "set relation" is originally given in [45] in which six type relations are well-defined; they consist of a variety of the pointwise ordering between elements from two sets by swapping "for all" and "for some." However, set orderings are not tractable comparing to vector orderings so that one sometimes needs quantification, materialization, or simplification before handling them. Tammer's approach [16–18] referred to as "sublinear scalarization" is one of the most popular and tractable tools for representation of set orderings. This concept especially realizes its great potential in optimality conditions: set-to-set comparisons are reduced to scalar calculations. Hamel [26] and Kuwano et al. [46] extended Tammer's scalarization to set functions and today, a number of papers describe them to utilize scalarization for sets. We have to note that this kind of simplification methods focusing narrow viewpoints simultaneously causes another loss of information. For example, the volume is a fundamental representation of objects, whereas it lacks their shapes or the number of vertices. The norm is an indispensable notion of vectors, whereas it fails to have their directions.

Motivation to this thesis is non-trivial calculability of Kuwano's functions [63,69,70]. Authors proved the functions are calculated by solving finitely many linear programming problems. To be honest, the evaluation functions are defined by the infimum or supremum of cone-like level sets ordered by the set relations. Therefore, it is a direct viewpoint that any approach to obtaining values of the funcitons is beyond the calculation of set relations. However, it is too early for us to think the scalarization is far behind simplification. Although the set relations are sometimes not tractable when one makes sure if they hold or not, the scalarization counterparts can be calculated by assuming some convexity conditions. This research discusses equivalent relationships between the set reltions and the scalarization functions. As a consequence, the set relations become practically maneuverable in operating calculation algorithms. We also introduce some theoretical application of our results to alternative theorems.

Speaking of Gordan's theorem of the alternative in [19], this represents fundamental parts of the linear structure. This theorem can be understood from various contexts: solutions of simultaneous equations and those of the dual inequalities are alternatively given; finitely many points are not contained in any half space when the linear combination of them with almost all positive coefficients represents the zero; a linear function which intersects the negative orthant cannot be linearly separated by any half space containing the positive orthant. There are several extension theorems of Gordan's beyond linear functions. To follow historical trends, authors would use the term "Gordan's type" in the literature in which whether a function has a "negative" value or not is alternatively considered and the alternative counterpart is given with a scalar function. This thesis presents new generalized Gordan's theorem for set-valued maps with respect to the set relations by using evaluation functions and reference sets. In the last part of this thesis, we utilize our results to describe robustness of feasible sets of optimization problems.

In robust optimization theory, authors investigate various sorts of optimization problems (e.g., linear robust optimization, semidefinite robust optimization, least-squares robust optimization, convex robust optimization, multiobjective robust optimization, nonlinear robust optimization) and several types of robust counterparts which define robustness (e.g., strictly robust counterpart, deviation robust counterpart, reliably robust counterpart, weighted robust counterpart). In general, nonlinear constraint conditions are not easy to deal with in well-defined robustness frameworks. We present simple robustness concepts of feasibility by calculating values of evaluation functions.

This thesis is structured as follows. Firstly in Section 2, we introduce basic frameworks of topological vector spaces and ordering properties with respect to convex cones. Section 3 deals with set relations given by Kuroiwa. The first part is devoted to fundamental notions. The latter half describes scalarization defined by the set relations. Evaluation functions are also defined and compared with Tammer's functional. In the last part of this section, we present calculation methods for evaluation functions under convexity assumptions. In Section 4, characterization theorems for set relations are introduced with a few examples and geometrical depictions. Section 5 contains generalization of Gordan's theorems of the alternative. Also, simple examples of the theorems are demonstrated. As application of the generalized Gordan's theorems, we introduce robustness of feasibility in the last section. Each section may contain short comments or observations about historical stories and surveys of existing researches.

Chapter 2

Ordered Topological Vector Space

To begin with, we introduce basic ideas specified in this thesis. This chapter is concerned with topological vector spaces, convexity notions, and partial order.

Set-valued analysis is usually treated in topological vector spaces in which topology and linearity are strongly connected. Actually, a convex solid cone plays important roles in considering set-to-set comparisons.

2.1 Convexity and Cone

Before giving the structure of topological vector spaces, we refer to some fundamental notations. The n-dimensional space is written as $\mathbb{R}^n$; $\mathbb{R}^n_+$ denotes the non-negative orthant of $\mathbb{R}^n$. Firstly, we introduce basic properties related to the linear structure. A vector space is defined as a set where the addition $(+)$ and the scalar multiplication $(\cdot)$ are defined; we use the real numbers $\mathbb{R}$ as a scalar in this thesis.

For sets S_1, S_2 in a vector space, we define addition and scalar multiplication as follows; $S_1 + S_2 := \{s_1 + s_2 : s_1 \in S_1 \text{ and } s_2 \in S_2\}$ and $\gamma S := \{\gamma s : s \in S\}$ $(\gamma \in \mathbb{R})$. We use simplified notiations $\{x\} + S = x + S$, $1S = S$, and $(-1)S = -S$. Particularly, $\emptyset + S = \emptyset$ and $\gamma\emptyset = \emptyset$.

Definition 2.1 (convexity)**.** A set S in a vector space is said to be *convex* if $\lambda S + (1-\lambda)S \subset S$ for all $\lambda \in (0, 1)$.

Definition 2.2 (cone)**.** A set S in a vector space is a *cone* if $\lambda S \subset S$ for all $\lambda > 0$.

Lemma 2.1. Let K be a convex cone. Then, $\gamma_1 K + \gamma_2 K \subset K$ for all $\gamma_1, \gamma_2 > 0$.

Proof. $\gamma_1 K + \gamma_2 K = (\gamma_1 + \gamma_2)\{(\gamma_1/(\gamma_1 + \gamma_2))K + (\gamma_2/(\gamma_1 + \gamma_2))K\} \subset K$. □

Note that if a set S satisfies $\gamma_1 S + \gamma_2 S \subset S$ for all $\gamma_1, \gamma_2 > 0$ conversely, S is a convex cone since S is obviously convex and $\gamma S \subset (\gamma/2)S + (\gamma/2)S \subset S$ for all $\gamma > 0$.

Generally, a convex cone $K \subset X$ having the zero defines a preorder on X as follows: $x \leq_K y :\Longleftrightarrow y - x \in K$ (*pointwise ordering*). We can see easily $\leq_K$ is *reflexive* ($x \leq_K x$ for all $x \in X$) and *transitive* ($x_1 \leq_K x_2$ and $x_2 \leq_K x_3$ imply $x_1 \leq_K x_3$). Also, it is invariant under the translation and the multiplication by positive scalars. If C is *pointed* (i.e., $K \cap (-K) = \{\mathbf{0}\}$), $\leq_K$ turns to be *antisymmetric* ($x_1 \leq_K x_2$ and $x_2 \leq_K x_1$ imply $x_1 = x_2$) and thus, it is a partial order.

Definition 2.3 (convex hull)**.** The *convex hull* of a set S denoted by $\operatorname{co} S$ is defined by the smallest convex set in X including S.

Definition 2.4 (conical hull)**.** The *conical hull* of a set S denoted by $\operatorname{cone} S$ is defined by the smallest cone in X including S.

An usual notaion of $\operatorname{cone} S$ does not represent convex cones; however in this thesis, we use $\operatorname{cone} S$ as the convex conical hull $\operatorname{cone} \operatorname{co} S$.

2.2 Convex Solid Cones

Next, we show the definition of topological vector spaces.

Definition 2.5 (topological vector space)**.** Let X be a vector space with a topology τ thereon. (X, τ), usually is a *topological vector space* if the addition is continuous on $X \times X$ and so is the scalar multiplication on $\mathbb{R} \times X$.

We would like to note that $\operatorname{int} S$, $\operatorname{cl} S$, and S^c are the interior of a set S, the closure of S, and the complement of S. S is said to be *solid* when $\operatorname{int} S \neq \emptyset$, *proper* when $S \neq X$ and *free-disposal* with respect to K when $S + K \subset S$. Moreover, S is K*-proper* when $S + K \neq X$ and K*-bounded* when for all open set $U \subset X$, $S \subset \gamma U + K$ for some $\gamma > 0$.

For a topological vector space X and a convex solid cone $K \subset X$, we show several lemmas which reflect important parts of the topological and linear structure.

Lemma 2.2. Let K be proper. Then, $\gamma k + K \subsetneq \operatorname{int} K$ for all $k \in \operatorname{int} K$ and all $\gamma > 0$.

Proof. Assume that $k \in \operatorname{int} K$ and $\gamma > 0$. Since $\gamma k \in \operatorname{int} K$, there is a neighborhood of the zero V such that $V + \gamma k \subset \operatorname{int} K$. Thus, $V + K + \gamma k \subset \operatorname{int} K + K \subset K + K \subset K$. Therefore, $\gamma k + K \subset \operatorname{int} K$.

There is $\zeta \in (0, \gamma)$ satisfying $(\gamma - \zeta)k \in \operatorname{int} K$. If $-\zeta k \in K$, then for all $d \in X$, there is $\mu > 0$ such that $(\mu/2)d = (1/2)(-\zeta k) + (1/2)(\zeta k + \mu d) \subset K + K \subset K$. Since d is arbitrary, K is no longer proper, which contradicts the assumption. Then, $(\gamma - \zeta)k \notin \gamma k + K$ and thus, $\gamma k + K \subsetneq \operatorname{int} K$. □

Lemma 2.3. For all vector $k \in \operatorname{int} K$ and $x \in X$, there is some $\gamma > 0$ such that $x \in \gamma k - \operatorname{int} K$.

Proof. Since $k \in \operatorname{int} K$, for all $x \in X$, there exists $\lambda > 0$ such that $k - \lambda x \in \operatorname{int} K$. Then, $x - (1/\lambda)k = (1/\lambda)(\lambda x - k) \in -(1/\lambda) \operatorname{int} K \subset - \operatorname{int} K$. □

Lemma 2.4. Let S be free-disposal with respect to K. Then, For any vector $k \in \operatorname{int} K$ and $x \in X$, there is some $\gamma > 0$ such that $x \in \gamma k - \operatorname{int} S$.

Proof. Let $s \in S$. It is clear that $X = \operatorname{int} K - \operatorname{int} K = \operatorname{int} K - \operatorname{int} K + s - s = \operatorname{int}(s+K) - \operatorname{int}(s+K) \subset \operatorname{int} S - \operatorname{int} S$ by Lemma 2.3. □

Lemma 2.5. Let S be free-disposal with respect to K and proper, and $k \in \operatorname{int} K$. Then, for all $x \in X$, there is some $\gamma \in \mathbb{R}$ such that $x \notin \gamma k + S$.

Proof. Assume that $x \in S$ (otherwise, it is obvious). Note that $\gamma k + S \subset S$ for all $\gamma \geq 0$ since S is free-disposal. If $x \in \gamma k + S$ for all $\gamma > 0$, then $x - \gamma k + K \subset S$. However, $X = \bigcup_{\gamma > 0}(x - \gamma k + K) \subset S$ by Lemma 2.3, and this contradicts properness of S. □

Lemma 2.6. Let $S \subset X$ be a compact set. Then, $\bigcap_{s \in S}(s + \operatorname{int} K)$ is open.

Proof. Let $S \subset X$ be compact and $T := \bigcap_{s \in S}(s + \operatorname{int} K)$. Then, $T^c = \bigcup_{s \in S}(s + \operatorname{int} K)^c = \bigcup_{s \in S}(s + (\operatorname{int} K)^c) = S + (\operatorname{int} K)^c$. By the assumption, T^c is closed. □

Unless otherwise referred to, in this thesis, let X be a topological vector space and $K \subset X$ a convex solid proper cone.

Chapter 3

Set Relations

In case of set-to-set comparisons, one may have a large difficulty seeing which one is preferred to the other. A pair of real numbers can be ordered by the total order "$\leq$" and that of vectors in n-dimensional space also is usually compared with the partial order "$\leq_{\mathbb{R}_+}$" similarly.

We may see some notions for set orderings in the literature. As first unified binary relations between sets, [45] is well known as one originally introducing this concept to set-valued analysis in 1997. They presented six types of set orderings which we usually call Kuroiwa's set relations. Today, we can find a lot of papers investigating set orderings with reference to his work (e.g., [8, 31, 32]). Honestly speaking, there had been several works using similar techniques before Kuroiwa's types. Nishnianidze [55] proposed fixed point theory via multi-valued operators by using a basic relation between sets. The same relation is used by Young [68] for sets of the real numbers with the notation "many-valuedness" to obtain relationship of upper and lower limits of real numbers. They are known as ones of the earliest works using set orderings. As a recent trend, Kuroiwa's third and fifth types which have important properties of the ordering structure commonly grab one's attention as "set order relations," and today a large number of papers relate to investigate these orderings. As a result, there are several derived forms of the relations (e.g., [8,26,28,60]) and associated new concepts are formed. Set relations are also studied in vigorous frameworks: for example, see [49, 50] for so-called "complete lattice approach" that differs from classic ideas and formulates another standard of set inclusion relations. For a systematical story of set relations, see [25, 27] and references cited therein.

In another side of research, the rest of Kuroiwa's relations have been carefully studied. In [46], one can see basic properties like invariance with respect to the addition and the scalar multiplication. Quasi convexity and concavity notions are shown in [45, 46]. [57] deals with alternative characterizations. Taking the whole six types into account contributes to systematize previous works and to unveil core ideas of them. In this chapter, Kuroiwa's six relations are targeted to introduce basic notions and invaluable utilization of them which authors call "scalarization."

3.1 Basic Notions

Definition 3.1 (Kuroiwa, Tanaka, Ha [45]). Let $S_1, S_2 \subset X$ be nonempty sets.

- $S_1 \preceq_K^{(1)} S_2$
 $:\Longleftrightarrow S_1 \subset \bigcap_{s_2 \in S_2}(s_2 - K)$
 $\Longleftrightarrow S_2 \subset \bigcap_{s_1 \in S_1}(s_1 + K)$;
- $S_1 \preceq_K^{(2)} S_2$
 $:\Longleftrightarrow S_1 \cap \bigcap_{s_2 \in S_2}(s_2 - K) \neq \emptyset$;
- $S_1 \preceq_K^{(3)} S_2$
 $:\Longleftrightarrow S_2 \subset S_1 + K$;
- $S_1 \preceq_K^{(4)} S_2$
 $:\Longleftrightarrow S_2 \cap \bigcap_{s_1 \in S_1}(s_1 + K) \neq \emptyset$;
- $S_1 \preceq_K^{(5)} S_2$
 $:\Longleftrightarrow S_1 \subset S_2 - K$;
- $S_1 \preceq_K^{(6)} S_2$
 $:\Longleftrightarrow S_1 \cap (S_2 - K) \neq \emptyset$
 $\Longleftrightarrow S_2 \cap (S_1 + K) \neq \emptyset$.

As pointed out before, $\preceq_K^{(3)}$ and $\preceq_K^{(5)}$ play wide roles as l-type $\preceq^l$ and u-type $\preceq^u$.

Lemma 3.1. Let $\mathbf{0} \in K$. Then, $\preceq_K^{(i)}$ is reflexive when $i = 3, 5, 6$.

This lemma follows from Definition 3.1.

Lemma 3.2. $\preceq_K^{(i)}$ is transitive when $i = 1, \ldots, 5$.

Proof. Let $S_1, S_2, S_3 \subset X$. Consider $i = 1$ and $S_1 \preceq_K^{(1)} S_2$ and $S_2 \preceq_K^{(1)} S_3$. Then, it follows that $S_1 \subset s_2 - K \subset S_2 - K \subset s_3 - K - K \subset s_3 - K$ for all $s_2 \in S_2$ and all $s_3 \in S_3$. Therefore, $S_1 \subset \bigcap_{s_3 \in S_3}(s_3 - K)$.

Next, we prove the assertion when $i = 2$. Assume that $S_1 \preceq_K^{(2)} S_2$, $S_2 \preceq_K^{(2)} S_3$. Then, there is $s_2 \in S_2$ satisfying $s_2 \in \bigcap_{s_3 \in S_3}(s_3 - K)$. It follows that $s_1 \in s_2 - K \subset \bigcap_{s_3 \in S_3}(s_3 - K)$ for some $s_1 \in S_1$, and thus, $S_1 \preceq_K^{(2)} S_3$. The case $i = 4$ shall be similarly shown.

Finally, we prove the case $i = 3$. By letting $S_1 \preceq_K^{(3)} S_2$ and $S_2 \preceq_K^{(3)} S_3$, $S_3 \subset S_2 + K \subset S_1 + K + K \subset S_1 + K$. Thus, we have $S_1 \preceq_K^{(3)} S_3$. It is also clear when $i = 5$. □

Lemma 3.3. For nonempty sets $S_1, S_2 \subset X$,

- $S_1 \preceq_K^{(1)} S_2$ implies $S_1 \preceq_K^{(2)} S_2$;
- $S_1 \preceq_K^{(2)} S_2$ implies $S_1 \preceq_K^{(3)} S_2$;
- $S_1 \preceq_K^{(3)} S_2$ implies $S_1 \preceq_K^{(6)} S_2$;
- $S_1 \preceq_K^{(1)} S_2$ implies $S_1 \preceq_K^{(4)} S_2$;
- $S_1 \preceq_K^{(4)} S_2$ implies $S_1 \preceq_K^{(5)} S_2$;
- $S_1 \preceq_K^{(5)} S_2$ implies $S_1 \preceq_K^{(6)} S_2$.

Lemma 3.4 ([46]). For nonempty sets $S_1, S_2 \subset X$ and $i = 1, \ldots, 6$, $S_1 \preceq_K^{(i)} S_2$ implies that

- $(S_1 + x) \preceq_K^{(i)} (S_2 + x)$ for $x \in X$;
- $\gamma S_1 \preceq_K^{(i)} \gamma S_2$ for $\gamma > 0$;
- $(S_1 + x_1) \preceq_K^{(i)} (S_2 + x_2)$ for $x_1, x_2 \in X$ satisfying $x_1 \leq_K x_2$.

Lemma 3.5 ([46])**.** For nonempty sets $S_1, S_2 \subset X$, if $\mathbf{0} \in K$, then

- $S_1 \preceq_K^{(i)} S_2$ iff $S_1 \preceq_K^{(i)} (S_2 + K)$ iff $(S_1 - K) \preceq_K^{(i)} S_2$ when $i = 1$;
- $S_1 \preceq_K^{(i)} S_2$ iff $S_1 \preceq_K^{(i)} (S_2 + K)$ iff $(S_1 + K) \preceq_K^{(i)} S_2$ when $i = 2, 3$;
- $S_1 \preceq_K^{(i)} S_2$ iff $S_1 \preceq_K^{(i)} (S_2 - K)$ iff $(S_1 - K) \preceq_K^{(i)} S_2$ when $i = 4, 5$;
- $S_1 \preceq_K^{(i)} S_2$ iff $S_1 \preceq_K^{(i)} (S_2 - K)$ iff $(S_1 + K) \preceq_K^{(i)} S_2$ when $i = 6$.

Lemma 3.6. For nonempty sets $S_1, S_2 \subset X$,

- $S_1 \preceq_K^{(1)} S_2$ iff $S_2 \preceq_{(-K)}^{(1)} S_1$;
- $S_1 \preceq_K^{(6)} S_2$ iff $S_2 \preceq_{(-K)}^{(6)} S_1$;
- $S_1 \preceq_K^{(2)} S_2$ iff $S_2 \preceq_{(-K)}^{(4)} S_1$;
- $S_1 \preceq_K^{(3)} S_2$ iff $S_2 \preceq_{(-K)}^{(5)} S_1$.

3.2 Scalarization Functionals

Scalarization is a fundamental technique generating a scalar to quantify an object. It can be seen in many important properties such as inner product, probability, determinant, or norm. On optimization theory, authors sometimes estimate the extreme values like sup or inf, of inner product to scale feasible sets.

In set optimization, scalarization usually characterizes "distance" between two sets by the translation toward a specific direction. Tammer's vector scalarization functional via a convex cone [16–18] is one of earlier frameworks that triggers introducing notations of nonlinear scalarization in set optimization or set-valued analysis. In [18, 35], this concept is geometrically given as a nonconvex separation theorem. Today, one can find nonlinear scalarization is a common tool in set optimization and most of them are based on Tammer's functional (e.g., [8, 10, 11, 15, 20, 21, 26, 28, 36, 40, 47, 51, 60, 66, 71]). Their utilizations have been studied for variational principles (e.g., [15, 22, 24, 26]), minimal element theorems (e.g., [26, 28, 60]), well-posedness (setness) (e.g., [11, 21, 71]), optimality conditions (e.g., [1, 8, 10, 36, 40, 51, 57, 66]), minimax theorems (e.g., [1, 47, 58, 59]), dual minimax expressions (e.g., [8, 20, 40, 63, 69, 70]) and so on. For detailed information, see [23]. As a further interesting approach, we would refer to [30] investigating fuzzy set relations.

This section focuses on scalarization from a viewpoint of set functions suggested in [26] in which the authors only refer to two cases using Kuroiwa's third and fifth types. This has encouraged others for treating the whole six types in a similar way (e.g., [46, 47, 57, 62, 63]).

Definition 3.2 (scalarization functional [16])**.** For vectors $x \in X$ and $k \in \operatorname{int} K$, *scalarization functional* $h_{K,k} : V \to \mathbb{R}$ is defined by $h_{K,k}(x) := \inf\{\gamma \in \mathbb{R} : x \in \gamma k - K\}$.

This functional is well-defined and utilizes good properties of convex cones in the topological structure: Lemma 2.3 implies that $h_{K,k}(x)$ cannot be $+\infty$ and also, it is clear that Lemma 2.5 says $h_{K,k}(x) \neq -\infty$. In finite dimensional spaces, the functional has interesting reformations on the bilinear structure.

Lemma 3.7. Let $k \in \mathbb{R}^n$ satisfying $\langle k, k \rangle = 1$ and $K := \{x' \in \mathbb{R}^n : \langle k, x' \rangle \geq 0\}$. Then, $h_{K,k}(x) = \langle k, x \rangle$ for $x \in \mathbb{R}^n$.

Proof. Note that $k \in \operatorname{int} K$. $h_{K,k}(x) = \inf\{\gamma : x \in \gamma k - K\} = \inf\{\gamma : \langle k, \gamma k - x \rangle \geq 0\} = \inf\{\gamma : \langle k, x \rangle \leq \gamma\} = \langle k, x \rangle$. □

Lemma 3.8 ([7, Proposition 1.53])**.** Let $k \in \operatorname{int} K$. Then, $h_{K,k}(x) = \sup_{z \in Z_k} \langle z, x \rangle$ for all $x \in \mathbb{R}^n$ where $Z_k := \{z \in \mathbb{R}^n : \langle k, z \rangle = 1\}$.

Lemma 3.9 ([65])**.** Let $p_1, \ldots, p_m \in \mathbb{R}^n$, $K := \{x \in \mathbb{R}^n : \langle p_j, x \rangle \geq 0 \text{ for } j = 1, \ldots, m\}$ solid and $k \in \operatorname{int} K$. Then, $h_{K,k}(x) = \max_{j \in \{1,\ldots,m\}}(\langle p_j, x \rangle / \langle p_j, k \rangle)$ for $x \in \mathbb{R}^n$.

Lemma 3.7 implies that this functional extends the inner product $\langle \cdot, \cdot \rangle$. In the usual convex separation, sets S_1 and S_2 are linearly separated if there is a vector p such that $\langle p, s_1 \rangle \leq \langle p, s_2 \rangle$ for all $s_1 \in S_1$ and all $s_2 \in S_2$. Likewise, a similar assertion can be applied to Lemma 3.7: S_1 and S_2 may be nonlinearly (or convex conically) separated as $h_{K,k}(s_1) \leq h_{K,k}(s_2)$ for all $s_1 \in S_1$ and all $s_2 \in S_2$. The geometrical structure of this fact is mainly studied in [18, Chapter 2.3] or [35, Chapter 5]. Lemmas 3.8 and 3.9 are dual forms of Tammer's functional and calculable approaches used in [63, 69, 70].

Lemma 3.10. For a vector $k \in \operatorname{int} K$,

- $h_{K,k}(x_1 + x_2) \leq h_{K,k}(x_1) + h_{K,k}(x_2)$ for $x_1, x_2 \in X$;
- $h_{K,k}(\gamma x) = \gamma h_{K,k}(x)$ for $\gamma \geq 0$ and $x \in X$.

Proof. At first, we prove the subadditivity. Assume that $x_1, x_2 \in X$. For any $\varepsilon > 0$ and each $i = 1, 2$, $x_i \in (h_{K,k}(x_i) + \varepsilon/2)k - K$. Thus, $x_1 + x_2 \in (h_{K,k}(x_1) + h_{K,k}(x_2) + \varepsilon)k - K$. That leads to $h_{K,k}(x_1 + x_2) \leq h_{K,k}(x_1) + h_{K,k}(x_2)$. Next, we let $\gamma \geq 0$ and $x \in X$. If $\gamma = 0$, then it is true since $h_{K,k}(\mathbf{0}) = 0$. Otherwise, $h_{K,k}(\gamma x) = \inf\{\zeta : x \in (\zeta/\gamma)k - K\} = \inf\{\gamma\eta : x \in \eta k - K\} = \gamma h_{K,k}(x)$ by setting $\eta := \zeta/\gamma$. □

By the above lemma, the functional in Definition 3.2 is sometimes called a sublinear scalarization functional. As a natural recasting, we may consider the following quantification by adding a secondary vector: $h_{K,k}(x_1, x_2) := \inf\{\gamma : x_1 \leq_K x_2 + \gamma k\}$; it is easily seen that $h_{K,k}(x_1 - x_2) = h_{K,k}(x_1, x_2)$. The following lemma deduces that this functional is an extension of metric.

Lemma 3.11. For $k \in \operatorname{int} K$,

- $h_{K,k}(x, x) = 0$ for $x \in X$;
- $h_{K,k}(x_1, x_3) \leq h_{K,k}(x_1, x_2) + h_{K,k}(x_2, x_3)$ for $x_1, x_2, x_3 \in X$.

Proof. The first assertion comes direct from the definition. Let $x_1, x_2, x_3 \in X$. Then, by Lemma 3.10, $h_{K,k}(x_1, x_3) = h_{K,k}(x_1 - x_3) = h_{K,k}((x_1 - x_2) + (x_2 - x_3)) \leq h_{K,k}(x_1 - x_2) + h_{K,k}(x_2 - x_3) = h_{K,k}(x_1, x_2) + h_{K,k}(x_2, x_3)$. □

3.3 Evaluation Functions

Next, we will show a generalization of this functional for scaling sets. We denote the set of all subsets of X by 2^X.

Definition 3.3 (evaluation functions, [57])**.** Let $S_1, S_2 \subset X$ be nonempty sets and $k \in \operatorname{int} K$. For each $i = 1, \dots, 6$, *evaluation function* $\Phi^{(i)}_{K,k} : 2^X \to \mathbb{R} \cup \{\pm\infty\}$ is defined by $\Phi^{(i)}_{K,k}(S_1, S_2) := \inf\{\gamma \in \mathbb{R} : S_1 \preceq^{(i)}_K S_2 + \gamma k\}$.

These functions quantify how far S_2 needs to be shifted toward k to be preferred to S_1. We will introduce basic properties related to the functions for nonempty sets $S_1, S_2 \subset X$ and $k \in \operatorname{int} K$.

Lemma 3.12. $\Phi^{(1)}_{K,k}(S_1, S_2) > -\infty$.

Proof. Note that $\bigcap_{s_2 \in S_2}(s_2 - K)$ is free-disposal with respect to $-K$ and proper for all $S_2 \subset X$ since if $\bigcap_{s_2 \in S_2}(s_2 - K) = X$, $X \subset s_2 - K$ for all $s_2 \in S_2$ and it contradicts the properness of K. By Lemma 2.5, for all $s_1 \in S_1$ there is $\gamma > 0$ such that $s_1 \notin -\gamma k + \bigcap_{s_2 \in S_2}(s_2 - K)$. This implies $\Phi^{(1)}_{K,k}(S_1, S_2) \geq -\gamma > -\infty$. □

Lemma 3.13. $\Phi^{(6)}_{K,k}(S_1, S_2) < \infty$.

Proof. $S_2 - K$ is free-disposal with respect to $-K$. By Lemma 2.4, for all $k \in \operatorname{int} K$ and for $s_1 \in S_1$, there is $\gamma > 0$ such that $s_1 \in \gamma k + (\operatorname{int} S_2 - K)$. Therefore, $\Phi^{(6)}_{K,k}(S_1, S_2) \leq \gamma < \infty$. □

Proposition 3.1 ([47])**.** $\Phi^{(i)}_{K,k}(S_1, S_2) \in \mathbb{R}$ if

- S_1 is $(-K)$-bounded and S_2 is K-bounded $(i = 1)$;
- S_1 is K-proper and S_2 is K-bounded $(i = 2, 3)$;
- S_1 is $(-K)$-bounded and S_2 is $(-K)$-proper $(i = 4, 5)$;
- S_1 is K-proper and S_2 is $(-K)$-proper $(i = 6)$.

Example 3.1. The following show counterexamples of Proposition 3.1.

- $\Phi^{(1)}_{K,k}(K, S_2) = \Phi^{(1)}_{K,k}(S_1, -K) = \infty$;
- $\Phi^{(2)}_{K,k}(-K, S_2) = -\infty$, $\Phi^{(2)}_{K,k}(S_1, -K) = \infty$;
- $\Phi^{(3)}_{K,k}(-K, S_2) = -\infty$, $\Phi^{(3)}_{K,k}(S_1, -K) = \infty$;
- $\Phi^{(4)}_{K,k}(S_1, K) = -\infty$, $\Phi^{(4)}_{K,k}(K, S_2) = \infty$;
- $\Phi^{(5)}_{K,k}(S_1, K) = -\infty$, $\Phi^{(5)}_{K,k}(K, S_2) = \infty$;
- $\Phi^{(6)}_{K,k}(-K, S_2) = \Phi^{(6)}_{K,k}(S_1, K) = -\infty$.

Lemma 3.14. The following assertions hold:

- $\Phi^{(1)}_{K,k}(S_1, S_2) \geq \Phi^{(2)}_{K,k}(S_1, S_2) \geq \Phi^{(3)}_{K,k}(S_1, S_2) \geq \Phi^{(6)}_{K,k}(S_1, S_2)$;

- $\Phi^{(1)}_{K,k}(S_1, S_2) \geq \Phi^{(4)}_{K,k}(S_1, S_2) \geq \Phi^{(5)}_{K,k}(S_1, S_2) \geq \Phi^{(6)}_{K,k}(S_1, S_2)$.

Lemma 3.15. The following assertions hold:

- when S_1 is a singleton, $\Phi^{(1)}_{K,k} = \Phi^{(2)}_{K,k} = \Phi^{(3)}_{K,k}$ and $\Phi^{(4)}_{K,k} = \Phi^{(5)}_{K,k} = \Phi^{(6)}_{K,k}$;
- when S_2 is a singleton, $\Phi^{(1)}_{K,k} = \Phi^{(4)}_{K,k} = \Phi^{(5)}_{K,k}$ and $\Phi^{(2)}_{K,k} = \Phi^{(3)}_{K,k} = \Phi^{(6)}_{K,k}$.

As is evident, the evaluation functions $\Phi^{(i)}_{K,k}(S_1, S_2)$ and the recasted Tammer's functional $h_{K,k}(s_1, s_2)$ coincide when $S_1 = \{s_1\}$ and $S_2 = \{s_2\}$.

Proposition 3.2. For $i = 1, \ldots, 6$, $\Phi^{(i)}_{\mathrm{int}\, K,k}(S_1, S_2) = \Phi^{(i)}_{K,k}(S_1, S_2) = \Phi^{(i)}_{\mathrm{cl}\, K,k}(S_1, S_2)$.

Proof. It is clear that $\Phi^{(i)}_{\mathrm{cl}\, K,k}(S_1, S_2) \leq \Phi^{(i)}_{K,k}(S_1, S_2) \leq \Phi^{(i)}_{\mathrm{int}\, K,k}(S_1, S_2)$. Therefore, We prove $\Phi^{(i)}_{\mathrm{int}\, K,k}(S_1, S_2) \leq \Phi^{(i)}_{\mathrm{cl}\, K,k}(S_1, S_2)$.

Case $i = 1$. Let $\zeta := \Phi^{(1)}_{\mathrm{cl}\, K,k}(S_1, S_2)$. For all $\zeta' > \zeta$, there is $\bar{\gamma} < \zeta'$ such that $S_1 \subset \bigcap_{s_2 \in S_2}(s_2 + \bar{\gamma}k - \mathrm{cl}\, K)$. Since $S_1 \subset \bigcap_{s_2 \in S_2}(s_2 + \bar{\gamma}k - \mathrm{cl}\, K) \subset \bigcap_{s_2 \in S_2}(s_2 + \bar{\gamma}k - \mathrm{int}\, K) + (\zeta' - \bar{\gamma})k = \bigcap_{s_2 \in S_2}(s_2 + (\zeta')k - \mathrm{int}\, K)$ by Lemma 2.2, we have $\inf\{\gamma : S_1 \subset \bigcap_{s_2 \in S_2}(s_2 + \gamma k - \mathrm{int}\, K)\} \leq \zeta'$. Thus, $\Phi^{(1)}_{\mathrm{int}\, K,k}(S_1, S_2) \leq \zeta$ as $\zeta' \to \zeta$.

Case $i = 2$. Let $\zeta := \Phi^{(2)}_{\mathrm{cl}\, K,k}(S_1, S_2)$. By the above proof of case $i = 1$, we have $S_1 \cap \bigcap_{s_2 \in S_2}(s_2 + (\zeta')k - \mathrm{int}\, K) \neq \emptyset$ for all $\zeta' > \zeta$. Therefore, $\Phi^{(2)}_{\mathrm{int}\, K,k}(S_1, S_2) \leq \zeta$.

To prove case $i = 4$, one shall follow a similar process.

Case $i = 3$. Let $\zeta := \Phi^{(3)}_{\mathrm{cl}\, K,k}(S_1, S_2)$. For all $\zeta' > \zeta$, there is $\bar{\gamma} < \zeta'$ such that $S_1 \subset S_2 + \bar{\gamma}k - \mathrm{cl}\, K$. Since $S_1 \subset S_2 + \bar{\gamma}k - \mathrm{cl}\, K \subset S_2 + \bar{\gamma}k - \mathrm{int}\, K + (\zeta' - \bar{\gamma})k = S_2 + (\zeta')k - \mathrm{int}\, K$ by Lemma 2.2, we have $\Phi^{(3)}_{\mathrm{int}\, K,k}(S_1, S_2) \leq \zeta$.

To prove case $i = 5$, one shall follow a similar process.

Case $i = 6$. Let $\zeta := \Phi^{(6)}_{\mathrm{cl}\, K,k}(S_1, S_2)$. By the above proof of case $i = 3$, we have $S_1 \cap (S_2 + (\zeta')k - \mathrm{int}\, K) \neq \emptyset$ for all $\zeta' > \zeta$. Therefore, $\Phi^{(6)}_{\mathrm{int}\, K,k}(S_1, S_2) \leq \zeta$. □

According to the above proposition, the evaluation functions are immune to closedness (or openness) of K since the infimum values are attained on the boundary of K. Finally, we introduce minimax (dual) expressions of the evaluation functions. The functions are usually described as combinations of "min" or "max" forms: minimax expressions are interestingly proved in [20] for third and fifth types and in [69] for whole six types.

Proposition 3.3 ([69])**.** Let $S_1, S_2 \subset X$ be nonempty sets. For $k \in \mathrm{int}\, K$,

- $\Phi^{(1)}_{K,k}(S_1, S_2) = \sup_{s_1 \in S_1} \sup_{s_2 \in S_2} h_{K,k}(s_1, s_2)$;
- $\Phi^{(2)}_{K,k}(S_1, S_2) = \inf_{s_1 \in S_1} \sup_{s_2 \in S_2} h_{K,k}(s_1, s_2)$;
- $\Phi^{(3)}_{K,k}(S_1, S_2) = \sup_{s_2 \in S_2} \inf_{s_1 \in S_1} h_{K,k}(s_1, s_2)$;
- $\Phi^{(4)}_{K,k}(S_1, S_2) = \inf_{s_2 \in S_2} \sup_{s_1 \in S_1} h_{K,k}(s_1, s_2)$;
- $\Phi^{(5)}_{K,k}(S_1, S_2) = \sup_{s_1 \in S_1} \inf_{s_2 \in S_2} h_{K,k}(s_1, s_2)$;
- $\Phi^{(6)}_{K,k}(S_1, S_2) = \inf_{s_1 \in S_1} \inf_{s_2 \in S_2} h_{K,k}(s_1, s_2)$.

Comments. The original form of the above functions was given in [26] as set functions and Kuwano [46, 47] systematized whole six types as a "unified approach." These functions usually called "sublinear-like" scalarization, of which [56] partially introduces a story of the reason, have a lot of fundamental properties. Here, the author mentions related works in the literature [38, 47] usually given with interesting properties (e.g., cone-convexity, cone-boundedness, cone-continuity). Moreover, we additionally refer to importance of inherited properties on compositions of the functions and set-valued maps (e.g., [38, 42, 53, 54, 62]).

3.4 Algorithmic Computing

It is recently found that values of the evaluation functions are algorithmically obtained by computational methods (e.g., [40, 69, 70]). Referring to some existing results, Tammer and Winkler [65] and Sonda et al. [63] presented computational reformations of Tammer's scalarization. One could see the dual form in Lemma 3.9 in the literature (e.g., [7, 40]). In 2017, Yu et al. [69] suggested generalized ideas calculating the six evaluation functions for convex polytopes. This work was soon generalized by [70] for convex polyhedra (not necessarily bounded).

The huge essentiality of Proposition 3.3 is confirmed by calculation methods given by Yu et al. [70]. They proved that finding values of the evaluation functions between polyhedral sets in a finite dimensional space is far reduced to addressing at most finitely many linear programming problems. This calculability has a potential to encourage further research of the functions. In this part, let us assume S_1, S_2 and K are polyhedral in $\mathbb{R}^n$.

Definition 3.4 (polyhedron)**.** A set $S \subset \mathbb{R}^n$ is said to be *polyhedral* if $S = \{x \in \mathbb{R}^n : Mx \leq v\}$ for some $M \in M^{\gamma \times n}$, which is the set of all $\gamma \times n$ matrices, and some $v \in \mathbb{R}^\gamma$.

In the last part of this chapter, we use the following notations: $S_1 = \{x \in \mathbb{R}^n : P_1 x \leq q_1\}$, $S_2 = \{x \in \mathbb{R}^n : P_2 x \leq q_2\}$, and $K = \{x \in \mathbb{R}^n : \langle p_j, x \rangle \leq 0 \text{ for all } j \in J(m)\}$ for $P_1 \in M^{\alpha \times n}$, $P_2 \in M^{\beta \times n}$, $q_1 \in \mathbb{R}^\alpha$, $q_2 \in \mathbb{R}^\beta$, $J(m) := \{1, \dots, m\}$, and $p_j \in \mathbb{R}^n$ for $j \in J(m)$.

Theorem 3.1 ([70])**.** For nonempty polyhedral sets $S_1, S_2 \subset \mathbb{R}^n$ and $k \in \operatorname{int} K$, each value of the evaluation functions comes by optimal values of the following corresponding linear programming problems.

- $\Phi^{(1)}_{K,k}(S_1, S_2) = \min_{j \in J(m)} \{\mathrm{Val}(\mathrm{P}_j(1))\}$;
- $\Phi^{(2)}_{K,k}(S_1, S_2) = \mathrm{Val}(\mathrm{P}(2))$;
- $\Phi^{(3)}_{K,k}(S_1, S_2) = \sup_{s \in S_2} \{\mathrm{Val}(\mathrm{P}_s(3))\}$;
- $\Phi^{(4)}_{K,k}(S_1, S_2) = \mathrm{Val}(\mathrm{P}(4))$;
- $\Phi^{(5)}_{K,k}(S_1, S_2) = \sup_{s \in S_1} \{\mathrm{Val}(\mathrm{P}_s(5))\}$;
- $\Phi^{(6)}_{K,k}(S_1, S_2) = \mathrm{Val}(\mathrm{P}(6))$.

where $\mathrm{Val}(\cdot)$ stands for the optimal values of the following specified problems:

$$\begin{array}{lll}
\mathrm{P}_j(1): & \text{Maximize} & \langle p_j, x_1 - x_2\rangle/\langle p_j, k\rangle \\
& \text{subject to} & P_1 x_1 \le q_1 \text{ and } P_2 x_2 \le q_2;
\end{array}$$

$$\begin{array}{lll}
\mathrm{P}(2): & \text{Minimize} & \gamma \in \mathbb{R} \\
& \text{subject to} & \langle p_j, x_1\rangle/\langle p_j, k\rangle + \sup_{s_2 \in S_2} (\langle p_j, -s_2\rangle/\langle p_j, k\rangle) \le \gamma \text{ for all } j \in J(m), \\
& & P_1 x_1 \le q_1;
\end{array}$$

$$\begin{array}{lll}
\mathrm{P}_s(3): & \text{Minimize} & \gamma \in \mathbb{R} \\
& \text{subject to} & \langle p_j, x_1 - s\rangle/\langle p_j, k\rangle \le \gamma \text{ for all } j \in J(m), \\
& & P_1 x_1 \le q_1;
\end{array}$$

$$\begin{array}{lll}
\mathrm{P}(4): & \text{Minimize} & \gamma \in \mathbb{R} \\
& \text{subject to} & \sup_{s_1 \in S_1} (\langle p_j, s_1\rangle/\langle p_j, k\rangle) + \langle p_j, -x_2\rangle/\langle p_j, k\rangle \le \gamma \text{ for all } j \in J(m), \\
& & P_2 x_2 \le q_2;
\end{array}$$

$$\begin{array}{lll}
\mathrm{P}_s(5): & \text{Minimize} & \gamma \in \mathbb{R} \\
& \text{subject to} & \langle p_j, s - x_2\rangle/\langle p_j, k\rangle \le \gamma \text{ for all } j \in J(m), \\
& & P_2 x_2 \le q_2;
\end{array}$$

$$\begin{array}{lll}
\mathrm{P}(6): & \text{Minimize} & \gamma \in \mathbb{R} \\
& \text{subject to} & \langle p_j, x_1 - x_2\rangle/\langle p_j, k\rangle \le \gamma \text{ for all } j \in J(m), \\
& & P_1 x_1 \le q_1 \text{ and } P_2 x_2 \le q_2.
\end{array}$$

Proof. Case 1. One can check by Lemma 3.9 and Proposition 3.3, $\Phi^{(1)}_{K,k}(S_1, S_2) = \sup_{s_1 \in S_1} \sup_{s_2 \in S_2} \max_{j \in J(m)} \langle p_j, s_1 - s_2\rangle/\langle p_j, k\rangle$. Therefore, it is clear that $\Phi^{(1)}_{K,k}(S_1, S_2) = \max_{j \in J(m)}\{\mathrm{Val}(\mathrm{P}_j(1))\}$.

Case 2. $\Phi^{(2)}_{K,k}(S_1, S_2) = \inf_{s_1 \in S_1} \sup_{s_2 \in S_2} \max_{j \in J(m)} \langle p_j, s_1 - s_2\rangle/\langle p_j, k\rangle$. It holds that $\Phi^{(2)}_{K,k}(S_1, S_2) = \inf_{s_1 \in S_1} \max_{j \in J(m)}\{\langle p_j, s_1\rangle/\langle p_j, k\rangle + \sup_{s_2 \in S_2}(\langle p_j, -s_2\rangle/\langle p_j, k\rangle)\}$. Therefore, $\Phi^{(2)}_{K,k}(S_1, S_2)$ is equal to the infimum of $\gamma \in \mathbb{R}$ satisfying $\gamma \ge \langle p_j, s_1\rangle/\langle p_j, k\rangle + \sup_{s_2 \in S_2}(\langle p_j, -s_2\rangle/\langle p_j, k\rangle)$ for all $j \in J(m)$, and $s_1 \in S_1$. This coincides with Val(P(2)). Case 4 is proved by switching roles of S_1 and S_2.

Case 3. $\Phi^{(3)}_{K,k}(S_1, S_2) = \sup_{s_2 \in S_2} \inf_{s_1 \in S_1} \max_{j \in J(m)} \langle p_j, s_1 - s_2\rangle/\langle p_j, k\rangle$. This can be reformed into $\sup_{s_2 \in S_2} \inf_{s_1 \in S_1} \inf\{\gamma \in \mathbb{R} : \gamma \ge \langle p_j, s_1 - s_2\rangle/\langle p_j, k\rangle \text{ for all } j \in J(m)\}$. Thus, $\Phi^{(3)}_{K,k}(S_1, S_2)$ coincides with $\sup_{s_2 \in S_2}\{\mathrm{Val}(\mathrm{P}_{s_2}(3))\}$. Case 5 is proved by switching roles of S_1 and S_2.

Case 6. $\Phi^{(6)}_{K,k}(S_1, S_2) = \inf_{s_1 \in S_1} \inf_{s_2 \in S_2} \max_{j \in J(m)} \langle p_j, s_1 - s_2\rangle/\langle p_j, k\rangle$. This can be reformed into $\inf_{s_1 \in S_1} \inf_{s_2 \in S_2} \inf\{\gamma \in \mathbb{R} : \gamma \ge \langle p_j, s_1 - s_2\rangle/\langle p_j, k\rangle \text{ for all } j \in J(m)\}$. Thus, $\Phi^{(6)}_{K,k}(S_1, S_2) = \mathrm{Val}(\mathrm{P}(6))$. □

Remark. $\Phi^{(1)}_{K,k}(\cdot, \cdot)$ is obtained by solving m problems and choosing the maximum from their optimal values. Each of $\Phi^{(2)}_{K,k}(\cdot, \cdot)$ and $\Phi^{(4)}_{K,k}(\cdot, \cdot)$ requires extra problems ($\sup_{x_2 \in S_2} \langle p_j, -x_2\rangle$ or $\sup_{x_1 \in S_1} \langle p_j, x_1\rangle$ for $j \in J(m)$). This implies we need $m + 1$ steps. On the other hand, $\Phi^{(6)}_{K,k}(\cdot, \cdot)$ is the simplest.

By the above theorem, the functions but third and fifth types are calculated by solving finitely many algorithms. To conclude this section, we show the rest two types are also obtained by finite calculations. We use the fact that if S_1, S_2, K are polyhedral, they are *finitely generated*: there are nonempty finite sets $S_1', S_1'', S_2', S_2'', K' \subset X$ such that $S_1 = \operatorname{co} S_1' + \operatorname{cone} S_1''$, $S_2 = \operatorname{co} S_2' + \operatorname{cone} S_2''$, $K = \operatorname{cone} K'$ and these transformations can be accomplished in finitely many steps by Theorem 2.1 of [72]. Note that S_1'', S_2'', K' contain the zero since if not, S_1, S_2, or K may not be closed and contradicts the polyhedrality.

Theorem 3.2 ([70])**.** Let $k \in \operatorname{int} K$. If $\operatorname{cone} S_2'' \not\subset \operatorname{cone} S_1'' + \operatorname{cone} K'$, then $\Phi^{(3)}_{K,k}(S_1, S_2) = \infty$, otherwise, $\sup_{s \in S_2}\{\mathrm{Val}(\mathrm{P}_s(3))\} = \max_{s \in S_2'}\{\mathrm{Val}(\mathrm{P}_s(3))\}$.

Proof. At first, the unboundedness is shown. Assume that $\operatorname{cone} S_2'' \not\subset C := \operatorname{cone} S_1'' + \operatorname{cone} K'$. Then we have $x \notin C$ for some $x \in \operatorname{cone} S_2''$. Since C is a closed convex cone and $x \neq \mathbf{0}$, there is $p \in \mathbb{R}^n$ such that $\langle p, x \rangle > 0 \geq \langle p, x' \rangle$ for all $x' \in C$. Thus, $\operatorname{co} S_1' + C \subset \bar{\gamma} k + C$ for some $\bar{\gamma} \in \mathbb{R}$ by Lemma 2.3 since S_1' is finite. For all $\gamma \in \mathbb{R}$ and all $z \in \operatorname{co} S_2'$, there is $\tilde{\gamma} > 0$ such that $\tilde{\gamma}\langle p, x \rangle > \langle p, -z - \gamma k + \bar{\gamma} k \rangle$, that is, $z + \tilde{\gamma} x + \gamma k - \bar{\gamma} k \notin C$. This leads to $S_2 + \gamma k = \operatorname{co} S_2' + \operatorname{cone} S_2'' + \gamma k \not\subset \bar{\gamma} k + C$. Therefore, $S_2 + \gamma k \not\subset \operatorname{co} S_1' + C = \operatorname{co} S_1' + \operatorname{cone} S_1'' + \operatorname{cone} K' = S_1 + K$. Since $\gamma \in \mathbb{R}$ is arbitrary, $\Phi^{(3)}_{K,k}(S_1, S_2) = \inf\{\gamma : S_2 + \gamma k \subset S_1 + K\} = \infty$.

For the next one, it is sufficient that we prove $\Phi^{(3)}_{K,k}(S_1, S_2) = \Phi^{(3)}_{K,k}(S_1, S_2')$, that is, $S_2 + \gamma k \subset S_1 + K$ if and only if $S_2' + \gamma k \subset S_1 + K$ for all $\gamma \in \mathbb{R}$. Consider $S_2' + \gamma k \subset S_1 + K$. Then, $\operatorname{co} S_2' + \gamma k \subset S_1 + K$. By the assumption, we have $S_2 + \gamma k = \operatorname{co} S_2' + \operatorname{cone} S_2'' + \gamma k \subset S_1 + \operatorname{cone} S_1'' + K + \operatorname{cone} K' = \operatorname{co} S_1' + \operatorname{cone} S_1'' + \operatorname{cone} S_1'' + K + K \subset \operatorname{co} S_1' + \operatorname{cone} S_1'' + K = S_1 + K$. The converse inclusion is obvious. □

Corollary 3.1. Let $k \in \operatorname{int} K$. If $\operatorname{cone} S_1'' \not\subset \operatorname{cone} S_2'' - \operatorname{cone} K'$, then $\Phi^{(5)}_{K,k}(S_1, S_2) = \infty$, otherwise, $\sup_{s \in S_1}\{\mathrm{Val}(\mathrm{P}_s(5))\} = \max_{s \in S_1'}\{\mathrm{Val}(\mathrm{P}_s(5))\}$.

Remark. The inclusion condition $\operatorname{cone} S_2'' \not\subset \operatorname{cone} S_1'' + \operatorname{cone} K'$ in Theorem 3.2 is easily tested. Actually, it is enough to check $S_2'' \subset \operatorname{cone}(S_1'' \cup K')$. Concretely, by letting $S_1'' \cup K' = \{c^1, \ldots, c^l\}$, solve the problem:

$$\begin{aligned} \text{Maximize } & -(\lambda_1 + \cdots + \lambda_n) \\ \text{subject to } & c_1^1 x_1 + \cdots + c_1^l x_l + \lambda_1 = s_1, \\ & \vdots \\ & c_n^1 x_1 + \cdots + c_n^l x_l + \lambda_n = s_n, \\ & x_1, \ldots, x_l, \lambda_1, \ldots, \lambda_n \geq 0 \end{aligned}$$

for all $s \in S_2''$. Since S_2'' is finite, this calculation is finished in $\#(S_2'')$ steps. To summarize, $\Phi^{(3)}_{K,k}(\cdot,\cdot)$ comes over three phases: firstly we transform S_1, S_2, K into finitely generated forms; next check the condition in Theorem 3.2 by the above problem; finally one shall solve $\mathrm{P}_s(3)$ for all $s \in S_2'$ and take the maximum of their optimal values.

Comments. A core context of the calculability is convexity of sets and compactness (boundedness). Convex polytopes are structured by the convex hull of discrete elements. polyhedra are composed of polytopes and extreme directions, which cannot be expressed by any conical combinations of emanating directions of sets from the zero. The evaluation functions are for the most well-behaved cases, sensitive to the boundary of sets. Therefore, values of the functions on simple structures can be calculated along finite vertices and edges of sets. As seen in Theorem 3.1, the four types are tractable in calculation. However, the third and fifth types require further steps for taking the supremum along all elements of given sets. In general cases, a given set may not be a polytope; some of edges or faces are unbounded. This thesis addresses to this issue with a technical process shown in Theorem 2.1 of [72]. Theorem 3.2 implies if one checks the inclusion condition by using the first one of the two-phase simplex method in the above remark, the complicated maximization is far reduced to polytope cases.

Example 3.2. We assume that a hexahedron $S_1 := \{x \in \mathbb{R}^3 : P_1 x \leq q_1\}$, a dodecahedron $S_2 := \{x \in \mathbb{R}^3 : P_2 x \leq q_2\}$, and a convex solid proper cone $K := \{x \in \mathbb{R}^3 : \langle p_j, x\rangle \leq 0 \text{ for all } j \in J(3)\}$ where

$$P_1 := \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 1 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix}, q_1 := \begin{pmatrix} 2 \\ 2 \\ -3 \\ 4 \\ 1 \\ 1 \end{pmatrix}, P_2 := \begin{pmatrix} -2 & -1 & 0 \\ -2 & 1 & 0 \\ 2 & 1 & 0 \\ 2 & -1 & 0 \\ 0 & -2 & -1 \\ 0 & -2 & 1 \\ 0 & 2 & 1 \\ 0 & 2 & -1 \\ -1 & 0 & -2 \\ 1 & 0 & -2 \\ 1 & 0 & 2 \\ -1 & 0 & 2 \end{pmatrix}, q_2 := \begin{pmatrix} -2 \\ 2 \\ 10 \\ 6 \\ -2 \\ 2 \\ 10 \\ 6 \\ -2 \\ 2 \\ 10 \\ 6 \end{pmatrix},$$

$$p_1 := \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, p_2 := \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}, p_3 := \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}, k := \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

One can easily check S_1 is a rectangular box and S_2 is covered with 12 symmetric pentagons. Also, $\langle p_j, k\rangle = -1$ for all $j \in J(3)$ and S_1 has the following eight vertices:

$$s^1_1 = -(1/2, 5/2, -1)^{\mathrm{T}},\ s^1_2 = -(5/2, 1/2, -1)^{\mathrm{T}},\ s^1_3 = -(1, 3, -1)^{\mathrm{T}},\ s^1_4 = -(3, 1, -1)^{\mathrm{T}},$$
$$s^1_5 = -(1/2, 5/2, 1)^{\mathrm{T}},\quad s^1_6 = -(5/2, 1/2, 1)^{\mathrm{T}},\quad s^1_7 = -(1, 3, 1)^{\mathrm{T}},\quad s^1_8 = -(3, 1, 1)^{\mathrm{T}}$$

and S_2 has the following 20 vertices:

$$\begin{aligned}
&s_1^2 = (0,2,1)^{\mathrm{T}}, \quad s_2^2 = (0,2,3)^{\mathrm{T}}, \quad s_3^2 = (1,0,2)^{\mathrm{T}}, \quad s_4^2 = (1,4,2)^{\mathrm{T}},\\
&s_5^2 = (2,1,0)^{\mathrm{T}}, \quad s_6^2 = (2,1,4)^{\mathrm{T}}, \quad s_7^2 = (2,3,0)^{\mathrm{T}}, \quad s_8^2 = (2,3,4)^{\mathrm{T}},\\
&s_9^2 = (3,0,2)^{\mathrm{T}}, \quad s_{10}^2 = (3,4,2)^{\mathrm{T}}, \quad s_{11}^2 = (4,2,1)^{\mathrm{T}}, \quad s_{12} = (4,2,3)^{\mathrm{T}},\\
&s_{13}^2 = (2/3,2/3,2/3)^{\mathrm{T}}, \quad s_{14}^2 = (2/3,2/3,10/3)^{\mathrm{T}}, \quad s_{15}^2 = (2/3,10/3,2/3)^{\mathrm{T}},\\
&s_{16}^2 = (2/3,10/3,10/3)^{\mathrm{T}}, \quad s_{17}^2 = (10/3,2/3,2/3)^{\mathrm{T}}, \quad s_{18}^2 = (10/3,2/3,10/3)^{\mathrm{T}},\\
&s_{19}^2 = (10/3,10/3,2/3)^{\mathrm{T}}, \quad s_{20}^2 = (10/3,10/3,10/3)^{\mathrm{T}}.
\end{aligned}$$

For the value of $\Phi_{K,k}^{(1)}(S_1, S_2)$, we shall solve the following problems with variables $x, y \in \mathbb{R}^3$ for $j \in J(3)$:

$$\mathrm{P}_j(1) : \text{Maximize } -\langle p_j, x - y\rangle \text{ subject to } P_1 x \leq q_1 \text{ and } P_2 y \leq q_1.$$

To obtain $\Phi_{K,k}^{(2)}(S_1, S_2)$, let us consider the following for $j \in J(3)$:

$$\begin{aligned}
\mathrm{P}(2) : \text{Minimize } \gamma \text{ subject to } & -\langle p_j, x\rangle - \gamma \leq -\operatorname{Val}(\mathrm{P}_j(2)) \text{ for all } j \in J(3),\\
& P_1 x \leq q_1;\\
\mathrm{P}_j(2) : \text{Maximize } \langle p_j, y\rangle & \text{ subject to } P_2 y \leq q_2.
\end{aligned}$$

Next, let us calculate $\Phi_{K,k}^{(3)}(S_1, S_2)$. Note that $\Phi_{K,k}^{(3)}(S_1, S_2) < \infty$ since $S_1 = \operatorname{co}\{s_1^1, \ldots, s_8^1\}$ and $S_2 = \operatorname{co}\{s_1^2, \ldots, s_{20}^2\}$. Then, we need to solve the following for $s \in \{s_1^2, \ldots, s_{20}^2\}$:

$$\begin{aligned}
\mathrm{P}_s(3) : \text{Minimize } \gamma \text{ subject to } & -\langle p_j, x - s\rangle - \gamma \leq 0 \text{ for all } j \in J(3),\\
& P_1 x \leq q_1.
\end{aligned}$$

Finally, $\Phi_{K,k}^{(6)}(S_1, S_2)$ is tested by solving the following:

$$\begin{aligned}
\mathrm{P}(6) : \text{Minimize } \gamma \text{ subject to } & -\langle p_j, x - y\rangle \leq \gamma \text{ for all } j \in J(3),\\
& P_1 x \leq q_1 \text{ and } P_2 y \leq q_1.
\end{aligned}$$

It is confirmed by the Tables 3.1–3.4 that we have results of the above calculation as follows:

- $\Phi_{K,k}^{(1)}(S_1, S_2) = \min_{j \in J(m)}\{\operatorname{Val}(\mathrm{P}_j(1))\} \approx 1.1429$;
- $\Phi_{K,k}^{(2)}(S_1, S_2) = \operatorname{Val}(\mathrm{P}(2)) = 1$;
- $\Phi_{K,k}^{(3)}(S_1, S_2) = \max_{s \in \{s_1^2, \ldots, s_{20}^2\}}\{\operatorname{Val}(\mathrm{P}_s(3))\} = -1$;
- $\Phi_{K,k}^{(6)}(S_1, S_2) = \operatorname{Val}(\mathrm{P}(6)) \approx -4.7778$.

To conclude this example, check $\Phi_{K,k}^{(1)}(S_1, S_2) \geq \Phi_{K,k}^{(2)}(S_1, S_2) \geq \Phi_{K,k}^{(3)}(S_1, S_2) \geq \Phi_{K,k}^{(6)}(S_1, S_2)$ and this truly follows Lemma 3.14.

Table 3.1. The optimal values of $P_j(1)$.

	Val(·)	x_1	x_2	x_3	y_1	y_2	y_3
$P_1(1)$	**5.0000**	-2.5000	-0.5000	1.0000	3.3333	0.6667	0.6667
$P_2(1)$	**5.0000**	-0.5000	-2.5000	1.0000	0.6667	3.3333	0.6667
$P_3(1)$	**1.1429**	-2.5000	-0.5000	-1.0000	0.2857	1.4286	4.8571

Table 3.2. The optimal values of P(2) and $P_j(2)$.

	Val(·)	γ	x_1	x_2	x_3	y_1	y_2	y_3
$P_1(2)$	**2.0000**	-	-	-	-	3.3333	0.6667	0.6667
$P_2(2)$	**2.0000**	-	-	-	-	0.6667	3.3333	0.6667
$P_3(2)$	**3.1429**	-	-	-	-	0.2857	1.4286	4.8571
P(2)	**1.0000**	1.0000	-1.5714	-1.5714	-1.0000	-	-	-

Table 3.3. The optimal values of $P_s(3)$.

	Val(·)	γ	x_1	x_2	x_3
$P_{s_1^2}(3)$	**-2.0000**	-2.0000	-2.5000	-0.5000	-1.0000
$P_{s_2^2}(3)$	**-3.0000**	-3.0000	-3.0000	-1.0000	0.0000
$P_{s_3^2}(3)$	**-2.5000**	-2.5000	-1.5000	-2.5000	-0.5000
$P_{s_4^2}(3)$	**-2.0000**	-2.0000	-2.5000	-0.5000	-1.0000
$P_{s_5^2}(3)$	**-1.0000**	-1.0000	-1.0000	-2.0000	-1.0000
$P_{s_6^2}(3)$	**-3.5000**	-3.5000	-1.5000	-2.5000	0.5000
$P_{s_7^2}(3)$	**-1.0000**	-1.0000	-2.0000	-1.0000	-1.0000
$P_{s_8^2}(3)$	**-4.5000**	-4.5000	-2.5000	-1.5000	-0.5000
$P_{s_9^2}(3)$	**-2.0000**	-2.0000	-0.5000	-2.5000	-1.0000
$P_{s_{10}^2}(3)$	**-3.0000**	-3.0000	-2.0000	-1.0000	-1.0000
$P_{s_{11}^2}(3)$	**-2.0000**	-2.0000	-0.5000	-2.5000	-1.0000
$P_{s_{12}^2}(3)$	**-4.0000**	-4.0000	-0.5000	-2.5000	-1.0000
$P_{s_{13}^2}(3)$	**-1.6667**	-1.6667	-1.5000	-1.5000	-1.0000
$P_{s_{14}^2}(3)$	**-2.6667**	-2.6667	-2.0000	-2.0000	0.6667
$P_{s_{15}^2}(3)$	**-1.0000**	-1.0000	-2.5000	-0.5000	-1.0000
$P_{s_{16}^2}(3)$	**-3.6667**	-3.6667	-3.0000	-1.0000	-1.0000
$P_{s_{17}^2}(3)$	**-1.0000**	-1.0000	-0.5000	-2.5000	-1.0000
$P_{s_{18}^2}(3)$	**-3.6667**	-3.6667	-1.0000	-3.0000	-1.0000
$P_{s_{19}^2}(3)$	**-1.6667**	-1.6667	-1.5000	-1.5000	-1.0000
$P_{s_{20}^2}(3)$	**-4.3333**	-4.3333	-1.5000	-1.5000	-1.0000

Table 3.4. The optimal value of P(6).

	Val(·)	γ	x_1	x_2	x_3	y_1	y_2	y_3
P(6)	**-4.7778**	-4.7778	-2.3333	-1.6667	-1.0000	2.4444	3.1111	3.7778

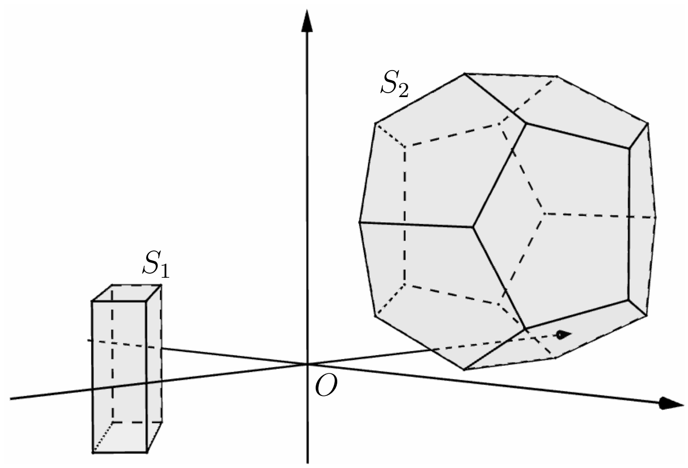

Figure 3.1. The sets S_1 and S_2 in Example 3.2.

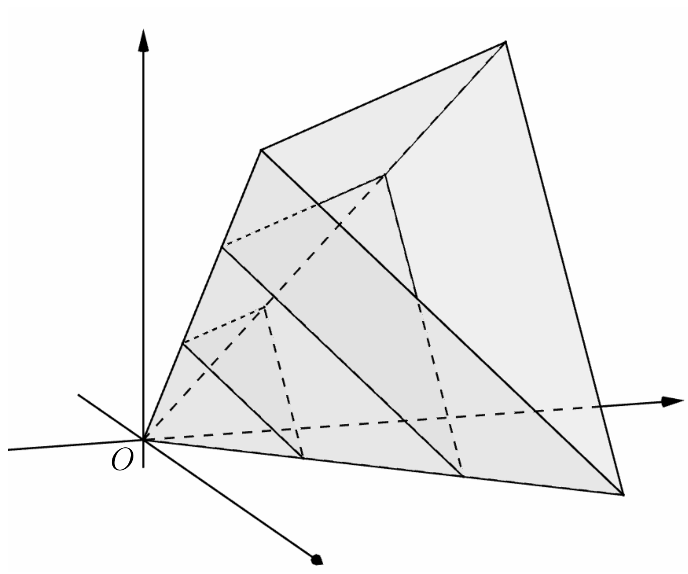

Figure 3.2. The convex cone K in Example 3.2.

Chapter 4

Characterization Theorems of the Alternative

This part is devoted to introducing a central result in this thesis. We show the set relations in Definition 3.1 are characterized by the positivity of evaluation functions. In the literature, it is not a special idea that we consider nonlinear quantification of set ordering relations and one can find similar alternative approaches. Our results are originally motivated by Nishizawa et al. [52] discussing generalized alternative theorems of Gordan in which generalized Tammer's functionals represent set orderings. As further works after Nishizawa, the set orderings and the functionals have been switched to Kuroiwa's set relations and Kuwano's scalarization functions, respectively. Especially, in our knowledge, third and fifth types of six also have been mainly studied for the next decade, similarly to set relations. Hernández and Rodríguez-Marín [28], in 2007, presented lower type cases under cone-closedness and cone-compactness. Representation of these two types was comprehensively systematized by Araya [1]. Finite dimensional cases are studied in [51]. Moreover, Kobayashi et al. [38] partially referred to strict relations (ordered by the interior of a cone). As a result, interesting properties of set optimization have grown mainly on the two types. This study focuses one's interest on the whole six types as a comprehensive generalization of [52] and as a complement of an existing trend. As another kind of alternative approach, we remark that set relations are characterized by oriented distance functions (e.g., [20, 66]).

4.1 Theorems of the Alternative

Theorem 4.1 (characterization theorem of the alternative)**.** Let X be a topological vector space and $K \subset X$ a convex solid cone. For $S_1, S_2 \subset X$ and $k \in \operatorname{int} K$, if

- S_1, S_2 are compact ($i = 1$);
- S_2 is compact ($i = 2, 3$);
- S_1 is compact ($i = 4, 5$),

then exactly one of the following two systems is consistent for $i = 1, \ldots, 6$:

(1) $S_1 \preceq^{(i)}_{\operatorname{int} K} S_2$;

(2) $\Phi^{(i)}_{K,k}(S_1, S_2) \geq 0$.

proof. [Not (1)⇒(2)]. For each $i = 1, \ldots, 6$, let us assume that (1) fails to be. If $\Phi^{(i)}_{\operatorname{int} K,k}(S_1, S_2) = \inf\{\gamma : S_1 \preceq^{(i)}_{\operatorname{int} K} S_2 + \gamma k\} < 0$, then $S_1 \preceq^{(i)}_{\operatorname{int} K} S_2 - \gamma k$ for some $\gamma > 0$. However, it directly follows that $S_1 \preceq^{(i)}_{\operatorname{int} K} S_2$ by the third assertion in Lemma 3.4 since $\mathbf{0} \leq_K \gamma k$. Therefore, Proposition 3.2 deduces $\Phi^{(i)}_{K,k}$ cannot be negative.

[(1)⇒not (2)]. Case $i = 1$. Assume that (1) is consistent. Then, $s_1 \in \bigcap_{s \in S_2}(s - \operatorname{int} K)$ for all $s_1 \in S_1$. Since $\bigcap_{s \in S_2}(s - \operatorname{int} K)$ is open by Lemma 2.6, there is $\gamma_{s_1} > 0$ satisfying $s_1 + \gamma_{s_1} k \in \bigcap_{s \in S_2}(s - \operatorname{int} K)$. Therefore, $s_1 \in U_{s_1} := \bigcap_{s \in -\gamma_{s_1} k + S_2}(s - \operatorname{int} K)$. This means $\{U_{s_1}\}_{s_1 \in S_1}$ is a cover of S_1. Since S_1 is compact, then there exists $\{\sigma_1, \ldots, \sigma_n\} \subset S_1$ such that $S_1 \subset U_{\sigma_1} \cup \cdots \cup U_{\sigma_n}$. Taking $\bar{\gamma} := \min\{\gamma_{\sigma_1}, \ldots, \gamma_{\sigma_n}\} > 0$, $S_1 \subset \bigcup_{j \in \{1,\ldots,n\}} \bigcap_{s \in -\gamma_{\sigma_j} k + S_2}(s - \operatorname{int} K) = \bigcap_{s \in -\bar{\gamma} k + S_2}(s - \operatorname{int} K) \subset \bigcap_{s \in -\bar{\gamma} k + S_2}(s - K)$. Thus, we have $\Phi^{(1)}_{K,k}(S_1, S_2) = \inf\{\gamma \in \mathbb{R} : S_1 \subset \bigcap_{s \in \gamma k + S_2}(s - K)\} \leq -\bar{\gamma} < 0$.

Case $i = 2$. Assume that (1) is consistent. Then, there is $\bar{s} \in S_1$ such that $\bar{s} \in \bigcap_{s \in S_2}(s - \operatorname{int} K)$. Since $\bigcap_{s \in S_2}(s - \operatorname{int} K)$ is open, we can find $\bar{\gamma} > 0$ such that $\bar{s} + \bar{\gamma} k \in \bigcap_{s \in S_2}(s - \operatorname{int} K) \subset \bigcap_{s \in S_2}(s - K)$. Therefore, $\bar{s} \in \bigcap_{s \in -\bar{\gamma} k + S_2}(s - K)$, and hence $\Phi^{(2)}_{K,k}(S_1, S_2) = \inf\{\gamma \in \mathbb{R} : S_1 \cap (\bigcap_{s_2 \in \gamma k + S_2}(s_2 - K)) \neq \emptyset\} \leq -\bar{\gamma} < 0$.

Case $i = 4$ can be clear by following a similar way.

Case $i = 3$. Assume that (1) is consistent. Then, $s_2 \in S_1 + \operatorname{int} K$ for all $s_2 \in S_2$. Since $S_1 + \operatorname{int} K$ is open, there is $\gamma_{s_2} > 0$ satisfying $s_2 - \gamma_{s_2} k \in S_1 + \operatorname{int} K$. Since $s_2 \in U_{s_2} := \gamma_{s_2} k + S_1 + \operatorname{int} K$, then $\{U_{s_2}\}_{s_2 \in S_2}$ is a cover of S_2. Since S_2 is compact, there exists $\{\sigma_1, \ldots, \sigma_n\} \subset S_2$ such that $S_2 \subset U_{\sigma_1} \cup \cdots \cup U_{\sigma_n}$. Taking $\bar{\gamma} := \min\{\gamma_{\sigma_1}, \ldots, \gamma_{\sigma_n}\} > 0$, $S_2 \subset \bigcup_{j \in \{1,\ldots,n\}}(\gamma_{\sigma_j} k + S_1 + \operatorname{int} K) = \bar{\gamma} k + S_1 + \operatorname{int} K \subset \bar{\gamma} k + S_1 + K$. Thus, $\Phi^{(3)}_{K,k}(S_1, S_2) = \inf\{\gamma \in \mathbb{R} : (\gamma k + S_2) \subset (S_1 + K)\} \leq -\bar{\gamma} < 0$.

Case $i = 5$ can be clear by following a similar way.

Case $i = 6$. Assume that (1) is consistent. Then, there is $\bar{s} \in S_1$ such that $\bar{s} \in S_2 - \operatorname{int} K$. It holds that there exists $\bar{\gamma} > 0$ satisfying $\bar{s} + \bar{\gamma} k \in S_2 - \operatorname{int} K \subset S_2 - K$. Then, $S_1 \cap (-\bar{\gamma} k + S_2 - K) \neq \emptyset$. Therefore, $\Phi^{(6)}_{K,k}(S_1, S_2) = \inf\{\gamma \in \mathbb{R} : S_1 \cap (\gamma k + S_2 - K) \neq \emptyset\} \leq -\bar{\gamma} < 0$.

□

Theorem 4.2. Let X be a topological vector space and $K \subset X$ a convex solid cone. For $S_1, S_2 \subset X$ and $k \in \operatorname{int} K$, if

- S_1 is compact ($i = 2, 3$);
- S_2 is compact ($i = 4, 5$);
- S_1 and S_2 are compact ($i = 6$),

then exactly one of the following two systems is consistent for $i = 1, \ldots, 6$:

(1) $S_1 \preceq^{(i)}_{\operatorname{cl} K} S_2$;

(2) $\Phi^{(i)}_{K,k}(S_1,S_2) > 0$.

Proof. [(1)⇒not (2)]. For each $i = 1,\dots,6$, assume that $S_1 \preceq^{(i)}_{\mathrm{cl}\,K} S_2$. Then, it is clear that $\Phi^{(i)}_{K,k}(S_1,S_2) = \Phi^{(i)}_{\mathrm{cl}\,K,k}(S_1,S_2) = \inf\{\gamma : S_1 \preceq^{(i)}_{\mathrm{cl}\,K} S_2 + \gamma k\} \le 0$ by Proposition 3.2.

[Not (1)⇒(2)]. Case $i = 1$. Assume that $S_1 \not\preceq^{(i)}_{\mathrm{cl}\,K} S_2$ and let $S := \bigcap_{s_2 \in S_2}(s_2 - \mathrm{cl}\,K)$. There exists $s_1 \in S_1$ such that $s_1 \in S^c$ and $s_1 - \bar{\gamma}k \in S^c$ for some $\bar{\gamma} > 0$ since S is closed. This implies $\inf\{\gamma : s_1 \in S + \gamma k\} \ge \bar{\gamma} > 0$. Thus, $\Phi^{(1)}_{K,k}(S_1,S_2) = \Phi^{(1)}_{\mathrm{cl}\,K,k}(S_1,S_2) = \sup_{s\in S_1} \inf\{\gamma : s \in S + \gamma k\} \ge \bar{\gamma} > 0$.

Case $i = 2$. Assume that $S_1 \not\preceq^{(2)}_{\mathrm{cl}\,K} S_2$ and $S := \bigcap_{s_2\in S_2}(s_2 - \mathrm{cl}\,K)$. Then, $S_1 \subset S^c$ so that for all $s_1 \in S_1$ there is $\gamma_{s_1} > 0$ such that $s_1 - \gamma_{s_1}k \in S^c$. Since $U_{s_1} := S^c + \gamma_{s_1}k$ is open, $\{U_{s_1}\}_{s_1\in S_1}$ is an open cover of S_1. Since S_1 is compact, there is $\{\sigma_1,\dots,\sigma_n\} \subset S_1$ such that $S_1 \subset U_{\sigma_1} \cup \cdots \cup U_{\sigma_n}$. Taking $\bar{\gamma} := \min\{\gamma_{\sigma_1},\dots,\gamma_{\sigma_n}\} > 0$, it holds that $S_1 - \bar{\gamma}k \subset S^c$, that is, $S_1 \cap (S + \bar{\gamma}k) = \emptyset$. This implies $S_1 \cap (S + \gamma k) = \emptyset$ for all $\gamma \le \bar{\gamma}$ Since S is free-disposal with respect to $-K$ (which means $S - \gamma k \subset S$ for all $\gamma > 0$). Therefore, $\Phi^{(2)}_{K,k}(S_1,S_2) = \Phi^{(2)}_{\mathrm{cl}\,K,k}(S_1,S_2) = \inf\{\gamma : S_1 \cap (S+\gamma k) \neq \emptyset\} \ge \bar{\gamma} > 0$.

One can prove with a similar way when $i = 4$.

Case $i = 3$. Assume that $S_1 \not\preceq^{(3)}_{\mathrm{cl}\,K} S_2$. Then, there is $s_2 \in S_2$ such that $s_2 \notin S_1 + \mathrm{cl}\,K$. Since $S_1 + \mathrm{cl}\,K$ is closed, there exists $\bar{\gamma} > 0$ such that $s_2 - \bar{\gamma}k \in (S_1 + \mathrm{cl}\,K)^c$. This implies $\inf\{\gamma : s_2 \in \gamma k + S_1 + \mathrm{cl}\,K\} \ge \bar{\gamma}$. Thus, $\Phi^{(3)}_{K,k}(S_1,S_2) = \Phi^{(3)}_{\mathrm{cl}\,K,k}(S_1,S_2) = \sup_{s\in S_2} \inf\{\gamma : s \in \gamma k + S_1 + \mathrm{cl}\,K\} \ge \bar{\gamma} > 0$.

One can prove with a similar way when $i = 5$.

Case $i = 6$. Assume that $S_1 \not\preceq^{(6)}_{\mathrm{cl}\,K} S_2$. Then, $S_2 \subset (S_1 + \mathrm{cl}\,K)^c$. Since S_1 is compact, $S_1 + \mathrm{cl}\,K$ is closed. Thus, for all $s_2 \in S_2$, there exists $\gamma_{s_2} > 0$ such that $s_2 + \gamma_{s_2}k \in (S_1 + \mathrm{cl}\,K)^c$. Since $U_{s_2} := (S_1 + \mathrm{cl}\,K)^c - \gamma_{s_2}k$ is open, $\{U_{s_2}\}_{s_2\in S_2}$ is an open cover of S_2. Since S_2 is compact, there exists $\{\sigma_1,\dots,\sigma_n\} \subset S_2$ such that $S_2 \subset U_{\sigma_1} \cup \cdots \cup U_{\sigma_n}$. Taking $\bar{\gamma} := \min\{\gamma_{\sigma_1},\dots,\gamma_{\sigma_n}\} > 0$, it holds that $(S_1 - \bar{\gamma}k) \cap (S_2 - \mathrm{cl}\,K) = \emptyset$. Therefore, $\Phi^{(6)}_{K,k}(S_1,S_2) = \Phi^{(6)}_{\mathrm{cl}\,K,k}(S_1,S_2) = \inf\{\gamma : S_1 \cap (S_2 + \gamma k - \mathrm{cl}\,K) \neq \emptyset\} \ge \bar{\gamma} > 0$. □

4.2 Geometry

Example 4.1. Let $A := \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 \le 4\}$, $B := \{x \in \mathbb{R}^2 : (x_1+2)^2 + (x_2+2)^2 \le 1\}$, and $k := (1,1)^{\mathrm{T}} \in \mathrm{int}\,\mathbb{R}^2_+$. Then, any relation $A \preceq^{(i)}_{\mathrm{int}\,\mathbb{R}^n_+} B$ does not hold but $i = 6$. Conversely, any relation $B \preceq^{(i)}_{\mathrm{int}\,\mathbb{R}^n_+} A$ holds but $i = 1$. Remark that the corresponding relations ($A \preceq^{(i)}_{\mathbb{R}^n_+} B$ or not) and ($B \preceq^{(i)}_{\mathbb{R}^n_+} A$ or not) hold.

Table 4.1. Characterization of $A \preceq^{(i)}_{\mathrm{int}\,\mathbb{R}^n_+} B$ in Example 4.1.

	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$
$A \preceq^{(i)}_{\mathrm{int}\,\mathbb{R}^n_+} B$	✗	✗	✗	✗	✗	✓
$\Phi^{(i)}_{\mathbb{R}^n_+,k}(A,B)$	5	$3-\sqrt{2}$	$2-\sqrt{2}/2$	$4-\sqrt{2}/2$	3	$2-3\sqrt{2}/2$

Table 4.2. Characterization of $B \preceq^{(i)}_{\operatorname{int} K} A$ in Example 4.1.

	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$
$B \preceq^{(i)}_{\operatorname{int} \mathbb{R}^n_+} A$	✗	✓	✓	✓	✓	✓
$\Phi^{(i)}_{\mathbb{R}^n_+,k}(B,A)$	1	$-\sqrt{2}/2$	$\sqrt{2}/2-2$	$-\sqrt{2}-1$	$-2-\sqrt{2}/2$	$-2-3\sqrt{2}/2$

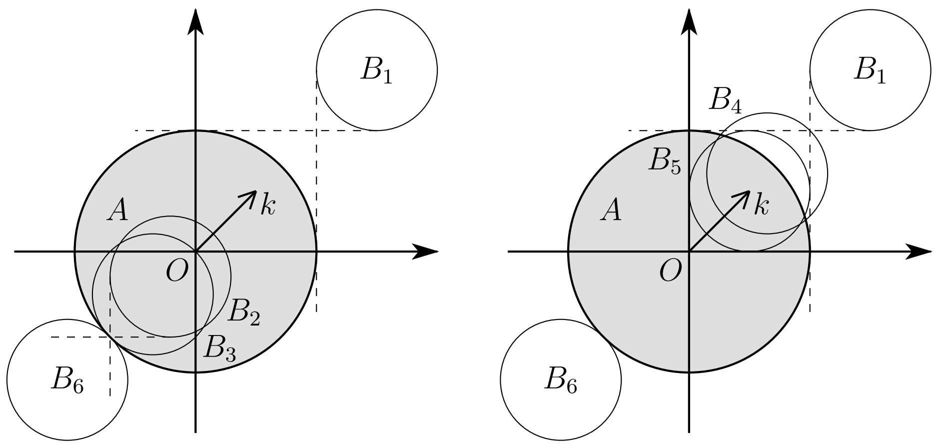

Figure 4.1. The sets A and $B_i := \Phi^{(i)}_{\mathbb{R}^n_+,k}(A,B)\cdot k + B$ in Example 4.1.

Example 4.2. Let $k \in \operatorname{int} \mathbb{R}^2_+$. Theorem 4.1 fails in the following circumstances since $A \preceq^{(1)}_{\operatorname{int} \mathbb{R}^2_+} B$ and $\Phi^{(1)}_{\mathbb{R}^n_+,k}(A,B) = 0$ coincide simultaneously.

(I) $A := \{x \in \mathbb{R}^2 : (x_1+1)^2 + (x_2+1)^2 \le 1\}$, $B := \{x \in \mathbb{R}^2 : x_2 \ge 1/x_1, x_1 > 0\}$.

(II) $A := \{x \in \mathbb{R}^2 : x_2 \le 1/x_1, x_1 < 0\}$, $B := \{x \in \mathbb{R}^2 : (x_1-1)^2 + (x_2-1)^2 \le 1\}$.

(III) $A := \{x \in \mathbb{R}^2 : (x_1+1)^2 + (x_2+1)^2 < 1\}$, $B := \{x \in \mathbb{R}^2 : (x_1-1)^2 + (x_2-1)^2 \le 1\}$.

Example 4.3. Let $k \in \operatorname{int} \mathbb{R}^2_+$. Theorem 4.2 fails in the following circumstances since $A \not\preceq^{(6)}_{\mathbb{R}^2_+} B$ and $\Phi^{(6)}_{K,k}(A,B) = 0$ coincide simultaneously.

(I) $A := \{x \in \mathbb{R}^2 : x_2 \ge -1/x_1, x_1 < 0\}$, $B := \{x \in \mathbb{R}^2 : (x_1-1)^2 + (x_2+1)^2 \ge 1\}$.

(II) $A := \{x \in \mathbb{R}^2 : (x_1+1)^2 + (x_2-1)^2 \le 1\}$, $B := \{x \in \mathbb{R}^2 : x_2 \le -1/x_1, x_1 > 0\}$.

(III) $A := \{x \in \mathbb{R}^2 : (x_1+1)^2 + (x_2-1)^2 < 1\}$, $B := \{x \in \mathbb{R}^2 : (x_1-1)^2 + (x_2+1)^2 \le 1\}$.

Proposition 4.1. Let $S_1, S_2 \subset X$ be compact and $k \in \operatorname{int} K$. Then, $S_1 \preceq^{(i)}_{\operatorname{cl} K} S_2$ and $S_1 \not\preceq^{(i)}_{\operatorname{int} K} S_2$ if and only if $\Phi^{(i)}_{K,k}(S_1,S_2) = 0$ for each $i = 1, \ldots, 6$.

This proposition follows from Theorems 4.1 and 4.2.

Comments. Usually in the literature, authors define a closed convex proper cone for discussing set orderings or scalarization functions. Closed ordering cones are well-behaved in optimality notions since the orders satisfy reflexivity as the pointwise ordering. Interestingly, closedness and openness of ordering cones require alternative

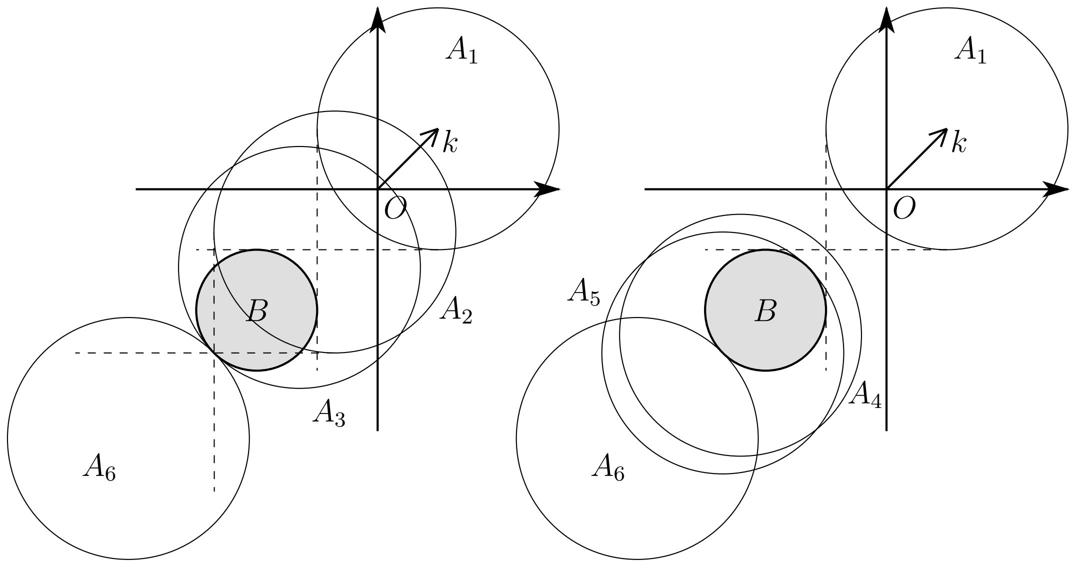

Figure 4.2. The sets $A_i := \Phi^{(i)}_{\mathbb{R}^n_+,k}(B, A) \cdot k + A$ and B in Example 4.1.

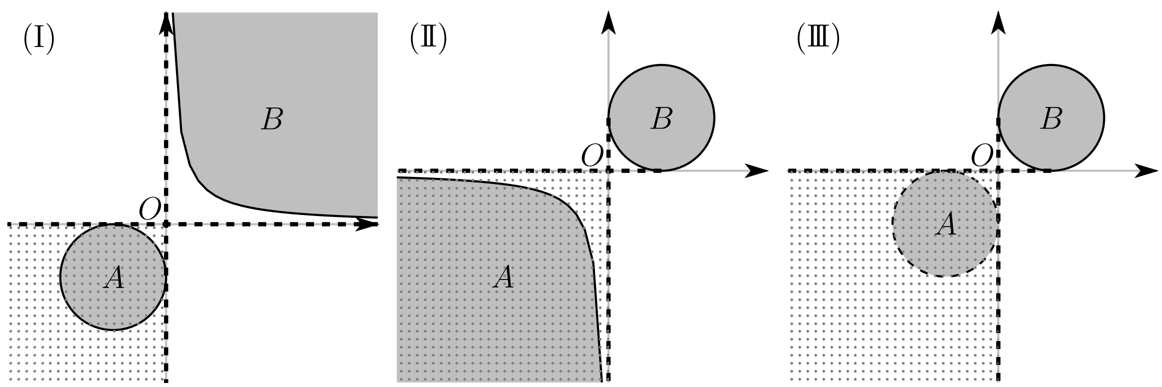

Figure 4.3. The sets A and B in Example 4.2.

compactness (compare Theorems 4.1 and 4.2). Some authors naturally state $\preceq^{(1)}_K$ is a strict or strong relation, whereas $\preceq^{(6)}_K$ is a weak one, however, by Theorem 4.2, the closed cone K requires the most strict restrictions of all six to connect the set relations to the corresponding evaluation functions. The other four types are not seriously affected by the topological structures on convex cones.

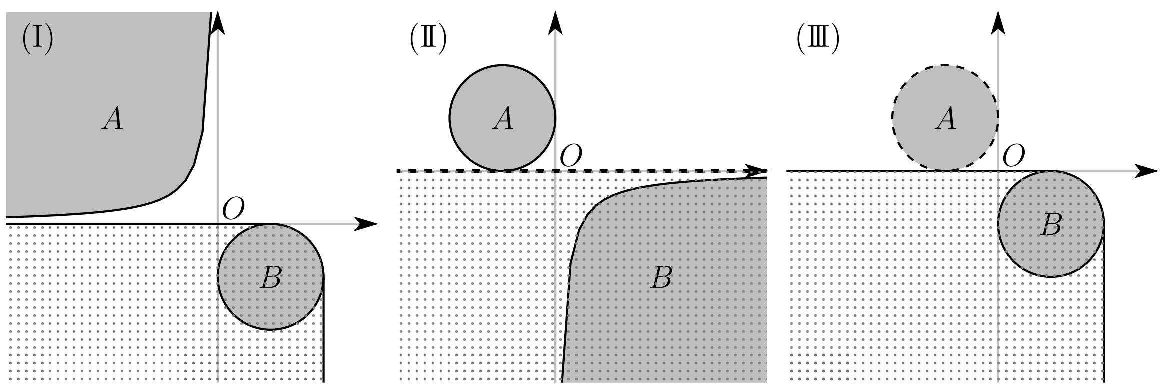

Figure 4.4. The sets A and B in Example 4.3.

Chapter 5

Application

Alternative equivalence of representing orderings has usually been a basic tool for identifying optimality conditions in set optimization. Precisely speaking, scalarization reduces set-to-set comparisons to scalar orderings. As we mentioned before, third and fifth types have already been of interest to authors (e.g., [1, 10, 36, 51, 52, 66]). In this part, we introduce new Gordan's theorems for set-valued maps as an applicable reformation of Theorems 4.1 and 4.2.

5.1 Generalized Gordan's Theorems

Gordan's theorem [19] was presented in 1873 and is well known as one of classic theorems of the alternative such as Farkas' lemma. This theorem addresses positional relations between finitely many (discrete) vectors and the zero in which solutions of simultaneous inequalities and those of the dual equations are alternatively given. This contains a lot of important aspects of linear programming problems and convex analysis. In 1986, Jeyakumar [33] proved generalized Gordan's theorem by replacing matrices with vector-valued functions to give a generalized minimax theorem. Later, Li [48] and Yang et al. [67] follow his alternative theorem by introducing set-valued maps. However, their theorems linearly separate values of functions from the negative (or positive) orthant, and so convexity assumptions are necessarily given. On the other hand, Nishizawa et al. [52] utilized Tammer's nonlinear scalarization to omit any convexity for set-valued alternative theorems.

Theorem 5.1 (generalized Gordan's theorem I, [57])**.** Let S be a nonempty set, X a topological vector space, and $K \subset X$ a convex solid cone. For a set-valued map $F : S \to 2^X \setminus \{\emptyset\}$ and a nonempty set $U \subset X$, if

- F is compact-valued and U is compact ($i = 1$);
- U is compact ($i = 2, 3$);
- F is compact-valued ($i = 4, 5$),

then exactly one of the following two systems is consistent for $i = 1, \ldots, 6$:

(1) there exists $s \in S$ such that $F(s) \preceq^{(i)}_{\operatorname{int} K} U$;

(2) there exists $k \in \operatorname{int} K$ such that $\Phi^{(i)}_{K,k}(F(s), U) \geq 0$ for all $s \in S$.

Theorem 5.2 (generalized Gordan's theorem II)**.** Let S be a nonempty set, X a topological vector space, and $K \subset X$ a convex solid cone. For a set-valued map $F : S \to 2^X \setminus \{\emptyset\}$ and a nonempty set $U \subset X$, if

- F is compact-valued $(i = 2, 3)$;
- U is compact $(i = 4, 5)$;
- F is compact-valued and U is compact $(i = 6)$,

then exactly one of the following two systems is consistent for $i = 1, \dots, 6$:

(1) there exists $s \in S$ such that $F(s) \preceq^{(i)}_{\operatorname{cl} K} U$;

(2) there exists $k \in \operatorname{int} K$ such that $\Phi^{(i)}_{K,k}(F(s), U) > 0$ for all $s \in S$.

Example 5.1. We will show an instant example of Theorems 5.1 and 5.2 for $S := [-\pi, \pi]$ and $F : S \to 2^{\mathbb{R}} \setminus \{\emptyset\}$ where $F(s) := \{x \in \mathbb{R}^2 : |x_1 + x_2 - s + \cos s + 1| + |x_1 + x_2 - s + \cos s| \leq 1\}$, $U := \{x \in \mathbb{R}^2 : x_1 \in [0, \pi/2], x_2 \in [0, 1]\}$, $k := (1, 1)^{\mathrm{T}} \in \operatorname{int} \mathbb{R}^2_+$. We will see the following:

$$\Phi^{(1)}_{\mathbb{R}^2_+,k}(F(s), U) = -\sqrt{2}\cos s \ (s \in [-\pi, \alpha)),\ \sqrt{2}s \ (s \in [\alpha, \pi]),$$

$$\Phi^{(2,3)}_{\mathbb{R}^2_+,k}(F(s), U) = -\sqrt{2}(\cos s + 1) \ (s \in [-\pi, \alpha)),\ \sqrt{2}(s - 1) \ (s \in [\alpha, \pi]),$$

$$\Phi^{(4,5)}_{\mathbb{R}^2_+,k}(F(s), U) = -\sqrt{2}(\cos s + 1) \ (s \in [-\pi, \beta)),\ \sqrt{2}(s - \pi/2) \ (s \in [\beta, \pi]),$$

$$\Phi^{(6)}_{\mathbb{R}^2_+,k}(F(s), U) = -\sqrt{2}(\cos s + 2) \ (s \in [-\pi, \beta)),\ \sqrt{2}\ (s - 1 - \pi/2) \ (s \in [\beta, \pi]),$$

where $\alpha = -\cos\alpha$ and $\beta = -\cos\beta - 1 + \pi/2$. Each value of the evaluation functions is depicted in Figure 5.2.

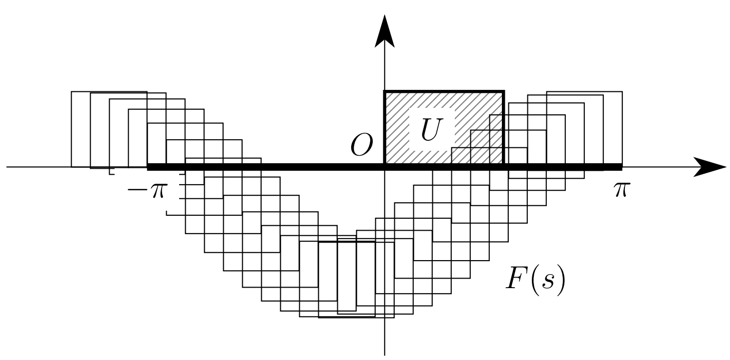

Figure 5.1. The sets $F(s)$ and U for $s \in S$ in Example 5.1.

We can easily see from the context of Theorem 5.1 that optimality conditions of set optimization are given as a direct viewpoint: we call $\bar{s} \in S$ an minimal solution if $F(s) \not\preceq^{(i)}_{\operatorname{int} K} F(\bar{s})$ for all $s \in S \setminus \{\bar{s}\}$. As special cases, by setting $U := \{\mathbf{0}\}$ or resetting $K := -K$ and $k := -k$, we obtain Theorems 3.1 to 3.4 in [52]. Furthermore, we will obtain theorems of the alternative in [19, 33, 48, 67] from Theorem 5.1 by giving convexity assumptions and slight reformations.

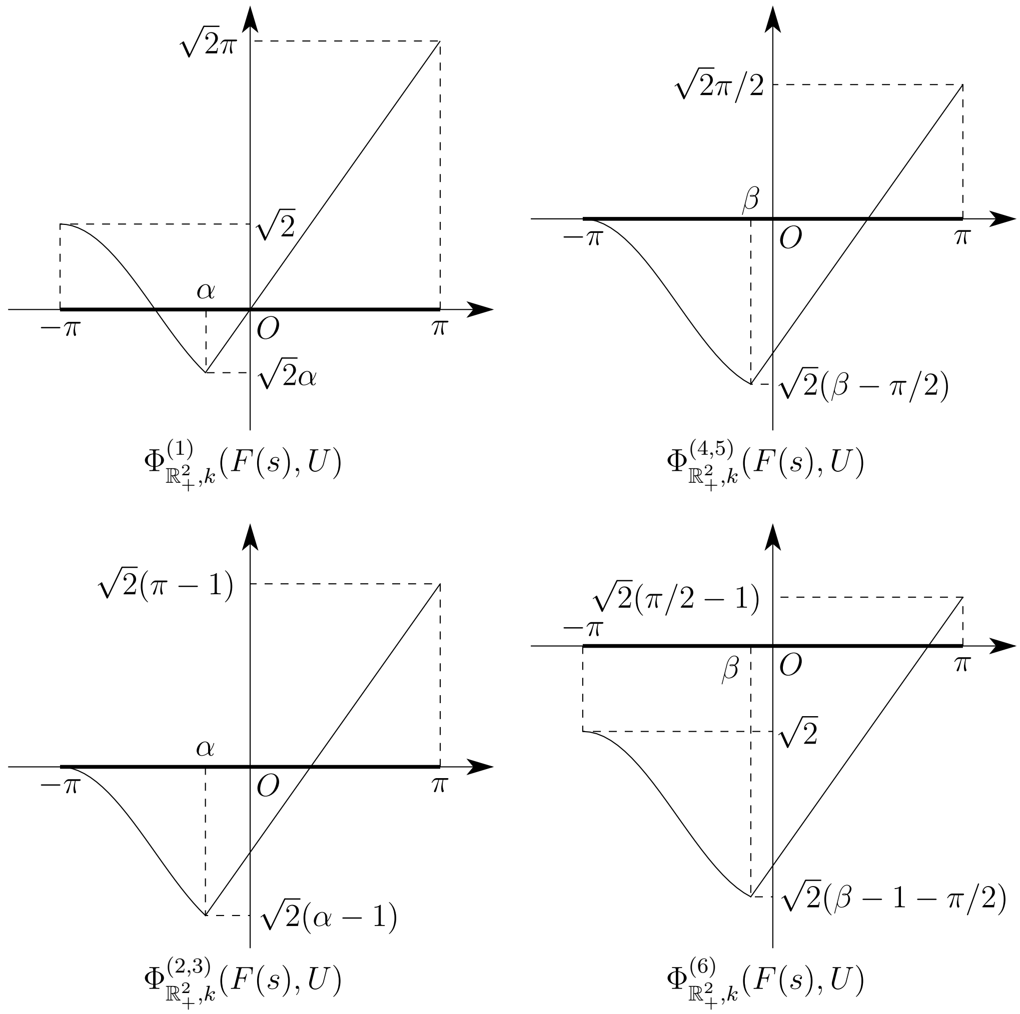

Figure 5.2. The graphs of $\Phi^{(i)}_{\mathbb{R}^2_+,k}(F(s),U)$ in Example 5.1.

5.2 Robustness of Optimization Problems

In optimization theory, we would face various kinds of uncertainty. As major interests, sensitivity, stability, robustness, or stochasticity is given. Sensitivity analysis investigates how a small deviation on variables affects the value of problems, which is used in mathematical economics like scheduling problems, cost reductions, and portfolio managements. Stability theory usually relates to some parts of physics in which behavior of physical equilibrium, that is usually given as solutions of differential equations, caused by a small unpredictable disturbance needs to be mathematically analyzed. Robust optimization and stochastic optimization focus on problems with uncertain data on objective functions and feasible sets. These two concepts deal with an invariance of basic structures against small deviation arising from natural errors or inevitable perturbation.

In this section, we introduce the above theorem of the alternative playing an interesting role. Robust optimization requests that we aim to reduce loss unless situations come far beyond our predictions in advance. If a solution of a problem becomes completely far from the original one under a small deviation (e.g., replacements of coefficients), the problem is sensitive and its solutions may not be credible. Moreover, if reformed problems no longer have a solution, we have to say the problem is not well modeled. Speaking the history of robustness, we would mention Soyster [64] giving uncertainty onto coefficients of linear programming problems. Falk [14] in 1976 and Singh [61] in 1982 follow. However, robust optimization has been remarkably encouraged since 2000 due to [5, 12, 13]. For a detailed flow of history, see [4] and references cited therein.

Soyster introduced a base framework of robust optimization [64] with linear scalarization in which the worst case is given as supremum of the inner product of the Euclidean standard basis and a vector in a deviation set. In 1997, Kouvelis and Yu proposed a robust discrete optimization framework in [41]. In 1998, Ben-Tal [5] proposed non-trivial ellipsoidal cases. Several kinds of approaches have been done for single-valued functions (e.g., [3, 6, 34, 37]).

Deb and Gupta [9] brought robustness concepts to multi-objective problems in 2006 and since then, multi-objective robust optimization has been vigorously studied (e.g., [2, 39, 43, 44]). Set relations indispensable in set optimization, usually play other roles in the robust optimization (e.g., [29]).

In this part, we mention criteria of robustness of feasibility on vector optimization (precisely, only vector-valued constraints are needed) by using Theorems 5.1 and 5.2.

Let S be a nonempty set, $f : S \times \mathcal{U} \to X$, and $g : \mathcal{V} \to X$ where $\mathcal{U}$ and $\mathcal{V}$ are taken to be uncertainty sets. Here, we assume K containing the zero, $F_{\mathcal{U}}(s) := \{f(s,u) : u \in \mathcal{U}\}$ for $s \in S$ and $G_{\mathcal{V}} := \{g(v) : v \in \mathcal{V}\}$.

By using Theorem 5.2, we give criteria of robust feasibility problems (RFP) defined by

$$\text{(RFP) Minimize } \phi(s) \text{ subject to } f(s,u) \leq_{\operatorname{cl} K} g(v)$$

where $u \in \mathcal{U}$ and $v \in \mathcal{V}$.

Theorem 5.3 (weak robustness)**.** Assume the similar conditions to those of Theorem 5.2. If there is $k \in \operatorname{int} K$ and $s \in S$ satisfying

- $\Phi^{(1)}_{K,k}(F_{\mathcal{U}}(s), G_{\mathcal{V}}) \leq 0$,
  then (RFP) is feasible for all $u \in \mathcal{U}$ and $v \in \mathcal{V}$;
- $\Phi^{(2)}_{K,k}(F_{\mathcal{U}}(s), G_{\mathcal{V}}) \leq 0$,
  then there is $u \in \mathcal{U}$ such that (RFP) is feasible for all $v \in \mathcal{V}$;
- $\Phi^{(3)}_{K,k}(F_{\mathcal{U}}(s), G_{\mathcal{V}}) \leq 0$,
  then for all $v \in \mathcal{V}$, there is $u \in \mathcal{U}$ letting (RFP) be feasible;
- $\Phi^{(4)}_{K,k}(F_{\mathcal{U}}(s), G_{\mathcal{V}}) \leq 0$,
  then there is $v \in \mathcal{V}$ such that (RFP) is feasible for all $u \in \mathcal{U}$;
- $\Phi^{(5)}_{K,k}(F_{\mathcal{U}}(s), G_{\mathcal{V}}) \leq 0$,
  then for all $u \in \mathcal{U}$, there is $v \in \mathcal{V}$ letting (RFP) be feasible;

- $\Phi^{(6)}_{K,k}(F_{\mathcal{U}}(s), G_{\mathcal{V}}) \leq 0$,
  then we could only see (RFP) is feasible for some points $u \in \mathcal{U}$ and $v \in \mathcal{V}$.

Proposition 5.1. Let $i = 1, \ldots, 6$ and $s \in S$. If there is $k \in \operatorname{int} K$ such that $\Phi^{(i)}_{K,k}(F_{\mathcal{U}}(s), G_{\mathcal{V}}) < 0$, it holds that $F_{\mathcal{U}}(s) \preceq^{(i)}_{\operatorname{int} K} G_{\mathcal{V}}$.

This proposition clearly holds by the proof of Theorem 4.1. Note that we need no longer compactness. Similarly to Theorem 5.3, the next theorem follows from Proposition 5.1.

Theorem 5.4 (strong robustness)**.** If there is $k \in \operatorname{int} K$ and $s \in S$ satisfying

- $\Phi^{(1)}_{K,k}(F_{\mathcal{U}}(s), G_{\mathcal{V}}) < 0$,
  then (RFP) is feasible for all $u \in \mathcal{U}$ and $v \in \mathcal{V}$;
- $\Phi^{(2)}_{K,k}(F_{\mathcal{U}}(s), G_{\mathcal{V}}) < 0$,
  then there is $u \in \mathcal{U}$ such that (RFP) is feasible for all $v \in \mathcal{V}$;
- $\Phi^{(3)}_{K,k}(F_{\mathcal{U}}(s), G_{\mathcal{V}}) < 0$,
  then for all $v \in \mathcal{V}$, there is $u \in \mathcal{U}$ letting (RFP) be feasible;
- $\Phi^{(4)}_{K,k}(F_{\mathcal{U}}(s), G_{\mathcal{V}}) < 0$,
  then there is $v \in \mathcal{V}$ such that (RFP) is feasible for all $u \in \mathcal{U}$;
- $\Phi^{(5)}_{K,k}(F_{\mathcal{U}}(s), G_{\mathcal{V}}) < 0$,
  then for all $u \in \mathcal{U}$, there is $v \in \mathcal{V}$ letting (RFP) be feasible;
- $\Phi^{(6)}_{K,k}(F_{\mathcal{U}}(s), G_{\mathcal{V}}) < 0$,
  then we could only see (RFP) is feasible for some points $u \in \mathcal{U}$ and $v \in \mathcal{V}$.

Theorem 5.4 guarantees the existence of an interior point in the feasible set of (RFP) when $f(\cdot, u)$ is continuous on S for all $u \in \mathcal{U}$, which means if one finds $s \in S$ satisfying $\Phi^{(i)}_{K,k}(F(s), G_{\mathcal{V}}) < 0$, the optimal solutions could be given with the interior-point method.

Comments. The robustness mentioned above is set slightly apart from historical standard of robustness, which focuses on uncertain objective functions and constraints together with several robust counterparts. We aim to check suitability or validity of optimization models, not to verify solvability of robust optimization problems. An ideal modeling of (RFP) is given as

$$\text{(P) Minimize } \phi(s) \text{ subject to } f(s) \leq_{\operatorname{cl} K} g$$

and its feasible set is fixed. However, logical models sometimes do not represent real problems under uncertainty. Theorems 5.3 and 5.4 indicate how essentially (P) is modeled. If it holds that $\Phi^{(1)}_{K,k}(F_{\mathcal{U}}(s), G_{\mathcal{V}}) \leq 0$, (P) is well-modeled. In case $\Phi^{(2)}_{K,k}(F_{\mathcal{U}}(s), G_{\mathcal{V}}) \leq 0$ or $\Phi^{(4)}_{K,k}(F_{\mathcal{U}}(s), G_{\mathcal{V}}) \leq 0$, then one shall pay attention to the stability of f or g to justify (P) is a proper model, respectively. When $\Phi^{(3)}_{K,k}(F_{\mathcal{U}}(s), G_{\mathcal{V}}) \leq 0$ or $\Phi^{(5)}_{K,k}(F_{\mathcal{U}}(s), G_{\mathcal{V}}) \leq 0$, there are quite narrow ways for (P) to prevent the deviations $\mathcal{U}$ and $\mathcal{V}$. Unfortunately, it is hard to see (P) represents the reality when only the sixth condition holds.

Chapter 6

Conclusion

This research investigates relationship between the set relations and the evaluation functions from a viewpoint of alternative approaches shown in Theorems 4.1 and 4.2. The utilizability of the results are attributed to the computing algorithms presented in Section 3.4. The alternative theorems can lead to generalized Gordan's theorems of the alternative. One can utilize these theorems to describe optimality notions for set optimization and robustness of feasibility for vector-valued constraints.

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