Calculation Methods for Scalarization of Polyhedral Sets Based on Set Relations

Hui Yu

Doctoral Program in Fundamental Sciences
Graduate School of Science and Technology
Niigata University

2018

Contents

Chapter 1

Introduction

Optimization is the methodology of making systems, programs, products, etc. as good or effective as possible under some constraints and applied in mathematics, engineering, economics and other areas. Set optimization is a vibrant and expanding branch of optimization where each image of the objective and/or constraint maps is a set. In this study, we consider set optimization. An important method in this research area is scalarization. Actually, we are usually required to find a set which precedes others to solve given problems by using preference relations of sets called set relations. The concept of set relations was originally stated with six types proposed by Kuroiwa, Tanaka and Ha in [16], and there are some variations of set relations having been studied (e.g., [13]) based on the pointwise ordering of vectors; we use the original six types in this thesis. However, it is quite difficult to see which set of two is preferred because a set may consist of infinitely many elements. If scalars characterize set relations, set-to-set comparisons would get far simplified. This is a reason why we should consider scalarization as a characterization of set relations.

Gerstewitz's scalarization function [6] is one of well-defined scalarization methods for vectors with which many researchers have produced various types of characterization functions [5] to scale sets. These functions have been used to describe optimality [1, 11, 14, 17], well-posedness (setness) [4, 9, 21] and so on. Recently, several interesting expressions

[8, 20] like oriented distance types are considered. In addition to these theoretical results, Sonda, Tanaka and Yamada [19] proposed a certain computational scheme to calculate practically the four derived forms via Gerstewitz's function for a given set.

In recent years, scalarization functions for sets are proposed in [17] based on the six set relations. These functions have been studied as characterizations of set relations in [1, 8, 18] and applied to fuzzy theory in [12]. However, there are no studies on concrete calculation process of the values of the scalarization functions whereas this is of great importance because some authors usually describe many properties on set-valued maps and set optimization by scalar for simplification (see [15]).

In this thesis, we propose a technical approach to calculate the scalarization functions when the set relations are given between two polyhedral sets. As a result, we show all types are reduced to a finite number of linear programming problems. The organization of the thesis is as follows. In Chapter 2, we recall basic definitions about set relations, scalarization functions, polyhedral sets and finitely generated sets. In Chapter 3, we give a reformation of the scalarization functions. In Chapter 4, we introduce some calculation algorithms for the scalarization functions between polyhedral sets. In Section 4.1, we give some algorithms for the special case of polytopes (or, equivalently, bounded polyhedral sets). In Section 4.2, we expand the applicable range of calculation methods from the polytope case to more general polyhedral case. In Section 4.3, a calculation example is described.

Chapter 2

Preliminaries

In this chapter, we recall some basic definitions which will be used in the thesis.

2.1 Basic Knowledge

Throughout this thesis, let X be a real topological vector space. We write the set of all subsets of X excluding the empty set $\emptyset$ as $\mathcal{P}(X)$. The topological interior, convex hull and convex conical hull of a set $A \subset X$ are denoted by $\operatorname{int} A$, $\operatorname{co} A$ and $\operatorname{cone} A$, respectively. For given A, $B \in \mathcal{P}(X)$ and $t \in \mathbb{R}$, the algebraic sum $A + B$ and the scalar multiplication tA are defined as follows:

$$A + B := \{a + b \mid a \in A,\ b \in B\}, \quad tA := \{ta \mid a \in A\}.$$

Particular, $x + A := \{x\} + A$ for $x \in X$ and $-A := (-1)A$.

Definition 2.1 (cone)**.** Let C be a nonempty subset of X. Then, the set C is said to be a cone if

$$\lambda x \in C \text{ for all } x \in C \text{ and } \lambda \geq 0.$$

Proposition 2.1. *Let $C \in \mathcal{P}(X)$ be a cone. Then, C is convex if and only if*

$$C + C = C.$$

Proof. Assume that cone C is a convex set. Then, for every $x,\, y \in C$, we have

$$x + y = 2\left(\frac{x}{2} + \frac{y}{2}\right) \in C,$$

and hence $x + y \in C$. Thus, we obtain $C + C \subset C$. To show the relation '$\supset$', let $x \in C$. We have

$$x = \frac{x}{2} + \frac{x}{2} \in C + C,$$

and we obtain $C + C \supset C$. Conversely, we assume that the set C satisfies $C + C = C$. For any $x,\, y \in C$ and $\lambda \in (0, 1)$, we obtain $\lambda x \in C$ and $(1 - \lambda)y \in C$ because C is a cone. Since $C + C = C$, we obtain $\lambda x + (1 - \lambda)y \in C$, that is, C is convex. ∎

Let C be a convex cone in X with $\operatorname{int} C \neq \emptyset$. Then we define the binary relation $\leq_C$ on X induced by C as follows: $x \leq_C y$ if $y - x \in C$ for $x,\, y \in X$. Since C is a convex cone, this relation $\leq_C$ has reflexivity and transitivity, which means C is a preorder.

2.2 Set Relations and Scalarization Functions

In this section, we define some binary relations between two sets using the relation $\leq_C$ and scalarization functions for sets.

Definition 2.2 (set relations, [16]). For $A,\, B \in \mathcal{P}(X)$,

(i) $A \leq_C^{(1)} B \overset{\text{def}}{\iff} \forall a \in A,\ \forall b \in B,\ a \leq_C b \iff A \subset \bigcap_{b \in B}(b - C)$;

(ii) $A \leq_C^{(2)} B \overset{\text{def}}{\iff} \exists a \in A$ s.t. $\forall b \in B,\ a \leq_C b \iff A \cap \left(\bigcap_{b \in B}(b - C)\right) \neq \emptyset$;

(iii) $A \leq_C^{(3)} B \overset{\text{def}}{\iff} \forall b \in B,\ \exists a \in A$ s.t. $a \leq_C b \iff B \subset A + C$;

(iv) $A \leq_C^{(4)} B \overset{\text{def}}{\iff} \exists b \in B$ s.t. $\forall a \in A,\ a \leq_C b \iff \left(\bigcap_{a \in A}(a + C)\right) \cap B \neq \emptyset$;

(v) $A \leq_C^{(5)} B \overset{\text{def}}{\iff} \forall a \in A,\ \exists b \in B$ s.t. $a \leq_C b \iff A \subset B - C$;

(vi) $A \leq_C^{(6)} B \overset{\text{def}}{\iff} \exists a \in A,\ \exists b \in B$ s.t. $a \leq_C b \iff A \cap (B - C) \neq \emptyset$.

Proposition 2.2. *For A, $B \in \mathcal{P}(X)$, the following statements hold:*

$$A \leq_C^{(1)} B \text{ iff } B \leq_{-C}^{(1)} A; \quad A \leq_C^{(2)} B \text{ iff } B \leq_{-C}^{(4)} A;$$
$$A \leq_C^{(3)} B \text{ iff } B \leq_{-C}^{(5)} A; \quad A \leq_C^{(4)} B \text{ iff } B \leq_{-C}^{(2)} A;$$
$$A \leq_C^{(5)} B \text{ iff } B \leq_{-C}^{(3)} A; \quad A \leq_C^{(6)} B \text{ iff } B \leq_{-C}^{(6)} A.$$

Proof. By Definition 2.2, the statements are clear. ∎

Definition 2.3 (scalarization functions, [17]). Let A, $B \in \mathcal{P}(X)$ and $k \in \operatorname{int} C$. For each $i = 1, \ldots, 6$, we define a scalarization function $E_{C,k}^{(i)} : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R} \cup \{\pm\infty\}$ by

$$E_{C,k}^{(i)}(A, B) := \inf \left\{ t \in \mathbb{R} \;\middle|\; A \leq_C^{(i)} (B + tk) \right\}.$$

These scalarization functions measure the difference between two given sets with respect to each set relation. By the definition, one can easily check that the following inequalities hold:

$$E_{C,k}^{(1)}(A, B) \geq E_{C,k}^{(2)}(A, B) \geq E_{C,k}^{(3)}(A, B) \geq E_{C,k}^{(6)}(A, B);$$
$$E_{C,k}^{(1)}(A, B) \geq E_{C,k}^{(4)}(A, B) \geq E_{C,k}^{(5)}(A, B) \geq E_{C,k}^{(6)}(A, B).$$

2.3 Polyhedral Sets and Finitely Generated Sets

In this section, we introduce basic concepts of polyhedral set and finitely generated set. Let X^* be the topological dual space of X and A° the negative polar cone of $A \subset X$. The set of all $m \times n$ real matrices is written as $M^{m\times n}$.

Definition 2.4 (polyhedral set). A set $A \subset X$ is said to be polyhedral if $A = \{x \in X \mid \langle p_i, x\rangle \leq q_i \ (i = 1, \ldots, m)\}$ for some $p_1, \ldots, p_m \in X^*$ and $q_1, \ldots, q_m \in \mathbb{R}$. Particularly, $A \subset \mathbb{R}^n$ is polyhedral if $A = \{x \in \mathbb{R}^n \mid Px \leq q\}$ for some $P \in M^{m\times n}$ and $q \in \mathbb{R}^m$.

Definition 2.5 (finitely generated set). A set $A \subset X$ is said to be finitely generated if $A = \operatorname{co} V + \operatorname{cone} W$ for some finite sets V, $W \subset X$.

We remark that a cone $C \subset X$ is polyhedral if $C = \{x \in X \mid \langle p_i, x\rangle \leq 0\ (i = 1, \dots, m)\}$ for some $p_1, \dots, p_m \in X^*$ and is finitely generated if $C = \operatorname{cone} W$ for some finite set $W \subset X$. Also, a set $A \subset X$ is called a polytope if $A = \operatorname{co} V$ for some finite set $V \subset X$.

The polyhedrality and the finitely generatedness of a set, in fact, coincide with each other in a finite-dimensional space. In Chapter 4, we utilize the transformation of a polyhedral form into a finitely generated form to obtain our main results. Therefore, we introduce here a detailed technique for the transformation, where the following Fourier–Motzkin elimination plays an important role.

Proposition 2.3 (Fourier–Motzkin elimination, e.g., see [3])**.** *Let*

$$\sum_{j=1}^{n} p_{ij} x_j \leq q_i \text{ for } i = 1, \dots, m$$

be a system of linear inequalities with variables $x_1, \dots, x_n$. Then, we can eliminate the variable x_1 and turn the system into another one

$$\sum_{j=2}^{n} p'_{ij} x_j \leq q'_i \text{ for } i = 1, \dots, m'$$

with variables $x_2, \dots, x_n$ such that both systems have the same solutions over the remaining variables. In particular, $q_i = 0$ for $i = 1, \dots, m$ implies $q'_i = 0$ for $i = 1, \dots, m'$.

Proposition 2.4 (Theorem 1.3 in [23])**.** *A cone $C \subset \mathbb{R}^n$ is polyhedral if and only if it is finitely generated.*

Proof. Assume that C is a finitely generated cone. Then, there exist $w_1, \dots, w_m \in \mathbb{R}^n$ such that

$$C = \left\{ x \in \mathbb{R}^n \;\middle|\; x = \sum_{i=1}^{m} \mu_i w_i,\ \mu_i \geq 0\ (i = 1, \dots, m) \right\}.$$

By using the Fourier–Motzkin elimination, we can eliminate the variables $\mu_1, \dots, \mu_m$ from the system

$$\begin{cases} x_j = \displaystyle\sum_{i=1}^{m} \mu_i w_{ij}\ (j = 1, \dots, n) \\ \mu_i \geq 0\ (i = 1, \dots, m) \end{cases}$$

(where x_j and w_{ij} are the j-th element of x and w_i, respectively) and turn it into a system of homogeneous linear inequalities with variables $x_1, \dots, x_n$. This means C is a polyhedral cone.

Conversely, assume that C is a polyhedral cone. Then, $C = \{x \in \mathbb{R}^n \mid Px \leq \mathbf{0}\}$ for some $P \in M^{m\times n}$. Now, we define a finitely generated cone $D := \{x \in \mathbb{R}^n \mid x = P^{\mathrm{T}}\mu,\ \mu \geq \mathbf{0}\}$ and deduce $C = D^\circ$. Since D is a closed convex cone, we have $D = D^{\circ\circ} = C^\circ$ by the bipolar theorem. Hence, C° is finitely generated. As we already know that a finitely generated cone is also a polyhedral cone, it follows C° is polyhedral. From the above argument (the polar of any polyhedral cone is finitely generated), we conclude that $C = C^{\circ\circ}$ is finitely generated. ∎

Proposition 2.5 (Theorem 1.2 in [23])**.** *A set $A \subset \mathbb{R}^n$ is polyhedral if and only if it is finitely generated.*

Proof. Assume that A is a finitely generated set. Then,

$$A = \left\{ x \in \mathbb{R}^n \;\middle|\; x = \sum_{i\in I} \lambda_i v_i + \sum_{j \in J} \mu_j w_j,\ \sum_{i \in I} \lambda_i = 1,\ \lambda_i, \mu_j \geq 0\ (i \in I,\ j \in J) \right\}$$

for finite sets $\{v_i \mid i \in I\}$, $\{w_j \mid j \in J\} \subset \mathbb{R}^n$. By using the Fourier–Motzkin elimination, we can eliminate the variables λ_i, μ_j $(i \in I,\ j \in J)$ and deduce that A is a polyhedral set.

Conversely, assume that A is polyhedral: $A = \{x \in \mathbb{R}^n \mid \langle p_i, x\rangle \leq q_i\ (i \in I)\}$ for finite sets $\{p_i \mid i \in I\} \subset \mathbb{R}^n$ and $\{q_i \mid i \in I\} \subset \mathbb{R}$. Consider a polyhedral cone

$$C_A := \left\{ \begin{pmatrix} x \\ r \end{pmatrix} \in \mathbb{R}^{n+1} \;\middle|\; -r \leq 0,\ \langle p_i, x\rangle - q_i r \leq 0\ (i \in I) \right\}.$$

By Proposition 2.4, C_A is finitely generated. Hence,

$$C_A = \left\{ \begin{pmatrix} x \\ r \end{pmatrix} \in \mathbb{R}^{n+1} \;\middle|\; \begin{pmatrix} x \\ r \end{pmatrix} = \begin{pmatrix} \sum_{j\in J} \mu_j w_j \\ \sum_{j \in J} \mu_j d_j \end{pmatrix},\ \mu_j \geq 0\ (j \in J) \right\}$$

for finite sets $\{w_j \mid j \in J\} \subset \mathbb{R}^n$ and $\{d_j \mid j \in J\} \subset \mathbb{R}$. Since $r \geq 0$, we have $d_j \geq 0$ for all $j \in J$, and thus $J = J^+ \cup J^0$ where $J^+ := \{j \in J \mid d_j > 0\}$ and $J^0 := \{j \in J \mid d_j = 0\}$.

Putting $v_i := (1/d_i)w_i$ and $\lambda_i := \mu_i d_i$ for $i \in J^+$, we obtain

$$C_A = \left\{ \begin{pmatrix} x \\ r \end{pmatrix} \in \mathbb{R}^{n+1} \;\middle|\; \begin{pmatrix} x \\ r \end{pmatrix} = \begin{pmatrix} \sum_{i\in J^+} \lambda_i v_i + \sum_{j \in J^0} \mu_j w_j \\ \sum_{i \in J^+} \lambda_i \end{pmatrix}, \right.$$
$$\left. \lambda_i, \mu_j \geq 0 \; (i \in J^+,\, j \in J^0) \right\}.$$

Therefore, we deduce

$$\begin{aligned} A &= \left\{ x \in \mathbb{R}^n \;\middle|\; \begin{pmatrix} x \\ 1 \end{pmatrix} \in C_A \right\} \\ &= \left\{ x \in \mathbb{R}^n \;\middle|\; x = \sum_{i \in J^+} \lambda_i v_i + \sum_{j \in J^0} \mu_j w_j,\ \sum_{i \in J^+} \lambda_i = 1,\ \lambda_i, \mu_j \geq 0 \; (i \in J^+, j \in J^0) \right\}, \end{aligned}$$

which means A is finitely generated. ∎

Chapter 3

Reformation of the Scalarization Functions

In this chapter, we reform the scalarization functions in Definition 2.3 to vector-oriented expressions by using the ordering $\leq_C$ for vectors.

Proposition 3.1. *The scalarization functions for sets can be reformed as follows:*

(i) $E^{(1)}_{C,k}(A,B) = \sup_{a\in A}\sup_{b\in B}\inf\{t \in \mathbb{R} \mid a \leq_C b + tk\}$;

(ii) $E^{(2)}_{C,k}(A,B) = \inf_{a\in A}\sup_{b\in B}\inf\{t \in \mathbb{R} \mid a \leq_C b + tk\}$;

(iii) $E^{(3)}_{C,k}(A,B) = \sup_{b\in B}\inf_{a\in A}\inf\{t \in \mathbb{R} \mid a \leq_C b + tk\}$;

(iv) $E^{(4)}_{C,k}(A,B) = \inf_{b\in B}\sup_{a\in A}\inf\{t \in \mathbb{R} \mid a \leq_C b + tk\}$;

(v) $E^{(5)}_{C,k}(A,B) = \sup_{a\in A}\inf_{b\in B}\inf\{t \in \mathbb{R} \mid a \leq_C b + tk\}$;

(vi) $E^{(6)}_{C,k}(A,B) = \inf_{a\in A}\inf_{b\in B}\inf\{t \in \mathbb{R} \mid a \leq_C b + tk\}$.

Proof. It follows from Definition 2.2 that the scalarization function on the left-hand side of each statement above is represented by an elementwise formula with $\leq_C$.

(i) We shall prove that

$$\inf\{t \mid \forall a \in A,\ \forall b \in B,\ a \leq_C b + tk\} = \sup_{a\in A}\sup_{b\in B}\inf\{t \mid a \leq_C b + tk\}. \tag{3.1}$$

By selecting elements $a' \in A$ and $b' \in B$, it is easy to see

$$\{t \mid \forall a \in A,\ \forall b \in B,\ a \leq_C b + tk\} \subset \{t \mid a' \leq_C b' + tk\}.$$

Taking the infimum for both sides of this formula, we have

$$\inf\{t \mid \forall a \in A,\ \forall b \in B,\ a \leq_C b + tk\} \geq \inf\{t \mid a' \leq_C b' + tk\}.$$

Since a' and b' are arbitrary,

$$\inf\{t \mid \forall a \in A,\ \forall b \in B,\ a \leq_C b + tk\} \geq \sup_{a\in A}\sup_{b\in B}\inf\{t \mid a \leq_C b + tk\}.$$

Now, we assume that there exists $\bar{s} \in \mathbb{R}$ such that

$$\inf\{t \mid \forall a \in A,\ \forall b \in B,\ a \leq_C b + tk\} > \bar{s} > \sup_{a\in A}\sup_{b\in B}\inf\{t \mid a \leq_C b + tk\}. \tag{3.2}$$

Since $\sup_{a\in A}\sup_{b\in B}\inf\{t \mid a \leq_C b + tk\} < \bar{s}$, for all $a \in A$ and $b \in B$ there exists $s_{a,b} < \bar{s}$ such that $a \leq_C b + s_{a,b}k$. It follows that $b - a \in C - s_{a,b}k \subset C - \bar{s}k$. This implies that $a \leq_C b + \bar{s}k$. Therefore, $\inf\{t \mid \forall a \in A,\ \forall b \in B,\ a \leq_C b + tk\} \leq \bar{s}$, which contradicts (3.2). Thus, (3.1) holds.

(ii) In a similar way to the proof of (i), we get

$$\inf\{t \mid \exists a \in A \text{ s.t. } \forall b \in B,\ a \leq_C b + tk\} = \inf_{a\in A}\inf\{t \mid \forall b \in B,\ a \leq_C b + tk\}. \tag{3.3}$$

From (3.1), we have

$$\inf\{t \mid \forall b \in B,\ a \leq_C b + tk\} = \sup_{b\in B}\inf\{t \mid a \leq_C b + tk\} \tag{3.4}$$

for each $a \in A$.

Hence, by (3.3) and (3.4), we obtain

$$\inf\{t \mid \exists a \in A \text{ s.t. } \forall b \in B,\ a \leq_C b + tk\} = \inf_{a\in A}\sup_{b\in B}\inf\{t \mid a \leq_C b + tk\}.$$

(iii) In a similar way to the proof of (i), we get

$$\inf\{t \mid \forall b \in B, \exists a \in A \text{ s.t. } a \leq_C v + tk\} = \sup_{b\in B} \inf\{t \mid \exists a \in A \text{ s.t. } a \leq_C b + tk\}. \quad (3.5)$$

From (3.3), we have

$$\inf\{t \mid \exists a \in A \text{ s.t. } a \leq_C b + tk\} = \inf_{a\in A} \inf\{t \mid a \leq_C b + tk\} \quad (3.6)$$

for each $b \in B$.

Thus, by (3.5) and (3.6), we complete the proof.

(iv) By Proposition 2.2,

$$\inf\left\{t \mid A \leq_C^{(4)} (B + tk)\right\} = \inf\left\{t \mid B \leq_{-C}^{(2)} (A + t(-k))\right\}. \quad (3.7)$$

From the result (ii), we have

$$\inf\left\{t \mid B \leq_{-C}^{(2)} (A + t(-k))\right\} = \inf_{b\in B} \sup_{a\in A} \inf\{t \mid b \leq_{-C} a + t(-k)\}. \quad (3.8)$$

By (3.7) and (3.8),

$$\begin{aligned}\inf\left\{t \mid A \leq_C^{(4)} (B + tk)\right\} &= \inf_{b\in B} \sup_{a\in A} \inf\{t \mid b \leq_{-C} a + t(-k)\} \\ &= \inf_{b\in B} \sup_{a\in A} \inf\{t \mid a \leq_C b + tk\},\end{aligned}$$

which completes the proof.

(v) By Proposition 2.2,

$$\inf\left\{t \mid A \leq_C^{(5)} (B + tk)\right\} = \inf\left\{t \mid B \leq_{-C}^{(3)} (A + t(-k))\right\}. \quad (3.9)$$

From the result (iii), we have

$$\inf\left\{t \mid B \leq_{-C}^{(3)} (A + t(-k))\right\} = \sup_{a\in A} \inf_{b\in B} \inf\{t \mid b \leq_{-C} a + t(-k)\}. \quad (3.10)$$

By (3.9) and (3.10),

$$\begin{aligned}\inf\left\{t \mid A \leq_C^{(5)} (B + tk)\right\} &= \sup_{a\in A} \inf_{b\in B} \inf\{t \mid b \leq_{-C} a + t(-k)\} \\ &= \sup_{a\in A} \inf_{b\in B} \inf\{t \mid a \leq_C b + tk\},\end{aligned}$$

which completes the proof.

(vi) In a similar way to the proof of (i), the equality

$$\inf\{t \mid \exists a \in A,\ \exists b \in B \text{ s.t. } a \leq_C b + tk\} = \inf_{a\in A}\inf_{b\in B}\inf\{t \mid a \leq_C b + tk\}$$

is clear.

$\blacksquare$

Chapter 4

Calculation Methods for the Scalarization Functions

In this chapter, we discuss how to compute values of the six types of scalarization functions under certain assumptions. Consider a Euclidean space $\mathbb{R}^n$. Assume that C is a polyhedral cone defined as $C := \{x \in \mathbb{R}^n \mid \langle p_l, x \rangle \leq 0 \ (l = 1, \dots, m)\}$ where $p_1, \dots, p_m \in \mathbb{R}^n$ and let $k \in \operatorname{int} C$.

4.1 Algorithms in the Case of Polytopes

In this section, we propose a technical approach to calculate the scalarization functions when the set relations are given between polytopes.

Firstly, we consider a sort of reformation of a scalarizing function for vectors using the polyhedral cone C.

Proposition 4.1 (see Proposition 1.44 and Corollary 1.45 of [2])**.** *Assume that $k \in \operatorname{int} C$. We have*

$$\inf\{t \in \mathbb{R} \mid x \leq_C tk\} = \max_{l=1,\dots,m} \left\langle \frac{p_l}{\langle p_l, k \rangle}, x \right\rangle \ \text{for } x \in \mathbb{R}^n.$$

Proof. Since $k \in \operatorname{int} C$, $\langle p_l, k \rangle > 0$ for each $l = 1, \dots, m$. It follows from the definition of C that for all $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$,

$$\begin{aligned} x \leq_C tk &\Longleftrightarrow \langle p_l, tk - x\rangle \geq 0,\ \forall l = 1, \ldots, m \\ &\Longleftrightarrow t \geq \left\langle \frac{p_l}{\langle p_l, k\rangle}, x \right\rangle,\ \forall l = 1, \ldots, m \\ &\Longleftrightarrow t \geq \max_{l=1,\ldots,m} \left\langle \frac{p_l}{\langle p_l, k\rangle}, x \right\rangle. \end{aligned}$$

Thus,

$$\inf\{t \in \mathbb{R} \mid x \leq_C tk\} = \max_{l=1,\ldots,m} \left\langle \frac{p_l}{\langle p_l, k\rangle}, x \right\rangle.$$

∎

Proposition 4.2. *Let A, $B \subset \mathbb{R}^n$. Then the following equalities hold:*

$$\text{(i)}\ \ E^{(1)}_{C,k}(A, B) = \sup_{a\in A} \sup_{b\in B} \max_{l=1,\ldots,m} \left\langle \frac{p_l}{\langle p_l, k\rangle}, a - b \right\rangle;$$

$$\text{(ii)}\ \ E^{(2)}_{C,k}(A, B) = \inf_{a\in A} \sup_{b\in B} \max_{l=1,\ldots,m} \left\langle \frac{p_l}{\langle p_l, k\rangle}, a - b \right\rangle;$$

$$\text{(iii)}\ \ E^{(3)}_{C,k}(A, B) = \sup_{b\in B} \inf_{a\in A} \max_{l=1,\ldots,m} \left\langle \frac{p_l}{\langle p_l, k\rangle}, a - b \right\rangle;$$

$$\text{(iv)}\ \ E^{(4)}_{C,k}(A, B) = \inf_{b\in B} \sup_{a\in A} \max_{l=1,\ldots,m} \left\langle \frac{p_l}{\langle p_l, k\rangle}, a - b \right\rangle;$$

$$\text{(v)}\ \ E^{(5)}_{C,k}(A, B) = \sup_{a\in A} \inf_{b\in B} \max_{l=1,\ldots,m} \left\langle \frac{p_l}{\langle p_l, k\rangle}, a - b \right\rangle;$$

$$\text{(vi)}\ \ E^{(6)}_{C,k}(A, B) = \inf_{a\in A} \inf_{b\in B} \max_{l=1,\ldots,m} \left\langle \frac{p_l}{\langle p_l, k\rangle}, a - b \right\rangle.$$

Proof. By replacing x in Proposition 4.1 with $a - b$ for $a \in A$ and $b \in B$, we can transform Proposition 3.1 into the above statements. ∎

Theorem 4.1. *Let A, $B \subset \mathbb{R}^n$ be polytopes defined as $A := \mathrm{co}\{a_1, \ldots, a_\alpha\}$, $B := \mathrm{co}\{b_1, \ldots, b_\beta\}$. For each $h \in \mathbb{N}$, define $I(h) := \{1, \ldots, h\}$ and $\Delta^h := \{(\lambda_1, \ldots, \lambda_h) \in \mathbb{R}^h \mid \sum_{i=1}^h \lambda_i = 1,\ \lambda_i \geq 0\ (i \in I(h))\}$. Then,*

(i) $E^{(1)}_{C,k}(A, B) = \max_{i\in I(\alpha)} \max_{j\in I(\beta)} \max_{l\in I(m)} \left\langle \frac{p_l}{\langle p_l, k\rangle}, a_i - b_j \right\rangle$;

(ii) $E^{(2)}_{C,k}(A,B) = \min\left\{ t \in \mathbb{R} \;\middle|\; \langle p_l, k\rangle\, t + \sum_{i=1}^{\alpha} \langle p_l, -a_i\rangle\, \lambda_i \geq \max_{j\in I(\beta)} \langle p_l, -b_j\rangle \;\; (l \in I(m)) \right.$

$\left. \textit{for some } \lambda \in \Delta^{\alpha} \right\};$

(iii) $E^{(3)}_{C,k}(A,B) = \max_{j\in I(\beta)} \min\left\{ t \in \mathbb{R} \;\middle|\; \langle p_l, k\rangle\, t + \sum_{i=1}^{\alpha} \langle p_l, -a_i\rangle\, \lambda_i \geq \langle p_l, -b_j\rangle \;\; (l \in I(m)) \right.$

$\left. \textit{for some } \lambda \in \Delta^{\alpha} \right\};$

(iv) $E^{(4)}_{C,k}(A,B) = \min\left\{ t \in \mathbb{R} \;\middle|\; \langle p_l, k\rangle\, t + \sum_{j=1}^{\beta} \langle p_l, b_j\rangle\, \mu_j \geq \max_{i\in I(\alpha)} \langle p_l, a_i\rangle \;\; (l \in I(m)) \right.$

$\left. \textit{for some } \mu \in \Delta^{\beta} \right\};$

(v) $E^{(5)}_{C,k}(A,B) = \max_{i\in I(\alpha)} \min\left\{ t \in \mathbb{R} \;\middle|\; \langle p_l, k\rangle\, t + \sum_{j=1}^{\beta} \langle p_l, b_j\rangle\, \mu_j \geq \langle p_l, a_i\rangle \;\; (l \in I(m)) \right.$

$\left. \textit{for some } \mu \in \Delta^{\beta} \right\};$

(vi) $E^{(6)}_{C,k}(A,B) = \min\left\{ t \in \mathbb{R} \;\middle|\; \langle p_l, k\rangle\, t + \sum_{i=1}^{\alpha} \langle p_l, -a_i\rangle\, \lambda_i + \sum_{j=1}^{\beta} \langle p_l, b_j\rangle\, \mu_j \geq 0 \;\; (l \in I(m)) \right.$

$\left. \textit{for some } \lambda \in \Delta^{\alpha},\; \mu \in \Delta^{\beta} \right\}.$

Proof. By Propositions 4.2, we have

$$E^{(1)}_{C,k}(A,B) = \sup_{a\in A}\sup_{b\in B}\max_{l\in I(m)} \left\langle \frac{p_l}{\langle p_l, k\rangle}, a-b \right\rangle;\; E^{(2)}_{C,k}(A,B) = \inf_{a\in A}\sup_{b\in B}\max_{l\in I(m)} \left\langle \frac{p_l}{\langle p_l, k\rangle}, a-b \right\rangle;$$

$$E^{(3)}_{C,k}(A,B) = \sup_{b\in B}\inf_{a\in A}\max_{l\in I(m)} \left\langle \frac{p_l}{\langle p_l, k\rangle}, a-b \right\rangle;\; E^{(4)}_{C,k}(A,B) = \inf_{b\in B}\sup_{a\in A}\max_{l\in I(m)} \left\langle \frac{p_l}{\langle p_l, k\rangle}, a-b \right\rangle;$$

$$E^{(5)}_{C,k}(A,B) = \sup_{a\in A}\inf_{b\in B}\max_{l\in I(m)} \left\langle \frac{p_l}{\langle p_l, k\rangle}, a-b \right\rangle;\; E^{(6)}_{C,k}(A,B) = \inf_{a\in A}\inf_{b\in B}\max_{l\in I(m)} \left\langle \frac{p_l}{\langle p_l, k\rangle}, a-b \right\rangle.$$

For all $a \in A$ and $b \in B$, there exist $\lambda \in \Delta^{\alpha}$ and $\mu \in \Delta^{\beta}$ such that $a = \sum_{i=1}^{\alpha} \lambda_i a_i$, $b = \sum_{j=1}^{\beta} \mu_j b_j$.

Firstly, we prove statement (i).

$$\begin{aligned}
E^{(1)}_{C,k}(A,B) &= \sup_{a\in A}\sup_{b\in B}\max_{l\in I(m)}\left\langle \frac{p_l}{\langle p_l,k\rangle}, a-b\right\rangle \\
&= \sup_{a\in A}\max_{l\in I(m)}\sup_{\mu\in\Delta^\beta}\left\langle \frac{p_l}{\langle p_l,k\rangle}, \sum_{j=1}^{\beta}\mu_j(a-b_j)\right\rangle \\
&= \sup_{a\in A}\max_{l\in I(m)}\max_{j\in I(\beta)}\left\langle \frac{p_l}{\langle p_l,k\rangle}, a-b_j\right\rangle \\
&= \max_{l\in I(m)}\max_{j\in I(\beta)}\sup_{\lambda\in\Delta^\alpha}\left\langle \frac{p_l}{\langle p_l,k\rangle}, \sum_{i=1}^{\alpha}\lambda_i(a_i-b_j)\right\rangle \\
&= \max_{l\in I(m)}\max_{j\in I(\beta)}\max_{i\in I(\alpha)}\left\langle \frac{p_l}{\langle p_l,k\rangle}, a_i-b_j\right\rangle \\
&= \max_{i\in I(\alpha)}\max_{j\in I(\beta)}\max_{l\in I(m)}\left\langle \frac{p_l}{\langle p_l,k\rangle}, a_i-b_j\right\rangle .
\end{aligned}$$

Next, we prove statement (ii).

$$\begin{aligned}
E^{(2)}_{C,k}(A,B) &= \inf_{a\in A}\sup_{b\in B}\max_{l\in I(m)}\left\langle \frac{p_l}{\langle p_l,k\rangle}, a-b\right\rangle \\
&= \inf_{a\in A}\max_{l\in I(m)}\sup_{\mu\in\Delta^\beta}\left\langle \frac{p_l}{\langle p_l,k\rangle}, \sum_{j=1}^{\beta}\mu_j(a-b_j)\right\rangle \\
&= \inf_{a\in A}\max_{l\in I(m)}\max_{j\in I(\beta)}\left\langle \frac{p_l}{\langle p_l,k\rangle}, a-b_j\right\rangle \\
&= \inf_{\lambda\in\Delta^\alpha}\max_{l\in I(m)}\max_{j\in I(\beta)}\left\langle \frac{p_l}{\langle p_l,k\rangle}, \sum_{i=1}^{\alpha}\lambda_i a_i-b_j\right\rangle \\
&= \min_{\lambda\in\Delta^\alpha}\max_{j\in I(\beta)}\max_{l\in I(m)}\left\langle \frac{p_l}{\langle p_l,k\rangle}, \sum_{i=1}^{\alpha}\lambda_i a_i-b_j\right\rangle .
\end{aligned}$$

Actually, since the real-valued function

$$f(\lambda) := \max_{j\in I(\beta)}\max_{l\in I(m)}\left\langle \frac{p_l}{\langle p_l,k\rangle}, \sum_{i=1}^{\alpha}\lambda_i a_i-b_j\right\rangle$$

is continuous on the compact set Δ^α, it must attain its minimum on Δ^α. Therefore, the last equality above holds.

Putting

$$T := \left\{ t \in \mathbb{R} \;\middle|\; t \geq \max_{j \in I(\beta)} \max_{l \in I(m)} \left\langle \frac{p_l}{\langle p_l, k \rangle}, \sum_{i=1}^{\alpha} \lambda_i a_i - b_j \right\rangle \text{ for some } \lambda \in \Delta^\alpha \right\},$$

we have $E^{(2)}_{C,k}(A, B) = \min T$. For all $t \in \mathbb{R}$, it holds

$$\begin{aligned} & t \geq \max_{j \in I(\beta)} \max_{l \in I(m)} \left\langle \frac{p_l}{\langle p_l, k \rangle}, \sum_{i=1}^{\alpha} \lambda_i a_i - b_j \right\rangle \text{ for some } \lambda \in \Delta^\alpha \\ \Longleftrightarrow\; & \langle p_l, k \rangle\, t + \sum_{i=1}^{\alpha} \langle p_l, -a_i \rangle\, \lambda_i \geq \max_{j \in I(\beta)} \langle p_l, -b_j \rangle \;\; (l \in I(m)) \text{ for some } \lambda \in \Delta^\alpha. \end{aligned}$$

Therefore, the conclusion follows.

Then, we can prove statements (iv) and (vi) similarly.

Next, we prove (iii). Using the convexity of $A + C$,

$$\inf\{t \in \mathbb{R} \mid (B + tk) \subset A + C\} = \max_{j \in I(\beta)} \inf\{t \in \mathbb{R} \mid b_j + tk \in A + C\}. \tag{4.1}$$

Then, for all $j \in I(\beta)$, the following equality holds:

$$\inf\{t \in \mathbb{R} \mid b_j + tk \in A + C\} = \inf\{t \in \mathbb{R} \mid A \cap (b_j - C + tk) \neq \emptyset\}. \tag{4.2}$$

Taking $B := \{b_j\}$ in (ii),

$$\inf\{t \in \mathbb{R} \mid A \cap (b_j - C + tk) \neq \emptyset\} = \min_{\lambda \in \Delta^\alpha} \max_{l \in I(m)} \left\langle \frac{p_l}{\langle p_l, k \rangle}, \sum_{i=1}^{\alpha} \lambda_i a_i - b_j \right\rangle. \tag{4.3}$$

By (4.1), (4.2) and (4.3), we have

$$E^{(3)}_{C,k}(A, B) = \max_{j \in I(\beta)} \min_{\lambda \in \Delta^\alpha} \max_{l \in I(m)} \left\langle \frac{p_l}{\langle p_l, k \rangle}, \sum_{i=1}^{\alpha} \lambda_i a_i - b_j \right\rangle.$$

For each $j \in I(\beta)$, putting

$$T_j := \left\{ t \in \mathbb{R} \;\middle|\; t \geq \max_{l \in I(m)} \left\langle \frac{p_l}{\langle p_l, k \rangle}, \sum_{i=1}^{\alpha} \lambda_i a_i - b_j \right\rangle \text{ for some } \lambda \in \Delta^\alpha \right\},$$

we have $E^{(3)}_{C,k}(A,B) = \max_{j\in I(\beta)} \min T_j$. For all $t \in \mathbb{R}$, it holds

$$t \geq \max_{l\in I(m)} \left\langle \frac{p_l}{\langle p_l, k\rangle}, \sum_{i=1}^{\alpha} \lambda_i a_i - b_j \right\rangle \text{ for some } \lambda \in \Delta^\alpha$$

$$\Longleftrightarrow \langle p_l, k\rangle\, t + \sum_{i=1}^{\alpha} \langle p_l, -a_i\rangle\, \lambda_i \geq \langle p_l, -b_j\rangle \;\; (l \in I(m)) \text{ for some } \lambda \in \Delta^\alpha.$$

Therefore, the conclusion follows.

At last, we can prove statement (v) similarly. ∎

This theorem reveals that the problem to calculate each scalarization function can be decomposed into a finite number of linear programming problems when A and B are polytopes.

4.2 Algorithms in the Case of Polyhedral Sets

In this section, we deal with a new case where A and B are polyhedral sets. It is a direct generalization of Theorem 4.1 since any polytope is a polyhedral set. Henceforth, let A, B be defined as $A := \{x \in \mathbb{R}^n \mid P_A x \leq q_A\}$, $B := \{x \in \mathbb{R}^n \mid P_B x \leq q_B\}$ where $P_A \in M^{\alpha\times n}$, $P_B \in M^{\beta\times n}$, $q_A \in \mathbb{R}^\alpha$, $q_B \in \mathbb{R}^\beta$.

By Proposition 4.2, we give methods for calculating types (1), (2), (4) and (6) of the scalarization functions. At first, we consider the following linear programming problems for $l = 1, \ldots, m$.

$$\text{LP}(1.l): \quad \boxed{\text{Maximize } \left\langle \frac{p_l}{\langle p_l, k\rangle}, x - y \right\rangle \text{ subject to } P_A x \leq q_A,\; P_B y \leq q_B,}$$

$$\text{LP}(2.l): \quad \boxed{\text{Maximize } \left\langle \frac{p_l}{\langle p_l, k\rangle}, -y \right\rangle \text{ subject to } P_B y \leq q_B}$$

and

$$\text{LP}(2): \quad \boxed{\begin{array}{ll} \text{Minimize} & t \in \mathbb{R} \\ \text{subject to} & t \geq \left\langle \frac{p_l}{\langle p_l, k\rangle}, x \right\rangle + \text{Val}(\text{LP}(2.l)) \text{ for } l = 1, \ldots, \text{m}, \\ & P_A x \leq q_A. \end{array}}$$

Theorem 4.2. *The values $E^{(1)}_{C,k}(A,B)$, $E^{(2)}_{C,k}(A,B)$ can be calculated by solving a finite number of linear programming problems.*

(i) *If problems* LP(1.1), $\cdots$, LP(1.m) *have optimal values,* $E^{(1)}_{C,k}(A,B)$ *is equal to the maximum of their m optimal values. Otherwise,* $E^{(1)}_{C,k}(A,B)$ *is unbounded above.*

(ii) *If problems* LP(2.1), $\cdots$, LP(2.m) *have optimal values,* $E^{(2)}_{C,k}(A,B)$ *is equal to the optimal value of* LP(2). *Otherwise,* $E^{(2)}_{C,k}(A,B)$ *is unbounded above.*

The values $E^{(4)}_{C,k}(A,B)$ and $E^{(6)}_{C,k}(A,B)$ are similarly obtained.

Finally, for computing types (3) and (5), we convert the polyhedral cone C and polyhedral sets A, B into the following forms by using Propositions 2.4 and 2.5:

- $A = \operatorname{co} V_A + \operatorname{cone} W_A$ for finite sets V_A, $W_A \subset \mathbb{R}^n$;
- $B = \operatorname{co} V_B + \operatorname{cone} W_B$ for finite sets V_B, $W_B \subset \mathbb{R}^n$;
- $C = \operatorname{cone} W_C$ for a finite set $W_C \subset \mathbb{R}^n$.

Next, we consider the following linear programming problems for $v \in V_B$.

$$\text{LP}(3.v): \quad \boxed{\begin{array}{ll} \text{Min} & t \in \mathbb{R} \\ \text{s.t.} & t \geq \left\langle \frac{p_l}{\langle p_l, k\rangle}, x - v \right\rangle \text{ for } l = 1, \ldots, m, \\ & P_A x \leq q_A. \end{array}}$$

Theorem 4.3. *For the calculation of* $E^{(3)}_{C,k}(A,B)$, *the following statements hold.*

(i) *If* $\operatorname{cone} W_B \not\subset \operatorname{cone} W_A + \operatorname{cone} W_C$, $E^{(3)}_{C,k}(A,B)$ *is unbounded above.*

(ii) *If* $\operatorname{cone} W_B \subset \operatorname{cone} W_A + \operatorname{cone} W_C$, $E^{(3)}_{C,k}(A,B)$ *is equal to the maximum of the optimal values of problems* $\text{LP}_3(v)$ *for all* $v \in V_B$.

Proof. (i) Let $D := \operatorname{cone} W_A + \operatorname{cone} W_C$. Then, we have $x \notin D$ for some $x \in \operatorname{cone} W_B$. Since D is a closed convex cone, by the separation theorem there exists nonzero $p \in \mathbb{R}^n$ such that $\langle p, x\rangle > 0 \geq \langle p, y\rangle$ for all $y \in D$. By the compactness of $\operatorname{co} V_A$, it holds

$\operatorname{co} V_A + D \subset sk + D$ for some $s \in \mathbb{R}$. Now, fix $t \in \mathbb{R}$ and $z \in \operatorname{co} V_B$. As $\langle p, x \rangle > 0$, it follows that there exists $s' > 0$ such that $s' \langle p, x \rangle > \langle p, x - z - tk + sk \rangle$. This implies $z + s'x + tk - sk \notin D$ and hence $B + tk \not\subset sk + D$. Therefore, we obtain $B + tk \not\subset A + C$ for all $t \in \mathbb{R}$, which means $E^{(3)}_{C,k}(A, B) = +\infty$.

(ii) Let $t \in \mathbb{R}$. We prove $B + tk \subset A + C \iff V_B + tk \subset A + C$. The necessity of this equivalence is clear. Assume that $V_B + tk \subset A + C$. Then, by the convexity of $A + C$, we have $\operatorname{co} V_B + tk \subset A + C$. Since $\operatorname{cone} W_B \subset \operatorname{cone} W_A + \operatorname{cone} W_C$, it follows $\operatorname{co} V_B + \operatorname{cone} W_B + tk \subset A + \operatorname{cone} W_A + C + \operatorname{cone} W_C$, and thus we obtain $B + tk \subset A + C$. Therefore, we have $E^{(3)}_{C,k}(A, B) = E^{(3)}_{C,k}(A, V_B)$. From (iii) of Proposition 4.2, we can calculate $E^{(3)}_{C,k}(A, V_B)$ by solving the linear programming problems $\mathrm{LP}_3(v)$ for all $v \in V_B$. $E^{(3)}_{C,k}(A, B)$ comes equal to the maximum of their optimal values.

∎

We can get the value $E^{(5)}_{C,k}(A, B)$ in a similar way.

4.3 Example

As the last section of this chapter, we show a calculation example to demonstrate how it goes with our methods.

To begin with, let

$$P_A := \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 1 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix}, q_A := \begin{pmatrix} 2 \\ 2 \\ -3 \\ 4 \\ 1 \\ 1 \end{pmatrix}, P_B := \begin{pmatrix} 1 & 1 & -3 \\ -1 & -1 & 2 \\ -1 & -1 & -1 \\ 2 & -3 & 0 \\ -3 & 2 & 0 \end{pmatrix}, q_B := \begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 1 \end{pmatrix},$$

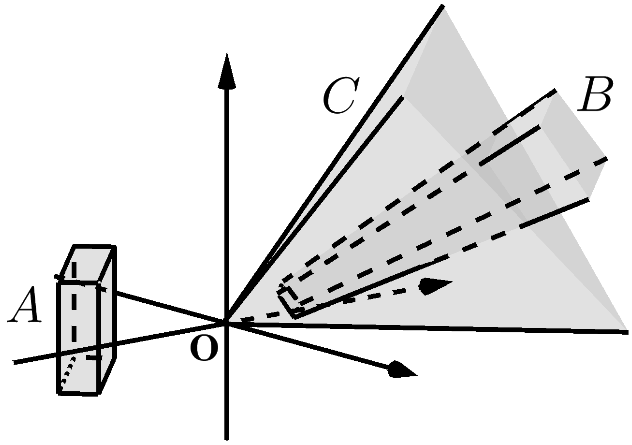

Figure 4.1: Illustration of the sets A, B and cone C.

$$p_1 := \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, p_2 := \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}, p_3 := \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}, k := \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

and define $A := \{x \in \mathbb{R}^3 \mid P_A x \leq q_A\}$, $B := \{x \in \mathbb{R}^3 \mid P_B x \leq q_B\}$, $C := \{x \in \mathbb{R}^3 \mid \langle p_l, x\rangle \leq 0 \ (l = 1, 2, 3)\}$. Note that B is not a polytope as opposed to A because $\{x \in \mathbb{R}^3 \mid x_1 = x_2 = x_3 \geq 1\} \subset B$, that is, B is not compact.

The value $E^{(1)}_{C,k}(A, B)$ is given by solving LP$(1.l)$ with variables $(x^{\mathrm{T}}, y^{\mathrm{T}}) \in \mathbb{R}^3 \times \mathbb{R}^3$ for $l = 1, 2, 3$.

$$\text{LP}(1.l) \quad \boxed{\text{Maximize } \left\langle \frac{p_l}{\langle p_l, k\rangle}, x - y \right\rangle \text{ subject to } \begin{pmatrix} P_A & \mathbf{0} \\ \mathbf{0} & P_B \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \leq \begin{pmatrix} q_A \\ q_B \end{pmatrix}.}$$

The numerical result is indicated in Table 4.1. Here, the symbol Val$(\cdot)$ stands for the optimal value of each specified problem. We derive $E^{(1)}_{C,k}(A, B) = \max\{7/2, -8/3\} = 7/2 > 0$ and hence $A \not\leq^{(1)}_C B$ because assuming $A \leq^{(1)}_C B$ implies $E^{(1)}_{C,k}(A, B) \leq 0$, a contradiction.

In order to obtain $E^{(2)}_{C,k}(A, B)$, we have to solve the two kinds of linear programming

Table 4.1: The optimal solutions and optimal values of LP$(1.l)$.

Type (1)	LP(1.1)	LP(1.2)	LP(1.3)
Val$(\cdot)$	**7/2**	**7/2**	**−8/3**
x_1	−5/2	−1/2	−5/2
x_2	−1/2	−5/2	−1/2
x_3	1	1	−1
y_1	5/4	1/2	1
y_2	1/2	5/4	1/3
y_3	1/4	1/4	2/3

problems below.

$$\text{LP}(2.l) \quad \boxed{\text{Maximize } \left\langle \frac{p_l}{\langle p_l, k\rangle}, -x \right\rangle \text{ subject to } P_B x \leq q_B.}$$

$$\text{LP}(2) \quad \boxed{\begin{array}{ll} \text{Minimize} & t \in \mathbb{R} \\ \text{subject to} & t \geq \left\langle \frac{p_l}{\langle p_l, k\rangle}, x \right\rangle + \text{Val}(\text{LP}(2.l)) \text{ for } l = 1,\ 2,\ 3, \\ & P_A x \leq q_A. \end{array}}$$

Table 4.2: The optimal solutions and optimal values of LP$(2.l)$ and LP(2).

Type (2)	LP(2.1)	LP(2.2)	LP(2.3)	LP(2)
Val$(\cdot)$	1/2	1/2	−2/3	**−1/2**
x_1	5/4	1/2	1	−3/2
x_2	1/2	5/4	1/3	−3/2
x_3	1/4	1/4	2/3	−1
t	-	-	-	−1/2

Table 4.2 shows $E^{(2)}_{C,k}(A,B) = -1/2 < 0$. From this outcome, we deduce $A \le^{(2)}_C B$ owing to the following property of the set relations: For each $i = 1, \dots, 6$, $A \le^{(i)}_C (B + tk)$ for some $t \in \mathbb{R}$ implies $A \le^{(i)}_C (B + t'k)$ for all $t' \in (t, +\infty)$.

Finally, we consider type (3) by following the calculation algorithm. Let

$$\begin{aligned}
&a_1 := (-1/2,\ -5/2,\ 1)^{\mathrm{T}}, && a_2 := (-5/2,\ -1/2,\ 1)^{\mathrm{T}}, && a_3 := (-1,\ -3,\ 1)^{\mathrm{T}},\\
&a_4 := (-3,\ -1,\ 1)^{\mathrm{T}}, && a_5 := (-1/2,\ -5/2,\ -1)^{\mathrm{T}}, && a_6 := (-5/2,\ -1/2,\ -1)^{\mathrm{T}},\\
&a_7 := (-1,\ -3,\ -1)^{\mathrm{T}}, && a_8 := (-3,\ -1,\ -1)^{\mathrm{T}}, && \\
&b_1 := (1,\ 1/3,\ 2/3)^{\mathrm{T}}, && b_2 := (1/3,\ 1,\ 2/3)^{\mathrm{T}}, && b_3 := (5/4,\ 1/2,\ 1/4)^{\mathrm{T}},\\
&b_4 := (1/2,\ 5/4,\ 1/4)^{\mathrm{T}}, && b_5 := (6,\ 9,\ 5)^{\mathrm{T}}, && b_6 := (9,\ 6,\ 5)^{\mathrm{T}},\\
&b_7 := (6,\ 4,\ 5)^{\mathrm{T}}, && b_8 := (4,\ 6,\ 5)^{\mathrm{T}}, && \\
&c_1 := (1,\ 1,\ 0)^{\mathrm{T}}, && c_2 := (1,\ 0,\ 1)^{\mathrm{T}}, && c_3 := (0,\ 1,\ 1)^{\mathrm{T}}.
\end{aligned}$$

Step 1: By using Propositions 2.4 and 2.5, we have $A = \mathrm{co}\{a_1, \dots, a_8\} + \mathrm{cone}\{\mathbf{0}\}$, $B = \mathrm{co}\{b_1, \dots, b_4\} + \mathrm{cone}\{b_5, \dots, b_8\}$ and $C = \mathrm{cone}\{c_1, \dots, c_3\}$.

Step 2: It is clear that $\mathrm{cone}\{b_5, \dots, b_8\} \subset \mathrm{cone}\{\mathbf{0}\} + \mathrm{cone}\{c_1, \dots, c_3\}$.

Step 3: For $j = 1, \dots, 4$, consider $\inf_{x \in A} \max_{l=1,2,3} \left\langle \frac{p_l}{\langle p_l, k\rangle}, x - b_j \right\rangle$, that is,

$$\mathrm{LP}(3.j)\quad \boxed{\begin{array}{ll} \text{Minimize} & t \in \mathbb{R} \\ \text{subject to} & t \ge \left\langle \frac{p_l}{\langle p_l, k\rangle}, x - b_j \right\rangle \ \text{for } l = 1,2,3, \\ & P_A x \le q_A. \end{array}}$$

According to Table 4.3, $E^{(3)}_{C,k}(A,B) = \max\{-5/3, -5/4\} = -5/4 < 0$. Also, we conclude $A \le^{(3)}_C B$ in analogy with the result of type (2).

Table 4.3: The optimal solutions and optimal values of LP($3.j$).

Type(3)	LP(3.1)	LP(3.2)	LP(3.3)	LP(3.4)
Val($\cdot$)	$\mathbf{-5/3}$	$\mathbf{-5/3}$	$\mathbf{-5/4}$	$\mathbf{-5/4}$
x_1	$-7/6$	$-11/6$	$-9/8$	$-15/8$
x_2	$-11/6$	$-7/6$	$-15/8$	$-9/8$
x_3	-1	-1	-1	-1
t	$-5/3$	$-5/3$	$-5/4$	$-5/4$

Chapter 5

Conclusion

In this thesis, we have given a new approach to getting values of the scalarization functions for set relations with finitely many linear programming problems. As shown in Section 4.3 through an example, one can calculate the values of the functions by following the algorithms stated in Theorems 4.2 and 4.3. In addition to the six types of scalarization functions (called "inf types"), we remark that other types (called "sup types") given in [17] can be applied to our reformation (by setting $C := -C,\ k := -k$). This implies that our result is a generalization of the computational methods proposed in [19].

Acknowledgments

I would like to express my deepest gratitude to Professor Tamaki Tanaka of Niigata University for his valuable advices and helps for improving this thesis.

Also, I am grateful to Professors Syuuji Yamada and Junichi Hirukawa of Niigata University for their comments and supports for me.

Moreover, I am grateful to Professors Kenji Kimura and Yutaka Saito for their useful suggestions and encouragement on my study.

Furthermore, I would like to thank the members of the laboratory of Tamaki Tanaka. Especially, I am very grateful to my colleagues Yuto Ogata and Koichiro Ike for their helpful comments and advices. Finally, I am thankful to my family for their encouragement and supports.

Bibliography

[1] Y. Araya, *Four types of nonlinear scalarizations and some applications in set optimization*, Nonlinear Anal. **75**, 2012, 3821–3835.

[2] G.-y. Chen, X. Huang, X. Yang, *Vector optimization: Set-valued and variational analysis*, Lecture Notes in Economics and Mathematical Systems, 541, Springer, Berlin, 2005.

[3] G. B. Dantzig, B. C. Eaves, *Fourier–Motzkin elimination and its dual*, J. Combinatorial Theory Ser. A **14**, 1973, 288–297.

[4] M. Dhingra, C. S. Lalitha, *Well-setness and scalarization in set optimization*, Optim. Lett. **10**, 2015, 1657–1667.

[5] P. Gr. Georgiev, T. Tanaka, *Vector-valued set-valued variants of Ky Fan's inequality*, J. Nonlinear Convex Anal. **1**, 2000, 245–254.

[6] C. Gerstewitz (C. Tammer), *Nichtkonvexe dualität in der vektoroptimierung*, Wiss. Z. Tech. Hochsch. Leuna-Merseburg **25**, 1983, 357–364.

[7] A. Göpfert, H. Riahi, C. Tammer, C. Zălinescu, *Variational Methods in Partially Ordered Spaces*, Springer, New York, 2003.

[8] C. Gutiérrez, B. Jiménez, E. Miglierina, E. Molho, *Scalarization in set optimization with solid and nonsolid ordering cones*, J. Global Optim. **61**, 2015, 525–552.

[9] C. Gutiérrez, E. Miglierina, E. Molho, V. Novo, *Pointwise well-posedness in set optimization with cone proper sets*, Nonlinear Anal. **75**, 2012, 1822–1833.

[10] A. H. Hamel, A. Löhne, *Minimal element theorems and Ekeland's principle with set relations*, J. Nonlinear Convex Anal. **7**, 2006, 19–37.

[11] E. Hernández, L. Rodríguez-Marín, *Nonconvex scalarization in set optimization with set-valued maps*, J. Math. Anal. Appl. **325**, 2007, 1–18.

[12] K. Ike, T. Tanaka, *Convex-cone-based comparisons of and difference evaluations for fuzzy sets*, Optimization **67**, 2018, 1051–1066.

[13] J. Jahn, T. X. D. Ha, *New order relations in set optimization*, J. Optim. Theory Appl. **148**, 2011, 209–236.

[14] S. Khoshkhabar-amiranloo, E. Khorram, M. Soleimani-damaneh, *Nonlinear scalarizaion functions and polar cone in set optimization*, Optim. Lett. **11**, 2017, 521–535.

[15] E. Köbis, M. A. Köbis, *Treatment of set order relations by means of a nonlinear scalarization functional: a full characterization*, Optimization **65**, 2016, 1805–1827.

[16] D. Kuroiwa, T. Tanaka, T. X. D. Ha, *On cone convexity of set-valued maps*, Nonlinear Anal. **30**, 1997, 1487–1496.

[17] I. Kuwano, T. Tanaka, S. Yamada, *Unified scalarization for sets and set-valued Ky Fan minimax inequality*, J. Nonlinear Convex Anal. **11**, 2010, 513–525.

[18] Y. Ogata, T. Tanaka, Y. Saito, G. M. Lee, J. H. Lee, *An alternative theorem for set-valued maps via set relations and its application to robustness of feasible sets*, Optimization **67**, 2018, 1067–1075.

[19] Y. Sonda, T. Tanaka, S. Yamada, *Computational methods on scalarizing functions for sets in a vector space*, Session ID: B3–5 in "Proceedings of the Japan Joint

Automatic Control Conference”, The Japan Joint Automatic Control Conference, Kyoto (Japan), 2010, DOI: 10.11511/jacc.52.0.137.0.

[20] Y. D. Xu, S. J. Li, *A new nonlinear scalarization function and applications*, Optimization **65**, 2016, 207–231.

[21] W. Y. Zhang, S. J. Li, K. L. Teo, *Well-posedness for set optimization problems*, Nonlinear Anal. **71**, 2009, 3769–3778.

[22] Z. A. Zhou, J. W. Peng, *Scalarization of set-valued optimization problems with generalized cone subconvexlikeness in real ordered linear spaces*, J. Optim. Theory Appl. **154**, 2012, 830–841.

[23] G. M. Ziegler, *Lectures on Polytopes*, Springer, New York, 1995.