Computational methods on scalarizing functions for sets in a vector space*

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Abstract In this study, we propose computational method on several scalarizing functions for sets in a vector spaces. Classically, we use some types of scalarization methods for multiobjective programming problems. For two decades, some papers devoted in the fields of vector optimization and set-valued optimization were appeared with interesting results using nonlinear scalarizing functions with respect to some partial ordering introduced by a convex cone in a vector space. The aim of this paper is to introduce several useful scalarizing functions for sets and propose certain computational methods for them by global optimization technique.

1 Introduction

In recent decades, several scalarization methods for vectors and sets in an ordered vector space are studied and utilized as one of important tools in multiobjective programming, vector optimization, and set-valued optimization. To evaluate vectors or calculate efficient solutions of a given set, we usually use several scalarization techniques for multiobjective programming problems; see [5, 9]. When we want to compare and evaluate two vectors in a vector space like a Euclidean space, we use the average value of the components of each vector, which is a special case of linear weighted sum for components, and the distance or norm of each vector from a certain reference point like the origin of the space or an aspiration level. They are referred to as a weighted sum approach and a weighted Chebyshev norm approach, respectively. However, both approaches are interpreted in a unified framework based on the idea of Minkowski functional. Recently, we find some interesting applications on generalizations of scalar problems like equilibrium problems by dealing with Gerstewitz's (Tammer's) scalarizing function for vectors and sets, which is a mathematical tool generalizing its approach; see [4].

The aim of this paper is to introduce some scalarization methods for sets in an ordered vector space and to show certain algorithms to scalarize sets in a Euclidean space by computational procedures. In this purpose, we define four types of scalarizing functions for sets by using the Gerstewitz's scalarizing function, and we show that each value of the four functions can be computed practically. Moreover, we construct a successive approximation algorithm for solving multicriteria optimization problems with a d.c. set (the difference of two convex sets).

The organization of the paper is as follows. In Section 2, we introduce Gerstewitz's (Tammer's) scalarizing function and four types of nonlinear scalarizing functions for sets in a vector space. In Section 3, we observe two types of computation algorithms in a Euclidean space for a simple case of polytopes with a non-negative orthant and a more general case of d.c. sets with a polyhedron (a finite intersection of closed half spaces). For the first case, we show that the four functions for a given polytope can be calculated with finite steps by minmax or maxmin type with respect to ratios on coordinates of each vertex of the polytope and coordinates of a direction vector in the non-negative orthant. For the second case, we propose a successive approximation algorithm to calculate the four functions for a given d.c. set by using a global optimization technique for d.c. programming problems.

*This work is based on research 21540121 supported by Grants-in-Aid for Scientific Research from the Japan Society for the Promotion of Science of Japan.

2 Mathematical Preliminaries

Throughout the paper, let Y be a real ordered topological vector space with the ordering $\leq_C$ induced by a nonempty convex cone C ($C + C = C$ and $\lambda C \subset C$ for all $\lambda \geq 0$) as follows:

$$x \leq_C y \text{ if } y - x \in C \text{ for } x, y \in Y.$$

It is well known that $\leq_C$ is reflexive and transitive where C is a convex cone, moreover, $\leq_C$ has invariant properties to vector space structure as translation and scalar multiplication. In particular, if C is pointed, then $\leq_C$ is antisymmetric, and hence Y is a partially ordered topological vector space. For any $A \subset Y$ we denote the interior, closure, complement, convex hull of A by $\operatorname{int} A$, $\operatorname{cl} A$, A^c, $\operatorname{co} A$, respectively.

We define the following function, called Gerstewitz's (Tammer's) scalarizing function of a vector $y \in Y$:

$$h_C(y; k) := \inf\{t \mid y \in tk - C\}$$

where $k \in \operatorname{int} C$; this function is essentially equivalent to the smallest strictly monotonic function defined by Luc [8]. For each $y \in Y$, $h_C(y; k) \cdot k$ corresponds the mimimum vector of upper bounds of y with respect to C restricted to direction k. Similarly, $-h_C(-y; k) \cdot k$ corresponds the maximum vector of lower bounds of y with respect to C restricted to direction k.

The idea of the function was dealt by Krasnosel'skij [6] in 1962 and by Rubinov [11] in 1977, and then it was applied to vector optimization with its concrete definition by Gerstewitz (Tammer) [2] in 1983, and to separation theorems for not necessary convex sets by Gerstewitz and Iwanow [3] in 1985. If C is a closed convex cone, $y \mapsto h_C(y;k)$ is a sublinear continuous function and the following relationship between level sets of the function and translations of convex cones:

$$\{z \in Y \mid h_C(z;k) \leq \lambda\} = \lambda k - C,$$

$$\{z \in Y \mid h_C(z;k) < \lambda\} = \lambda k - \operatorname{int} C.$$

For more detail, see [4].

Now, we consider a scalarization of subset $A \subset Y$ with respect to convex cone C and direction vector $k \in \operatorname{int} C$. By use of the Gerstewitz's scalarizing function, we define

$$\varphi_C^k(A) := \inf_{y \in A} h_C(y;k) \tag{1}$$

$$\psi_C^k(A) := \sup_{y \in A} h_C(y;k). \tag{2}$$

By $-h_C(-y;k) = \sup\{t \mid y \in tk + C\}$, we define another ones:

$$-\psi_C^k(-A) = \inf_{y \in A} -h_C(-y;k) \tag{3}$$

$$-\varphi_C^k(-A) = \sup_{y \in A} -h_C(-y;k) \tag{4}$$

where $-A = \{-a \in Y | a \in A\}$.

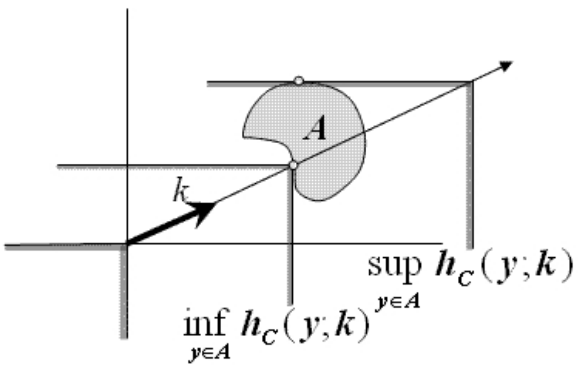

Figure 1: Scalarizations $\boldsymbol{\varphi_C^k(A)}$ and $\boldsymbol{\psi_C^k(A)}$

The first and fourth functions in (1) and (4) and the second and third ones in (2) and (3) have symmetric properties, respectively. These four scalarizing functions for set $A \subset Y$ can be regarded as an evaluation approach with 4-tuple of Chebyshev type scalarizations, as illustrated in Figures 1 and 2. These functions have been introduced by Georgiev and Tanaka [1] in 2000 for generalizing the classical Fan's inequality. Then they have been studied by Nishizawa, Tanaka and Georgiev [10] in 2003 from the viewpoint of cone convexity and cone semicontinuity as inherited properties for set-valued maps, and also applied into characterizations of optimality conditions for efficient solutions of set-valued optimization problems by Shimizu, Nishizawa and Tanaka [12] in 2007. Thus, we have the question whether they can be computed practically or not.

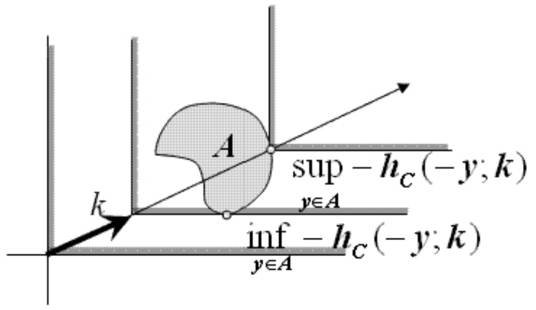

Figure 2: Scalarizations $\boldsymbol{-\psi_C^k(-A)}$ and $\boldsymbol{-\varphi_C^k(-A)}$

3 Computation Algorithm

At first, we consider a scalarization of a polytope when $Y = R^n$ and $C = R^n_+$.

Lemma 1 Let $k \in \operatorname{int} R^n_+$. For $z = (z_1, \ldots, z_n)^{\mathrm{T}} \in R^n$, we have

$$h_{R^n_+}(z;k) = \max\left\{\frac{z_1}{k_1}, \ldots, \frac{z_n}{k_n}\right\} \tag{5}$$

and

$$-h_{R^n_+}(-z;k) = \min\left\{\frac{z_1}{k_1}, \ldots, \frac{z_n}{k_n}\right\}. \tag{6}$$

Proof For $h_{R^n_+}(z;k)$ and $-h_{R^n_+}(-z;k)$, we consider the following two scalar optimization problems.

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & z \in tk - R^n_+, \end{array}$$

$$\begin{array}{ll} \text{maximize} & t \\ \text{subject to} & z \in tk + R^n_+. \end{array}$$

Since $k_i > 0 (i = 1, \ldots, n)$ and constraints $z \in tk - R^n_+$ and $z \in tk + R^n_+$ are equivalent to $t \geq z_i/k_i$ and $t \leq z_i/k_i$ for all $i = 1, \ldots, n$, respectively, we get (5) and (6). □

Theorem 1 Let $k \in \operatorname{int} R^n_+$. For nonempty polytope $A = \operatorname{co}\{a^{(1)}, \ldots, a^{(m)}\}$, where $a^{(1)}, \ldots, a^{(m)} \in R^n$, we get

$$\varphi^k_{R^n_+}(A) \leq \min_j \max_i \frac{a_i^{(j)}}{k_i}, \tag{7}$$

$$-\varphi^k_{R^n_+}(-A) \geq \max_j \min_i \frac{a_i^{(j)}}{k_i}. \tag{8}$$

Moreover we calculate scalarizing functions (2) and (3):

$$\psi^k_{R^n_+}(A) = \max_j \max_i \frac{a_i^{(j)}}{k_i}, \tag{9}$$

$$-\psi^k_{R^n_+}(-A) = \min_j \min_i \frac{a_i^{(j)}}{k_i}. \tag{10}$$

Proof By Lemma 1, (7) and (8) are obvious. For $z = (z_1, \ldots, z_n)^{\mathrm{T}} \in A$, there are some nonnegative coefficients λ_j $(j = 1, \ldots, m)$ such that $z = \sum_{j=1}^m \lambda_j a^{(j)}$ and $\sum_{j=1}^m \lambda_j = 1$. Hence, by the sublinearity of $h_{R^n_+}(\cdot; k)$, we get

$$\begin{aligned} h_{R^n_+}(z;k) &= h_{R^n_+}\left(\sum_{j=1}^m \lambda_j a^{(j)}; k\right) \\ &\leq \sum_{j=1}^m \lambda_j h_{R^n_+}(a^{(j)};k) \\ &\leq \max_{j=1,\ldots,p} h_{R^n_+}(a^{(j)};k) \\ &\leq \sup_{y\in A} h_{R^n_+}(y;k) = \psi^k_{R^n_+}(A). \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} -h_{R^n_+}(-z;k) &= -h_{R^n_+}\left(-\sum_{j=1}^m \lambda_j a^{(j)}; k\right) \\ &\geq \sum_{j=1}^m \lambda_j \left(-h_{R^n_+}(-a^{(j)};k)\right) \\ &\geq \min_{j=1,\ldots,p} \left(-h_{R^n_+}(-a^{(j)};k)\right) \\ &\geq \inf_{y\in A} \left(-h_{R^n_+}(-y;k)\right) = -\psi^k_{R^n_+}(-A) \end{aligned}$$

Therefore, (9) and (10) hold. □

In order to calculate the values of scalarizing functions (1) and (4), we consider the following linear programming problems.

$$\begin{array}{lll} (\mathrm{LP_{min}}) & \inf & t \\ (\text{resp. } (\mathrm{LP_{max}})\ \sup) & & \\ \text{subject to} & & tk = \sum_{j=1}^m \lambda_j \boldsymbol{a}^{(j)} \\ & & \sum_{j=1}^m \lambda_j = 1 \\ & & \lambda_i \geq 0 \quad (i = 1, \ldots, m) \end{array}$$

Finite optimal values of problems ($\mathrm{LP_{min}}$) and ($\mathrm{LP_{max}}$) coincide with the values of scalarizing functions (1) and (4), respectively. If both problems are infeasible, then each equality in (7) and (8) holds.

Next, we consider a more general case of d.c. set,

$$A = G_1 \setminus G_2$$

where G_1 is a compact convex set and G_2 is an open convex set, with a polyhedron

$$C = \left\{ z \;\middle|\; \left\langle c^{(i)}, z \right\rangle \geq 0, i = 1, \ldots, p \right\},$$

and we give a certain successive approximation algorithm for the values of scalarizing functions (1)–(4). Since

$$\begin{aligned} C &= \bigcap_{i=1}^p \left\{ z \in Y \;\middle|\; \left\langle c^{(i)}, z \right\rangle \geq 0 \right\} \\ &= \left\{ z \in Y \;\middle|\; \min_{i=1,\ldots,p} \left\langle c^{(i)}, z \right\rangle \geq 0 \right\} \end{aligned} \tag{11}$$

and $k \in \operatorname{int} C$, $\left\langle c^{(i)}, k \right\rangle > 0$ $(i = 1, \ldots, p)$. Let

$$c^{(i)}(k) := \frac{1}{\left\langle c^{(i)}, k \right\rangle} c^{(i)} \quad (i = 1, \ldots, p),$$

and then

$$\begin{aligned} h_C(z;k) &= \max_{i=1,\ldots,p} \left\langle c^{(i)}(k), z \right\rangle \\ -h_C(-z;k) &= \min_{i=1,\ldots,p} \left\langle c^{(i)}(k), z \right\rangle, \end{aligned}$$

which are convex and concave functions with respect to z, respectively. We denote

$$H^1_C(y;k) := \max_{i=1,\ldots,p} \left\langle c^{(i)}(k), y \right\rangle$$

and

$$H^2_C(y;k) := \min_{i=1,\ldots,p} \left\langle c^{(i)}(k), y \right\rangle.$$

Then, the values of (1)–(4) are

$$\varphi^k_C(A) = \inf_{y\in A} \left(H^1_C(y;k)\right), \tag{12}$$

$$\psi^k_C(A) = \sup_{y\in A} \left(H^1_C(y;k)\right), \tag{13}$$

$$-\psi^k_C(-A) = \inf_{y\in A} \left(H^2_C(y;k)\right), \tag{14}$$

$$-\varphi^k_C(-A) = \sup_{y\in A} \left(H^2_C(y;k)\right). \tag{15}$$

When we consider the following d.c. set

$$A = \{z \in Y | g_1(z) \leq 0\} \setminus \{z \in Y | g_2(z) < 0\}$$

where $g_1, g_2 : Y \to R$ are continuous convex functions, the d.c. programming problems (12)–(15) above are equivalent to the followings, respectively.

$$\left\{ \begin{array}{ll} \text{minimize} & H^1_C(y;k) \\ \text{subject to} & g_1(y) \leq 0,\ g_2(y) \geq 0, \end{array} \right. \tag{16}$$

$$\left\{\begin{array}{ll}\text{maximize} & H_C^1(y;k)\\ \text{subject to} & g_1(y)\le 0,\ g_2(y)\ge 0,\end{array}\right. \tag{17}$$

$$\left\{\begin{array}{ll}\text{minimize} & H_C^2(y;k)\\ \text{subject to} & g_1(y)\le 0,\ g_2(y)\ge 0.\end{array}\right. \tag{18}$$

$$\left\{\begin{array}{ll}\text{maximize} & H_C^2(y;k)\\ \text{subject to} & g_1(y)\le 0,\ g_2(y)\ge 0,\end{array}\right. \tag{19}$$

Then problems (16), (17), (18), and (19) can be reformulated as follows.

$$\left\{\begin{array}{ll}\text{maximize} & \alpha,\\ \text{subject to} & g_1(y)\le 0,\\ & \min\left\{g_2(y), H_C^1(y;k)-\alpha\right\}\ge 0\ ,\end{array}\right. \tag{20}$$

$$\left\{\begin{array}{ll}\text{minimize} & \alpha,\\ \text{subject to} & g_2(y)\ge 0,\\ & \max\left\{g_1(y), H_C^1(y;k)-\alpha\right\}\le 0\ ,\end{array}\right. \tag{21}$$

$$\left\{\begin{array}{ll}\text{maximize} & \alpha,\\ \text{subject to} & g_2(y)\ge 0,\\ & \max\left\{g_1(y), -H_C^2(y;k)+\alpha\right\}\le 0\ ,\end{array}\right. \tag{22}$$

$$\left\{\begin{array}{ll}\text{minimize} & \alpha,\\ \text{subject to} & g_1(y)\le 0,\\ & \min\left\{g_2(y), -H_C^2(y;k)+\alpha\right\}\ge 0\ .\end{array}\right. \tag{23}$$

Problems (21) and (22) are canonical d.c. programming problems, and they can be solved by some global optimization technique. However, problems (20) and (23) have d.c. constraint functions. Hence, in order to transform problems (20) and (23) into canonical d.c. programming problems, by using the basic property of d.c. functions, we replace d.c. constraints in problems (20) and (23) in the following manner:

$$\begin{aligned}&\left\{y\,\middle|\min\left\{g_2(y), H_C^1(y;k)-\alpha\right\}\ge 0\right\}\\ &=\Big\{y\,\Big|\,g_2(y)+H_C^1(y)-\alpha\\ &\qquad -\max\left\{g_2(y), H_C^1(y;k)-\alpha\right\}\ge 0\Big\}\\ &=\Big\{y\,\Big|\,g_2(y)+H_C^1(y)-\alpha\\ &\qquad \ge\max\left\{g_2(y), H_C^1(y;k)-\alpha\right\}\Big\}\\ &=\Big\{y\,\Big|\,g_2(y)+H_C^1(y)-\alpha\ge\beta\\ &\qquad \ge\max\left\{g_2(y), H_C^1(y;k)-\alpha\right\}\Big\}\\ &=\Big\{y\,\Big|\max\left\{g_2(y), H_C^1(y;k)-\alpha\right\}\le\beta,\\ &\qquad g_2(y)+H_C^1(y;k)-\alpha\ge\beta\Big\}\end{aligned}$$

and

$$\begin{aligned}&\left\{y\,\middle|\min\left\{g_2(y), -H_C^2(y;k)+\alpha\right\}\ge 0\right\}\\ &=\Big\{y\,\Big|\,g_2(y)-H_C^2(y;k)+\alpha\\ &\qquad -\max\left\{g_2(y), -H_C^2(y;k)+\alpha\right\}\ge 0\Big\}\\ &=\Big\{y\,\Big|\,g_2(y)-H_C^2(y;k)+\alpha\\ &\qquad \ge\max\left\{g_2(y), -H_C^2(y;k)+\alpha\right\}\Big\}\\ &=\Big\{y\,\Big|\,g_2(y)-H_C^2(y;k)+\alpha\ge\beta\\ &\qquad \ge\max\left\{g_2(y), -H_C^2(y;k)+\alpha\right\}\Big\}\\ &=\Big\{y\,\Big|\max\left\{g_2(y), -H_C^2(y;k)+\alpha\right\}\le\beta,\\ &\qquad g_2(y)-H_C^2(y;k)+\alpha\ge\beta\Big\}.\end{aligned}$$

References

[1] P. Gr. Georgiev and T. Tanaka, Vector-valued set-valued variants of Ky Fan's inequality, Journal of Nonlinear and Convex Analysis, **1(3)**, pp.245–254, 2000.

[2] Chr. Gerstewitz (Tammer), Nichtkonvexe dualität in der vektoroptimierung, Wiss. Zeitschr. TH Leuna-Merseburg, **25**, pp. 357–364 (1983)

[3] Chr. Gerstewitz (Tammer) and E. Iwanow, Dualität für nichtkonvexe vektoroptimierungs probleme, Wiss. Z. Tech. Hochsch Ilmenau,**2**, pp. 61–81 (1985)

[4] A. Göpfert, H. Riahi, C. Tammer, and C. Zălinescu, Variational methods in partially ordered spaces, Springer-Verlag, New York (2003)

[5] J. Jahn, Vector optimization. Theory, applications, and extensions, Springer-Verlag, Berlin (2004)

[6] M. A. Krasnosel'skij, Positive solutions of operator equations, (Russian) Fizmatgiz, Moskow (1962)

[7] D. Kuroiwa, T. Tanaka, and T. X. D. Ha, On cone convexity of set-valued maps, Nonlinear Analysis: Theory, methods and Applications, **30**, pp. 1487–1496 (1997)

[8] D. T. Luc, Theory of vector optimization, Lecture Notes in Economics and Mathematical Systems, Springer-Verlag, Berlin (1989)

[9] 中山弘隆, 谷野哲三: 多目的計画法の理論と応用, 計測自動制御学会 (1994)

[10] S. Nishizawa, T. Tanaka, and P. Gr. Georgiev, On inherited properties of set-valued maps, Proceedings of the second International Conference on Nonlinear Analysis and Convex Analysis, edited by W. Takahashi and T. Tanaka, Yokohama Publishers, Yokohama (Japan), pp. 341–350 (2003)

[11] A. M. Rubinov, Sublinear operators and their applications. (Russian) Russian Math. Surveys 32, No. 4, pp. 115–175, (1977)

[12] A. Shimizu, S. Nishizawa and T. Tanaka, Optimality conditions in set-valued optimization using nonlinear scalarization methods, Proceedings of the Fourth International Conference on Nonlinear Analysis and Convex Analysis, edited by W. Takahashi and T. Tanaka, Yokohama Publishers, Yokohama (Japan), pp. 565–574 (2007)