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Fisher-Cramér-Rao Lower Bound and MUSIC Standard Deviation Formulation for ESPAR Antennas

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1. Introduction

The problem of finding the directions of arrival (DoAs) of several waves by using an electronically steerable parasitic array radiator [1] (ESPAR) antenna has gained considerable interest because of the attractive low cost and low power consumption features of the antenna. Recently, the reactance domain (RD) MUSIC algorithm has been proposed for the ESPAR antenna [2]. So far, the theoretical performance expected for the ESPAR antenna, thus also for the RD-MUSIC estimator, with regard to the DoA estimation problem has not been examined. This paper proposes a Fisher-Cramér-Rao (FCR) lower bound on the variance of the estimation error. First, the formulation of the FCR lower bound and the RD-MUSIC estimation error variance (both derived from [3]) are presented. Then, using these formulations in a computer simulation, the efficiency of the RD-MUSIC estimator in the DoA resolution of two sources is studied.

2. ESPAR antenna

2.1. Antenna signal model

The $M+1$-element ESPAR antenna has a central element surrounded by M parasitic elements uniformly distributed in a circle with radius $R = \lambda/4$. Thus, the array steering vector is expressed by:

$$\mathbf{a}(\theta) = \left[1, e^{j\frac{\pi}{2}\cos(\theta-\phi_1)}, \ldots, e^{j\frac{\pi}{2}\cos(\theta-\phi_M)}\right]^T, \text{ with } \phi_m = \frac{2\pi}{M}(m-1),\ m = 1\ldots M. \quad (1)$$

Let us suppose that Q signals impinge on the $M+1$-element ESPAR antenna at a time t. We denote $\boldsymbol{\theta} = [\theta_1, \ldots, \theta_Q]$ as the impinging signal DoA parameter vector (or parameters), then, $\mathbf{A} = [\mathbf{a}(\theta_1) \ \ldots \ \mathbf{a}(\theta_Q)]$ as the steering matrix (or conventional steering matrix), and, $\mathbf{u}(t) = [u_1(t), \ldots, u_Q(t)]^T$ as the impinging signal vector. Assuming a linear receiving system, the ESPAR antenna output is expressed by:

$$y(t) = \mathbf{w}^T\mathbf{A}\mathbf{u}(t) + n(t), \quad (2)$$

where $n(t)$ is an additive noise. Moreover the vector $\mathbf{w}$ is the RF current weight vector, which is expressed by:

$$\mathbf{w} = 2z_s(\mathbf{Z}+\mathbf{X})^{-1}\mathbf{u}_0. \quad (3)$$

The constant z_s is the receiver input impedance. The matrix $\mathbf{Z}$ is the mutual coupling on the element's impedance matrix. The matrix $X = \mathrm{diag}\,[z_2, jx_1, \ldots, jx_M]$ is the reactance matrix ($j^2 = -1$). In the following a set of reactance $\mathbf{x}_{(n)}$ stands for the reactance values $\{x_1^n, \ldots, x_M^n\}$.

2.2. Reactance domain technique

As shown in (2), the ESPAR antenna output scalar is a combination of the mutual coupling between parasitic elements ($\mathbf{Z}$) and adjustable reactance settings ($\mathbf{X}$) included in the vector $\mathbf{w}$. However, almost all available array signal processing methods require a correlation matrix of the element output.

One method for obtaining a usable correlation matrix is to use the *reactance domain* technique for the ESPAR antenna [2]. This technique consists in choosing N different set of reactances $\{\mathbf{x}_{(1)}, \ldots, \mathbf{x}_{(N)}\}$, then, for each set getting the output $y_n(t_n)$ of the antenna , $n = 1\ldots N$. We assumed that the same signal ($\mathbf{u}(t_1) = \cdots = \mathbf{u}(t_N) = \mathbf{u}$) is sent N consecutive times.

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By denoting $\mathbf{W} = [\mathbf{w}_{(1)}, \ldots, \mathbf{w}_{(N)}]$ the RF weight matrix corresponding to the selected reactance sets, and $\mathbf{n} = [n(t_1), \ldots, n(t_N)]^T$ the additive noise vector, the output vector $\mathbf{y} = [y_1(t_1), \ldots, y_N(t_N)]^T$ is expressed by:

$$\mathbf{y} \equiv \mathbf{W}^T \mathbf{A}\mathbf{u} + \mathbf{n}. \tag{4}$$

Then the ESPAR antenna reactance domain correlation matrix is defined by $\mathbf{R}_{yy} \equiv \mathrm{E}[\mathbf{y}\mathbf{y}^H]$.

3. A Fisher-Cramér-Rao lower bound for ESPAR antennas

The following is based on the assumption that the $\mathbf{W}$ matrix is a known constant. First, let us denote $\mathbf{A}_{mod} \equiv \mathbf{W}^T\mathbf{A} = [\mathbf{a}_{mod}(\theta_1) \; \ldots \; \mathbf{a}_{mod}(\theta_Q)]$ and $\mathbf{D}_{mod} \equiv \mathbf{W}^T\mathbf{D} = \mathbf{W}^T\frac{\partial \mathbf{A}}{\partial \boldsymbol{\theta}} = [\mathbf{d}_{mod}(\theta_1) \; \ldots \; \mathbf{d}_{mod}(\theta_Q)]$. The FCR bound gives the ultimate lower bound of any unbiased estimator of the DoA parameter $\boldsymbol{\theta}$, *i.e.*, $\forall i \in [1, Q]$, $\mathrm{Var}_{\mathrm{CR}}(\hat{\theta}_i) \leq \mathrm{E}\left[(\hat{\theta}_i - \theta_i)^2\right]$. This bound, denoted $\mathrm{Var}_{\mathrm{CR}}$, has been derived from [3] by replacing the conventional steering matrix $\mathbf{A}$ with the reactance domain steering matrix $\mathbf{A}_{mod}$, and is expressed by:

$$\mathrm{Var}_{\mathrm{CR}}(\hat{\theta}_i) = \frac{\sigma^2}{2K}\left[\left\{\mathrm{Re}\left[\left\{\mathbf{D}_{mod}^H \mathbf{P}_{A_{mod}}^{\perp} \mathbf{D}_{mod}\right\} \odot \mathbf{P}^T\right]\right\}^{-1}\right]_{ii}. \tag{5}$$

Where the matrix $\mathbf{P}_{A_{mod}}^{\perp} = \mathbf{I}_N - \mathbf{A}_{mod}\left(\mathbf{A}_{mod}^H\mathbf{A}_{mod}\right)^{-1}\mathbf{A}_{mod}^H$, with $\mathbf{I}_N$ the identity matrix of size $N \times N$. Furthermore, $\odot$ denotes the Hadamard product (element-wise multiplication) and the matrix $\mathbf{P} = \mathrm{E}[\mathbf{u}(t)\mathbf{u}^H(t)]$ is the correlation matrix of the incoming signals. (Note that $\mathbf{P}$ is a diagonal matrix when the signals are uncorrelated.). The constant K is the number of signal samples used for getting one of the N outputs y_n ($n = 1 \ldots N$) of the ESPAR antenna in accordance with the reactance domain technique. The value σ^2 is the noise power and the operator $[\cdot]_{ij}$ denotes the selection of the i, j-th element of the matrix.

4. Reactance domain MUSIC estimation error variance

The RD-MUSIC for the ESPAR antenna [2] provides estimate $\hat{\boldsymbol{\theta}}$ of the DoA parameter $\boldsymbol{\theta}$ by searching for the maxima of the DoA spectrum P_{MU}^{ESPAR} as shown in its normalized version (6). The noise sub-space matrix $\mathbf{E}_n$ is composed of the $N - Q$ eigen vectors corresponding to the $N - Q$ smallest eigen values of the eigen decomposition of the correlation matrix $\mathbf{R}_{yy}$, with $N > Q$.

$$\hat{\boldsymbol{\theta}} = \max_{0 \leq \theta < 2\pi}\left[\mathrm{P}_{MU}^{ESPAR}(\theta)\right] \text{ with } \mathrm{P}_{MU}^{ESPAR}(\theta) = \frac{\mathbf{a}^H(\theta)\mathbf{W}^*\mathbf{W}^T\mathbf{a}(\theta)}{\mathbf{a}^H(\theta)\mathbf{W}^*\mathbf{E}_n\mathbf{E}_n^H\mathbf{W}^T\mathbf{a}(\theta)} \tag{6}$$

However, an asymptotic expression of the RD-MUSIC estimation error variance $\mathrm{E}\left[(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})^2\right]$, which is more suitable for performance analysis, could be derived in the same way as the FCR lower bound [3]. This estimation error variance is expressed by:

$$\mathrm{Var}_{\mathrm{MU}}(\hat{\theta}_i) = \mathrm{E}\left[(\hat{\theta}_i - \theta_i)^2\right] = \frac{\sigma^2}{2K}\frac{\left\{\left[\mathbf{P}^{-1}\right]_{ii} + \sigma^2\left[\mathbf{P}^{-1}(\mathbf{A}_{mod}^H\mathbf{A}_{mod})^{-1}\mathbf{P}^{-1}\right]_{ii}\right\}}{\mathbf{d}_{mod}^H(\theta_i)\mathbf{P}_{A_{mod}}^{\perp}\mathbf{d}_{mod}(\theta_i)} \tag{7}$$

Consequently, the statistical efficiency of the reactance domain MUSIC for ESPAR antenna estimator can be asymptotically assessed by studying the ratio $\mathrm{Std}_{\mathrm{CR}}/\mathrm{Std}_{\mathrm{MU}}$ (with $\mathrm{Std} \equiv \sqrt{\mathrm{Var}}$). The closer to one this ratio is, the more efficient the RD-MUSIC estimator is, *i.e.*, the estimator becomes close to the FCR bound. Notice that the efficiency does not depend on K.

5. Computer simulations

In the following simulation, the RF weight matrix $\mathbf{W}$ is assumed to be perfectly known. Also the antenna is a 7-element ESPAR antenna as shown in Fig. 1. Note that in practical applications, the matrix $\mathbf{W}$ must be obtained by a calibration method.

The simulation purpose is to study the RD-MUSIC estimator efficiency in a scenario with two equipowered impinging signals, where the signal DoA angular separation is Δ_θ. The performance against number of reactance sets N ($\neq$ number of antenna elements), noise level SNR and the two signals correlation coefficient ρ is studied. Notice that only the efficiency of $\hat{\theta}_1$ is of interest.

First, Figs. 2 and 3 show that the performance increases when N increases. Especially, in a strongly correlated case (Fig. 3), for closed sources $\Delta_\theta < 30^o$, increasing N can slightly improve the performance of the RD-MUSIC estimator; for widely separated sources $\Delta_\theta > 30^o$, there is less benefit from increasing N. However there is a limitation on N because the obtained performance benefit is not linear with N, but is bounded. Secondly, the relative fluctuation of the efficiency curves is a specific point of the ESPAR antenna's structure. Indeed, due to the strong mutual coupling between the elements embedded in the matrix $\mathbf{W}$, the array response is no longer uniform and depends on the angle of arrival of the desired signal [4]. Thus, the parameters embedded in the matrix $\mathbf{W}$ (*e.g.*, mutual coupling, $\mathbf{x}_{(1)}, \ldots, \mathbf{x}_{(N)}$, *etc.*) will directly influence the fluctuation of the efficiency. Consequently, the performance strongly depends on the signal DoAs and the parameters included in $\mathbf{W}$. Also, in the particular case of correlated signals, the curves show that $\mathrm{Std}_{\mathrm{CR}}/\mathrm{Std}_{\mathrm{MU}}$ does not converge with Δ_θ. Finally, as expected, the results of Fig. 3 show that performance increases when noise level decreases (*i.e.*, SNR increases).

6. Concluding comments

An RD-FCR bound and an RD-MUSIC estimation error variance formulation have been derived for the ESPAR antenna. The efficiency of the RD-MUSIC estimator has been studied for the resolution of two correlated sources. The results show that the resolution performance increases when SNR decreases, or when N increases. Also, due to the strong mutual coupling between the elements, the performance does not always converge with Δ_θ (*cf.* the case $\rho > 0$). The RD-MUSIC estimator is expected to provide optimal DoA resolution performance for non-correlated sources (*i.e.*, $\rho = 0$) and for a suitable number of reactance sets N and reactance values.

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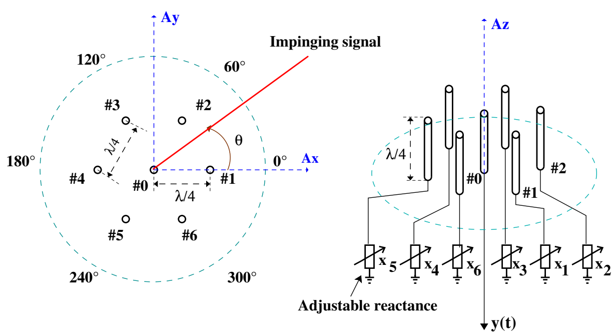
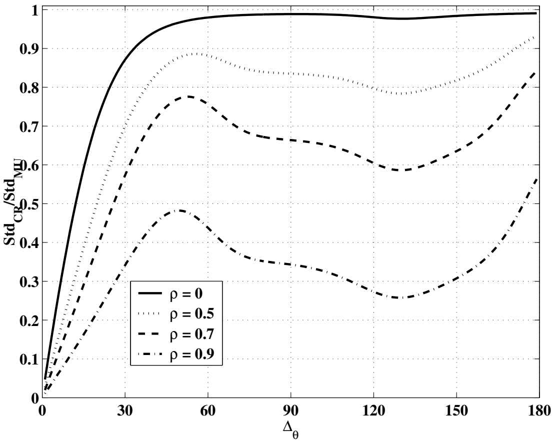
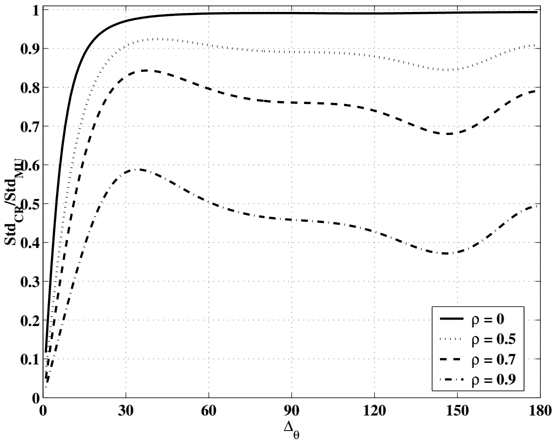


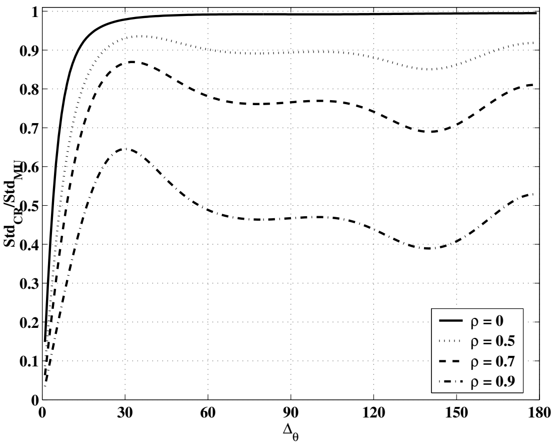
Figure 1: 7-element ESPAR antenna schemes



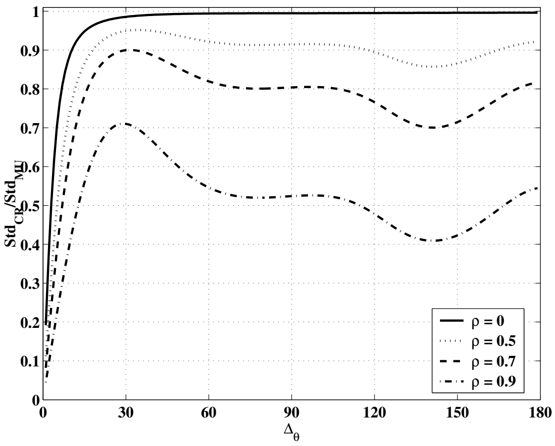
(a) $N = 4$.



(b) $N = 7$.

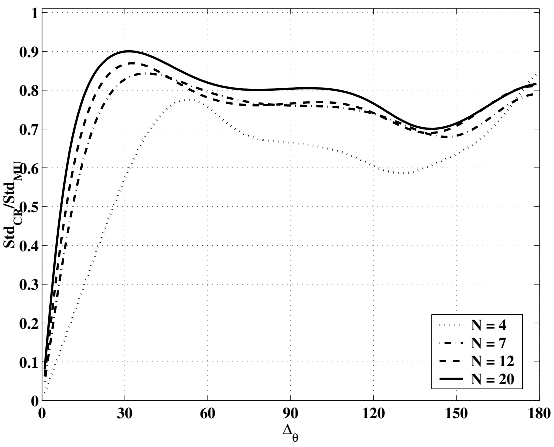


(c) $N = 12$.

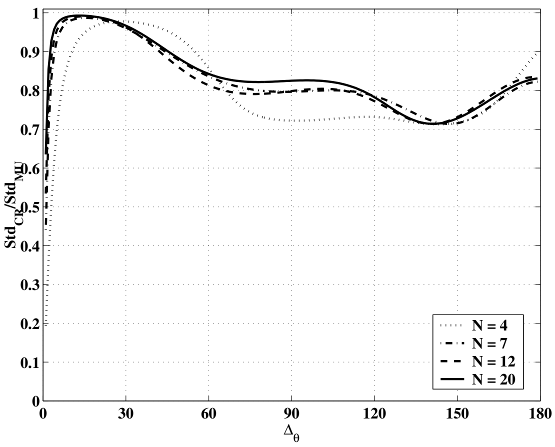


(d) $N = 20$.

Figure 2: Efficiency $\mathrm{Std}_{\mathrm{CR}}/\mathrm{Std}_{\mathrm{MU}}$, for Δ_θ separated two sources and SNR= 0 dB.



(a) SNR= 0 dB.



(b) SNR= 20 dB.

Figure 3: Efficiency $\mathrm{Std}_{\mathrm{CR}}/\mathrm{Std}_{\mathrm{MU}}$, for Δ_θ separated two sources and $\rho = 0.7$.