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On the Unique Existence of Pure-Strategy Nash Equilibrium in Smooth Asymmetric Contests

Takeshi Yamazaki*

Abstract

There can be three types of heterogeneity among players in a rent-seeking contest. First, the effectiveness of agent's effort on the winning probabilities may differ among players. Secondly, players may evaluate the rent or prize of the rent-seeking contest differently. Thirdly, players may face different financial constraints. Without assuming that each player's production function for lotteries is differentiable, Yamazaki (2008) proves under standard assumptions in the literature that there exists a unique pure-strategy Nash equilibrium in a general rent-seeking contest with these three types of heterogeneity among players. In this article, assuming the differentiability of each player's production function for lotteries, we give a simple proof of Yamazaki's (2008) result.

JEL Classification Numbers: C72, D43, L13.
Key Words: Contest, Rent Seeking, Nash Equilibrium, Uniqueness

1 Introduction

A rent-seeking contest is a strategic game in which player i expends effort x_i to increase p_i, player i's probability of winning a prize (rent). In the seminal work of Tullock (1980), player i's contest success function is given by $p_i = x_i \Big/ \left(\sum_{j=1}^{n} x_j \right)$. Tullock's contest has been extended in many directions. Perez-Castrillo and Verdier (1992) explore the implications of a contest success function of the form $p_i = x_i^r \Big/ \left(\sum_{j=1}^{n} x_j^r \right)$, $r > 0$. They prove that if their contest success function satisfies $r \leq 1$, there exists a unique pure-strategy Nash equilibrium. Skaperdas (1996) prove that

* Faculty of Economics, Niigata University, 8050 Ikarashi 2-no-cho, Nishi-ku, Niigata-shi 950-2181, Japan. E-mail: tyamazak@econ.niigata-u.ac.jp
This research was in part supported by JSPS Kakenhi, Grant-in-Aid for Scientific Research (C), Grant Number 19530151 (Rent Seeking and Endogenous Tariff Rate) and 24530194 (Theory and Applications of Dynamic Rent Seeking).

player i's probability of winning a prize satisfies five axioms if and only if player i's contest success function is specified as $p_i = f(x_i) \Big/ \left(\sum_{j=1}^{n} f(x_j) \right)$, where $f(\bullet)$ is a positive increasing function of its argument. He also prove that player i's probability of winning a prize satisfies five axioms together with homogeneity axiom if and only if player i's contest success function is specified as $p_i = \alpha x_i^r \Big/ \left(\sum_{j=1}^{n} \alpha x_j^r \right)$, $\alpha > 0$. Clark and Riis (1998) give an axiomatic foundation to an asymmetric and unfair contest success function. They prove that player i's probability of winning a prize satisfies four axioms, which include Skaperdas' homogeneity axiom but does not include his "symmetric axiom" (A3), if and only if player i's contest success function is specified as $p_i = \alpha_i x_i^r \Big/ \left(\sum_{j=1}^{n} \alpha_j x_j^r \right)$, $\alpha_i > 0$ for any i. Szidarovszky and Okuguchi (1997) assume that player i's contest success function has an asymmetric form of $p_i = f_i(x_i) \Big/ \left(\sum_{j=1}^{n} f_j(x_j) \right)$. Assuming that player i's production function for lotteries $f_i(x_i)$ is strictly increasing and strictly concave in x_i, Szidarovszky and Okuguchi (1997) prove that there exists a unique pure-strategy Nash equilibrium. Hillman and Riley (1989), Baik (1994), Nti (1999), Stein (2002) among others allow each player to have a player-specific valuation on the prize in rent-seeking contests. Che and Gale (1997) prove that there exists a unique pure-strategy Nash equilibrium in the budget-constrained Tullock's contest.

Except for Baik (1994) and Stein (2002), previous works mentioned above study only one of three types of heterogeneity among players in a rent-seeking contest, that is, heterogeneity of contest success functions, heterogeneity of players' valuations on the prize, and heterogeneity of financial constraints.[1] However, these three types of heterogeneities seem mutually related. For example, if all players have the same budget to finance their rent-seeking activities, then players' financial constraints are likely to bind for players with high valuation. Using the graphical techniques similar to Watts (1996), without assuming that each player's production function for lotteries is differentiable, Yamazaki (2008) proves that there exists a unique pure-strategy Nash equilibrium in a general asymmetric rent-seeking contest where each player's production function for lotteries is increasing and concave, each player places a player-specific value on the prize, and each player is budget-constrained.[2] Unless Skaperdas' (1996) six axioms or Clark and Riis' (1998) four axioms are satisfied, it is not certain if the production

[1] Baik (1994) and Stein (2002) analyze asymmetric contests with heterogeneity of contest success functions and of players' valuations on the prize. However, models of Baik and Stein are rather specific. Baik's model has only two players. In Stein's model, player i's contest success function is Tullock's type; $p_i = \alpha_i x_i \Big/ \left(\sum_{j=1}^{n} \alpha_j x_j \right)$, $\alpha_i > 0$ for any i.

[2] Watts (1996) proves some theorems that ensure uniqueness of Nash equilibrium in sharing models and applies one of her theorems to obtain a uniqueness result for the Cournot oligopoly model without product differentiation. However, her theorems are not directly applicable to rent-seeking contests.

function for lotteries is differentiable or not. However, assuming differentiability simplifies analysis very much and a smooth model of the rent seeking contest with three types of heterogeneity seems useful for applied works. In this article, assuming that each player's production function for lotteries is differentiable, we give a simple proof of Yamazaki's (2008) result.[3]

2 Model

Let n be the number of players who participate in a rent-seeking contest ($n \geq 2$). Players are assumed to be risk neutral. Player i is assumed to place a value V_i on the prize (rent), which the player seeks. Player i expends x_i to win the prize, $i = 1, 2, ..., n$. It is assumed that player i's probability of winning the rent is determined as a function of $x = (x_1, x_2, ..., x_n)$:

$$p_i = P_i(\mathbf{x}). \tag{1}$$

The function in (1) is called a contest success function. Many economists assume that the contest success function has a specific functional form.

Assumption 1: $p_i = \dfrac{f_i(x_i)}{\sum_{j=1}^{n} f_j(x_j)}$, $f_i(0) = 0$, $f_i'(x_i) > 0$, $f_i''(x_i) \leq 0$ *for* $i = 1, 2, ..., n$.

Szidarovszky and Okuguchi (1997) call the function $f_i(x_i)$ in Assumption 1 player i's production function for lotteries.

Player i's expected payoff can be written as

$$\pi_i(\mathbf{x}) = \frac{f_i(x_i)}{\sum_{j=1}^{n} f_j(x_j)} V_i - x_i. \tag{2}$$

If all $x_i = 0$, then π_i is defined to be zero.[4] Player i is assumed to maximize (2) with respect to $x_i (\geq 0)$ subject to his or her budget constraint: $x_i \leq I_i$, where $I_i (> 0)$ is player i's income or budget for his or her rent-seeking activities. As seen in the next section, if there is no budget constraint, the expected payoff is always positive, which implies $x_i < V_i$. Hence, if $I_i > V_i$ for all i, that is, all players have income or budget larger than the prize, the budget constraint does not

[3] Chapter 5 of Yamazaki (2013) ignores the budget constraints of players and proves the result of Yamazaki (2008) in a rent-seeking contest where each player's production function for lotteries is differentiable. Hence, this article can be interpreted as a more rigorous version of Chapter 5 of Yamazaki (2013), although Chapter 5 of Yamazaki (2013) contains other related materials.

[4] Assuming π_i to be $1/n$ instead of zero does not affect the following analysis.

bind at all. It can be interpreted that all previous works except for Che and Gale (1997) implicitly assume that all players have income or budget large enough.

Effective lobbying efforts $y_i = f_i(x_i)$ transform (2) into a function of y_i and $Y = \sum_{j=1}^{n} y_j$:

$$\Pi^i(y_i, Y) = \frac{y_i}{Y} V_i - g_i(y_i), \tag{3}$$

where $g_i = f_i^{-1}$.[5] The assumptions on f_i in Assumption 1 imply

$$g_i(0) = 0,\ g_i'(x_i) > 0,\ g_i''(x_i) \geq 0 \ \ for\ \ i = 1, 2, ..., n. \tag{4}$$

Player i's original maximization problem is equivalent to the one of maximizing (3) with respect to y_i subject to $y_i \leq f_i(I_i)$.

3 Equilibrium Analysis

Forget the non-negativity and budget constraints on x_i or equivalently on y_i for a while. The first order condition of maximizing (3) is

$$h^i(y_i, Y) = \frac{Y_{-i}}{Y^2} V_i - g_i'(y_i) = 0, \tag{5}$$

where $Y_{-i} = \sum_{j \neq i} y_j$. This first order condition can be rewritten as

$$h^i(y_i, y_i + Y_{-i}) = \frac{Y_{-i}}{(y_i + Y_{-i})^2} V_i - g_i'(y_i) = 0. \tag{6}$$

The total derivative of $h^i(y_i, y_i + Y_{-i})$ with respect to y_i given a fixed number $Y_{-i} \geq 0$ can be calculated as

$$\frac{\delta}{\delta y_i} h^i(y_i, y_i + Y_{-i}) = -\frac{2Y_{-i}}{(y_i + Y_{-i})^3} V_i - g_i''(y_i).$$

Assumption 1 ensures the second order condition of maximizing $\Pi^i(y_i, Y)$ with respect to y_i for $Y_{-i} \geq 0$, that is,

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5 This transformation, due to Szidarovszky and Okuguchi (1997), converts the original game into an aggregative game where each player's objective function is a function of his or her own choice variable and the sum of all players' choice variables.

$$\frac{\delta}{\delta y_i} h^i\left(y_i, y_i + Y_{-i}\right) = -\frac{2Y_{-i}}{\left(y_i + Y_{-i}\right)^3} V_i - g_i''\left(y_i\right) < 0 . \tag{7}$$

Since the total derivative $\frac{\delta}{\delta y_i} h^i\left(y_i, y_i + Y_{-i}\right)$ is not equal to zero, a unique solution y_i to (6) given $Y_{-i} \geq 0$ is a well defined and finite function of Y_{-i}. Denote the function as $\phi_i(Y_{-i})$.

The function $\phi_i(Y_{-i})$ derived above does not take into account the non-negativity and budget constraints on y_i. With taking these constraints into account, we can construct player i's best reply function $\hat{\phi}_i(Y_{-i})$ from the function $\phi_i(Y_{-i})$. First consider the non-negativity constraint. If

$$\lim_{y_i \to 0}\left[\frac{Y_{-i}}{\left(y_i + Y_{-i}\right)^2} V_i - g_i'\left(y_i\right)\right] = \frac{1}{Y_{-i}} V_i - \lim_{y_i \to 0}\left[g_i'\left(y_i\right)\right] > 0 , \tag{8}$$

which is satisfied for any $Y_{-i} > 0$ sufficiently close to zero, $\phi_i(Y_{-i})$ is positive. On the contrary, if

$$\lim_{y_i \to 0}\left[\frac{Y_{-i}}{\left(y_i + Y_{-i}\right)^2} V_i - g_i'\left(y_i\right)\right] = \frac{1}{Y_{-i}} V_i - \lim_{y_i \to 0}\left[g_i'\left(y_i\right)\right] \leq 0 , \tag{9}$$

we can set player i's best reply function $\hat{\phi}_i(Y_{-i})$ to be zero. Next consider the budget constraint. If

$$\lim_{y_i \to f_i(I_i)}\left[\frac{Y_{-i}}{\left(y_i + Y_{-i}\right)^2} V_i - g_i'\left(y_i\right)\right] = \frac{Y_{-i}}{\left(f_i\left(I_i\right) + Y_{-i}\right)^2} V_i - \lim_{y_i \to f_i(I_i)}\left[g_i'\left(y_i\right)\right] < 0 , \tag{10}$$

which is satisfied for any $Y_{-i} > 0$ sufficiently close to zero, $\phi_i(Y_{-i}) < f_i(I_i)$, that is, player i's budget constraint does not bind. On the contrary, if

$$\lim_{y_i \to f_i(I_i)}\left[\frac{Y_{-i}}{\left(y_i + Y_{-i}\right)^2} V_i - g_i'\left(y_i\right)\right] = \frac{Y_{-i}}{\left(f_i\left(I_i\right) + Y_{-i}\right)^2} V_i - \lim_{y_i \to f_i(I_i)}\left[g_i'\left(y_i\right)\right] \geq 0 , \tag{11}$$

we can set $\hat{\phi}_i(Y_{-i}) = f_i(I_i)$, that is, player i's budget constraint does bind. Summing up,

$$\begin{cases} \hat{\phi}_i\left(Y_{-i}\right) = 0 & \text{if } (8) \text{ is not satisfied,} \\ \hat{\phi}_i\left(Y_{-i}\right) = \phi_i\left(Y_{-i}\right) & \text{if } (8) \text{ and } (10) \text{ are satisfied,} \\ \hat{\phi}_i\left(Y_{-i}\right) = f_i\left(I_i\right) & \text{if } (10) \text{ is not satisfied.} \end{cases} \tag{12}$$

Define $\underline{Y}^i \equiv \lim_{Y_{-i} \to 0} \hat{\phi}_i\left(Y_{-i}\right)$. We can prove the following lemma.

Lemma 1: *Under Assumption 1*, $\underline{Y}^i \equiv \lim_{Y_{-i} \to 0} \hat{\phi}_i\left(Y_{-i}\right) = 0$, $i = 1, 2, \ldots, n$.

Proof Since $\lim_{x_i\to 0} f_i'(x_i)$ is a positive real number, $\lim_{y_i\to 0} g_i'(y_i)$ is also a positive real number. On the other hand, $\lim_{Y_{-i}\to 0}(1/Y_{-i})=\infty$. Hence,

$$\lim_{Y_{-i}\to 0} h^i(\alpha Y_{-i}, \alpha Y_{-i}+Y_{-i}) = \lim_{Y_{-i}\to 0}\left[\frac{1}{(1+\alpha)^2 Y_{-i}}V_i - g_i'(\alpha Y_{-i})\right]=\infty .$$

for any $\alpha\geq 0$. This implies that for any $\alpha\geq 0$, $\phi_i(Y_{-i})>\alpha Y_{-i}$ for any $Y_{-i}\geq 0$ sufficiently close to zero. Since for any $\alpha\geq 0$, $\phi_i(Y_{-i})>\alpha Y_{-i}$ for any $Y_{-i}\geq 0$ sufficiently close to zero,

$$\underline{Y}^i \equiv \lim_{Y_{-i}\to 0}\hat{\phi}_i(Y_{-i}) = \lim_{Y_{-i}\to 0}\phi_i(Y_{-i}) \geq \lim_{Y_{-i}\to 0}\alpha Y_{-i}=0 .$$

On the contrary, suppose $\underline{Y}^i>0$. If so,

$$\lim_{Y_{-i}\to 0} h^i(\underline{Y}^i-\varepsilon, \underline{Y}^i-\varepsilon+Y_{-i}) = \lim_{Y_{-i}\to 0}\left[\frac{Y_{-i}}{(\underline{Y}^i-\varepsilon+Y_{-i})^2}V_i - g_i'(\underline{Y}^i-\varepsilon)\right]$$
$$= -g_i'(\underline{Y}^i-\varepsilon)<0$$

holds for any $\varepsilon\in(0, \underline{Y}^i)$. This implies

$$h_i(\phi(Y_{-i})-\varepsilon,\ \phi_i(Y_{-i})-\varepsilon+Y_{-i})\leq 0$$

for any $Y_{-i}\geq 0$ sufficiently close to zero. Since this result holds for any $\varepsilon\in(0, \underline{Y}^i)$. This contradicts the fact ensured by Assumption 1 that $\phi_i(Y_{-i})$ is a unique solution to the first order condition (5) for $Y_{-i}\geq 0$. Hence, $\underline{Y}^i=0$. ■

Since for any $n>1$, $(1-(1/n))(1/Y_{-i})V_i$ is positive and decreasing in Y_{-i} and it approaches to zero as Y_{-i} goes to infinity and since $g_i'(Y_{-i}/(n-1))$ is positive and increasing in Y_{-i} by Assumption 1, there exists a non-negative real number $Y_{-i}\geq\overline{Y}_{-i}$ such that

$$\lim_{y_i\to Y_{-i}/(n-1)} h^i(y_i, y_i+Y_{-i}) = \lim_{y_i\to Y_{-i}/(n-1)}\left[\frac{Y_{-i}}{(y_i+Y_{-i})^2}V_i - g_i'(y_i)\right]$$
$$=\left(1-\frac{1}{n}\right)\frac{1}{Y_{-i}}V_i - g_i'\left(\frac{Y_{-i}}{n-1}\right)<0$$

for any given $Y_{-i}\geq\overline{Y}_{-i}$. This implies the following lemma.

Lemma 2: *Under Assumption 1, there exists a non-negative real number* $\bar{Y}_{-i}$ *such that* $\hat{\phi}_i(Y_{-i}) < \frac{1}{n-1}Y_{-i}$ *for any* $Y_{-i} \geq \bar{Y}_{-i}$, $i = 1, 2, \ldots, n$.

The function $\phi_i(Y_{-i})$ satisfies

$$f^i(Y_{-i}) \equiv \Pi_1^i(\phi_i(Y_{-i}), \phi_i(Y_{-i}) + Y_{-i}) + \Pi_2^i(\phi_i(Y_{-i}), \phi_i(Y_{-i}) + Y_{-i})$$
$$= \frac{Y_{-i}}{(\phi_i(Y_{-i}) + Y_{-i})^2}V_i - g_i'(\phi_i(Y_{-i})) = 0. \tag{13}$$

Differentiating (13) with respect to Y_{-i} shows

$$\phi_i'(Y_{-i}) = \frac{\frac{V_i}{(y_i + Y_{-i})^3}\{2y_i - (y_i + Y_{-i})\}}{\frac{2Y_{-i}}{(y_i + Y_{-i})^3}V_i + g_i''(y_i)}. \tag{14}$$

Since the denominator of (14) is positive under Assumption 1 and since the numerator of (14) is positive (zero, negative) if and only if $2y_i - (y_i + Y_{-i})$ is positive (zero, negative),

$$\phi_i'(Y_{-i}) \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} 0 \iff \phi_i(Y_{-i}) \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} Y_{-i}.$$

Now consider the sign of $h^i(y_i, y_i + Y_{-i})$ for $y_i = Y_{-i}$.

$$h^i(Y_{-i}, Y_{-i} + Y_{-i}) = \frac{Y_{-i}}{(Y_{-i} + Y_{-i})^2}V_i - g_i'(Y_{-i})$$
$$= \frac{1}{4Y_{-i}}V_i - g_i'(Y_{-i}).$$

The function $\phi_i(Y_{-i})$ is larger (equal to, smaller) than Y_{-i} if and only if $h^i(Y_{-i}, Y_{-i} + Y_{-i})$ above is negative (zero, positive). If $f_i(x_i)$ is concave as in Assumption 1, since $f_i'(x_i) > 0$ for some $x_i \geq 0$, $\lim_{x_i \to 0} f_i'(x_i) \geq 0$ cannot be zero.[6] Since $\lim_{x_i \to 0} f_i'(x_i)$ is a positive real number, $\lim_{y_i \to 0} g_i'(y_i)$ is also a positive real number. On the other hand, $\lim_{Y_{-i} \to 0}(1/Y_{-i}) = \infty$. Hence,

$$\lim_{Y_{-i} \to 0} h^i(Y_{-i}, Y_{-i} + Y_{-i}) = \lim_{Y_{-i} \to 0}\left[\frac{1}{4Y_{-i}}V_i - g_i'(Y_{-i})\right] = \infty.$$

[6] For the model to be meaningful, we need to require $f_i'(x_i) > 0$ for some $x_i \geq 0$. If $\lim_{x_i \to 0} f_i'(x_i) = 0$, the weak concavity of $f_i(x_i)$ implies $f_i'(x_i) \leq 0$ for any $x_i \geq 0$.

This implies that $\phi_i(Y_{-i}) > Y_{-i}$ for any $Y_{-i} \geq 0$ sufficiently close to zero.

Since $\lim_{x_i \to \infty} f_i'(x_i) \geq 0$, $\lim_{y_i \to 0} g_i'(y_i)$ is a positive real number. Since $\lim_{Y_{-i} \to \infty} (1/Y_{-i}) = 0$,

$$\lim_{Y_{-i} \to \infty} h^i(Y_{-i}, Y_{-i} + Y_{-i}) = \lim_{Y_{-i} \to \infty} \left[\frac{1}{4Y_{-i}} V_i - g_i'(Y_{-i}) \right] < 0 .$$

This implies that $\phi_i(Y_{-i}) < Y_{-i}$ for any Y_{-i} large enough. These observations inform us that under Assumption 1. $\phi_i(Y_{-i})$ has a single peak on the 45 degree line as in Figure 1. Remember Lemma 1, that is, $\underline{Y}^i \equiv \lim_{Y_{-i} \to 0} \phi_i(Y_{-i}) = 0$. In Figure 1, we assume that there exists a positive real number $\hat{Y}_{-i}$ such that $\phi_i(Y_{-i}) > 0$ for any $Y_{-i} \in [0, \hat{Y}_{-i})$ and $\phi_i(Y_{-i}) = 0$ for any $Y_{-i} \geq \hat{Y}_{-i}$. We can easily draw the graph of $\hat{\phi}_i(Y_{-i})$ from the one of $\phi_i(Y_{-i})$. If player i's budget constraint does bind for some Y_{-i}, the graph of $\hat{\phi}_i(Y_{-i})$ can be depicted as in Figure 2.

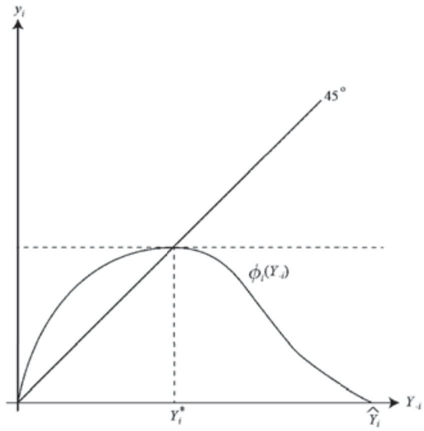

Figure 1 Graph of $\phi_i(Y_{-i})$

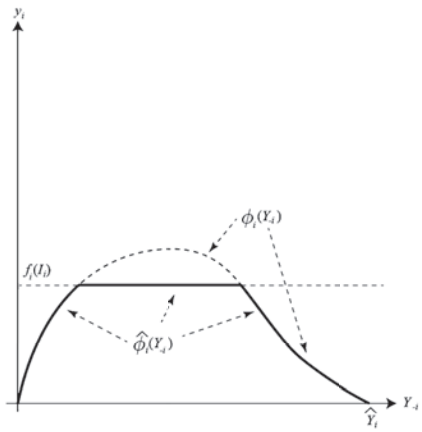

Figure 2 Graph of $\hat{\phi}_i(Y_{-i})$

The first order condition (5) can be treated as an implicit function of y_i and Y, that is,

$$h^i\left(y_i,Y\right)=\frac{Y-y_i}{Y^2}V_i-g_i'\left(y_i\right)=0. \tag{15}$$

The partial derivative of $h^i(y_i, Y)$ with respect to y_i for some $Y\geq 0$ can be calculated as

$$\frac{\partial}{\partial y_i}h^i\left(y_i,Y\right)=-\frac{1}{Y^2}V_i-g_i''\left(y_i\right), \tag{16}$$

which is negative under Assumption 1. Since the partial derivative $\frac{\partial}{\partial y_i}h^i\left(y_i,Y\right)$ is not equal to zero, Assumption 1 ensures that a unique solution y_i to (15) for given $Y\geq 0$ can be written as a well defined and finite function of $Y\geq \underline{Y}^i=0$. Denote the function as $\Phi_i(Y)$.

The function $\Phi_i(Y)$ derived above does not take into account the non-negativity and budget constraints on y_i. With taking these constraints into account, we can construct player i's cumulative best reply function $\hat{\Phi}_i(Y)$ from the function $\Phi_i(Y)$.[7] First consider the non-negativity constraint. If

$$\lim_{y_i\to 0}\left[\frac{Y-y_i}{Y^2}V_i-g_i'\left(y_i\right)\right]=\frac{1}{Y}V_i-\lim_{y_i\to 0}\left[g_i'\left(y_i\right)\right]>0, \tag{17}$$

which is satisfied for any $Y>0$ sufficiently close to zero, $\Phi_i(Y)$ is positive. On the contrary, if

$$\lim_{y_i\to 0}\left[\frac{Y-y_i}{Y^2}V_i-g_i'\left(y_i\right)\right]=\frac{1}{Y}V_i-\lim_{y_i\to 0}\left[g_i'\left(y_i\right)\right]\leq 0, \tag{18}$$

we can set player i's cumulative best reply $\hat{\Phi}_i(Y)$ to be zero. Next consider the budget constraint. If

$$\lim_{y_i\to f_i(I_i)}\left[\frac{Y-y_i}{Y^2}V_i-g_i'\left(y_i\right)\right]=\frac{Y-f_i\left(I_i\right)}{Y^2}V_i-\lim_{y_i\to f_i(I_i)}\left[g_i'\left(y_i\right)\right]<0, \tag{19}$$

which is satisfied for any $Y>0$ sufficiently close to zero, $\Phi_i(Y)<f_i(I_i)$, that is, player i's budget constraint does not bind. On the contrary, if

$$\lim_{y_i\to f_i(I_i)}\left[\frac{Y-y_i}{Y^2}V_i-g_i'\left(y_i\right)\right]=\frac{Y-f_i\left(I_i\right)}{Y^2}V_i-\lim_{y_i\to f_i(I_i)}\left[g_i'\left(y_i\right)\right]\geq 0, \tag{20}$$

[7] The terminology "cumulative best reply" is cited from Vives (1999). Cornes and Hartly (2003, 2005) call the cumulative best reply function a replacement function. Wolfstetter (1999) calls it an inclusive reaction function.

we can set $\hat{\Phi}_i(Y)=f_i(I_i)$, that is, player i's budget constraint does bind. Summing up,

$$\begin{cases} \hat{\Phi}_i(Y)=0 & \text{if (17) is not satisfied,} \\ \hat{\Phi}_i(Y)=\Phi_i(Y) & \text{if (17) and (19) are satisfied,} \\ \hat{\Phi}_i(Y)=f_i(I_i) & \text{if (17) is not satisfied.} \end{cases} \tag{21}$$

Lemma 1 proves the following lemma.

Lemma 1': *Under Assumption 1,* $\underline{Y}^i \equiv \lim_{Y\to 0}\hat{\Phi}_i(Y)=0$, $i=1, 2, ..., n$.

By Lemma 2, there exists a non-negative real number $\bar{Y}_{-i}$ such that

$$\hat{\phi}_i(Y_{-i})<\frac{1}{n-1}Y_{-i} \quad \text{for any } Y_{-i}\geq \bar{Y}_{-i} \quad ,$$

$i=1, 2, ..., n$. By adding $\hat{\phi}_i(Y_{-i})/(n-1)$ to the both sides of the above inequality,

$$\frac{n}{n-1}\hat{\phi}_i(Y_{-i})<\frac{1}{n-1}\left(Y_{-i}+\hat{\phi}_i(Y_{-i})\right) \quad \text{for any } Y_{-i}\geq \bar{Y}_{-i} \quad ,$$

$i=1, 2, ..., n$. Hence, Lemma 2 proves the following lemma.

Lemma 2': *Under Assumption 1, there exists a non-negative real number* $\bar{Y}_{-i}$ *such that* $\hat{\Phi}_i(Y)<\frac{1}{n}Y$ *for any* $Y_{-i}\geq \bar{Y}_{-i}$, $i=1, 2, ..., n$.

The function $\Phi_i(Y)$ satisfies

$$F^i(Y)=\frac{Y-\Phi_i(Y)}{Y^2}V_i-g_i'\left(\Phi_i(Y)\right)=0\,. \tag{22}$$

Differentiating (22) with respect to Y, we get

$$\frac{1}{Y^4}\left\{\left(1-\Phi_i'(Y)\right)Y^2-2\left(Y-\Phi_i(Y)\right)Y\right\}V_i-g_i''\left(\Phi_i(Y)\right)\Phi_i'(Y)=0\,.$$

From this equality, we can get

$$\Phi_i'(Y)=\frac{\left(2\Phi_i(Y)-Y\right)V_i}{V_iY+g_i''\left(\Phi_i(Y)\right)Y^3}\,. \tag{23}$$

Since the denominator of (23) is positive under Assumption 1 and since the numerator of (23) is positive (zero, negative) if and only if $2\Phi_i(Y)-Y$ is positive (zero, negative),

$$\Phi_i'(Y)\begin{Bmatrix}>\\=\\<\end{Bmatrix}0 \iff \Phi_i(Y)\begin{Bmatrix}>\\=\\<\end{Bmatrix}\frac{1}{2}Y.$$

Now consider the sign of $h^i(y_i, Y)$ for $y_i=Y/2$.

$$h^i(Y/2,Y)=\frac{Y/2}{Y^2}V_i-g_i'(Y/2)$$
$$=\frac{1}{2Y}V_i-g_i'(Y/2).$$

The function $\Phi_i(Y)$ is larger (equal to, smaller) than $Y/2$ if and only if $h^i(Y/2, Y)$ above is negative (zero, positive). If $f_i(x_i)$ is concave as in Assumption 1, since $f_i'(x_i)>0$ for some $x_i\geq 0$, $\lim_{x_i\to 0}f_i'(x_i)$ cannot be zero. Since $\lim_{x_i\to 0}f_i'(x_i)$ is a positive real number, $\lim_{Y\to 0}g_i'(Y/2)$ is also a positive real number. On the other hand, $\lim_{Y\to 0}(1/Y)=\infty$. Hence,

$$\lim_{Y\to 0}h^i(Y/2,Y)=\lim_{Y\to 0}\left[\frac{1}{2Y}V_i-g_i'(Y/2)\right]=\infty.$$

This implies that $\Phi_i(Y)>Y/2$ for any $Y\geq 0$ sufficiently close to zero. Since $\lim_{x_i\to\infty}f_i'(x_i)\geq 0$, $\lim_{y_i\to 0}g_i'(y_i)$ is a positive real number. Since $\lim_{Y\to\infty}(1/Y)=0$,

$$\lim_{Y\to\infty}h^i(Y/2,Y)=\lim_{Y\to 0}\left[\frac{1}{2Y}V_i-g_i'(Y/2)\right]<0.$$

This implies that $\Phi_i(Y)<Y/2$ for any Y large enough.

These observations inform us that under Assumption 1 the function $\Phi_i(Y)$ has a single peak on the line $Y/2$ as in Figure 3. If player i's budget constraint binds for some Y, player i's cumulative best reply function $\hat{\Phi}_i(Y)$ can be depicted as in Figure 4. Player i's cumulative best reply function $\hat{\Phi}_i(Y)$ in Figure 4 corresponds to player i's best reply function $\hat{\phi}_i(Y_{-i})$ in Figure 2 where it is assumed that there exists a positive real number $\hat{Y}_{-i}$ such that $\phi_i(Y_{-i})>0$ for any $Y_{-i}\in[0, \hat{Y}_{-i})$ and $\phi_i(Y_{-i})=0$ for any $Y_{-i}\geq\hat{Y}_{-i}$.

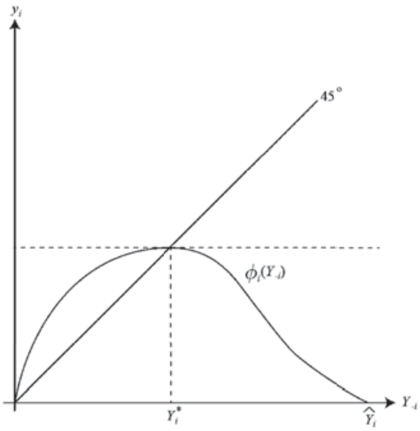

Figure 3 Graph of $\Phi_i(Y)$

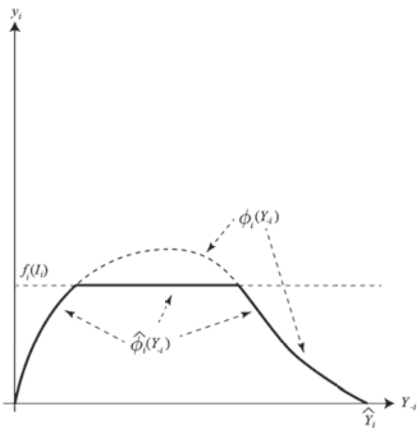

Figure 4 Graph of $\hat{\Phi}_i(Y)$

As in Cornes and Hartley (2003, 2005), define player i's share function $\sigma_i(Y) \equiv \hat{\Phi}_i(Y)/Y$, which is defined on $(\underline{Y}^i, \infty) = (0, \infty)$. Differentiating $\sigma_i(Y) = \hat{\Phi}_i(Y)/Y$ with respect to Y from the right-hand or left-hand side, we get

$$\begin{aligned} D_k\sigma_i(Y) &= \frac{1}{Y^2}\left(D_k\hat{\Phi}_i(Y)Y - \hat{\Phi}_i(Y)\right) \\ &= \frac{1}{Y}\left(D_k\hat{\Phi}_i(Y) - \frac{\hat{\Phi}_i(Y)}{Y}\right), \quad k = +, -, \quad i = 1, 2, \cdots, n, \end{aligned} \tag{24}$$

where $D_+f(x)$ and $D_-f(x)$ stand for the right-hand and left-hand derivatives of a function $f(x)$, respectively. As seen in Figure 4, the right-hand and left-hand derivatives of $\hat{\Phi}_i(Y)$ at $Y = Y^0 > 0$, the slope of $\hat{\Phi}_i(Y)$ at $Y = Y^0$, is always smaller than $\hat{\Phi}_i(Y)/Y$, the slope of the ray passing through the origin and the point $(Y^0,\ \hat{\Phi}_i(Y^0))$. That is,

$$D_k\hat{\Phi}_i(Y) < \frac{\hat{\Phi}_i(Y)}{Y} \text{ for any } Y>0,\ k=+,-,\ i=1,2,\cdots,n. \tag{25}$$

Hence, this fact together with (24) proves the following lemma.

Lemma 3: *$\sigma_i(Y)$ is strictly decreasing in Y, $i=1, 2, ..., n$.*

We have already shown $\lim_{Y\to 0}\Phi_i(Y)=0$ under Assumption 1. Since $\sigma_i(Y)=\hat{\Phi}_i(Y)/Y$ and $\hat{\Phi}_i(Y)\le Y$, $\lim_{Y\to 0}\sigma_i(Y)\le 1$. Since $\lim_{Y\to 0}\hat{\Phi}_i(Y)=0$, $\lim_{Y\to 0}Y=0$ and $\hat{\Phi}_i(Y)=\Phi_i(Y)$ for any positive Y sufficiently close to zero, by L'Hospital's rule,

$$\lim_{Y\to 0}\frac{\hat{\Phi}_i(Y)}{Y}=\lim_{Y\to 0}\Phi_i'(Y). \tag{26}$$

On the other hand, from (23), we can get the following relation.

$$\begin{aligned}\lim_{Y\to 0}\Phi_i'(Y)&=\lim_{Y\to 0}\frac{V_i\left(2\frac{\Phi_i(Y)}{Y}-1\right)}{g_i''(\Phi_i(Y))Y^2+V_i}\\&=\frac{V_i\left(2\lim_{Y\to 0}\frac{\Phi_i(Y)}{Y}-1\right)}{V_i}\\&=2\lim_{Y\to 0}\frac{\Phi_i(Y)}{Y}-1.\end{aligned} \tag{27}$$

Substituting (27) into (26), we can get

$$\lim_{Y\to 0}\sigma_i(Y)=\lim_{Y\to 0}\frac{\hat{\Phi}_i(Y)}{Y}=\lim_{Y\to 0}\Phi_i'(Y)=1.$$

We have already shown under Assumption 1 that there exists a non-negative real number $\bar{Y}_{-i}$ such that $\hat{\Phi}_i(Y)<\frac{1}{n}Y$ for any $Y>\bar{Y}_{-i}$. Since $\sigma_i(Y)=\hat{\Phi}_i(Y)/Y$, this fact implies that $\sigma_i(Y)<\frac{1}{n}$ for any $Y>\bar{Y}_i\equiv\frac{n+1}{n}\bar{Y}_{-i}$. Summing up, we have proved the following lemma.

Lemma 4: $\lim_{Y\to 0}\sigma_i(Y)=1$ *and there exists a non-negative real number $\bar{Y}_i$ such that $\sigma_i(Y)<\frac{1}{n}$ for any $Y>\bar{Y}_i$, $i=1, 2, ..., n$.*

Figure 5 depicts the graph of player i's share function $\sigma_i(Y)$, which satisfies the properties in

Lemmas 3 and 4. Player i's share function $\sigma_i(Y)$ in Figure 5 corresponds to player i's best reply function $\phi_i(Y_{-i})$ in Figure 2 where it is assumed that there exists a positive real number $\hat{Y}_{-i}$ such that $\phi_i(Y_{-i})>0$ for any $Y_{-i}\in[0,\hat{Y}_{-i})$ and $\phi_i(Y_{-i})=0$ for any $Y_{-i}\geq\hat{Y}_{-i}$.

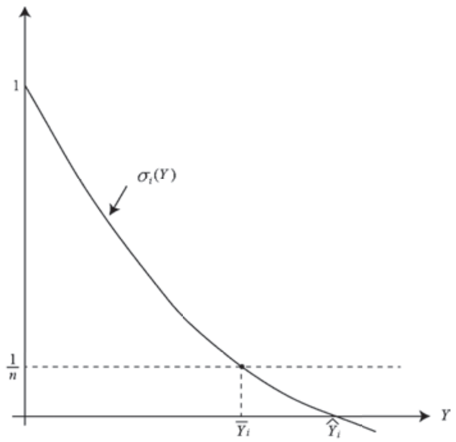

Figure 5 Graph of $\sigma_i(Y)$

A Nash equilibrium Y corresponds to a solution to

$$1=\sum_{i=1}^{n}\sigma_i\left(Y\right). \tag{28}$$

Figure 6 graphically shows how the graph of $\sum_{i=1}^{n}\sigma_i\left(Y\right)$ can be depicted in a special case where there are only two players in the game, that is, n=2. As seen in Figure 6, Lemma 3 ensures that $\sum_{i=1}^{n}\sigma_i\left(Y\right)$ is strictly decreasing in Y, while Lemma 4 ensures

$$\lim_{Y\to 0}\sum_{i=1}^{n}\sigma_i\left(Y\right)=n$$

and

$$\sum_{i=1}^{n}\sigma_i\left(Y\right)<\sum_{i=1}^{n}\frac{1}{n}=1$$

for any $Y>\overline{Y}\equiv\max\left\{\overline{Y}_i\right\}_{i=1}^{n}>0$. Hence, as seen in Figure 7, there exists a unique positive solution Y^E to the equation (28). This equilibrium Y^E uniquely determines all relevant equilibrium values: $p_i^E=\sigma_i(Y^E)$, $y_i^E=\sigma_i(Y^E)\,Y$ and $x_i^E=g_i(y_i^E)$, for all i. Hence, we have proved the following theorem.

Theorem: *Under Assumption 1, there exists a unique pure-strategy Nash equilibrium in a smooth rent-seeking contest with asymmetric valuations.*

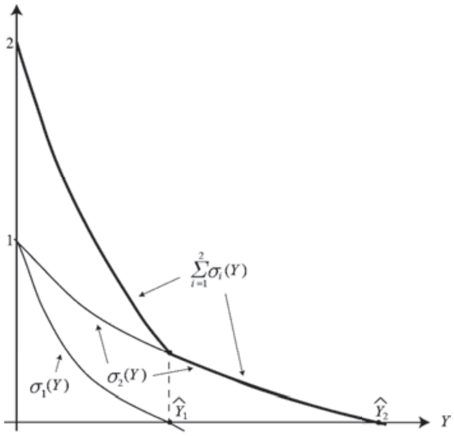

Figure 6 Graph of $\sum_{i=1}^{n}\sigma_i(Y)$, $n=2$

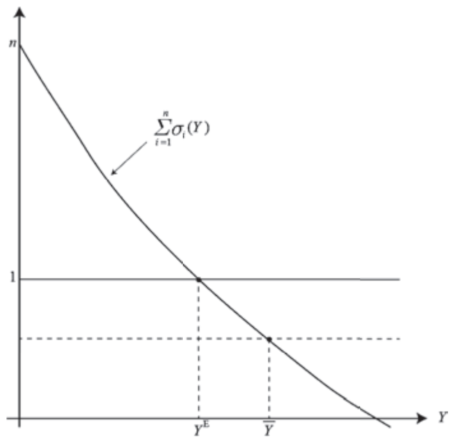

Figure 7 Unique Existence of Nash Equilibrium

4 Concluding Remarks

Without assuming that each player's production function for lotteries is differentiable, Yamazaki (2008) proves that there exists a unique pure-strategy Nash equilibrium in a general asymmetric rent-seeking contest where each player's production function for lotteries is increasing and concave, each player places a player-specific value on the prize, and each player is budget-constrained. In this article, with assuming that each player's production function for lotteries is differentiable, we have given a simple proof of Yamazaki's (2008) result. The asymmetric contest of this paper and Yamazaki (2008) is very general among risk-neutral rent-seeking contests, which are standard in the literature. The result of this article can be generalized in several directions. For a contest with risk-averse players, Cornes and Hartley (2003) analyze a risk-averse rent-seeking contest where the Arrow-Pratt measure of absolute risk aversion of players is assumed to be constant. They prove that their risk-averse rent-seeking contest possesses a unique pure-strategy Nash equilibrium. Yamazaki (2009, 2010) proves that there exists a unique pure strategy Nash equilibrium in a rent seeking contest if the Arrow-Pratt measure of absolute risk aversion of players is non-increasing.

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