A NOTE ON GRADED BETTI NUMBERS OF COMPLETELY $\mathfrak{m}$-FULL IDEALS

TADAHITO HARIMA, SATORU ISOGAWA AND JUNZO WATANABE

Abstract. We give a new proof of the characterization theorem for completely $\mathfrak{m}$-full ideals stated as follows: completely $\mathfrak{m}$-full ideals are characterized as the ideals that have the same graded Betti numbers as those of their generic initial ideals with respect to the reverse lexicographic order, provided their generic initial ideals are stable.

1. Introduction

Let I be a homogeneous ideal of a polynomial ring R and $\mathrm{Gin}(I)$ the generic initial ideal of I with respect to the reverse lexicographic order. In [1] Aramova, Herzog and Hibi proved the theorem which states that, in characteristic zero, I and $\mathrm{Gin}(I)$ have the same graded Betti numbers if and only if I is componentwise linear. For the case of positive characteristic, this theorem was proved by Nagel and Römer [10] with the additional assumption that $\mathrm{Gin}(I)$ is stable, which is automatically satisfied in characteristic zero. On the other hand, the authors [8] recently showed the result which states that the notion of componentwise linearity and completely $\mathfrak{m}$-fullness are equivalent in the class of ideals whose generic initial ideals are stable. Hence, combining those results, one immediately gets the following theorem on graded Betti numbers of completely $\mathfrak{m}$-full ideals.

Theorem 1. *Let I be a homogeneous ideal of $R = K[x_1, \ldots, x_n]$ and assume that the generic initial ideal $\mathrm{Gin}(I)$ of I is stable. Then the following conditions are equivalent:*

(i) *I is completely $\mathfrak{m}$-full.*
(ii) *$\beta_{i,i+j}(I) = \beta_{i,i+j}(\mathrm{Gin}(I))$ for all i and j.*
(iii) *$\beta_i(I) = \beta_i(\mathrm{Gin}(I))$ for all i.*
(iv) *$\beta_{0,j}(I) = \beta_{0,j}(\mathrm{Gin}(I))$ for all j.*
(v) *$\beta_0(I) = \beta_0(\mathrm{Gin}(I))$.*

The purpose of this note is to give a new proof of the theorem above. It is a direct proof obtained by *an inductive argument using the notion of "$\mathfrak{m}$-fullness"*. Our techniques are completely different from those of the papers [1] and [10] cited above. The notion of an $\mathfrak{m}$-full ideal was introduced by

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D. Rees. The $\mathfrak{m}$-full ideals form an important class of ideals having various interesting properties (cf. [3], [4], [5], [7], [8], [9], [11], [12], [13]).

In Section 2 we state some remarks on a minimal generating set of an $\mathfrak{m}$-full ideal, and review a result on graded Betti numbers obtained in [12]. These are needed for our proof of the main theorem. The implications (ii) $\Rightarrow$ (iii) $\Rightarrow$ (v) and (ii) $\Rightarrow$ (iv) $\Rightarrow$ (v) in Theorem 1 are obvious. We will give a proof of (i) $\Rightarrow$ (ii) in Section 3, and a proof of (v) $\Rightarrow$ (i) in Section 4.

Throughout this note, we let $R = K[x_1, \ldots, x_n]$ be the polynomial ring in n variables over an infinite field K with the standard grading, and $\mathfrak{m} = (x_1, \ldots, x_n)$ the homogeneous maximal ideal. Let $\mathrm{Gin}(I)$ denote the generic initial ideal of an ideal I of R with respect to the graded reverse lexicographic order induced by $x_1 > \cdots > x_n$.

2. Preliminaries on $\mathfrak{m}$-full ideals

Definition 2 ([11], Definition 4)**.** An ideal I of R is said to be $\mathfrak{m}$-*full* if there exists a linear form z in R such that $\mathfrak{m}I : z = I$.

Remark 3. Suppose that I is an $\mathfrak{m}$-full ideal of R. Then the equality $\mathfrak{m}I : z = I$ holds for a general linear form z in R ([11], Remark 2 (i)). Moreover, it is easy to see that the equality $\mathfrak{m}I : z = I$ implies that $I : \mathfrak{m} = I : z$ for any $z \in R$. Let $f_1, \ldots, f_r$ be homogeneous forms in $I : \mathfrak{m}$ such that $\{\overline{f_1}, \ldots, \overline{f_r}\}$ is a minimal generating set of $(I : \mathfrak{m})/I$, where $\overline{f_i}$ is the image of f_i in R/I. Then Proposition 2.2 in [5] implies that $\{zf_1, \ldots, zf_r\}$ can be a part of a minimal generating set of I.

Let $\beta_{i,j}(I)$ be the (i,j)th graded Betti number of I as an R-module, and $\beta_i(I) = \sum_j \beta_{i,j}(I)$ the ith total Betti number of I. Note that $\beta_0(I)$ is the minimal number of generators of I.

Proposition 4 ([12] Corollary 8 and [8] Proposition 5)**.** *Let I be an $\mathfrak{m}$-full ideal of R and let z be a general linear form of R satisfying $\mathfrak{m}I : z = I$. Let $\overline{I}$ be the image of I in R/zR and let $\beta_{i,j}(\overline{I})$ be the (i,j)th graded Betti number of $\overline{I}$ as an R/zR-module. With the same notation as Remark 3, set $c_j = \#\{i \mid 1 \le i \le l, \deg(zf_i) = j\}$ for all j. Then*

$$\beta_{i,i+j}(I) = \beta_{i,i+j}(\overline{I}) + \binom{n-1}{i} c_j$$

for $i = 0, 1, \ldots, n-1$ and $j = 0, 1, 2, \ldots$.

Remark 5. With the same notation as Proposition 4, let

$$H((I : \mathfrak{m})/I, j) = \dim_K [(I : \mathfrak{m})/I]_j$$

be the Hilbert function of $(I : \mathfrak{m})/I$. Since $c_j = H((I : \mathfrak{m})/I, j-1)$, the formula in Proposition 4 can be rewritten as:

$$\beta_{i,i+j}(I) = \beta_{i,i+j}(\overline{I}) + \binom{n-1}{i} H((I : \mathfrak{m})/I, j-1)$$

for all i and j. In particular we see that

$$\beta_0(I) = \beta_0(\overline{I}) + \mathrm{length}((I : \mathfrak{m})/I).$$

Definition 6 ([12], Definition 2)**.** Let I be a homogeneous ideal of $R = K[x_1, \ldots, x_n]$. We say that I is *completely* $\mathfrak{m}$*-full* if there exist general linear forms $z_n, z_{n-1}, \ldots, z_1$ of R satisfying the following conditions:

(i) $\mathfrak{m}I : z_n = I$, i.e., I is $\mathfrak{m}$-full.
(ii) $\overline{\mathfrak{m}}\overline{I} : \overline{z_{n-i+1}} = \overline{I}$ in $\overline{R} = R/(I, z_n, \ldots, z_{n-i+2})$ for all $i = 2, 3, \ldots, n$, where $\overline{*}$ denotes the image $*$ in $\overline{R}$.

In this case we say that $(I; z_n, z_{n-1}, \ldots, z_1)$ has the complete $\mathfrak{m}$-full property.

Definition 7. A monomial ideal I of $R = K[x_1, \ldots, x_n]$ is said to be *stable* if I satisfies the following condition: for each monomial $u \in I$, the monomial $x_i u/x_{m(u)}$ belongs to I for every $i < m(u)$, where $m(u)$ is the largest index j such that x_j divides u.

Remark 8. Stable ideals $(I; x_n, x_{n-1}, \ldots, x_1)$ have the complete $\mathfrak{m}$-full property ([8], Example 17).

3. Proof of (i) $\Rightarrow$ (ii)

For the proof of (i) $\Rightarrow$ (ii) we need the following lemma.

Lemma 9. *Let I be an $\mathfrak{m}$-full ideal of R, and assume that* $\mathrm{Gin}(I)$ *is stable. Then the equalities*

$$\mathrm{H}((I : \mathfrak{m})/I, j) = \mathrm{H}((\mathrm{Gin}(I) : \mathfrak{m})/\mathrm{Gin}(I), j)$$

hold for all j.

Proof. Set $J = \mathrm{Gin}(I)$. Since I and J are $\mathfrak{m}$-full, there exists a general linear form z of R satisfying $\mathfrak{m}I : z = I$ and $\mathfrak{m}J : z = J$. Then it is easy to see that $I : \mathfrak{m} = I : z$ and $J : \mathfrak{m} = J : z$. Furthermore, from the exact sequence

$$0 \to (I : \mathfrak{m})/I \to R/I \stackrel{\times z}{\to} R/I \to R/(I + zR) \to 0,$$

we have

$$\mathrm{H}((I : \mathfrak{m})/I, j-1) = \mathrm{H}(R/I + zR, j) - \mathrm{H}(R/I, j) + \mathrm{H}(R/I, j-1)$$

for all j. Similarly, we have

$$\mathrm{H}((J : \mathfrak{m})/J, j-1) = \mathrm{H}(R/J + zR, j) - \mathrm{H}(R/J, j) + \mathrm{H}(R/J, j-1).$$

for all j. Here recall the well-known facts:

- $\mathrm{H}(R/I, j) = \mathrm{H}(R/J, j)$ for all j.

- $\mathrm{H}(R/(I+zR),j) = \mathrm{H}(R/(J+zR),j)$ for all j ([2], Lemma 1.2).

Hence we get the desired equalities. $\square$

Proof of (i) $\Rightarrow$ (ii) in Theorem 1. After a generic linear change of variables we may assume that $(I; x_n, x_{n-1}, \ldots, x_1)$ has the complete $\mathfrak{m}$-full property. Since I and $\mathrm{Gin}(I)$ are $\mathfrak{m}$-full, it follows by Remark 5 that

$$\beta_{i,i+j}(I) = \beta_{i,i+j}(\overline{I}) + \binom{n-1}{i} \times \mathrm{H}((I:\mathfrak{m})/I, j-1)$$

and

$$\beta_{i,i+j}(J) = \beta_{i,i+j}(\overline{J}) + \binom{n-1}{i} \times \mathrm{H}((J:\mathfrak{m})/J, j-1)$$

for all i and j. Since $\overline{J}$ is the generic initial ideal of $\overline{I}$ ([6], Corollary 2.15), it follows by an inductive argument on the number of variables that

$$\beta_{i,i+j}(\overline{I}) = \beta_{i,i+j}(\overline{J})$$

for all i and j. Hence, by Lemma 9, we get $\beta_{i,i+j}(I) = \beta_{i,i+j}(J)$ for all i and j. $\square$

4. Proof of (v) $\Rightarrow$ (i)

For the proof of (v) $\Rightarrow$ (i) we need the following lemma.

Lemma 10. *Let I be a homogeneous ideal of R, and assume that* $\mathrm{Gin}(I)$ *is stable. If $\beta_0(I) = \beta_0(\mathrm{Gin}(I))$, then* $\mathrm{Gin}(\mathfrak{m}I) = \mathfrak{m}\mathrm{Gin}(I)$.

Proof. There exists a generic linear change of variables φ such that $\mathrm{Gin}(I)$ coincides with the initial ideal $\mathrm{In}(\varphi(I))$ of $\varphi(I)$. We first show that $R/\mathrm{In}(\mathfrak{m}\varphi(I))$ and $R/\mathfrak{m}\mathrm{In}(\varphi(I))$ have the same Hilbert function. Since $\beta_0(\varphi(I)) = \beta_0(\mathrm{In}\varphi(I))$ holds by $\beta_0(I) = \beta_0(\mathrm{Gin}(I))$, there exists a minimal generating set $\{f_1.\ldots, f_r\}$ of $\varphi(I)$ such that $\{\mathrm{In}(f_1), \ldots, \mathrm{In}(f_r)\}$ is a minimal generating set of $\mathrm{In}(\varphi(I))$, where $\mathrm{In}(f_j)$ denotes the initial monomial of f_j. Hence, it gives that

$$H(\varphi(I)/\mathfrak{m}\varphi(I), j) = H(\mathrm{In}(\varphi(I))/\mathfrak{m}\mathrm{In}(\varphi(I)), j)$$

for all j, Thus we get that

$$\begin{array}{rcl} H(R/\mathrm{In}(\mathfrak{m}\varphi(I)), j) & = & H(R/\mathfrak{m}\varphi(I), j) \\ & = & H(R/\varphi(I), j) + H(\varphi(I)/\mathfrak{m}\varphi(I), j) \\ & = & H(R/\mathrm{In}(\varphi(I)), j) + H(\mathrm{In}(\varphi(I))/\mathfrak{m}\mathrm{In}(\varphi(I)), j) \\ & = & H(R/\mathfrak{m}\mathrm{In}(\varphi(I)), j) \end{array}$$

for all j.

Since the inclusion $\mathfrak{m}\mathrm{In}(\varphi(I)) \subset \mathrm{In}(\mathfrak{m}\varphi(I))$ is obvious, we have proved the equality $\mathfrak{m}\mathrm{In}(\varphi(I)) = \mathrm{In}(\mathfrak{m}\varphi(I))$. Thus, for a sufficient generic linear change of variables φ, we have that

$$\begin{aligned}\mathrm{Gin}(\mathfrak{m}I) &= \mathrm{In}(\varphi(\mathfrak{m}I)) = \mathrm{In}(\varphi(\mathfrak{m})\varphi(I)) \\ &= \mathrm{In}(\mathfrak{m}\varphi(I)) = \mathfrak{m}\mathrm{In}(\varphi(I)) = \mathfrak{m}\mathrm{Gin}(I).\end{aligned}$$

□

Proof of (v) ⇒ (i) in Theorem 1. Set $J = \mathrm{Gin}(I)$. Let z be a general linear form of R. We first show that I is $\mathfrak{m}$-full. From the exact sequence

$$0 \to (\mathfrak{m}I : z)/\mathfrak{m}I \to R/\mathfrak{m}I \stackrel{\times z}{\to} R/\mathfrak{m}I \to R/(\mathfrak{m}I + zR) \to 0,$$

it follows that

$$\mathrm{H}((\mathfrak{m}I : z)/\mathfrak{m}I, j) = \mathrm{H}(R/\mathfrak{m}I, j) - \mathrm{H}(R/\mathfrak{m}I, j+1) + \mathrm{H}(R/(\mathfrak{m}I + zR), j+1)$$

for all j. Similarly, we have

$$\mathrm{H}((\mathfrak{m}J : z)/\mathfrak{m}J, j) = \mathrm{H}(R/\mathfrak{m}J, j) - \mathrm{H}(R/\mathfrak{m}J, j+1) + \mathrm{H}(R/(\mathfrak{m}J + zR), j+1)$$

for all j. Furthermore, by Lemma 10 above and Lemma 1.2 in [2], it follows that

$$\mathrm{H}(R/\mathfrak{m}I, j) = \mathrm{H}(R/\mathfrak{m}J, j), \quad \mathrm{H}(R/(\mathfrak{m}I + zR), j) = \mathrm{H}(R/(\mathfrak{m}J + zR), j)$$

for all j. Hence we have

$$\mathrm{H}((\mathfrak{m}I : z)/\mathfrak{m}I, j) = \mathrm{H}((\mathfrak{m}J : z)/\mathfrak{m}J, j)$$

for all j. Moreover, since J is $\mathfrak{m}$-full, this implies that

$$\mathrm{length}(\mathfrak{m}J : z/\mathfrak{m}J) = \mathrm{length}(J/\mathfrak{m}J) = \beta_0(J).$$

Hence we get that

$$\beta_0(I) = \mathrm{length}(I/\mathfrak{m}I) \leq \mathrm{length}(\mathfrak{m}I : z/\mathfrak{m}I) = \mathrm{length}(\mathfrak{m}J : z/\mathfrak{m}J) = \beta_0(J).$$

Thus, by the assumption $\beta_0(I) = \beta_0(J)$, it follows that $\mathfrak{m}I : z = I$.

Since I and J are $\mathfrak{m}$-full, it follows by Remark 5 that

$$\beta_0(I) = \beta_0(\overline{I}) + \mathrm{length}((I : \mathfrak{m})/I)$$

and

$$\beta_0(J) = \beta_0(\overline{J}) + \mathrm{length}((J : \mathfrak{m})/J).$$

Hence we have the equality $\beta_0(\overline{I}) = \beta_0(\overline{J})$ by Lemma 9 and $\beta_0(I) = \beta_0(J)$. Thus, our assertion can be obtained by an inductive argument on the number of variables, since $\overline{J}$ is the generic initial ideal of $\overline{I}$ ([6], Corollary 2.15). □

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Department of Mathematics Education, Niigata University, Niigata 950-2181, JAPAN
E-mail address: harima@ed.niigata-u.ac.jp

Faculty of Liberal Studies, Kumamoto National College of Technology, Yatsushiro 866-8501, JAPAN
E-mail address: isogawa@kumamoto-nct.ac.jp

Department of Mathematics, Tokai University, Hiratsuka 259-1292, JAPAN
E-mail address: watanabe.junzo@tokai-u.jp