GALOIS EMBEDDING OF ALGEBRAIC VARIETY AND ITS APPLICATION TO ABELIAN SURFACE

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Abstract. We define a Galois embedding of a projective variety V and give a criterion whether an embedding is Galois or not. Then we consider several representations of the Galois group. Following the method developed in the first half, we consider the structure of an abelian surface with the Galois embedding in the latter half. We give a complete list of all possible groups and show that the abelian surface is isogenous to the square of an elliptic curve.

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1. Introduction

The main purpose of this article is to present a new viewpoint for the study of projective varieties. Let k be the ground field of our discussion, we assume it to be the field of complex numbers, however most results hold also for an algebraically closed field of characteristic zero. Let V be a nonsingular projective algebraic variety of dimension n with a very ample divisor D; we denote this by a pair (V, D). Let $f = f_D : V \hookrightarrow \mathbb{P}^N$ be the embedding of V associated with the complete linear system $|D|$, where $N + 1 = \dim H^0(V, \mathcal{O}(D))$. Suppose that W is a linear subvariety of $\mathbb{P}^N$ satisfying $\dim W = N - n - 1$ and $W \cap f(V) = \emptyset$. Consider the projection π_W with the center W, $\pi_W : \mathbb{P}^N \dashrightarrow W_0$, where W_0 is an n-dimensional linear subvariety not meeting W. The composition $\pi = \pi_W \cdot f$ is a surjective morphism from V to $W_0 \cong \mathbb{P}^n$.

Let $K = k(V)$ and $K_0 = k(W_0)$ be the function fields of V and W_0 respectively. The covering map π induces a finite extension of fields $\pi^* : K_0 \hookrightarrow K$ of degree $d = \deg f(V) = D^n$, which is the self-intersection number of D. It is easy to see that the structure of this extension does not depend on the choice of W_0 but only on W, hence we denote by K_W the Galois closure of this extension and by $G_W = Gal(K_W/K_0)$ the Galois group of K_W/K_0. Note that G_W is isomorphic to the monodromy group of the covering $\pi : V \longrightarrow W_0$, see [4].

Definition 1.1. In the above situation we call G_W the Galois group at W. If the extension K/K_0 is Galois, we call f and W a Galois embedding and a Galois subspace for the embedding respectively.

Definition 1.2. A nonsingular projective algebraic variety V is said to have a Galois embedding if there exist a very ample divisor D satisfying that the embedding associated with $|D|$ has a Galois subspace. In this case the pair (V, D) is said to define a Galois embedding.

If W is the Galois subspace and T is a projective transformation of $\mathbb{P}^N$, then $T(W)$ is a Galois subspace of the embedding $T \cdot f$. Therefore the existence of Galois subspace does not depend on the choice of the basis giving the embedding.

Remark 1.3. For a projective variety V, by taking a linear subvariety, we can define a Galois subspace and Galois group similarly as above. Suppose that V is not normally embedded and there exists a linear subvariety W such that the projection π_W induces a Galois extension of fields . Then, taking D as a hyperplane section of V in the embedding, we infer readily that (V, D) defines a Galois embedding with the same Galois group in the above sense.

By this remark, for the study of Galois subspaces, it is sufficient to consider the case where V is normally embedded.

We have studied Galois subspaces and Galois groups for hypersurfaces in [13], [14] and [15] and space curves in [2] and [17]. The idea introduced above is a generalization of the ones used in these studies. In what follows, we give a criterion for an embedding to be Galois or not and show the existence of several representations of the Galois group when (V, D) defines a Galois embedding. Second we study the structure of abelian surfaces and their Galois groups when the surface has Galois embedding.

Hereafter we use the following notation and convention:

- $Aut(V)$: the automorphism group of a variety V
- $\langle a_1, \cdots, a_m \rangle$: the subgroup generated by $a_1, \cdots, a_m$
- Z_m : the cyclic group of order m
- D_m : the dihedral group of order $2m$
- (f) : the divisor defined by a function f
- $|G|$: the order of a group G
- $\sim$: the linear equivalence of divisors
- $\mathbf{1}_m$: the unit matrix of size m
- $\mathbb{G}(k, m)$: the Grassmanian parametrizing k-planes in m-dimensional projective space $\mathbb{P}^m$
- $e_m := \exp(2\pi\sqrt{-1}/m)$
- $\rho := e_6$
- $D_1 \cdot D_2$: the intersection number of two divisors D_1 and D_2 on a surface
- D^2 : the self-intersection number of a divisor D on a surface
- $[\alpha_1, \ldots, \alpha_m]$: the diagonal matrix with entries $\alpha_1, \ldots, \alpha_m$

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2. Statement of results

By definition, if W is the Galois subspace, then each element σ of G_W is an automorphism of $K = K_W$ over K_0. Therefore it induces a birational transformation of V over V_0. This implies that G_W can be viewed as a subgroup of $Bir(V/W_0)$, the group of birational transformations of V over W_0. Further we can say the following:

Representation 1. *The element of G_W turns out to be regular on V, hence we have a representation*

$$\alpha : G_W \hookrightarrow Aut(V). \tag{1}$$

Therefore, if the order of $Aut(V)$ is small, then V cannot have a Galois embedding. On the other hand, we have examples such that there exist infinitely many distinct Galois embeddings, see Example 4.1.

When (V, D) defines a Galois embedding, we identify $f(V)$ with V. Let H be a hyperplane of $\mathbb{P}^N$ containing W and D' be the intersection divisor of V and H. Since $D' \sim D$ and $\sigma^*(D') = D'$, for any $\sigma \in G_W$, we see that σ induces an automorphism of $H^0(V, \mathcal{O}(D))$.

Representation 2. *We have a second representation*

$$\beta : G_W \hookrightarrow PGL(N, \mathbb{C}). \tag{2}$$

In the case where W is a Galois subspace we identify $\sigma \in G_W$ with $\beta(\sigma) \in PGL(N, \mathbb{C})$ hereafter. Since G_W is a finite subgroup of $Aut(V)$, we can consider the quotient V/G_W and let π_G be the quotient morphism, $\pi_G : V \longrightarrow V/G_W$.

Proposition 2.1. *If (V, D) defines a Galois embedding with the Galois subspace W such that the projection is $\pi_W : \mathbb{P}^N \dashrightarrow W_0$, then there exists an isomorphism $g : V/G_W \longrightarrow W_0$ satisfying $g \cdot \pi_G = \pi$. Hence the projection π is a finite morphism and the fixed loci of G_W consist of only divisors.*

Therefore, π turns out to be a Galois covering in the sense of Namba [7]. Now we present the criterion that (V, D) defines a Galois embedding.

Theorem 2.2. *The pair (V, D) defines a Galois embedding if and only if the following conditions hold*:

(1) *There exists a subgroup G of $Aut(V)$ satisfying that $|G| = D^n$.*

(2) *There exists a G-invariant linear subspace $\mathcal{L}$ of $H^0(V, \mathcal{O}(D))$ of dimension $n+1$ such that, for any $\sigma \in G$, the restriction $\sigma^*|_{\mathcal{L}}$ is a multiple of the identity.*
(3) *The linear system $\mathcal{L}$ has no base points.*

It is easy to see that $\sigma \in G_W$ induces an automorphism of W, hence we obtain another representation of G_W as follows. Take a basis $\{f_0, f_1, \dots, f_N\}$ of $H^0(V, \mathcal{O}(D))$ satisfying that $\{f_0, f_1, \dots, f_n\}$ is a basis of $\mathcal{L}$ in Theorem 2.2. Then we have the representation

$$\beta_1(\sigma) = \begin{pmatrix} \lambda_\sigma & & & \vdots & \\ & \ddots & & \vdots & * \\ & & \lambda_\sigma & \vdots & \\ \cdots & \cdots & \cdots & \vdots & \cdots \\ & \mathbf{0} & & \vdots & M_\sigma \end{pmatrix}.$$

Since the representation is completely reducible, we get another representation using a direct sum decomposition:

$$\beta_2(\sigma) = \lambda_\sigma \cdot \mathbf{1}_{n+1} \oplus M'_\sigma.$$

Thus we can define

$$\gamma(\sigma) = M'_\sigma \in PGL(N-n-1, \mathbb{C}).$$

Therefore σ induces an automorphism on W given by M'_σ.

Representation 3. *We get a third representation*

$$\gamma : G_W \longrightarrow PGL(N-n-1, \mathbb{C}). \qquad (3)$$

Let G_1 and G_2 be the kernel and image of γ respectively.

Theorem 2.3. *We have an exact sequence of groups*

$$1 \longrightarrow G_1 \longrightarrow G \xrightarrow{\gamma} G_2 \longrightarrow 1,$$

where G_1 is a cyclic group.

Corollary 2.4. *If $N = n+1$, then G is a cyclic group.*

This implies that the Galois group is cyclic if the codimension of the embedding is one (cf. [15]). Furthermore we have another representation.

Representation 4. *We have a fourth representation*

$$\delta : G_W \hookrightarrow Aut(C), \qquad (4)$$

where C is a smooth curve in V given by $V \cap L$ such that L is a general linear subvariety of $\mathbb{P}^N$ with dimension $N-n+1$ containing W.

Note that there may exist several Galois subspaces and Galois groups for one embedding (see, for example [2]). Generally we have the following.

Proposition 2.5. *Suppose that (V, D) defines a Galois embedding and let W_i $(i = 1, 2)$ be Galois subspaces such that $W_1 \neq W_2$. Then $G_1 \neq G_2$ in $Aut(V)$, where G_i is the Galois group at W_i.*

Corollary 2.6. *If V is of general type, then there are at most finitely many Galois subspaces for one embedding.*

Remark 2.7. It may happen that there exist infinitely many Galois subspaces for one embedding if the Kodaira dimension of V is small. For example, if $V = \mathbb{P}^1$ and $\deg D = 3$, i.e., $f(V)$ is a twisted cubic, then the Galois lines form two dimensional locally closed subvariety of the Grassmannian $\mathbb{G}(1, 3)$ (cf. [17]).

3. Abelian Surfaces

We apply the methods developed in the previous sections to the study of abelian varieties A. In the case where $\dim A = 1$ and $\deg D = 3$ or 4, we have studied in detail in [2]. *Hereafter we restrict ourselves to the case where* $\dim A \geq 2$.
First we consider several necessary conditions that (A, D) defines a Galois embedding.

Proposition 3.1. *Suppose that A has the Galois embedding with a Galois group G. Let R_π be the ramification divisor for the projection $\pi : A \longrightarrow W_0$. Then, each component of R_π is a translation of an abelian subvariety of dimension $n-1$ and $R_\pi \sim (n+1)D$. In particular, R_π is very ample and ${R_\pi}^n = (n+1)^n|G|$.*

Corollary 3.2. *Simple abelian varieties do not have Galois embeddings.*

We have further results. To state them, we need to prepare some notation. Let G be a subgroup of $Aut(A)$. Then $\sigma \in G$ has the complex representation $\tilde{\sigma} z = M(\sigma)z + t(\sigma)$, where $M(\sigma) \in GL(n, \mathbb{C})$, $z \in \mathbb{C}^n$ and $t(\sigma) \in \mathbb{C}^n$. Fixing the representation, we put $G_0 = \{\ \sigma \in G \mid M(\sigma) = \mathbf{1}_n\}$ and $H = \{\ M(\sigma) \mid \sigma \in G\}$. Then, we have the following exact sequence of groups

$$1 \longrightarrow G_0 \longrightarrow G \longrightarrow H \longrightarrow 1.$$

Suppose that A has a Galois embedding with the Galois group G. Then, $B = A/G_0$ is also an abelian variety and $H \cong G/G_0$ is a subgroup of $Aut(B)$. Moreover, it is easy to see that B/H is isomorphic to $A/G \cong W_0$. Using the notation above, we have the following.

Theorem 3.3. *If A has a Galois embedding, then H is not an abelian group.*

Corollary 3.4. *There exists no abelian embedding of an abelian variety of dimension ≥ 2, i.e., the Galois embedding whose Galois group is abelian does not exist.*

Note that these phenomena do not appear for elliptic curves. In fact, for an elliptic curve we have $H \cong Z_m$, where $m = 2, 3, 4$ or 6. Moreover it may happen that G is an abelian group (cf. [2]). We study the structure of H in detail in the case of abelian surfaces.

Lemma 3.5. *Let A be an abelian surface. Assume that G is a finite automorphism group of A satisfying that A/G is isomorphic to $\mathbb{P}^2$ and let $\pi : A \longrightarrow \mathbb{P}^2$ be the quotient morphism. If $\deg \pi \geq 10$, then $\pi^*(l) = D$ is very ample for each line l in $\mathbb{P}^2$.*

Combining Theorem 2.2 and Lemma 3.5 together, we infer readily the following.

Corollary 3.6. *Under the same assumption and notation of Lemma 3.5, the pair (A, D) defines a Galois embedding.*

Making use of the results of [5], we obtain the following theorem.

Theorem 3.7. *If an abelian surface A has the Galois embedding, then H is isomorphic to one of the following*:

(1) D_3
(2) D_4
(3) *the semi-direct product of groups*: $Z_2 \ltimes H'$, *where* $H' \cong D_4$ *or* $Z_m \times Z_m$ $(m = 3, 4, 6)$

To state case (3) *more precisely, we put $Z_2 = \langle a \rangle$ and $H' = \langle b, c \rangle$. Then the actions of Z_2 on H' are as follows*: *In the former case $H' \cong D_4$, we have $aba = bc^2,\ aca = c,\ c^4 = 1,\ b^2 = 1$ and $bcb = c^{-1}$. In the latter case $H' \cong Z_m \times Z_m$, we have $aba = b^{-1},\ aca = c^{-1},\ b^m = c^m = 1$ and $bc = cb$.*

Corollary 3.8. *If A has a Galois embedding, then the abelian surface $B = A/G_0$ is isomorphic to $E \times E$ for some elliptic curve E.*

Remark 3.9. Although we have $|H| \leq 72$, the order $|G|$ can be arbitrarily large. In fact, for any $m \geq 2$ there exist examples with $|G| = 8m^2$ in Example 4.5 below.

Remark 3.10. In the case where $W \cap f(A) \neq \emptyset$ there exist examples of Galois embeddings such that H is an abelian group, see Remark 4.8 below.

4. Examples

Example 4.1. Let E be the elliptic curve $\mathbb{C}/\Omega$, where $\Omega = (1, \tau)$ is a period matrix such that $\Im\tau > 0$. Let a and b be the automorphisms of E defined by $a(z) = -z$ and $b(z) = z + 1/m$ respectively, where $z \in \mathbb{C}$ and m is a positive integer ≥ 2. Let G be the subgroup of $Aut(E)$ generated by a, b. Then $G = \langle a, b \rangle \cong D_m$; the dihedral group of order $2m$. Let $y^2 = 4x^3 + px + q$ be the Weierstrass normal form of E and $K = \mathbb{C}(x, y)$. Then the fixed field of K by G is rational $\mathbb{C}(t)$, where $t \in \mathbb{C}(x)$. Putting $D = (t)_\infty$; the divisor of poles of t, we infer readily that $\deg D = 2m$ and (E, D) defines a Galois embedding for each m.

We present several examples of abelian surfaces, to which we can apply Corollary 3.6. In order to check that A/G is isomorphic to $\mathbb{P}^2$, we can use the following lemmas. Using the notation in Section 3, we put $H_1 = \{ M(\sigma) \in H \mid \det M(\sigma) = 1 \}$ and let $F(G)$ be the set of the fixed points of G, i.e., the set $\{ x \in A \mid \exists\, \sigma \in G,\ \sigma \neq id,\ \sigma(x) = x\}$. Then the lemmas can be stated as follows (cf. [12, Theorem 2.1]):

Lemma 4.2. *If H satisfies all of the following conditions, then A/G is rational.*

(1) $H \neq H_1$.
(2) *If H is cyclic, then no eigenvalue of the generator is* 1.
(3) *If H has the representation* $\langle \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \rangle$ *or* $\langle \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \rangle$, *then $F(G)$ is not a finite set.*

Since we have a morphism $\pi : A \longrightarrow A/G$, if A/G is smooth, then $C^2 \geq 0$ for each curve C on A/G. Moreover we can say $C^2 > 0$ if H has an irreducible representation (cf. [11, Corollary 3.3.3]).

Lemma 4.3. *If A/G is smooth and rational and H is irreducible, then A/G is isomorphic to $\mathbb{P}^2$.*

Now, let us state examples.

Example 4.4. Let A be the abelian surface with the period matrix

$$\Omega = \begin{pmatrix} -1 & \rho^2 & -\tau & \tau\rho^2 \\ 1 & \rho & \tau & \tau\rho \end{pmatrix} = \begin{pmatrix} -1 & \rho^2 \\ 1 & \rho \end{pmatrix} \begin{pmatrix} 1 & 0 & \tau & 0 \\ 0 & 1 & 0 & \tau \end{pmatrix},$$

where $\Im\tau > 0$. Clearly we have $A \cong E \times E$ where $E = \mathbb{C}/(1, \tau)$. Letting $z \in \mathbb{C}^2$ and $\boldsymbol{v}_i$ be the i-th column vector of Ω $(1 \leq i \leq 4)$, we define t_i to be the translation on A such that $t_i z = z + \boldsymbol{v}_i/m$, where m is an integer ≥ 2. Let a and b be the automorphism of A such that the complex representations are

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} -\rho & 0 \\ 0 & \rho^2 \end{pmatrix}$$

respectively. Put $G_0 = \langle t_1, \ldots, t_4 \rangle$ and $G = \langle G_0, a, b \rangle$. Then G_0 is a normal subgroup of G and $G/G_0 \cong D_3$. Clearly we have $|G| = 6m^4$. Looking at the fixed loci of H, we infer that A is smooth.

Example 4.5. Let E be the elliptic curve in Example 4.1, and let A be the abelian surface $E \times E$. We define several automorphisms on A as follows: let a, b and c be the homomorphisms whose complex representations are

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

respectively. Let t_1 and t_2 be translations on A defined by

$$t_1 z = z + {}^t\left(\frac{1}{m},\ 0\right) \text{ and } t_2 z = z + {}^t\left(0,\ \frac{1}{m}\right),$$

where $z \in \mathbb{C}^2$ and m is an integer ≥ 2. Put $G_0 = \langle t_1, t_2 \rangle$ and $G = \langle G_0, a, b \rangle$. Then G_0 is a normal subgroup of G and $G/G_0 \cong D_4$. Clearly $|G| = 8m^2$. We can make use of Corollary 3.6 and Lemma 4.2 to prove that (A, D) define a Galois embedding. In another way we can show this as follows (the proof will be done at the end of the paper): let Δ and Γ be the elliptic curves in A defined by

$$\Delta = \left\{ \begin{pmatrix} z \\ z \end{pmatrix} \mid\ z \in E \right\} \text{ and } \Gamma = \left\{ \begin{pmatrix} z \\ -z \end{pmatrix} \mid z \in E \right\},$$

respectively. Clearly $\Delta^2 = \Gamma^2 = 0$ and it is easy to see that $\Delta \cdot \Gamma = 4$. Put $\Delta_i = {t_1}^i(\Delta)$ and $\Gamma_j = {t_1}^j(\Gamma)$, where $1 \leq i, j \leq m-1$. Note that $\Delta_i = {t_2}^{m-i}(\Delta)$ and $\Gamma_j = {t_2}^j(\Gamma)$. Then put

$$D = \Delta_0 + \Delta_1 + \cdots + \Delta_{m-1} + \Gamma_0 + \Gamma_1 + \cdots + \Gamma_{m-1},$$

where we assume $\Delta_0 = \Delta$ and $\Gamma_0 = \Gamma$. We have $D^2 = 8m^2$ and infer that D is very ample if $m \geq 2$. We see from Theorem 2.2 that (A, D) defines a Galois embedding with Galois group G.

Example 4.6. Let A be the abelian surface $\mathbb{C}^2/\Omega$ such that Ω is the period matrix

$$\begin{pmatrix} 1 & 0 & i & (1+i)/2 \\ 0 & 1 & 0 & (1+i)/2 \end{pmatrix}, \text{where } i = e_4.$$

Recalling Theorem 3.7, we define the homomorphisms a, b and c, whose complex representations are

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

respectively. Put $G = \langle a, b, c \rangle$. Then $G \cong Z_2 \ltimes D_4$ and $G \subset Aut(A)$ and $A/G \cong \mathbb{P}^2$.

Example 4.7. Let E be the elliptic curve E in Example 4.1 such that $\tau = e_m,\ m = 3,\ 4$ or 6. Let A be the abelian surface $E \times E$. We define automorphisms on A as follows: let a, b and c be the homomorphisms whose complex representations are

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} \tau & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & \tau \end{pmatrix}$$

respectively. Let $G = \langle a, b, c \rangle$. Clearly we have $a^2 = b^m = c^m = 1,\ bc = cb,\ ca = ab$ and $ba = ac$, and $|G| = 2m^2$. Moreover we have $G \cong Z_2 \ltimes (Z_m \times Z_m)$. Put $E_1 = E \times \{0\}$ and $E_2 = \{0\} \times E$, where 0 is the zero element of E, then put $D = m(E_1 + E_2)$, clearly we have $D^2 = 2m^2$. It is well known that

D is very ample if $m \geq 3$. We see from Theorem 2.2 that (A, D) defines a Galois embedding whose Galois group is isomorphic to G.

Let us examine the case $m = 3$ in a different point of view. Since E is defined by the Weierstrass normal form $y^2 = 4x^3 + 1$, we have that $\mathbb{C}(A) = \mathbb{C}(x, y, x', y')$, where ${y'}^2 = 4{x'}^3 + 1$. The automorphisms a, b and c induce the ones of $\mathbb{C}(A)$ as follows:

(i) a^* interchanges x and x', y and y'.
(ii) $b^*(x) = \rho^2 x$ and b^* fixes y, x' and y'.
(iii) $c^*(x') = \rho^2 x'$ and c^* fixes x, y and y'.

Therefore, the fixed field $\mathbb{C}(A)^G$ is $\mathbb{C}(y+y', yy')$, and we have $(y+y')+D \geq 0$ and $(yy') + D \geq 0$. Embedding by $3(E_1 + E_2)$ is the composition of the embedding $E \times E \hookrightarrow \mathbb{P}^2 \times \mathbb{P}^2$ followed by the Segre embedding $\mathbb{P}^2 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^8$. Using homogeneous coordinates (X, Y, Z) [resp. (X', Y', Z')] satisfying that $x = X/Z,\ y = Y/Z$ [resp. $x' = X'/Z', y' = Y'/Z'$], we can express this embedding as

$$f(X, Y, Z, X', Y', Z') = (XX', YX', ZX', XY', \dots, ZZ').$$

Letting $(T_0, \cdots, T_8)$ be a set of homogeneous coordinates of $\mathbb{P}^8$, we can express the Galois subspace by $T_5 + T_7 = T_4 = T_8 = 0$.

Remark 4.8. In the situation of the latter half of Example 4.7, let W be the linear subspace defined by $T_5 = T_7 = T_8 = 0$. Consider the projection π_W with the center W. In this case $f(A) \cap W$ consists of nine sets of points. The projection $\pi_W|_A$ induces the extension of fields $\mathbb{C}(A)/\mathbb{C}(W_0)$, which is a Galois extension with the Galois group isomorphic to $Z_3 \times Z_3$.

Example 4.9. Take the elliptic curve E in Example 4.7 such that $m = 3$. If we take different actions of b and c, then we can get different Galois subspaces. To show this, first we recall the result of Galois subspaces (i.e., Galois points) for plane elliptic curves. Let E be a smooth cubic in $\mathbb{P}^2$. If there exists a Galois point, then E is projectively equivalent to the curve defined by $Y^2Z = 4X^3 + Z^3$ and it has just three Galois points $(X : Y : Z) = (1 : 0 : 0), (0 : -\sqrt{-3} : 1)$ and $(0 : \sqrt{-3} : 1)$. Then we have three projections $\pi : \mathbb{P}^2 \cdots \to \mathbb{P}^1$ given by $\pi(X : Y : Z) = (Y : Z)$, $(X : Y + \sqrt{-3}Z)$ and $(X : Y - \sqrt{-3}Z)$, which yields Galois covering $\pi|_E : E \longrightarrow \mathbb{P}^1$. By taking these projections, we can find nine Galois subspaces for the embedding $E \times E \subset \mathbb{P}^8$. For example, if we take $\pi_1(X : Y : Z) = (Y : Z)$ and
$\pi_2(X : Y : Z) = (X : Y + \sqrt{-3}Z)$, then the fixed field of $\mathbb{C}(A)$ by G is equal to $\mathbb{C}(y + u, yu)$, where $u = (y' + \sqrt{-3})/x'$. Hence the Galois subspace is defined by $T_1 + T_5 + \sqrt{-3}T_8 = T_4 + \sqrt{-3}T_7 = T_2 = 0$ using the coordinates in the last part of Example 4.7.

5. Proofs

We use the notation in Introduction. First we prove Representation 1. Each element $\sigma \in G_W$ induces a birational transformation of V over W_0, we denote it by the same letter σ. In the case where $\dim V = 1$, σ is a morphism, so we assume $\dim V \geq 2$. For a point $P \in V$ we denote by $\langle W, P \rangle$ the linear subvariety spanned by W and P.

Lemma 5.1. *If $\langle W, P \rangle$ meets V at d distinct points, then σ is regular at P and $\sigma(P)$ is one of the points in $\langle W, P \rangle \cap V$.*

Proof. The projection π is a finite morphism near $\pi(P)$ by the hypothesis. Since σ is a birational transformation of V over W_0 and π is finite near $\pi(P)$ and V is smooth (normal is enough), by Zariski's Main Theorem, we see that σ is a morphism (necessarily an isomorphism) from a suitable open set of the form $\pi^{-1}(U)$ containing P. $\square$

If P does not fulfill the assumption of Lemma 5.1, then $\pi(P) \in \Delta_\pi$, where Δ_π is the divisor of the discriminant of π. Let U_P be a small neighborhood of P. Then $Z_P = U_P \cap \pi^{-1}(\Delta_\pi)$ is a set of zero points of some holomorphic function in U_P. Let $\sigma = (\sigma_1, \dots, \sigma_n)$ be an expression of σ on U_P. Then each σ_i is regular and bounded on $U_P \setminus Z_P$ by Lemma 5.1. By Riemann's Extension Theorem ([3, p. 9]), σ is regular at P. Thus σ is regular on V. Hence it becomes an automorphism of V.

Next we prove Proposition 2.1. By definition we have a birational map $g := \pi \cdot {\pi_G}^{-1} : V/G_W \dashrightarrow W_0$. Let W_1 be a general hyperplane of W_0 and let $V_1 = \pi^*(W_1)$ be a very ample divisor. Then $\pi_G(V_1)$ is ample, hence V/G_W is a projective variety. Moreover it is normal. Therefore we have only to prove, by Zariski's Main Theorem, that the total transform $g(x)$ [resp. $g^{-1}(y)$], where $x \in V/G_W$ [resp. $y \in W_0$] do not contain curves. Since π_G is a finite morphism, $g(x)$ contains no curves. Thus, it is sufficient to prove that $\pi^{-1}(x)$ does not contain a curve. Suppose the contrary. Then, let Γ be the curve and W_1 be a hyperplane in W_0 not passing through x. Since $\pi^*(W_1) \sim D$, it is a very ample divisor, hence we have $\pi^*(W_1) \cap \Gamma \neq \emptyset$, which is a contradiction. Thus g is an isomorphism. Since π_G is a finite morphism, so is π. By the theorem of purity of branch loci, due to Zariski, each component of the loci has codimension one.

Now we prove Theorem 2.2. Suppose that (V, D) defines the Galois embedding $f : V \hookrightarrow \mathbb{P}^N$ with the Galois subspace W. Then, by definition and Representation 1, the first assertion (1) is clear. Let W_1 be a general hyperplane in W_0. Then $V_1 = \pi^*(W_1)$ is an irreducible smooth subvariety of codimension one in V by Bertini's theorem, and $\sigma^*(V_1) = V_1$ for any $\sigma \in G_W$. Take $n+1$ independent general hyperplanes W_{1i} of W_0 ($i = 0, 1, \dots, n$) and let f_i be the section of $H^0(V, \mathcal{O}(D))$ defined by the pullback of W_{1i} by π. Then the linear subspace $\mathcal{L}$ generated by $\{f_0, f_1, \dots, f_n\}$ satisfies conditions (2) and (3).

Conversely, we assume that conditions (1), (2) and (3) hold.
Let $\{f_0, f_1, \ldots, f_N\}$ be a basis of $H^0(V, \mathcal{O}(D))$, where $f_0, f_1, \ldots, f_n$ are given in (2). Then $f = (f_0, f_1, \ldots, f_N)$ defines an embedding $V \hookrightarrow \mathbb{P}^N$.
Let $(X_0, X_1, \ldots, X_N)$ be the corresponding set of homogeneous coordinates on $\mathbb{P}^N$ and let W be the linear subspace defined by $X_0 = X_1 = \ldots = X_n = 0$. Since the linear system corresponding to $\mathcal{L}$ has no base points, we infer that $W \cap f(V) = \emptyset$. Therefore the projection induces a morphism $\pi : V \longrightarrow W_0$. Since D is ample, π is surjective. Hence K is a finite extension of $K_0 := k(f_1/f_0, f_2/f_0, \ldots, f_n/f_0)$ and $[K : K_0] = D^n$. Let K^G be the fixed field of K by G. Then, since $\sigma^*(f_i/f_0) = f_i/f_0$ for any $\sigma \in G$, we see that $K^G \supset K_0$. By the assumption $|G| = D^n$ we have that $K^G = K_0$, hence W turns out to be a Galois subspace. This completes the proof.

The proof of Theorem 2.3 is simple. For any $\sigma \in G_1$, define a homomorphism $h : G_1 \longrightarrow k^* = k \setminus \{0\}$ by $h(\sigma) = \lambda_\sigma/\mu_\sigma$, where $\gamma(\sigma) = \lambda_\sigma \cdot \mathbf{1}_{n+1} \oplus \mu_\sigma \cdot \mathbf{1}_{N-n}$. Since h is injective and G_1 is a finite multiplicative subgroup of k^*, G_1 is cyclic.

The proof of Representation 4 is done inductively. Suppose that (V, D) defines a Galois embedding and let G be a Galois group for some Galois subspace W. Then, take a general hyperplane W_1 of W_0 and put $V_1 = \pi^*(W_1)$. The divisor V_1 has the following properties:

(i) If $n \geq 2$, then V_1 is a smooth irreducible variety.
(ii) $V_1 \sim D$.
(iii) $\sigma^*(V_1) = V_1$ for any $\sigma \in G$.
(iv) V_1/G is isomorphic to W_1.

Put $D_1 = V_1 \cap H_1$, where H_1 is a general hyperplane of $\mathbb{P}^N$. Then (V_1, D_1) defines a Galois embedding with the Galois group G (cf. Remark 1.3). Iterating the above procedures, we get a sequence of pairs (V_i, D_i) such that

$$(V, D) \supset (V_1, D_1) \supset \cdots \supset (V_{n-1}, D_{n-1}).$$

These pairs satisfy the following properties:

(a) V_i is a smooth subvariety of V_{i-1}, which is a hyperplane section of V_{i-1}, where $D_i = V_{i+1}$, $V = V_0$ and $D = V_1$ $(1 \leq i \leq n-1)$.
(b) (V_i, D_i) defines a Galois embedding, with the same Galois group G.

In particular, letting C be the curve V_{n-1}, we get the fourth representation.

Next we prove Proposition 2.5. Let $\langle W, P \rangle$ denote the linear subvariety spanned by W and a point P. Suppose the contrary, i.e., $G_1 = G_2$. Then, we infer readily that $\langle W_1, P \rangle \cap V = \langle W_2, P \rangle \cap V$ for any $P \in V$. This implies that $\langle W_1 \cap W_2, P \rangle \cap V = \langle W_1, P \rangle \cap V$. Since $W_1 \neq W_2$, we have $W_1 \cap W_2 \subsetneqq W_1$, therefore we have $\dim(W_1 \cap W_2) \leq N - n - 2$. Thus V is contained in a linear subvariety $\widetilde{W}$ such that $\dim(\pi_W^{-1}(Q) \cap \widetilde{W}) \leq N - n - 1$, where Q is a general point in W_0. Hence we conclude that V is contained in a hyperplane, which is a contradiction.

We proceed with the proof of the assertions in Section 3. First we consider Proposition 3.1. By the ramification formula we have $K_A \sim \pi^*(-(n +$

$1)W_1) + R_\pi$, where K_A is a canonical divisor of A and W_1 is a hyperplane in W_0. Since A is an abelian variety, we have $K_A \sim 0$, hence $R_\pi \sim (n+1)D$.

By the way, we note that if $\sigma \in G$, where $\widetilde{\sigma} z = M(\sigma)z + t(\sigma)$, then the matrix part $M(\sigma)$ induces an automorphism $\widehat{\sigma}$ of A defined by $\widehat{\sigma} z = M(\sigma)z$.

Let R be a reduced part of the irreducible component of R_π. Then there exists $\sigma \in G$ $(\sigma \neq id)$ such that $\sigma|_R = id$. This implies that some translation of R is contained in the kernel of the homomorphism $\widehat{\sigma} - id$, where $\widehat{\sigma}$ is the homomorphism defined above. This means that R is a translation of an abelian subvariety.

Now the proof of Corollary 3.2 is clear, because $\pi : A \longrightarrow W_0$ is not an unramified covering.

Now we consider the proof of Theorem 3.3. Referring to [11, Corollary 3.2.2], since $B/H \cong \mathbb{P}^n$, we see that H is generated by reflections. Suppose that H is an abelian group. Then it can be generated by elements whose complex representations are diagonal matrices $\sigma_i = [\alpha_{i1}, \dots, \alpha_{in}]$ such that $\alpha_{ij} = 1$ if $i \neq j$ and α_{ii} is a root of unity and $\neq 1$. Take an element σ_1 and consider the homomorphism $h : B \longrightarrow B$ defined by $h(z) = \sigma_1(z) - z$, where $z \in \mathbb{C}^n$. Then $h(B)$ is an elliptic curve E. Since H is assumed to be an abelian group, we have $\bar{\sigma} \cdot h = h \cdot \sigma$, where $\bar{\sigma}$ is an automorphism on E induced by σ. We infer from this that there exists a morphism $\bar{h} : B/H \longrightarrow \mathbb{P}^1$, which is a contradiction, since $B/H \cong \mathbb{P}^n$.

In order to prove Lemma 3.5, we make use of the following lemma (cf. [9], [1]).

Lemma 5.2. *Let (A, D) be a polarized abelian surface, where D is an ample divisor on A with $D^2 \geq 4d + 6$. Then D is d-very ample unless there exists an effective divisor Δ on A such that*

$$D \cdot \Delta - d - 1 \leq \Delta^2 < D \cdot \Delta/2 < d + 1.$$

By this lemma it is sufficient to prove that there exists no effective divisor Δ on A satisfying that $\Delta^2 = 0$ and $D \cdot \Delta = 1$ or 2. Put $\Delta = \sum_{i=1}^r m_i C_i$, where m_i is an integer > 0 and C_i is an irreducible curve $(1 \leq i \leq r)$. Since $C_i \cdot C_j \geq 0$ for any i, j, we have that $C_i^2 = 0$ and $C_i \cap C_j = \emptyset$ if $i \neq j$. Thus C_i is an elliptic curve and we infer readily, by taking a translation, that Δ is algebraically equivalent to mE, where E is an elliptic curve and $m > 0$. Since $D \cdot \Delta = 1$ or 2, we get $D \cdot E = 1$ or 2. In the former case, we consider the morphism $g : A \longrightarrow A/E_0$, where E_0 is a subabelian variety, which is a translation of E. Then D will be a section of g, which is a contradiction, since the genus of D is not less than 6. In the latter case, by taking a translation of E, we may assume that it does not belong to the ramification divisor of π. Put $D^2 = d$, which is an even number, and $G_E = \{ \sigma \in G \mid \sigma(E) = E \}$, which is a subgroup of G. Let f be the rational map associated with $|D|$. Then $f(E)$ is a rational curve, since $D \cdot E = 2$. Thus there exists $\sigma \in G$ such that $\sigma(E) = E$ and $\sigma \neq id$, therefore $G_E \neq \{id\}$. Put $G/G_E = \langle \bar{\sigma}_1, \cdots, \bar{\sigma}_r \rangle$, where $\sigma_i \in G$ $(1 \leq i \leq r)$. Then $r < d$ and r is a factor of d. If $\deg \pi(E) = d'$, then $\pi^*(\pi(E)) \sim d'D$.

We can put $\pi^*(\pi(E)) = E_1 + \cdots + E_r$, where $E = E_1$, $E_i = \sigma_i(E)$. Since $\sigma(D) = D$, we have $D \cdot E_i = 2$ for $1 \le i \le r$. Therefore we get $2r = d'd$, hence $d' = 1$ and $r = d/2$. For $1 \le i \le r$ we have $E_1 \cdot E_i + \cdots + E_r \cdot E_i = 2$ and ${E_i}^2 = 0$. Since $r \ge 5$, there exists j satisfying that $(E_1 + E_2) \cdot E_j = 0$ and $E_1 \cdot E_2 \ge 1$. This is a contradiction, because, since $(E_1 + E_2)^2 \ge 2$, the divisor $E_1 + E_2$ is ample (cf. [6, Ch.4, (5.2)]).

The proof of Corollary 3.8 is as follows. Looking at Theorem 1 in [5], we infer easily that except in the case $(4, 2)_1$ B is a product type. Concerning the exceptional case, the period matrix of the abelian surface $B = \mathbb{C}^2/\Omega$ has the expression

$$\Omega = \begin{pmatrix} 1 & 0 & i & (1+i)/2 \\ 0 & 1 & 0 & (1+i)/2 \end{pmatrix}, \text{where } i = e_4.$$

Put $B_0 = E_i \times E_i$, where $E_i = \mathbb{C}/(1, i)$. Then B is isogenous to B_0, hence B is a singular abelian surface, this implies $B \cong F_1 \times F_2$, where F_1 and F_2 are elliptic curves (cf. [10]). Since the scalar matrix $[i, i]$ acts on B, F_1 and F_2 have the automorphism defined by $z \mapsto iz$, where $z \in \mathbb{C}$, hence we have $F_1 \cong F_2 \cong E_i$.

Finally we mention the proof of the last assertion of Example 4.5. Thanks to Lemma 5.2 we have only to prove that there does not exist an elliptic curve F satisfying that $D \cdot F = 2$ or $D \cdot F = 1$. If $\Delta_i \cdot F = 0$ for some i and for some elliptic curve F, then F is a translation of Δ_i, hence $F \cdot \Gamma_j = 4$ $(j = 0, \ldots, m-1)$. This implies $D \cdot F \ge 4m$. If $\Delta_i \cdot F \ge 1$ and $\Gamma_j \cdot F \ge 1$, then we have $D \cdot F \ge 2m$. Therefore we conclude that $D \cdot F \ge 2m$. Clearly we have $A/G \cong \mathbb{P}^2$. Take three general lines l_i $(i = 0, 1, 2)$ on A/G and let f_i be the pull back of l_i by $\pi : A \longrightarrow A/G$. We infer from Theorem 2.2 that (A, D) defines a Galois embedding.

Finally we raise problems.

Problems.

(1) In the situation of Introduction find the set
$\{\, W \in \mathbb{G}(N-n-1, N) \mid G_W \cong S_d \}$.
In particular, is it true that the codimension of the complement of the set is at least two (cf. [8])?

(2) Suppose that $\dim Lin(V) = 0$, W is close to W' (in the Grassmanian) and $W \ne W'$. Then is it true that K_W is not isomorphic to $K_{W'}$? (cf. [16])

(3) For an embedding (V, D) find the structure of Galois group G_W for each $W \in \mathbb{G}(N-n-1, N)$.

(5) Find all the Galois subspaces for one Galois embedding. Especially find the rule of arrangements of Galois subspaces (cf. [2], [15]).

(6) Consider the similar subject in the case where $f(V) \cap W \ne \emptyset$.

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