

## Localization of Eigenstates in One-Dimensional Infinite Disordered Systems with Off-Diagonal Randomness

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Problem of localization of eigenstates is examined for one-dimensional infinite disordered systems with off-diagonal randomness. For this purpose Matsuda and Ishii's theory, based on Furstenberg's convergent theorem on products of random matrices, is generalized by introducing "irreducible sequences"  $S^{(i)}$  and "irreducible transfer matrices"  $Q^{*(i)}$  as useful mathematical tools.

A Furstenberg-type theorem is established for the product of matrices associated with a Markov-chain. This theorem leads to some conclusions about the localization of eigenstates, which are very similar, except for some minor differences, to those obtained by Matsuda and Ishii for systems with diagonal-randomness only.

### § 1. Introduction

Recently Weissman & Cohan<sup>1)</sup> and Bush<sup>2)</sup> have discussed the density of states and the extendedness of an eigenstate of some one-dimensional infinite systems with nearest-neighbour random interaction. They predicted that there are some anomalous features at the middle of the energy band ( $E=0$ ) if the system has off-diagonal randomness (ODR). Theodorou and Cohen<sup>3)</sup> have given a rather general proof, on the basis of the central limiting theorem, that the eigenstate is extended at  $E=0$ . They predicted further that there exists an example in which all states are extended, by using a perturbation expansion of the Green function and the relation proposed by Herbert & Jones<sup>4)</sup> and Thouless.<sup>5)</sup>

However, it has been known that rigorous investigations sometimes bring us to the conclusions which are at variance with those obtained by approximate methods. Theoretical criticisms on Economou and Cohen's work<sup>6)</sup> have been given most clearly by Ishii.<sup>7)</sup> Ishii has cast doubts also to Herbert & Jones and Thouless's relation.<sup>7)</sup> A prediction has been given on the extended state mentioned above by Fleishman and Licciardello.<sup>8)</sup> In a recent paper Odagaki and Yonezawa<sup>9)</sup> have noted that the  $L(E)$  method<sup>10)</sup> should be used very carefully for discussing the localization problem.

The purpose of this paper is to discuss the problem of localization of eigenstates through Matsuda and Ishii's rigorous approach (hereafter referred to as MI<sup>11)</sup> and I<sup>7)</sup>). It becomes then necessary to generalize the Furstenberg convergent theorem<sup>12)</sup> to the case of product of matrices associated with a Markov-chain. The Furstenberg-type theorem thus obtained plays an essential role in this paper.

Our formulation is general in the sense that it can treat the systems with both kinds of randomness: diagonal one (DR) and ODR. For simplicity's sake, however, we confine ourselves in this paper to systems with ODR only and to those with mutually independent DR and ODR.

In the next section the general formulation is given. In §§ 3 and 4 the Furstenberg theorem is generalized to the case of matrices associated with a Markov-chain, for the system with ODR only. The concepts "irreducible sequence" and "irreducible transfer matrix" are introduced in these sections. In § 5 it is shown that the related theorems given in MI and I can also be extended easily. In § 6 it is shown that the same considerations can be made also for systems with ODR and DR. The final section is devoted to conclusions and discussion.

### § 2. Formulation of the problem

The system considered in this paper is an infinite linear chain described by the Hamiltonian

$$H = \sum_n |n\rangle \varepsilon_n \langle n| + \sum_n (|n\rangle t_{n,n+1} \langle n+1| + |n+1\rangle t_{n+1,n} \langle n|), \quad (-\infty \leq n \leq \infty) \tag{2.1}$$

where  $t_{n,n+1}$  and  $t_{n+1,n}$  are assumed to be nonzero, real and bounded:

$$t_{n,n+1} = t_{n+1,n} \quad \text{and} \quad 0 < e < |t_{n,n+1}| < T < \infty, \quad (\text{for all } n) \tag{2.2}$$

The system would become an assembly of separate pieces if some of  $t$ 's vanish. We assume further that the transfer integrals  $\{t_{n,n+1}\}$  can take, mutually independently,  $r$  different values with a common probability distribution:

$$P(t_{n,n+1} = t^{(i)}) = P^{(i)} = P_i, \quad (\text{independent of } n) \quad i = 1, \dots, r, \\ \sum_{i=1}^r P^{(i)} = 1, \tag{2.3}$$

For simplicity it is here assumed that the diagonal elements  $\{\varepsilon_n\}$  can take, independently of  $\{t_{n,n+1}\}$ ,  $r'$  kinds of different values with a common probability:

$$P'(\varepsilon_n = \varepsilon^j) = P'^{(j)} = P'_j, \quad (\text{independent of } n) \quad j = 1, \dots, r', \\ \sum_{j=1}^{r'} P'^{(j)} = 1. \tag{2.4}$$

The eigenvalue equation of our system can be written in the form of a set of recurrence relations:

$$E \cdot a_n = \varepsilon_n \cdot a_n + t_{n,n+1} \cdot a_{n+1} + t_{n,n-1} \cdot a_{n-1}, \tag{2.5}$$

where  $a_n$  is the amplitude of the eigenstate at the site  $n$ . Transfer matrix  $T_n$  is then defined as follows:

$$\begin{pmatrix} a_{n+1} \\ a_n \end{pmatrix} = \begin{pmatrix} (E - \varepsilon_n)/t_{n,n+1} & -t_{n,n-1}/t_{n,n+1} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_n \\ a_{n-1} \end{pmatrix} = T_n \begin{pmatrix} a_n \\ a_{n-1} \end{pmatrix}, \quad (2.6)$$

$$\begin{aligned} T_n &= \sqrt{\pm \frac{t_{n,n-1}}{t_{n,n+1}}} \begin{pmatrix} \frac{(E - \varepsilon_n) \operatorname{sign}(t_{n,n+1})}{\sqrt{\pm t_{n,n+1} \cdot t_{n,n-1}}} \mp \sqrt{\pm \frac{t_{n,n-1}}{t_{n,n+1}}} \\ \sqrt{\pm \frac{t_{n,n+1}}{t_{n,n-1}}} & 0 \end{pmatrix} \\ &= \sqrt{\pm \frac{t_{n,n-1}}{t_{n,n+1}}} \cdot Q_n(t_{n,n+1}, \varepsilon_n, t_{n,n-1}), \end{aligned} \quad (2.7)$$

where the signs  $\pm$  and  $\mp$  correspond to the cases  $t_{n,n+1} \cdot t_{n,n-1} > 0$  and  $< 0$ , respectively. The sign of  $\det Q_n (= \pm 1)$  coincides with that of  $t_{n,n+1} \cdot t_{n,n-1}$ . The following relation holds about the exponential growth  $\|X_n\|$ , ( $X_0 \in R^2$  and  $\|X_0\| \neq 0$ ):

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \left\| \left( \prod_{l=1}^n T_l \right) X_0 \right\| = \lim_{n \rightarrow \infty} \frac{1}{n} \log \left\| \left( \prod_{l=1}^n Q_l \right) X_0 \right\|. \quad (2.8)$$

Our final purpose is to discuss whether the limit (2.8) exists, the value of which is finite and positive, i.e.,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \left\| \left( \prod_{l=1}^n Q_l \right) X_0 \right\| = 2\gamma > 0, \quad (2.9)$$

or not, independent of  $X_0 \in (R^2 - \{0\})$  and of sample systems. For this it is necessary to generalize the Furstenberg convergent theorem to the case of a product of matrices representing a Markov-chain. In the following two sections discussion is made for the systems with ODR only. Discussion for the systems with ODR and DR will be given briefly in § 6.

### § 3. Random chains with ODR only

3.1. A set  $\mathcal{Q}^0$  is defined as an aggregation of all sample systems of the type defined in § 1 and with ODR only. A physically reasonable measure  $\mu_i^0$  can be introduced on  $\mathcal{Q}^0$  in the same way as in I;  $\mu_i^0$  can be extended to a complete measure on the whole Borel sets of the interval  $\mathcal{Z}[0, 1]$ . It will be seen later that it is more convenient to omit from  $\mathcal{Q}^0$  a set of special sample systems  $\{\omega_j^0\}$  with a sufficiently small measure  $\varepsilon_N > 0$  in order to avoid a mathematical difficulty which occurs when we apply the Furstenberg convergent theorem. It will be seen there that we can make the measure  $\varepsilon_N > 0$  as small as we hope so that  $\varepsilon_N$  does not affect physical phenomena. We thus define  $\mathcal{Q} = \mathcal{Q}^0 - \{\omega_j^0\}$ .

Obviously a complete measure  $\mu = \mu^0 / (1 - \varepsilon_N)$  is meaningful (as  $\mu^0(\{\sigma_j^0\}) = \varepsilon_N$ ) also on the smallest Borel sets including the intervals  $\mathcal{Z}[0, 1] - \{\sigma_j^0\}$ , where  $\{\sigma_j^0\}$  is a set of intervals corresponding to the set  $\{\omega_j^0\}$ . We will use the expres-

sion  $\mu^0(\{\omega_j^0\})$  for  $\mu^0(\{\sigma_j^0\})$  in this paper. In the cases where we have no mathematical difficulty we understand that  $\{\omega_j^0\} = \phi$  and  $\varepsilon_N = 0$ .

The set of sample systems  $\mathcal{Q}$  can be decomposed, in one way, as follows:

$$\mathcal{Q} = \sum_{i=1}^r \mathcal{Q}^{(i)},$$

$$\mathcal{Q}^{(i)} = \{\omega; t_{0,1}(\omega) = t^{(i)} \quad \text{for } \forall \omega \in \mathcal{Q}^{(i)}\}, \quad (i=1 \sim r) \tag{3.1}$$

$\mathcal{Q}^{(i)}$  is a subset of  $\mathcal{Q}$  composed of systems for which

$$t_{0,1}(\omega) = t^{(i)}. \tag{3.2}$$

When we write

$$\mu(\mathcal{Q}^{(i)}) = \nu^{(i)} = \mu^0(\mathcal{Q}^{(i)}) / (1 - \varepsilon_N) = P^{(i)}(1 - \varepsilon_N^{(i)}) / (1 - \varepsilon_N), \quad (i=1 \sim r) \tag{3.3}$$

each  $(\mathcal{Q}^{(i)}, \beta^{(i)}, \mu^{(i)} = \mu/\nu^{(i)})$  becomes a probability space.

3.2. On each sample  $\omega \in \mathcal{Q}^{(i)} (i=1 \sim r)$ , the  $l$ -th "irreducible sequence" of the  $i$ -th kind  $S_l^{(i)}$ , and the corresponding  $l$ -th "irreducible transfer matrix" of the  $i$ -th kind  $Q_l^{*(i)}$ , are defined as follows: An "irreducible sequence" of the  $i$ -th kind  $S^{(i)}$  is a sequence of  $t_{n,n+1}$ 's which fulfills the conditions that (1) the preceding  $t$  is equal to  $t^{(i)}$ , (2) it ends with  $t^{(i)}$  and (3) no other  $t$ 's in the sequence are equal to  $t^{(i)}$ . An "irreducible transfer matrix" of the  $i$ -th kind  $Q^{*(i)}$  is a product of transfer matrices  $Q$  corresponding to an "irreducible sequence"  $S^{(i)}$  of  $t$ . The irreducible sequences and transfer matrices of the  $i$ -th kind are introduced in order to describe the right semi-infinite chain starting from  $t_{0,1} = t^{(i)}$ . Obviously the left semi-infinite chain can also be described with the correspondingly defined sequences and matrices. More precisely, a right semi-infinite chain can be represented, under the condition that  $t_{0,1} = t^{(i)}$ , by an infinite sequence of the irreducible sequences

$$(S_1^{(i)}, S_2^{(i)}, \dots, S_l^{(i)}, \dots) \tag{3.4}$$

with probability 1, and also by a product of the irreducible transfer matrices

$$(\dots Q_3^{*(i)} \cdot Q_2^{*(i)} \cdot Q_1^{*(i)}) \tag{3.5}$$

with probability 1. As an example some possible sets of the irreducible sequences and the irreducible transfer matrices are shown in Table I, for the case  $r=2$  and  $i=1$ , with their values of probability distribution  $\mu^{(i)}(Q^{*(i)}) = \mu^{(i)}(S^{(i)})$ .

It is now obvious that  $\det Q^{*(i)} = 1$  and  $Q^{*(i)} \in SL(2, R)$ . We define  $G^{(i)}$  as the smallest closed subgroup of  $SL(2, R)$  including all kinds of the irreducible transfer matrices of the  $i$ -th kind  $Q^{*(i)}$ . Then an infinite sequence of  $Q^{*(i)} \{Q_l^{*(i)}; l=1, 2, 3, \dots\}$  can be regarded as a sequence of mutually independent  $G^{(i)}$ -valued random variables with a common distribution  $\mu^{(i)}$ . Now we apply the Furstenberg convergent theorem to our set of irreducible transfer matrices to obtain the follow-

Table I. An example of the irreducible sequences, the corresponding irreducible transfer matrices and the values of the distribution in the case  $r=2$  and  $i=1$ . It is shown in § 4 that  $\varepsilon_N^{(1)}$  is equal to the sum of  $(1-P^{(1)})^l P^{(1)}$  ( $l=N^{(1)} \sim \infty$ ) in this case and thus equal to  $(1-P^{(1)})N^{(1)}$ .

$S^{(1)}$	$Q^{*(1)}(S^{(1)})$	$\mu^{(1)}(Q^{*(1)}) = \mu^{(1)}(I(S^{(1)}))$
$\{t^{(1)}\} \setminus t^{(1)}$	$Q(t^{(1)}, t^{(1)})$	$P^{(1)}/(1-\varepsilon_N^{(1)})$
$\{t^{(1)}\} \setminus \{t^{(2)} t^{(1)}\}$	$Q(t^{(1)}, t^{(2)}) Q(t^{(2)}, t^{(1)})$	$(1-P^{(1)}) \cdot P^{(1)}/(1-\varepsilon_N^{(1)})$
$\vdots$	$\vdots$	$\vdots$
$\{t^{(1)}\} \setminus \overbrace{\{t^{(2)} t^{(2)} \dots t^{(2)} t^{(1)}\}}^{n-1}$	$Q(t^{(1)}, t^{(2)}) (Q(t^{(2)}, t^{(2)}))^{(n-2)} Q(t^{(2)}, t^{(1)})$	$(1-P^{(1)})^{(n-1)} \cdot P^{(1)}/(1-\varepsilon_N^{(1)})$
$\vdots$	$\vdots$	$\vdots$
$\{t^{(1)}\} \setminus \overbrace{\{t^{(2)} t^{(2)} \dots t^{(2)} t^{(1)}\}}^{N^{(1)}-1}$	$Q(t^{(1)}, t^{(2)}) (Q(t^{(2)}, t^{(2)}))^{(N^{(1)}-2)} Q(t^{(2)}, t^{(1)})$	$(1-P^{(1)})^{(N^{(1)}-1)} \cdot P^{(1)}/(1-\varepsilon_N^{(1)})$

ing conclusion:

If  $G^{(i)}$  satisfies  $F$ -condition<sup>11)</sup> for an energy  $E$  with the condition that  $\int \|Q^{*(i)}\| d\mu^{(i)}(Q^{*(i)}) < \infty$  ( $\|Q^{*(i)}\| = \sup \|Q^{*(i)} X\|$ ,  $X \in R^2$  and  $\|X\| = 1$ ), then

$$\lim_{m \rightarrow \infty} \frac{1}{m} \log \left\| \left( \prod_{i=1}^m Q_i^{*(i)} \right) X_0 \right\| = 2\gamma^{*(i)} > 0 \tag{3.6}$$

with probability 1 on the sample space  $\mathcal{Q}^{(i)}$  for all  $X_0 \in (R^2 - \{0\})$ . We thus have

$$\lim_{m \rightarrow \infty} \left( \frac{m}{n^{(i)}(m)} \right) \frac{1}{m} \log \left\| \left( \prod_{i=1}^m Q_i^{*(i)} \right) X_0 \right\| = P^{(i)} 2\gamma^{*(i)} = 2\gamma^{(i)} > 0 \tag{3.7}$$

with probability 1 on  $\mathcal{Q}^{(i)}$  for all  $X_0 (\neq 0)$ , where  $n^{(i)} = n^{(i)}(m)$  is the number of  $Q$  contained in the product of the sequence of  $Q^{*(i)}$ 's, i.e.,

$$\left( \prod_{j=1}^{n^{(i)}(m)} Q_j \right) = \prod_{i=1}^m Q_i^{*(i)}. \tag{3.8}$$

3.3. It has been shown that there exists a positive number  $\gamma^{(i)}$  defined in (3.7), if  $G^{(i)}$  satisfies  $F$ -condition for an energy  $E$  with the condition  $\int \|Q^{*(i)}\| d\mu^{(i)}(Q^{*(i)}) < \infty$ . Consequently, it is apparent in this case that for an sufficiently small  $\varepsilon$  there exists an integer  $N^{(i)}$  such as for  $n \geq N^{(i)}$ ,

$$\alpha^{(i)} e^{2(\gamma^{(i)} - \varepsilon)n^{(i)}(m)} < a_{n^{(i)}(m)}^2 + a_{n^{(i)}(m)-1}^2 < \beta^{(i)} e^{2(\gamma^{(i)} + \varepsilon)n^{(i)}(m)} \tag{3.9}$$

for each  $n^{(i)}(m)$  where  $t_{n, n+1} = t^{(i)}$ . Quantities  $\alpha^{(i)}$  and  $\beta^{(i)}$  are some positive finite numbers.

The relation (3.9) provides us, however, with only partial information about the exponential growth of the wave function; it does not guarantee the existence of  $\gamma$  in (2.9) with probability 1 on  $\mathcal{Q}$  for any  $X_0 \neq 0$ . There are three cases.

Case a) Existence of  $\gamma^{(i)}$  is guaranteed for every subsets ( $i=1 \sim r$ ) (for an energy  $E$  with probability 1).

Case b) Existence of  $\gamma^{(i)}$  is guaranteed for at least one subset but at the same

time it is not guaranteed for at least one of another subsets.

Case c) Existence of  $\gamma^{(i)}$  cannot be guaranteed for any subset.

It is shown in Appendix A that at least in the case a) the existence of  $\gamma$  is guaranteed for the energy with probability 1 on  $\mathcal{Q}$  for any  $X_0 \neq 0$  and

$$\gamma = \gamma^{(1)} = \gamma^{(2)} = \dots = \gamma^{(r)}, \tag{3.10}$$

that is, the limiting value (2.9) exists for the energy with probability 1 on  $\mathcal{Q}$  for any  $X_0 \neq 0$ .

We thus reach the following conclusion: A sufficient condition for the exponential growth of the wave function is that the property (3.6) is proved for every subset  $\mathcal{Q}^{(i)}$  ( $i=1 \sim r$ ).

It is noted that for the model adopted in this paper only the cases a) and c) appear.

#### § 4. A Furstenberg-type theorem for the systems with ODR only

As mentioned just above, a sufficient condition for the exponential growth of the wave function is that the property (3.6) is proved for every subset  $\mathcal{Q}^{(i)}$  ( $i=1 \sim r$ ) for the energy  $E$  with probability 1. We call it the *GF-condition* (generalized *F-condition*). A sufficient condition for the property (3.6) to be valid (for a given energy  $E$ ) for one subset  $\mathcal{Q}^{(i)}$  is

- 1)  $G^{(i)}$  satisfies *F-condition*,
- 2)  $\int_{SL(\alpha, \beta)} \|Q^{*(i)}\| d\mu^{(i)}(Q^{*(i)}) < \infty$ .

The second condition can be made to be satisfied when we omit from  $G^{(i)}$  the irreducible transfer matrices corresponding to the irreducible sequences the lengths of which are larger than a sufficiently large integer  $N^{(i)} \geq N/P^{(i)}$ . The finite integer  $N^{(i)}$  can be made as large as one hopes so that the integer does not affect physical phenomena, that is, the probability distribution of the subtracted set can be made as small as one hopes by taking a large integer  $N$ . It can be done by constructing the ensemble defined in § 2 by subtracting from  $\mathcal{Q}^0$  a set  $\{\omega_j^0\}$  with a sufficiently small measure  $\varepsilon_N > 0$ . Now apparently, it is adequate to understand

$$\begin{aligned} \{\omega_j^0\}^{(i)} \in \mathcal{Q}^{0(i)}, \quad \mu^0(\{\omega_j^0\}^{(i)}) = P^{(i)} \varepsilon_N^{(i)} \quad (i=1 \sim r) \\ \{\omega_j^0\} = \sum_{i=1}^r \{\omega_j^0\}^{(i)}, \quad \varepsilon_N = \sum_{i=1}^r P^{(i)} \varepsilon_N^{(i)}, \end{aligned} \tag{4.1}$$

and each  $\{\omega_j^0\}^{(i)}$  consists of sample systems, each sample of which includes at least one irreducible sequence of the  $i$ -th kind, the length of which is greater than  $N^{(i)}$ . The ensemble  $\mathcal{Q}$  should therefore be understood, when it is necessary, as an aggregation of all sample systems in which each sample  $\omega \in \mathcal{Q}^{(i)}$  ( $i=1 \sim r$ ) can be represented by two infinite sequences of the irreducible sequences, the lengths of which are less than or equal to  $N^{(i)}$ , describing the right and the left

parts of the chain.

After omitting the irreducible sequences of this kind, we have only to consider the condition 1) for each subset  $\mathcal{Q}^{(i)}$ . It is evident that we can make the finite integer  $N$  as large as we hope and accordingly the positive value  $\epsilon_N$  can be made as small as we hope so that the existence of  $\epsilon_N$  does not affect physical phenomena.

The following results are obtained about the  $GF$ -condition, with a result in Appendix B,

case (1)  $r=2$  and  $|t^{(2)}| \neq |t^{(1)}|$

The  $GF$ -condition is satisfied for all the possible energies except  $E=0$ .

case (2)  $r=2$  and  $t^{(2)} = -t^{(1)}$

The  $GF$ -condition is not necessarily fulfilled for every energy in the band.

case (3)  $r \geq 3$  ( $r < \infty$ )

The  $GF$ -condition is satisfied for all of the possible energies except  $E=0$  because  $G_{(r)}^{(i)}$  of the system includes at least one subgroup  $G_{(2)}^{(i)}$  which corresponds to that in the case (1).

### § 5. Extension of other theorems in MI and I

Once the  $GF$ -condition is proved to be satisfied, it is straightforward to derive the conclusions about the localization of eigenstates; it suffices to establish some auxiliary theorems. These theorems can easily be derived by slightly modifying the corresponding theorems in MI and I.

The theorems corresponding to the theorems 3, 4 and 6 in MI can be obtained by extending the relation (4.1) in MI as

$$(\phi^+(n) \cdot \phi^+(n+1) - \phi^+(n+1) \cdot \phi^+(n)) = t_{0,1}/t_{n,n+1}, \quad (5.1)$$

and modifying the relations (4.5)  $\sim$  (4.9), (2.9)  $\sim$  (2.11) and (4.13)  $\sim$  (4.15) in MI in an entirely similar way. The theorem 9.3 in I can be extended, in the same way, by extending the relations (9.29) and (9.30) in I as

$$X_{n+1}Y_n - Y_{n+1}X_n = t_{0,1}/t_{n,n+1}, \quad (5.2)$$

$$G_{n,m}(E) = Y_n(E)Y_m(E) \sum_{l=m}^{\infty} \frac{t_{0,1}/t_{l,l+1}}{Y_l(E)Y_{l+1}(E)}, \quad (5.3)$$

respectively, and modifying similarly the relations (A5.6), (A5.8)  $\sim$  (A5.11) and (A5.14) in I.

Thus we can conclude that eigenfunctions of the systems with ODR only are localized with probability 1 on  $\mathcal{Q}$  if the  $GF$ -condition is satisfied.

It is important to note that the extension indicated above is independent of whether the system under consideration has only ODR or has both ODR and



DR. Thus the same conclusions about the localization in MI and I can readily be obtained also for the systems with ODR and DR, if the *GF*-condition is proved to be valid for such systems. We shall prove it in the next section.

**§ 6. Validity of the *GF*-condition for the chains with ODR and DR**

In this section we shall prove that the *GF*-condition is fulfilled by the systems with ODR and DR, if those are independent of each other. To do this it is only necessary to introduce a slightly different measure  $\mu_1'$  for the sample space and to define the "irreducible sequences" more carefully.

6.1. Each sample system characterized by a sequence  $\{\epsilon_n, t_{n,n-1}; n=1, 2, 3, \dots\}$  can be represented by an  $r$  and  $r'$ -adic number

$$\sigma = 0_{.q_1' q_2' q_3' \dots} \tag{6.1}$$

or

$$\sigma = 0_{.q_1 q_2 \dots} \quad (\text{and } \epsilon_1) \tag{6.1'}$$

contained in  $\Sigma = [0, 1]$ , where one-to-one correspondence is established by

$$\epsilon_n = \epsilon(q_n') \quad \text{and} \quad t_{n-1,n} = t(q_n). \quad (n=1, 2, 3, \dots) \tag{6.2}$$

Then a suitable measure  $\mu_1^{0'}$  can be introduced on the ensemble of systems  $\Omega^0$  in the almost same way as that in I.

6.2. The "irreducible sequence" of the  $i$ -th kind  $S^{(i)}$  should be defined more carefully in this stage. It is the sequence of  $\epsilon_n$  and  $t_{n,n-1}$  which fulfills the conditions that (1) the preceding  $t$  is equal to  $t^{(i)}$ , (2) it ends with  $t^{(i)}$  and (3) there appear no  $t^{(i)}$  in any other position in the sequence. Obviously each  $\epsilon_n$  can take all possible values in the sequence. Then the right half of the chain from  $t_{0,1}(=t^{(i)})$  can be represented, with probability 1, by an infinite sequence of the irreducible sequence of the  $i$ -th kind

$$(S_1^{(i)}, S_2^{(i)}, S_3^{(i)}, \dots) \tag{6.3}$$

6.3. Now the *GF*-condition of the system can be discussed in the almost same way as that in §§ 3 and 4. The following result is readily obtained, as each  $G^{(i)}$  contains the subgroup  $G_D^{(i)}$ , which corresponds to systems with DR only. Each subgroup  $G_D^{(i)}$  is the same as what has been discussed extensively in MI and I. Case (4): General case where both ODR and DR exist. The *GF*-condition is satisfied for all of the possible energies including  $E=0$ .

**§ 7. Concluding remarks**

The first conclusion obtained in this paper is that a Furstenberg-type theorem can be established for products of random matrices representing a Markov-chain;



the *GF*-condition is a sufficient condition for the convergence of the quantity (2.8). It should be remarked here that our method will be effective also for systems which are more general than those treated in this paper, so that the Furstenberg-type theorems can be established also for these systems.

Our second conclusion is that any infinite chain which belongs to the category considered can be made to have an exponentially localized solution for a given energy  $E$  for which the *GF*-condition is satisfied, by modifying a transfer integral  $t_{0,1}$  (or an atomic energy  $\epsilon_0$ ) such that it gets a suitable value, except for the chains with measure zero on  $\mathcal{Q}$ .

The third conclusion is that almost all of the eigenstates (for the energy  $E$ ) are exponentially localized, in infinite systems, with probability 1 on  $\mathcal{Q}$ ,\* in the sense that the following relation hold with probability 1 on  $\mathcal{Q}$ ,

$$|G_{n,m}(E-i0)| < O(\exp\{-\gamma(E)|n-m|\}), \text{ in the limit } |n-m| \rightarrow \infty.$$

The fourth conclusion is that the weak absence of diffusion<sup>7)</sup> takes place also with probability 1 on  $\mathcal{Q}$ , for above mentioned energies.

The fifth conclusion is drawn in the case (2) that all of the eigenstates, except those for  $E = E_{\pm} = \pm 2|t^{(1)}|$  (the values of band edges of the regular system with  $t^{(1)}$  or  $t^{(2)}$ ) are extended. This is because every irreducible transfer matrix in  $G^{(1)}$  (and  $G^{(2)}$ ) can be diagonalized by a non-unitary transformation, so that the diagonal matrix elements have the form  $e^{i\theta}$  and  $e^{-i\theta}$  ( $\theta$ : real). In this case the randomness of phase of the transfer energies does not seem to play any role for the localization of the eigenstates. The value of  $\gamma$  is obviously 0 in this case except for  $E = E_{\pm}$ .

It will be interesting to discuss the rate of exponential growth  $\gamma$  and the feature of spectral densities. Here we confine ourselves to note, however, that some recent developments in the theory of spectral densities<sup>13)</sup> (on some different model from ours) are possibly helpful to obtain some further conclusions on the localization problem.

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### Appendix A

—A Proof of the Statement Given in (3.3) for the Case a)—

In the case a) existence of  $\gamma^{(i)}$  is guaranteed on each subset  $\mathcal{Q}^{(i)}$  ( $i=1 \sim r$ ).

\* It should be mentioned that probability 1 on  $\mathcal{Q}$  represents probability  $1 - \epsilon_N$  on  $\mathcal{Q}^0$  and measure zero on  $\mathcal{Q}$  does measure  $\epsilon_N$  on  $\mathcal{Q}^0$ .

We rearrange here the indices  $i$  such that

$$i^{(1)} \leq i^{(2)} \leq i^{(3)} \leq \dots \leq i^{(r)}. \tag{A.1}$$

We have for each  $\omega \in \mathcal{Q}^{(i)} (i=1 \sim r)$  a sequence of numbers  $n^{(i)}(m)$  appeared in (3.7):

$$n^{(i)}(m), \quad (m=1, 2, 3, \dots) \tag{A.2}$$

From the exponential properties (3.6) and (3.7) it follows that for sufficiently small positive number  $\varepsilon > 0$  there exist integers  $N^{(i)}$  such that

$$\begin{aligned} t^{(i)} e^{n^{(i)}(m) \{2\gamma^{(i)}(E) - \varepsilon\}} &< a_{n^{(i)}(m)}^2(E) + a_{n^{(i)}(m)+1}^2(E) \\ &< \beta^{(i)} e^{n^{(i)}(m) \{2\gamma^{(i)}(E) + \varepsilon\}} \quad \text{for } n^{(i)}(m) \geq N^{(i)} (i=1 \sim r) \end{aligned} \tag{A.3}$$

independent of the initial condition  $a_0$  and  $a_1$ . Obviously the site corresponding to  $n^{(i)}(m)$  has  $t_{n^{(i)}(m), n^{(i)}(m)-1} = t^{(i)}$ . From the independence of the property (A.3) of the initial condition  $X_0 (\neq 0)$  we can conclude that the exponential properties (3.6) and (3.7) and thus (A.3) are valid with probability 1 on the set of systems  $\mathcal{Q}$ . It is apparent from (A.3) that at least one of the quantities,  $a_{n^{(i)}(m)}(E)$  and  $a_{n^{(i)}(m)-1}(E)$ , satisfies the following inequalities

$$\left( \frac{1}{2} \alpha^{(r)} e^{n^{(r)}(m) \{2\gamma^{(r)}(E) - \varepsilon\}} \right) < a_{n^{(r)}(m)}^2(E), \tag{A.4}$$

$$\left( \frac{1}{2} \alpha^{(r)} e^{n^{(r)}(m) \{2\gamma^{(r)}(E) - \varepsilon\}} \right) < a_{n^{(r)}(m)+1}^2(E) \tag{A.4'}$$

for each  $n^{(r)}(m) (m=1, 2, 3, \dots)$ .

When the inequality (A.4) is satisfied we have

$$\left( \frac{1}{2} \alpha^{(r)} e^{\{2\gamma^{(r)}(E) - \varepsilon\}} \right) e^{n^{(r)}(m)-1 \{2\gamma^{(r)}(E) - \varepsilon\}} < a_{n^{(r)}(m)-1}^2(E) + a_{n^{(r)}(m)}^2(E),$$

that is,

$$\left( \frac{1}{2} \alpha^{(r)} e^{\{2\gamma^{(r)}(E) - \varepsilon\}} \right) e^{n^{(r)}(m') \{2\gamma^{(r)}(E) - \varepsilon\}} < a_{n^{(r)}(m')}^2(E) + a_{n^{(r)}(m')+1}^2(E) \tag{A.5}$$

independent of the kind of the corresponding (the preceding) transfer integral  $t^{(r)}$ . When the inequality (A.4') is satisfied, we have

$$\left( \frac{1}{2} \alpha^{(r)} e^{-\{2\gamma^{(r)}(E) - \varepsilon\}} \right) e^{n^{(r)}(m)+1 \{2\gamma^{(r)}(E) - \varepsilon\}} < a_{n^{(r)}(m)+1}^2 + a_{n^{(r)}(m)+2}^2,$$

that is,

$$\left( \frac{1}{2} \alpha^{(r)} e^{-\{2\gamma^{(r)}(E) - \varepsilon\}} \right) e^{n^{(r)}(m'') \{2\gamma^{(r)}(E) - \varepsilon\}} < a_{n^{(r)}(m'')}^2 + a_{n^{(r)}(m'')+1}^2 \tag{A.5'}$$

independent of the kind of the corresponding (the next) transfer integral  $t^{(r)}$ .

These inequalities show that

$$\gamma^{(r)} \leq \gamma^{(1)} \leq \gamma^{(2)} \leq \dots \leq \gamma^{(r)} \quad (\text{A}\cdot 6)$$

and hence

$$\gamma^{(1)} = \gamma^{(2)} = \dots = \gamma^{(r)} = \gamma. \quad (\text{A}\cdot 7)$$

This implies that the limiting value (2.9) exists (with  $\gamma$  in (3.9)) for the energy with probability 1 on  $\mathcal{Q}$  for any  $X_0 \neq 0$ .

### Appendix B

—A Proof of the Result about the GF-Condition for  $E=0$   
in the Cases (1), (2) and (3)—

It is shown in this Appendix that each  $G^{(i)}$  does not necessarily satisfy the GF-condition for  $E=0$  in the cases (1), (2) and (3).

First consider the cases (1) and (2). In these cases it is easily seen that the closed subgroup  $G^{(1)}$  (or  $G^{(2)}$ ) is composed of the elements

$$\pm \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \pm \begin{pmatrix} \alpha^2 & 0 \\ 0 & \alpha^{-2} \end{pmatrix} \right\}, \quad (\text{B}\cdot 1)$$

where  $\alpha = \sqrt{|t^{(1)}/t^{(2)}|}$  (or  $\sqrt{|t^{(2)}/t^{(1)}|}$ ). In the case (2) ( $\alpha=1$ ), it is obvious that the closed subgroup is compact. This completes the proof. In the case (1) ( $0 < \alpha < \infty$  and  $\alpha \neq 1$ ), the following reducible non-compact subgroup  $R$  of  $G^{(1)}$  (or  $G^{(2)}$ ) can be constructed

$$R = \left\{ \pm \begin{pmatrix} \alpha^{2n} & 0 \\ 0 & \alpha^{-2n} \end{pmatrix}; n: \text{all integers} \right\}. \quad (\text{B}\cdot 2)$$

The (left and right) co-set of  $R$  on the elements  $\pm \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$  is

$$C = \left\{ \pm \begin{pmatrix} 0 & -\alpha^{-2n} \\ \alpha^{2n} & 0 \end{pmatrix}; n: \text{all integers} \right\}. \quad (\text{B}\cdot 3)$$

It is seen that the product of any elements  $c, c' \in C$  is an element in  $R$  and the coset of  $R$  on any element  $c \in C$  is  $C$ . Accordingly it is concluded that

$$G^{(i)} = R^{(i)} + C^{(i)}, \quad (i=1, 2) \quad (\text{B}\cdot 4)$$

that is, there exists a reducible non-compact subgroup  $R^{(i)}$  of  $G^{(i)}$  with the index 2.

It is now apparent that the essential feature is the same also in the case (3). The difference lies only in rather complex expressions which appear on constructing the non-compact reducible subgroup (of the type  $R$ ) with the index 2.

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