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On a Random System Which Reveals Anomalous Localization of Wave Functions

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The anomalous localization of wave functions is numerically studied using a one-dimensional chaotic field with a long range correlation. The level statistics of energy eigenvalues and the distribution of Lyapunov exponents are discussed from the viewpoint of multi-fractal analysis and large deviation theory.

1. *Introduction*

Recently, the localization of wave functions in the systems without translational invariance has been studied theoretically and experimentally, with keen interest given to their scaling properties.[1] In purely one-dimensional random systems, the exponential localization is a universal phenomenon which is characterized by the fact that the energy eigenvalues are point spectra. On the other hand, the spectrum may become singular continuous when the potential field is quasi-periodic[2] (e.g., Fibonacci lattice) and/or ultra-metric.[3] The correlation length of potential fields directly reflects the localization properties of wave functions.

In this article, we will study the effect of long range correlation in potential fields. For a systematic approach to this problem, we will use a tightly binding model with an analytically tractable random field of one-dimension, which is generated in a deterministic way by means of a simple chaotic map, the so-called modified Bernoulli shift ($|X_i|\leq 1$),[4]

$$X_{i+1}=\phi(X_i)\equiv\begin{cases} X_i-\dfrac{1+C-\epsilon}{(1-C)^B}(1-X_i)^B-\epsilon\,, & (X_i\geq C)\\[2ex] X_i+\dfrac{1-C-\epsilon}{(1+C)^B}(1+X_i)^B+\epsilon\,, & (X_i< C)\end{cases} \tag{1}$$

where B and C are system parameters satisfying $B\geq 0$ and $-1<C<1$, and ϵ is a small parameter describing a perturbation.

From this mapping the potential field σ_i is defined on a one-dimensional lattice as follows; $\sigma_i=+1$ (for $X_i\geq 0$) and $\sigma_i=-1$ (for $X_i<0$), where the integer i stands for the lattice site ($-\infty<i<\infty$).

Random fields generated as above by deterministic ways have a remarkable difference from the usual random fields which are generated by purely probabilistic ways; the deterministic random fields cannot be characterized by a unique ergodic measure, but include many other ergodic measures. This property comes from the fact that a chaotic dynamical system such as described by Eq. (1) permits the coexistence of inherent multiple ergodic measures.[5] Even if there is only one most dominant measure μ_1 in the system, the statistical properties of the random field cannot be uniquely determined solely by μ_1 alone, and the higher order correlations should be decided by taking account of the non-dominant (singular) measures μ_i $(i=2, 3, \cdots)$. In this sense the deterministic chaotic field is quite different from the purely probabilistic field described by only one ergodic measure μ_1. Furthermore, when several dominant measures $(\mu_1{}^1, \mu_1{}^2, \cdots, \mu_1{}^k)$ are coexisting, the phase separation into the corresponding k-phases can be induced in the system under consideration. Then the system is called multi-ergodic.[6),7)]

Indeed, the model system treated in this article permits such a kind of phase separation at a certain value of the bifurcation parameter, $B=2$. The purpose is to show typical aspects of the localization of wave functions in a systematic way by changing the coherent length of the chaotic field, and to characterize the transition at the critical point $B=2$.

2. *Characterization of the chaotic field generated by the modified Bernoulli shift*[4)]

Since the mapping defined by Eq. (1) is a linear circle map at $B=0$ and $\epsilon=0$, the random field obtained for $\{\sigma_i\}$ becomes periodic (or quasi-periodic) for the rational (or irrational) values of C respectively. The parameter C controls the mean value $\langle\sigma\rangle\equiv\lim_{N\to\infty}(1/2N)\sum_{i=-N}^{i=N}\sigma_i=-2C$, but in what follows we fix the value of C to be zero and $\epsilon=10^{-10}$, and limit our main discussion to the case $B>1$. When $B=1$ is satisfied, the mapping ϕ of Eq. (1) is the Bernoulli shift, and the spatial correlation is short range for $1\leq B\leq 3/2$. But long range correlation (LRC) is obtained for $3/2\leq B\leq 2(\epsilon\to 0)$,

$$C(k)\equiv\langle\sigma_i\sigma_{i+k}\rangle\simeq k^{-\alpha}. \quad \left(\alpha=\frac{2-B}{B-1} \text{ for large } k\right) \tag{2}$$

Furthermore, the correlation becomes quite singular for $B\geq 2$

$$C(k)\simeq 1-(\epsilon^* k)^{-\alpha}, \tag{3}$$

which becomes constant in the asymptotic limit $\epsilon\to 0$, since $\epsilon^*\cong O(\epsilon^{1/\beta})$ $(\beta=B/(B-1))$ is satisfied. This situation was understood as the onset of the asymptotic non-stationary correlation (ANSC) regime in previous papers.[4)] In the LRC regime $(B\geq 3/2)$, the cluster size m defined by $m=$integer $\{\sigma_{i+j}>0$ $(j=1, 2, \cdots, m)$: $\sigma_{i+m+1}<0$, $\sigma_{i-1}<0\}$ becomes very large. Then the distribution of the cluster size $P(m)$ obeys an inverse power law $P(m)\simeq m^{-\beta}$. The appearance of the ANSC regime is accompanied by the divergence of the mean cluster size, $\langle m\rangle\simeq\sum_{m=1}^{\infty}mP(m)\simeq\sum_{m=1}^{\infty}m^{-1/(B-1)}$, which goes to infinity for $B\geq 2$, while in the stationary long range correlation regime $(3/2\leq B\leq 2)$ the variance is divergent, $\langle m^2\rangle=\sum_{m=1}^{\infty}m^{-(2-B)/(B-1)}$, though the mean value $\langle m\rangle$ is finite. Other statistical properties of the chaotic field $\{\sigma_i\}$ were studied in the previous papers,[4),7)] but the details will be skipped here, and we next explain the localization of wave functions under the chaotic field $\{\sigma_i\}$ by using the tightly binding model.

3. *Localization of wave functions*

Denoting the value of the wave function on the site i by ψ_i, the mapping $(\psi_{i-1}, \psi_i) \mapsto (\psi_i, \psi_{i+1})$ is described by

$$\begin{pmatrix} \psi_{i+1} \\ \psi_i \end{pmatrix} = M_i \begin{pmatrix} \psi_i \\ \psi_{i-1} \end{pmatrix} = \begin{pmatrix} E-\sigma_i & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \psi_i \\ \psi_{i-1} \end{pmatrix}. \tag{4}$$

The localization length l is defined by the inverse of the Lyapunov exponent (L-exponent) λ for large system size N,

$$l_N{}^{-1} = \lambda_N = \frac{1}{N} \log \left[\text{max. eigenvalue of } \prod_{j=1}^{N} M_j\right]. \tag{5}$$

The distribution of the L-exponent $P(\lambda_N)$ is obtained from the ensemble of sample processes (Fig. 1).

When the potential field $\{\sigma_i\}$ is short-ranged, the distribution is approximately gaussian and the exponential localization is correctly realized. But for the LRC regime ($B \geq 3/2$) the distribution becomes non-gaussian and a singular peak grows up at $\lambda \cong 0$. This singular peak reflects the appearance of large clusters ($\sigma = 1$ or $\sigma = -1$). In the thermodynamical limit $N \to \infty$, the weight of the singular peak is less than the normal background distribution for $3/2 \leq B < 2$ and the mean value $\langle \lambda_\infty \rangle$ is non-zero finite. However, for ANSC ($B \geq 2$) the singular peak is strongly enhanced and the normal background gradually suppressed. Then the mean localization length goes to infinity, i.e., the wave function becomes extended in a statistical sense.

In the short-range correlation regime ($1 \leq B \leq 3/2$), exponential localization is in excellent agreement with data; but the anomalous localization seems to gradually appear for $B \geq 3/2$, and the fluctuation of the L-exponent becomes large. Furthermore, in the ANSC regime ($B \geq 2$), the L-exponent $\langle \lambda_\infty \rangle$ seems to approach zero, though there remains a small finite value $\langle \lambda_\infty \rangle$ in the numerical calculations due to the effect of the perturbation ϵ ($\sim 10^{-10}$). These are clearly shown in Figs. 2 and 3. Figure 2 shows the anomalous convergence of the variance,

$$\langle \Delta\lambda_N{}^2 \rangle \equiv \langle \lambda_N{}^2 \rangle - \langle \lambda_N \rangle^2 \cong N^{-\gamma}, \tag{6}$$

and Fig. 3 shows the L-exponent. If the convergence of the L-exponent obeys a normal central limit theorem, the index γ should be unity ($\gamma = 1$). However, the fluctuation of the L-exponent is quite abnormal in our system. The index γ gradually decreases in the LRC regime ($B \geq 3/2$), and indeed is almost constant

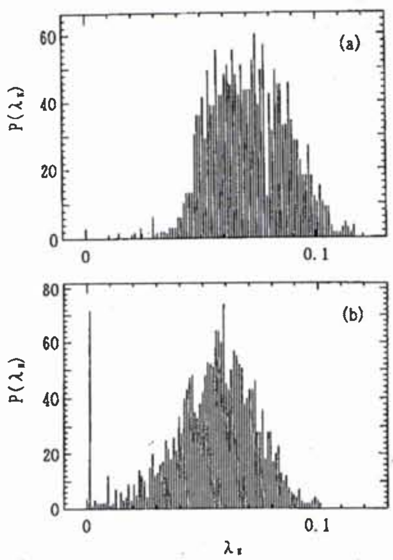

Fig. 1. Histogram of L-exponents ($N = 2^{10}$). (a) $B = 1.5$, (b) $B = 1.7$.

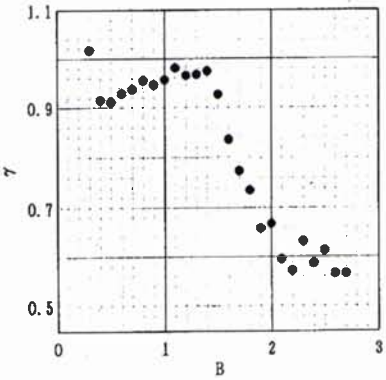

Fig. 2. The parameter B dependence of the index γ defined by Eq. (6) ($N=2^{10}$).

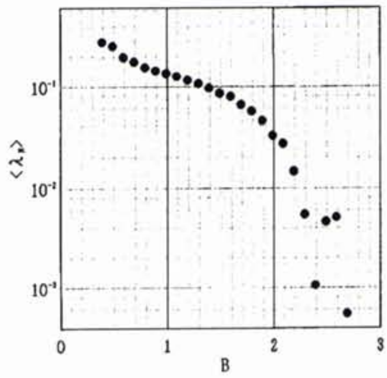

Fig. 3. The parameter B dependence of the mean Lyapunov exponent $\langle\lambda_N\rangle$ ($N=2^{10}$).

($\gamma\cong 0.6$) in the finite range calculation for the asymptotic non-stationary regime ($B \geq 2$).

We have no rigorous theory to explain the behavior obtained in Figs. 2 and 3. But the following large deviation analysis might be useful;[7] the distribution of the L-exponent should be expressed by

$$P(\lambda_N)\simeq N^{\gamma/2}\exp[-N^{\gamma}\psi(\lambda_N)]\,, \tag{7}$$

where the entropy function $\psi(\lambda_N)$ is approximated as $\psi(\lambda_N)\propto(\lambda_N-\langle\lambda_N\rangle)^2$. Then the variance satisfies $\langle\Delta\lambda_N{}^2\rangle\cong N^{-\gamma}$. In the forthcoming papers, the localization of wave functions in our system will be proved by using the Frustenberg theorem, and the transmission coefficient as well as the Thouless number will be precisely analysed.[11]

4. *Level statistics of energy eigenvalues*

The density of state $P(E)$ is illustrated in Fig. 4. As the localization of wave functions becomes weak when the value of B increases, the spectrum is surmised to change from the point spectrum to the absolutely continuous one. To see this transition, multi-fractal analysis is used for the energy spectrum. As we treat finite-size systems in what follows, we cannot clearly identify each spectral type.

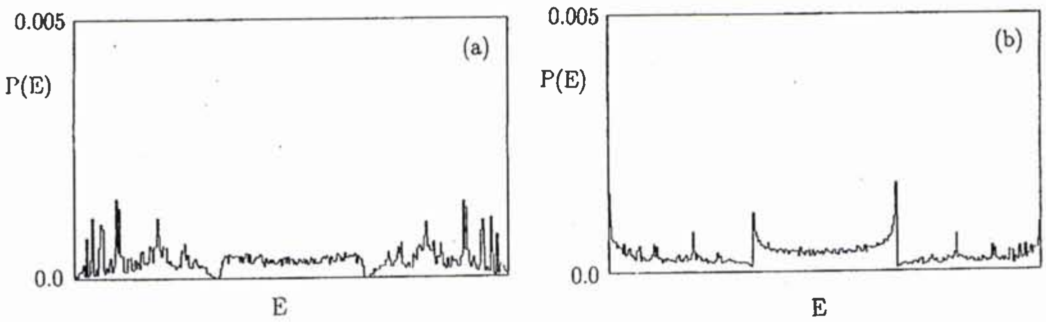

Fig. 4. The density of state $P(E)$ for one sample ($N=1.6\times10^4$). (a) $B=1.0$, (b) $B=2.1$.

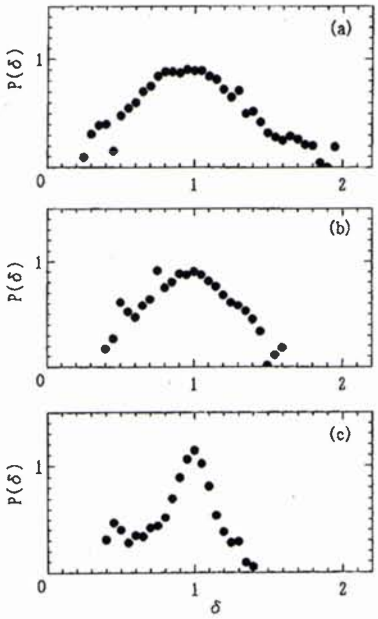

Fig. 5. Distribution of the local dimensions defined by Eq. (8) ($P(\delta)=\exp(f(\delta))$; $\Delta\to 2^{-10}$, $N=1.6\times 10^4$). (a) $B=1.0$, (b) $B=2.0$ and (c) $B=2.3$.

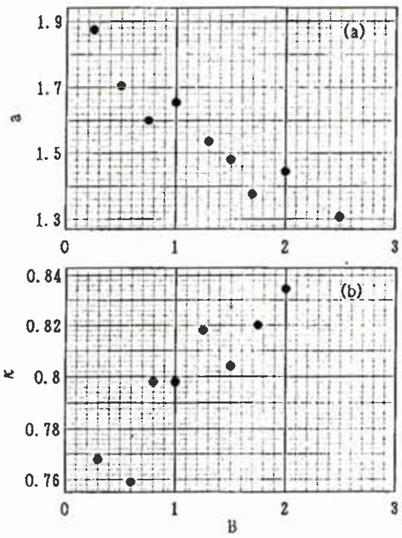

Fig. 6. The parameter B dependence of the fitting exponent: (a) Exponential fitting for small s (<0.5) by $P(s)\sim e^{-as}$, (b) stretched exponential fitting for large s (>0.5) by $P(s)\sim e^{-bs^{\kappa}}$.

However, a kind of systematic changes can be observed in the dimension spectrum. Instead of the usual technique based on the Legendre transformation from the generalized Renyi dimension, we adopt a direct method to determine the local dimension δ. First, we divide the energy axis into boxes with size Δ, and count the total number of energy eigenstate in each box $M(\Delta)$. The local dimension δ is defined in each box,[8]

$$\delta=\lim_{\Delta\to 0}\frac{\ln M(\Delta)}{\ln\Delta}\,. \tag{8}$$

The linearity of the $\log M$-$\log\Delta$ relation was observed near $\Delta\sim 2^{-10}$ at the system size $N=1.6\times 10^4$. The distribution of the dimension $P(\delta)$ is shown in Fig. 5. The distribution is unimodal in the stationary regime ($1\leq B<2$), though the variance is large. On the other hand, in the ANSC regime ($B\geq 2$) double peaks come to appear. These strange situations may be called *fat fractal* spectra, which support the superposition of the absolutely continuous and the singular continuous components. At the critical point ($B=2$), the distribution reveals the anomalous steepness. There is no theoretical explanation for these remarkable changes, but these seem to reflect the onset of multi-ergodicity[7] as was discussed in the beginning of this article. A similar result is shown in Ref. 9). As the results observed here include the finite-size effect, the detailed analysis must be continued. Further calculations will be reported in other papers.

The statistics of level spacing also represents the fat fractal property mentioned above. Denote the level spacing between two nearby eigenvalues by $s=E_n-E_{n-1}$. The distribution of the spacing $P(s)$ is well approximated by the exponential function

$P(s)=e^{-as}$ in the short-ranged correlation regime ($1\leq B\leq 3/2$) for small s (<0.5). This is consistent with the fact that the level statistics is Poissonian for the point spectra.[10] However, in the LRC regime ($B\geq 3/2$) the distribution does not obey the exponential fitting for large s (≥ 0.5) but is rather adjustable by a stretched exponential function $P(s)\cong\exp[-bs^{\kappa}]$ with $\kappa<1$. Such cross-over phenomena are often observed in our system (Fig. 6). The detailed analysis of this paper will be studied in the forthcoming papers.[11]

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