Long-range correlations in quantum systems with aperiodic Hamiltonians

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An efficient algorithm for the computation of correlation function (CF) at very long distances is presented for quantum systems whose Hamiltonian is formed by the substitution aperiodic sequence alternating over unit intervals in time or space. The algorithm reorganizes the expression of the CF in such a way that the evaluation of the CF at distances equal to some special numbers is related to a family of graphs generated recursively. As examples of applications, we evaluate the CF, over unprecedentedly long time intervals up to order of 10^{12}, for aperiodic two-level systems subject to kicking perturbations that are in the Thue-Morse, the period-doubling, and the Rudin-Shapiro sequences, respectively. Our results show the presence of long-range correlations in all these aperiodic quantum systems. [S1063-651X(97)16103-7]

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Considerable attention has been recently devoted to the long-time behavior of correlation functions (CF's) in quasiperiodically or, more generally, aperiodically driven quantum systems [1–6]. The question of particular interest is whether the quantum suppression of chaos, typical of time-independent and time-periodic Hamiltonians, can be circumvented, or at least strongly weakened, in Hamiltonians that are aperiodic in time. This question arises from the following background. In periodically driven systems, as the time-evolution operator U is periodic in time, according to the Floquet theorem, wave packets evolve as superpositions of periodic functions multiplied by $\exp(i\omega_\nu t)$, where ω_ν are the eigenvalues of the Floquet operator. In bounded systems, the spectrum of ω_ν is discrete, therefore the correlation functions do not decay. On the other hand, when the system is subject to an aperiodic driving, which represents an intermediate situation between the random and the periodic extremes, the Floquet theorem is no longer applicable, so that the time evolution need not be periodic or deterministically aperiodic. One might expect that the correlation functions decay to zero for sufficiently long time or, at least, that this happens when the perturbation is strong enough.

A number of papers in this aspect seemed to result in somewhat contradictory conclusions [1]. While some authors believed that there occurs a transition from the nondecaying correlations (nonmixing behavior) to decaying correlations (mixing behavior) when perturbation is increased [2,3], others questioned the existence of such a transition [4,5]. The basic difficulty in obtaining a clearcut answer to this question lies in the absence of an analytical solution, and the practical impossibility, up to now, of evaluating the autocorrelation functions over sufficiently *long*-time intervals to establish or exclude the sporadic revivals of the CF, the presence of which means the existence of nondecaying correlations.

In this paper, an efficient algorithm is presented to alleviate this difficulty for some aperiodically driven systems where the aperiodicity of the perturbation is generated by some substitution rules. We consider here as examples only two-level systems subject to three specific types of aperiodically modulated kicking perturbation that follow, respectively, the Thue-Morse (TM), period-doubling (PD) and Rudin-Shapiro (RS) sequences. The algorithm is, however, expected to be applicable to other aperiodically driven systems that are based on the substitution rules or possess self-similarity.

The two-level system subject to a kicking perturbation that is aperiodic over unit time intervals is described by the following Hamiltonian

$$H(t)=\tfrac{1}{2}E\sigma_z+\tfrac{1}{2}\sigma_x\sum_{n=1}^{\infty} a_{\nu(n)}\delta(t-n), \tag{1}$$

where σ_x and σ_z are Pauli matrices, $\nu(n)$, being either 1 or 0, follows an aperiodic sequence that is generated by the specific substitution rule, and $a_{\nu(n)}=a_1$ or a_0 are two different kickings. Considering the discrete times, one can write the solution of the Shrödinger equation as follows [5,6]:

$$|\psi(n)\rangle=U_{\nu(n)}U_{\nu(n-1)}\cdots U_{\nu(2)}U_{\nu(1)}|\psi(0)\rangle\equiv W_n|\psi(0)\rangle, \tag{2}$$

where

$$U_{\nu(n)}=e^{-(i/2)a_{\nu(n)}\sigma_x}e^{-(i/2)E\sigma_z}. \tag{3}$$

In the present study, we are interested in the long-time behavior of the CF [6,7],

$$C(r)=\lim_{N\to\infty}\frac{1}{N-r}\sum_{k=1}^{N-r}\langle\psi(k)|\psi(k+r)\rangle=\lim_{N\to\infty}C_N(r), \tag{4}$$

the decaying of which, for large r, indicates mixing behavior and ergodicity, whereas the revivals of which would signal the presence of long-range correlations and nonmixing behavior. However, as is well known, the direct numerical evaluation of $C_N(r)$ for $r>10^5$ and $N>10^7$ is not economi-

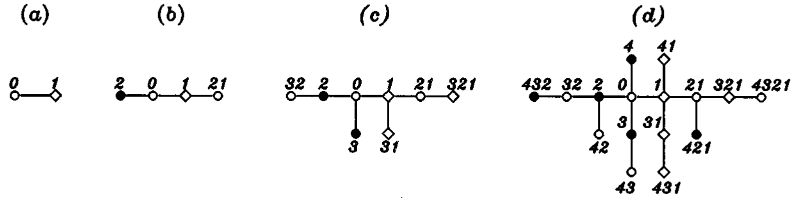

FIG. 1. The first four members of the graph family, G_1–G_4, serving to calculate the correlation functions $C_{2^{L+1}}(2^L)$ to $C_{2^{L+4}}(2^L)$ for the TM system at arbitrary integer L.

cal, and in fact rather difficult due to the numerical error accumulated. Fortunately, for aperiodic quantum systems whose aperiodicity is constructed with the substitution rule, the self-similar nature of the Hamiltonians allows a recursive reorganization of expression (4) with the help of a family of tree graphs, by which it becomes possible to evaluate $C_N(r)$ for very large r (and even much larger N to have convergence) equal to some special numbers. Although the decaying of $C_N(r)$ at such peculiar values of r may not provide any useful information, the revivals of the CF at such values of r would be a signal of nondecaying (long-range) correlations. For the cases with the TM, PD, and RS aperiodic Hamiltonians, such special numbers for r are found to be 2^L, while those for N are 2^{L+M}, with L and M positive integers. We shall now describe in some detail the algorithm for evaluating $C_N(r)$ at $r=2^L$ for the TM, PD, and RS two-level systems.

It should be emphasized that the reason for choosing the TM, PD, and RS sequences as our first working examples is that these aperiodic sequences display rather different types of aperiodic modulations, in the sense that they have the ''wandering exponents'' $\omega<0$, $\omega=0$, and $\omega>0$, respectively [8]. Here the wandering exponent ω is defined by [8–10]

$$\Delta(N)=\sum_{n=1}^{N}\,(a_{\nu(n)}-\overline{a})\sim N^{\omega}, \tag{5}$$

where N is the length of the time series, and $\overline{a}$ is the averaged value of modulation

$$\overline{a}=\lim_{N\to\infty}\,\frac{1}{N}\sum_{n=1}^{N}\,a_{\nu(n)}\,, \tag{6}$$

which is set to be zero for all three cases studied here.

TM SYSTEM

The Hamiltonian describing a two-level system with kicking perturbation modulated in the TM sequence is given by Eq. (1) with $a_{\nu(n)}=a_1=A$ if $\nu(n)=1$ and $a_{\nu(n)}=a_0=-A$ for $\nu(n)=0$. Here $\nu(n)$ follows the binary TM sequence deduced from the substitution rule $1\to10$, $0\to01$, and started with 1. Notice that a_1 and a_0 are so chosen as to make $\overline{a}$ vanish. The method of evaluating $C(r)$ at $r=2^L$ and $N=2^{L+M}$ is based on a family of graph G_M, with $M=1,2,3,\ldots$, the recursive law for which is interestingly related to that of the TM sequence itself. Figure 1 shows the first four members of the graph family, whose growth rules are described below. Similar growth rules serve to compute the CF for a large class of TM-driven quantum systems.

Each member of the graph family consists of nodes and branches that connect nodes in a treelike way. Each node is labeled by a finite string of distinct integers that are arranged in decreasing order from the left to the right. In addition, the nodes are classified into three types, distinguished by solid circles (●), empty circles, (○) and diamonds (◇) [see Fig. 1]. The first member G_1, shown in Fig. 1(a), generates the entire family of graphs G_M by the recursive laws described below:

(1) The M-stage graph G_M is generated from its previous stage G_{M-1}.

(2) When generating the graph G_M (see also below) from the graph G_{M-1}, every node, labeled, say, by $n_pn_{p-1}\cdots n_1$ in G_{M-1}, will survive and, in addition, grow a node denoted by $Mn_pn_{p-1}\cdots n_1$. For simplicity we omit the rightmost integer 0 for all nodes except for the one labeled by a single integer 0.

(3) Each node of G_{M-1} grows a new node in G_M of the type based on the following rules: (a) a node of type ○ yields a node of type ●; (b) a node of type ● produces a node of type ○; and (c) a node of type ◇ grows a node of the same type ◇. The only exception lies in the fact that the node labeled by $(M-1)\,\cdots 21$ and of the type ◇ (○) in G_{M-1} will give birth to a node of the type ○ (◇) in G_M rather than of type ◇ (●). See, e.g., the node ○ 4321 in G_4 [Fig. 1(d)] grown by the node ◇ 321 in G_3 [Fig. 1(c)] for the case with $M=4$. The graphs G_2, G_3, G_4 in Fig. 1 and, in fact, the graphs up to any stage M are easily made from the growth laws given above.

Now we turn to the rules relating the graph G_M and the autocorrelation function $C_N(r)$. Rewriting Eq. (4) as

$$C_N(r)=\langle\psi(0)|\hat{C}_N(r)|\psi(0)\rangle, \tag{7}$$

with

$$\hat{C}_N(r)=\frac{1}{N-r}\sum_{k=1}^{N-r}W_k^{\dagger}W_{k+r} \tag{8}$$

for $r=2^L$ and $N=2^{L+M}$, one has

$$\hat{C}_{2^{L+M}}(2^L)=\frac{1}{2^{L+M}-2^L}\hat{S}_M(2^L). \tag{9}$$

Here $\hat{S}_M(2^L)$ is an operator that can be obtained by summing up all nodes in the graph G_M except the one labeled by $M(M-1)\cdots21$. The string of distinct integers labeling each node, together with the type of the node (denoted by ○, ●, and ◇), determines the operator with which the node makes contributions to $\hat{S}_M(2^L)$. In the following, we illus-

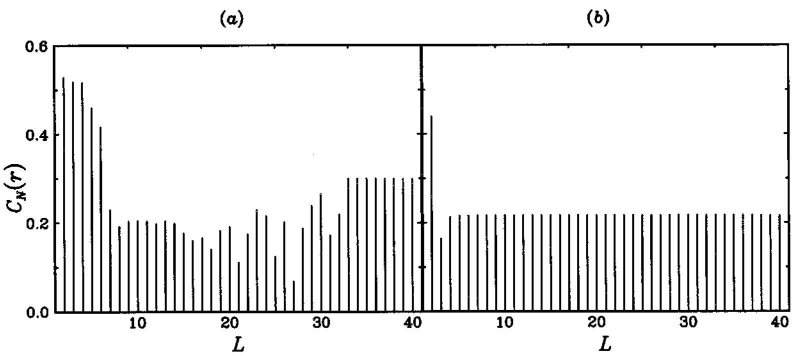

FIG. 2. Absolute value of the autocorrelation functions $C_N(r)$, with $N=2^{L+7}$ and $r=2^L$, as a function of L for the TM two-level system with $E=2\pi/3$ and (a) $A=0.05(\pi/2)$, (b) $A=0.5(\pi/2)$.

trate the contribution by each node; the derivation of such correspondence for the TM system will be presented as an example in the Appendix.

The operator contributed by a node labeled by $n_p n_{p-1} \cdots n_2 n_1$ is

$$W^{\dagger}_{n^* \times 2^L} X^*_L W_{(n^*+1)\times 2^L},$$

where $n^* = \lfloor (2^{n_p} + 2^{n_{p-1}} + \cdots + 2^{n_2} + 2^{n_1})/2 \rfloor + 1$, with $\lfloor x \rfloor$ denoting the integer part of x, and

$$X^*_L = \begin{cases} 2^L & \text{for a node of type } \diamondsuit \\ X_L & \text{for a node of type } \bigcirc \\ \overline{X_L} & \text{for a node of type } \bullet . \end{cases} \tag{10}$$

Here the operators X_L and $\overline{X_L}$ are given by the recursion laws

$$X_{L+1} = \overline{W_{2^L}} X_L W^{\dagger}_{2^L} + \overline{X_L},$$
$$\overline{X_{L+1}} = W_{2^L} \overline{X_L}\, \overline{W^{\dagger}_{2^L}} + X_L \tag{11}$$

and the initial conditions

$$X_1 = \overline{W_1} W^{\dagger}_1 + 1,$$
$$\overline{X_1} = W_1 \overline{W^{\dagger}_1} + 1, \tag{12}$$

with $\overline{W_{2^L}}$ given by W_{2^L} after the exchanging of each $\nu(n)=1$ $(a_{\nu(n)}=A)$ and $\nu(n)=0$ $(a_{\nu(n)}=-A)$. The recursion relations for W_{2^L} and W_{2^L} themselves are governed by the substitution rule of the TM sequence,

$$W_{2^{L+1}} = \overline{W_{2^L}} W_{2^L}, \quad \overline{W_{2^{L+1}}} = W_{2^L} \overline{W_{2^L}}. \tag{13}$$

As an example, we present the expression for $S_3(2^L)$, which is obtained from the graph G_3, shown in Fig. 1(c), and thus consists of seven terms:

$$\begin{aligned} S_3(2^L) &= (\bigcirc\ 0) + (\diamondsuit\ 1) + (\bullet\ 2) + (\bigcirc\ 21) \\ &\quad + (\bullet\ 3) + (\diamondsuit\ 31) + (\bigcirc\ 32) \\ &= W^{\dagger}_{2^L} X_L W_{2\times 2^L} + 2^L W^{\dagger}_{2\times 2^L} W_{3\times 2^L} + W^{\dagger}_{3\times 2^L} \overline{X_L} W_{4\times 2^L} \\ &\quad + W^{\dagger}_{4\times 2^L} X_L W_{5\times 2^L} + W^{\dagger}_{5\times 2^L} \overline{X_L} W_{6\times 2^L} \\ &\quad + 2^L W^{\dagger}_{6\times 2^L} W_{7\times 2^L} + W^{\dagger}_{7\times 2^L} X_L W_{8\times 2^L}. \end{aligned} \tag{14}$$

Here the symbols (○ 0), (◇ 1), (● 2), etc. denote the contributions made by the corresponding nodes in the graph G_3 [see Fig. 1(c)]. Similar symbols are used later in Eqs. (15) and (20) to denote contributions to $S_3(2^L)$ by the nodes in the graph. Notice that the node ◇ 321 in graph G_3 does not contribute to $S_3(2^L)$.

We have computed $C_{2^{L+M}}(2^L)$ for all $2 \leqslant L \leqslant 40$ with $M=7$, where the time lag $r=2^L$ is less than 1% of the sample length 2^{L+M}, and convergence has been reached in M. The calculation is performed using graph G_7, so that $S_7(2^L)$ is a sum of 127 terms. The results are plotted in Fig. 2 with two typical values of free parameters E and A. With the algorithm given above, one is able to evaluate $C(r)$ for very long-time delays up to order of $2^{40} \approx 10^{12}$. The difficulty of losing numerical accuracy when computing the CF with increasing r is avoided, since the total length N of the time series could be simultaneously increased, while always choosing graph G_7 and keeping $S_7(2^L)$ a sum of 127 terms. For example, when $L=40$, one has $r=2^{40} \approx 10^{12}$ and $N=2^{47} \approx 10^{14}$, an unprecedentedly long-time delay and time series, the direct evaluation of $C_N(r)$ needs to sum up 10^{14} terms. With the present algorithm, however, $C_N(r)$ stays to be a sum of 127 terms. As a consequence, the results shown in Fig. 2, although numerical in nature, are believed to be rather accurate and reliable. The revivals of the CF over long-time intervals is established with reasonable certainty, implying the presence of the long-range correlations in the corresponding quantum system.

PD SYSTEM

The Hamiltonian of the PD two-level system is given by Eq. (1) with $a_{\nu(n)}=A$ for $\nu(n)=1$ and $a_{\nu(n)}=-2A$ for $\nu(n)=0$, while $\nu(n)$ is the binary PD sequence composed of 1 and 0. The PD sequence is made from the substitution rule $1 \to 10$, $0 \to 11$ and started with 1 [9,10]. Notice here that a_1 and a_0 are so chosen as to make $\overline{a}$ vanish. Similar to the TM case, the method of evaluating $C_N(r)$ at $r=2^L$ and $N=2^{L+M}$ for the PD system is also based on a family of graph G_M, $M=1,2,3,\ldots$. Figure 3 shows the first four members of the graph family, the growth rules of which are described below.

Each member of the graph family is composed of nodes and branches that connect nodes in a treelike way. The nodes are labeled by a finite string of distinct integers with the rightmost integer being 0 or $\overline{n_0}$. Here n_0 is also an integer (see Fig. 3). The first member G_1, shown in Fig. 3(a), generates the entire family of graphs G_M by the recursive laws described below:

(1) The $(M+1)$-stage graph G_{M+1} is produced from its previous stage G_M.

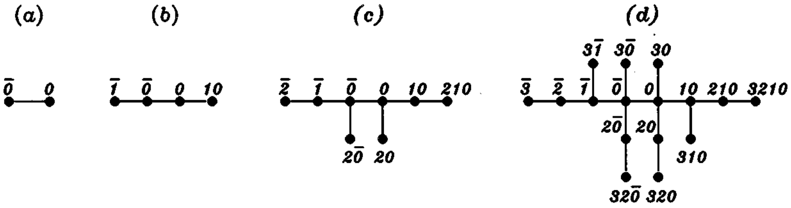

FIG. 3. The first four members of the graph family, G_1–G_4, determining the correlation functions $C_{2^{L+1}}(2^L)$ to $C_{2^{L+4}}(2^L)$ for the PD system at arbitrary integer L.

(2) When generating the graph G_{M+1} from the graph G_M, the nodes labeled by a string of integers ended up with 0, say, $n_p n_{p-1} \cdots n_1 0$ in G_M, will survive and grow a node denoted by $M n_p n_{p-1} \cdots n_1 0$, while a node associated with $n_q n_{q-1} \cdots n_1 \overline{n_0}$ also survives and gives birth to a node $M n_q n_{q-1} \cdots n_1 \overline{n_0}$. The only exception is that the node associated with a single integer $\overline{(M-1)}$ in the graph G_M should survive, and generate a node labeled by M instead of $M\overline{(M-1)}$. The graphs G_2, G_3, G_4 in Fig. 3 and, in fact, the graphs up to any stage M are available from the recursive laws given above.

The autocorrelation function $C_N(r)$ is also given by Eqs. (7)–(9). The operator $\hat{S}_M(2^L)$ is found by summing over all nodes in the graph G_M. The string of integers associated with each node governs the operator as well as its prefactor with which the node contributes to $\hat{S}_M(2^L)$.

The operator contributed by a node labeled by $n_p n_{p-1} \cdots n_2 n_1 0$ is

$$W^\dagger(n_p) W^\dagger(n_{p-1}) \cdots W^\dagger(n_2) W^\dagger(n_1) W(0) W(n_1) \times W(n_2) W(n_{p-1}) W(n_p),$$

while the contribution to $\hat{S}_M(2^L)$ of a node denoted by $n_q n_{q-1} \cdots n_2 n_1 \overline{n_0}$ is

$$W^\dagger(n_q) W^\dagger(n_{q-1}) \cdots W^\dagger(n_2) W^\dagger(n_1) A(n_0) W(n_1) \times W(n_2) W(n_{q-1}) W(n_q),$$

with

$$A(n_0) = W^\dagger(n_0) W^\dagger(n_0 - 1) \cdots W^\dagger(1) W^\dagger(0) W(n_0 + 1).$$

Here

$$W(n) = W_{2^L + n},$$

which can be easily found by using the substitution rule of the PD sequence. The prefactor carried by each node is 2^L, with the following two exceptions: (a) the node 0 carries a prefactor $2^L - 1$; and (b) in the graph G_M, the node $\overline{(M-1)}$ carries a prefactor 1.

As an instance, we give the expression for $S_3(2^L)$, which is generated by the graph G_3, shown in Fig. 3(c), and has eight terms:

$$\begin{aligned} S_3(2^L) &= (\bullet\, 0) + (\bullet\, 10) + (\bullet\, 20) + (\bullet\, 210) + (\bullet\, \bar{0}) \\ &\quad + (\bullet\, \bar{1}) + (\bullet\, 2\bar{0}) + (\bullet\, \bar{2}) \\ &= (2^L - 1) W_{2^L} + 2^L W^\dagger_{2^L+1} W_{2^L} W_{2^L+1} \\ &\quad + 2^L W^\dagger_{2^L+2} W_{2^L} W_{2^L+2} \\ &\quad + 2^L W^\dagger_{2^L+2} W^\dagger_{2^L+1} W_{2^L} W_{2^L+1} W_{2^L+2} + 2^L W^\dagger_{2^L} W_{2^L+1} \\ &\quad + 2^L W^\dagger_{2^L+1} W^\dagger_{2^L} W_{2^L+2} + 2^L W^\dagger_{2^L+2} W^\dagger_{2^L} W_{2^L+1} W_{2^L+2} \\ &\quad + W^\dagger_{2^L+2} W^\dagger_{2^L+1} W^\dagger_{2^L} W_{2^L+3}. \end{aligned} \tag{15}$$

As in the TM case, we have computed $C_{2^{L+M}}(2^L)$ for all $2 \leqslant L \leqslant 40$ with $M = 7$ to have convergence in M. The calculation is performed using graph G_7, so $S_7(2^L)$ is a sum of 128 terms, regardless of the value of L. The results are plotted in Fig. 4 with two typical values of free parameters E and A. The nondecay behavior of the CF over long-time intervals is clearly seen.

RS SYSTEM

The Hamiltonian of the RS two-level system is once again given by Eq. (1) with $a_{\nu(n)} = A$ for $\nu(n) = 1$ and $a_{\nu(n)} = -A$ for $\nu(n) = 0$, but $\nu(n)$ is the binary RS sequence composed of 1 and 0. The RS sequence is made from a more complicated substitution rule $00 \to 0001$, $01 \to 0010$, $10 \to 1101$, $11 \to 1110$, and started with 00 [9,10]. Similar to the former cases, the method of evaluating $C_N(r)$ at $r = 2^L$ and $N = 2^{L+M}$ for the RS system can also be based on a family of graph G_M, with $M = 1,2,3,\ldots$. Figure 5 shows the first three members (denoted by G_2, G_3, and G_4) of the

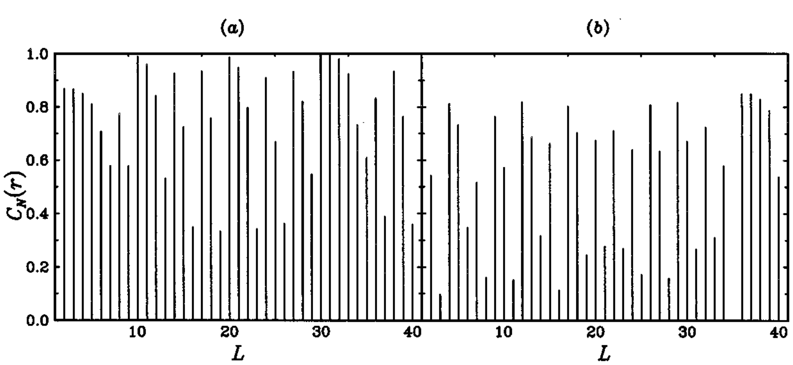

FIG. 4. Absolute value of the autocorrelation functions $C_N(r)$, with $N = 2^{L+7}$ and $r = 2^L$, as a function of L for the PD two-level system with $E = 2\pi/3$ and (a) $A = 0.05(\pi/2)$, (b) $A = 0.5(\pi/2)$.

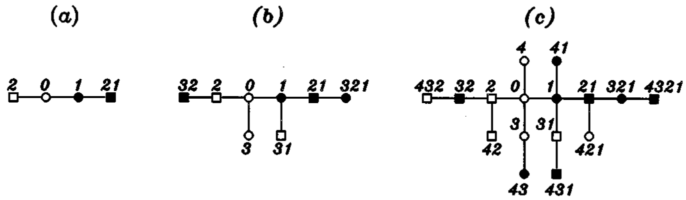

FIG. 5. The first three members of the graph family, G_2–G_4, governing the correlation functions $C_{2^{L+2}}(2^L)$ to $C_{2^{L+4}}(2^L)$ for the RS system at arbitrary integer L.

graph family. The reason for starting with G_2 consisting of four nodes instead of G_1 with two nodes is that the RS sequence is constructed from a string of two digit, 00, rather than a single digit 1 in the TM and the PD cases. As a consequence, the growth rules of the graph G_M turn out to be more complex. For the sake of completeness, they are presented below.

Each member of the graph family are composed of nodes and branches that connect nodes in a treelike way. There are four types of nodes, distinguished by solid circles (●), empty circles (○), solid square (■), and empty square (□). Each node is labeled by a finite string of distinct integers in decreasing order from the left to the right. The first member, or the second-stage graph G_2 shown in Fig. 5(a), produces the whole family of graph by the recursive laws given below:

(1) The M-stage graph G_M is made from its previous stage G_{M-1}.

(2) When generating the graph G_M from the graph G_{M-1}, every node, denoted, say, by $n_p n_{p-1}\cdots n_2 n_1$ in G_{M-1}, will survive and grow a node labeled by $M n_p n_{p-1}\cdots n_2 n_1$. For simplicity we omit the rightmost integer 0 for all nodes except the one associated with a single integer 0. Notice that these two rules are the same as in the TM case.

(3) The node labeled, say, by $n_p n_{p-1}\cdots n_2 n_1$ in G_{M-1}, grows a node in G_M of the type based on the following rules: (a) for $n_p < M-1$, the node grows a node of the same type; and (b) for $n_p = M-1$, on the other hand, a node of type ○ (●) will produce a node of type ● (○), while a node of type □ (■) yields a node of type ■ (□). There are two exceptional cases. The first one is that in G_{M-1}, the node denoted by the largest number of integers, $(M-1)(M-2)\cdots 21$, and of the type ● (■) will give birth to a node of the type ■(●). See, e.g., the node ■ 4321 in G_4 [Fig. 5(c)] grown by the node ● 321 in G_3 [Fig. 5(b)] for the case with $M=4$. The second exception lies in the fact that in G_{M-1}, the node labeled by $(M-2)(M-3)\cdots 21$ and of the type ● (■) should yield a node of the type □ (○). See, e.g., the node ○ 421 in G_4 [Fig. 5(c)] produced by the node ■ 21 in G_3 for the case with $M=4$. The graphs G_3 and G_4 in Fig. 5 and, in fact, the graphs up to any stage M are obtainable from the recursive laws presented above.

The autocorrelation function $C_N(r)$, given by Eqs. (7)–(9), can be found from the operator $\hat{S}_M(2^L)$, which is worked out by summing over all nodes in the graph G_M except the one labeled by $M(M-1)\cdots 21$. The string of integers associated with each node, as well as the type of the node (denoted by ○, ●, □, and ■), governs the operator with which the node contributes to $\hat{S}_M(2^L)$.

The operator $\mathcal{N}_{n_p n_{p-1}\cdots n_2 n_1}$ contributed by the node $n_p n_{p-1}\cdots n_2 n_1$ in G_M, is

$$\mathcal{N}_{n_p n_{p-1}\ldots n_2 n_1} = \begin{cases} 2^{L-1} W^\dagger_{m_1} W_{m_2} + W^\dagger_{m_3} X_L W_{m_4} & \text{for the node of type } \bigcirc \\ 2^{L-1} W^\dagger_{m_1} W_{m_2} + W^\dagger_{m_3} \overline{X_L} W_{m_4} & \text{for the node of type } \bullet \\ W^\dagger_{m_1} Y_L W_{m_2} + 2^{L-1} W^\dagger_{m_3} W_{m_4} & \text{for the node of type } \square \\ W^\dagger_{m_1} \overline{Y_L} W_{m_2} + 2^{L-1} W^\dagger_{m_3} W_{m_4} & \text{for the node of type } \blacksquare, \end{cases} \tag{16}$$

with the following exception:

$$\mathcal{N}_0 = (2^{L-1}-1) W_{2^L} + W^\dagger_{2^{L-1}} X_L W_{2^L+2^{L-1}} + W^\dagger_{(2^M-2)\times 2^L+2^{L-1}} W_{(2^M-1)\times 2^L+2^{L-1}}. \tag{17}$$

Here

$$m_1 = (2^{n_p} + 2^{n_{p-1}} + \cdots + 2^{n_2} + 2^{n_1}) \times 2^{L-1}, \quad m_2 = m_1 + 2^L,$$

$$m_3 = m_1 + 2^{L-1}, \quad m_4 = m_3 + 2^L.$$

The operators X_L, $\overline{X_L}$, Y_L, and $\overline{Y_L}$ are given by the recursion relations, for $L \geqslant 2$,

$$X_{L+1} = W^\dagger_{2^{L-1}} \overline{X_L}\, \overline{W_{2^{L-1}}} + Y_L,$$

$$\overline{X_{L+1}} = \overline{W^\dagger_{2^{L-1}}} X_L W_{2^{L-1}} + \overline{Y_L},$$

$$Y_{L+1} = W^\dagger_{2^{L-1}} X_L \overline{W_{2^{L-1}}} + Y_L,$$

$$\overline{Y_{L+1}} = \overline{W^\dagger_{2^{L-1}}}\, \overline{X_L} W_{2^{L-1}} + \overline{Y_L}, \tag{18}$$

and the initial conditions

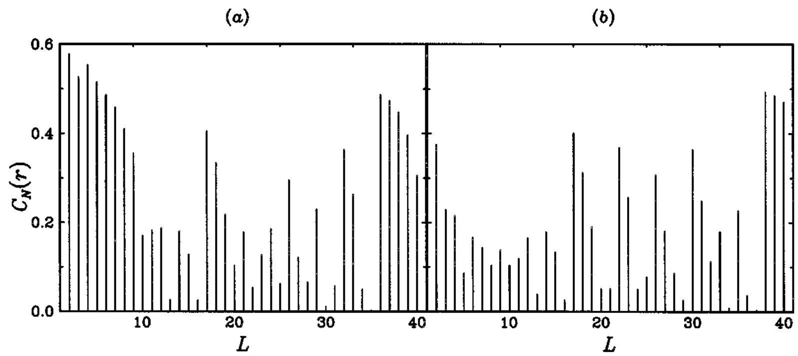

FIG. 6. Absolute value of the autocorrelation functions $C_N(r)$, with $N=2^{L+7}$ and $r=2^L$, as a function of L for the RS two-level system with $E=2\pi/3$ and (a) $A=0.05(\pi/2)$, (b) $A=0.5(\pi/2)$.

$$X_2=Y_2=U_0^\dagger U_1+1,$$

$$\overline{X_2}=\overline{Y_2}=U_1^\dagger U_0+1, \tag{19}$$

where $\overline{W_{2^L}}$ are obtained from W_{2^L} by the exchanging of each $\nu(n)=1$ ($a_{\nu(n)}=A$) and $\nu(n)=0$ ($a_{\nu(n)}=-A$), and W_{2^L} can be found from the substitution rule of the RS sequence.

As an example, we give the expression for $S_3(2^L)$, which is generated by the graph G_3, shown in Fig. 5(b). Notice that G_3 consists of eight nodes, seven among which (saving the node ●321) contribute to $S_3(2^L)$. So $S_3(2^L)$ is composed of 15 terms, i.e.,

$$\begin{aligned} S_3(2^L)&=(\bigcirc\ 0)+(\bullet\ 1)+(\square\ 2)+(\blacksquare\ 21)+(\bigcirc\ 3)+(\square\ 31)+(\blacksquare\ 32)\\ &=[(2^{L-1}-1)W_{2^L}+W^\dagger_{2^{L-1}}X_LW_{2^L+2^{L-1}}+W^\dagger_{6\times2^L+2^{L-1}}W_{7\times2^L+2^{L-1}}]+[2^{L-1}W^\dagger_{2^L}W_{2\times2^L}+W^\dagger_{2^L+2^{L-1}}\overline{X_L}W_{2\times2^L+2^{L-1}}]\\ &\quad+[W^\dagger_{2\times2^L}Y_LW_{3\times2^L}+2^{L-1}W^\dagger_{2\times2^L+2^{L-1}}W_{3\times2^L+2^{L-1}}]+[W^\dagger_{3\times2^L}\overline{Y_L}W_{4\times2^L}+2^{L-1}W^\dagger_{3\times2^L+2^{L-1}}W_{4\times2^L+2^{L-1}}]\\ &\quad+[2^{L-1}W^\dagger_{4\times2^L}W_{5\times2^L}+W^\dagger_{4\times2^L+2^{L-1}}X_LW_{5\times2^L+2^{L-1}}]+[W^\dagger_{5\times2^L}Y_LW_{6\times2^L}+2^{L-1}W^\dagger_{5\times2^L+2^{L-1}}W_{6\times2^L+2^{L-1}}]\\ &\quad+[W^\dagger_{6\times2^L}\overline{Y_L}W_{7\times2^L}+2^{L-1}W^\dagger_{6\times2^L+2^{L-1}}W_{7\times2^L+2^{L-1}}]. \end{aligned} \tag{20}$$

Notice that the first three terms on the right-hand side of the above equation correspond to the node 0 in G_3, whereas every two terms within a pair of square brackets correspond to each of the other six nodes in G_3 [see Eqs. (16) and (17), and remember that node 321 in G_3 does not contribute to $S_3(2^L)$].

We have evaluated $C_{2^{L+M}}(2^L)$ for all $2\leqslant L\leqslant 40$ with $M=7$ to obtain convergence in M. The calculation is performed using graph G_7, so that $S_7(2^L)$ is a sum of 255 terms, independent of the value of L. The results are plotted in Fig. 6 with two typical values of free parameters E and A, from which one observes the nondecay behavior of the CF over long-time intervals up to order of 10^{12}.

Finally, we would like to point out that the results shown in Figs. 2, 4, and 6 are obtained with the initial state $|\psi(0)\rangle$ satisfying $\sigma_z|\psi(0)\rangle=|\psi(0)\rangle$, as in Ref. [6]. We have evaluated the CF for a variety of initial states as well. It is found that although the quantitative behavior of the CF depends on the initial state, for all three systems considered here the long-range correlations exist regardless of the choice of the initial state.

In summary, we have presented an effective algorithm for the evaluation of the autocorrelation functions for aperiodic quantum systems where the aperiodicity in perturbation is based on the substitution rule. The algorithm reorganizes the expression of the CF such that the computation of the CF is related to a family of graphs that are generated recursively. As examples of applications, we have calculated the CF over unprecedentedly long-time intervals up to order 10^{12} for the TM, PD, and RS aperiodic two-level systems. Our results show that there exist long-range correlations in all these aperiodic quantum systems, despite their different characteristics in the aperiodic modulation in the sense of different wandering exponent ω.

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APPENDIX

In this appendix, we present, as an example, the brief derivation that reduces the CF [Eq. (8)] to the contributions of nodes in the graph [Eq. (9)] for the TM system. Other aperiodic systems based on the substitution sequences can be treated in a similar way.

From the self-similar nature of the TM sequence, it follows that if one replaces, respectively, each 1 and 0 in the

M-stage TM sequence S_M with the sequence S_L and its alternative S_L, one obtains an $(L+M)$-stage sequence S_{L+M}. Here the L-stage sequence S_L and its alternative $\overline{S_L}$ are given by applying L times TM substitution $1\rightarrow 10$, $0\rightarrow 01$ to a single element 1 and 0, respectively, namely,

$$S_0=1,\quad \overline{S_0}=0,$$

$$S_1=10,\quad \overline{S_1}=01,$$

$$S_2=1001,\quad \overline{S_2}=0110,$$

$$S_3=10\,010\,110,\quad \overline{S_3}=01\,101\,001,$$

$$\ldots .$$

As a result, we may start our derivation for the operator $\hat{C}_{2^{L+M}}(2^L)$ in Eq. (8) with $L=1$ and $M=2,3,4,\ldots$. For a general L, one can simply replace each quantity associated with stage 1 by the quantity corresponding to stage L.

For $L=1$ and $M=2$, it follows from Eq. (8) that $\hat{S}_M(2)=Q_2=\Sigma_{k=1}^{6}W_k^{\dagger}W_{k+2}$, which can be rewritten as

$$Q_2=W_{1\times 2}^{\dagger}X_1W_{2\times 2}+2W_{2\times 2}^{\dagger}W_{3\times 2}+W_{3\times 2}^{\dagger}\overline{X_1}W_{4\times 2}, \tag{A1}$$

where the first term is the sum of $W_k^{\dagger}W_{k+2}$ for $k=1$–2, the second term is that for $k=3$–4, and the third term that for $k=5$–6. The operators X_1 and X_1 are given by

$$X_1=\overline{W_1}W_1^{\dagger}+1,\quad \overline{X_1}=W_1\overline{W_1^{\dagger}}+1. \tag{A2}$$

Notice that $\overline{X_1}$ is defined by interchanging each W ($W^{\dagger}$) and W ($W^{\dagger}$) in X_1. Note also that $X_1=X_1^{\dagger}$.

When $L=1$ and $M=3$, denoting $Q_3=\Sigma_{k=7}^{14}W_k^{\dagger}W_{k+2}$, one is ready to have $\hat{S}_M(2)=Q_2+Q_3$, where Q_3 can be cast into

$$Q_3=W_{4\times 2}^{\dagger}X_1W_{5\times 2}+W_{5\times 2}^{\dagger}\overline{X_1}W_{6\times 2}+2W_{6\times 2}^{\dagger}W_{7\times 2}+W_{7\times 2}^{\dagger}X_1W_{8\times 2}. \tag{A3}$$

As one proceeds further, it will be straightforward to find that $\hat{S}_M(2^L)$ for $L=1$ is simply a sum of 2^M-1 terms,

$$\hat{S}_M(2)=\sum_{k=1}^{2^M-1}W_{k\times 2}^{\dagger}X_1^{*}(k)W_{(k+1)\times 2}, \tag{A4}$$

where $X_1^{*}(k)$ may be 2, X_1 or $\overline{X_1}$. It follows from Eq. (A4) that increasing the value of M from M to $M+1$ needs to sum up 2^M more terms. Based on these facts, it is not difficult to obtain the recursive laws 1 and 2 for the graph, and establish the correspondence between each term in Eq. (A4) and each node in the graph. In particular, the finite string of distinct integers labeling each node in the graph serves to determine the value of k in expression (A4) of the node-corresponding term. More explicitly, a node labeled by $n_pn_{p-1}\cdots n_2n_1$ corresponds to a term $W_{n^{*}\times 2}^{\dagger}X_1^{*}(n^{*})W_{(n^{*}+1)\times 2}$ in expression (A4), where $n^{*}=\lfloor(2^{n_p}+2^{n_{p-1}}+\cdots+2^{n_2}+2^{n_1})/2\rfloor+1$, with $\lfloor x\rfloor$ denoting the integer part of x. The difficulty then lies in determining the k dependence of the node type in the graph, which governs the operator $X_1^{*}(k)$ and is dependent on the structure correlation of the TM sequence.

Noticing first that $X_1^{*}(k)$ may be 2, X_1, or $\overline{X_1}$, one classifies the nodes in the graph into three types, denoted by $\diamondsuit$, $\bigcirc$, and $\bullet$, corresponding respectively to $X_1^{*}(k)=2$, X_1, and X_1. Consider the TM sequence as a sequence made of two basic blocks S_1 and S_1. $X_1^{*}(k)$ in Eq. (A4) can then be specified,

$$X_1^{*}(k)=\begin{cases}2 & \text{if the } k\text{th and the } (k+1)\text{th blocks are the same}\\ X_1 & \text{if the } k\text{th block is } S_1 \text{ and the } (k+1)\text{th block is } \overline{S_1}\\ \overline{X_1} & \text{if the } k\text{th block is } \overline{S_1} \text{ and the } (k+1)\text{th block is } S_1.\end{cases} \tag{A5}$$

Finally, one can easily prove that, among others, the TM sequence possesses the following two characteristics: (i) if the k block is S_1 (S_1), then the (2^j+k)th block, with any integer j satisfying $2^j>k$, will be S_1 (S_1); (ii) the 2^ith block toggles between S_1 and S_1, while the (2^i+1)th block stays to be S_1, as the integer i increases from 1 to any large integer. Based on the first characteristic, it is concluded that if the kth and the $(k+1)$th blocks are respectively S_1 and S_1 (S_1 and S_1), then the (2^j+k)th and the (2^j+k+1)th will be S_1 and S_1 (S_1 and S_1), whereas the (2^j+k)th and the (2^j+k+1)th blocks should be the same whenever the kth and the $(k+1)$th blocks are the same. As a consequence,

$$X_1^{*}(2^j+k)=\begin{cases}X_1 & \text{if } X_1^{*}(k)=\overline{X_1}\\ \overline{X_1} & \text{if } X_1^{*}(k)=X_1\\ 2 & \text{if } X_1^{*}(k)=2,\end{cases} \tag{A6}$$

where j can be any integer satisfying $2^j>k$. Equation (A6) leads to rules (a), (b), and (c) in recursive law (3), concerning the type of the node, for the graph. On the other hand, the second characteristic results in the fact that the 2^ith and the (2^i+1)th blocks switch between S_1S_1 and S_1S_1, as the integer i increases. To be specific, the second and third blocks are both S_1, the fourth and fifth blocks turn, respectively, to

S_1 and $\overline{S_1}$; the eighth and ninth blocks both return to $\overline{S_1}$; the 16th and 17th blocks become respectively S_1 and $\overline{S_1}$; and so on. Therefore, $X_1^*(2^i)$ should be either 2 or X_1, and, in addition, for any positive integer i,

$$X_1^*(2^{i+1})=\begin{cases}2 & \text{if } X_1^*(2^i)=X_1\\ X_1 & \text{if } X_1^*(2^i)=2.\end{cases} \tag{A7}$$

This gives rise to the exceptional case in the recursive law (3) for the graph.

Now we turn to the study for the case with a general L. In this case, one may regard the TM sequence as constructed by two basic blocks, S_L and $\overline{S_L}$. Therefore, $\hat{S}_M(2^L)$ should read

$$\hat{S}_M(2^L)=\sum_{k=1}^{2^M-1} W^{\dagger}_{k\times 2^L} X_L^*(k) W_{(k+1)\times 2^L} \tag{A8}$$

in place of Eq. (A4), where, similar to Eq. (A5),

$$X_L^*(k)=\begin{cases}2^L & \text{if the } k\text{th and the } (k+1)\text{th blocks are the same}\\ X_L & \text{if the } k\text{th block is } S_L \text{ and the } (k+1)\text{th block is } \overline{S_L}\\ \overline{X_L} & \text{if the } k\text{th block is } \overline{S_L} \text{ and the } (k+1)\text{th block is } S_L.\end{cases} \tag{A9}$$

By paying attention to the inflation rules $S_{L+1}=S_L\overline{S_L}$ and $\overline{S_{L+1}}=\overline{S_L}S_L$ for the TM sequence, it is not difficult to work out the recursion laws (11) for X_L and $\overline{X_L}$. Then, the derivation that reduces Eq. (8) to Eq. (9) for a general L is similar to that following Eq. (A5).

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