Isentropic equation of state of 2-flavor QCD

S. Ejiri,[1] F. Karsch,[2,3] E. Laermann,[3] and C. Schmidt[2]

[1]*Department of Physics, The University of Tokyo, Tokyo 113-0033, Japan*
[2]*Physics Department, Brookhaven National Laboratory, Upton, New York 11973, USA*
[3]*Fakultät für Physik, Universität Bielefeld, D-33615 Bielefeld, Germany*

Using Taylor expansions of the pressure obtained previously in studies of 2-flavor QCD at nonzero chemical potential we calculate expansion coefficients for the energy and entropy densities up to $\mathcal{O}(\mu_q^6)$ in the quark chemical potential. We use these series in μ_q/T to determine lines of constant entropy per baryon number (S/N_B) that characterize the expansion of dense matter created in heavy ion collisions. In the high temperature regime these lines are found to be well approximated by lines of constant μ_q/T. In the low temperature phase, however, the quark chemical potential is found to increase with decreasing temperature. This is in accordance with resonance gas model calculations. Along the lines of constant S/N_B we calculate the energy density and pressure. Within the accuracy of our present analysis we find that the ratio p/ϵ for $T > T_0$ as well as the softest point of the equation of state, $(p/\epsilon)_{\min} \simeq 0.075$, show no significant dependence on S/N_B.

I. INTRODUCTION

Recently studies of QCD thermodynamics on the lattice have successfully been extended to nonvanishing quark (or baryon) chemical potential using Taylor expansions [1,2] as well as reweighting techniques [3] around the limit of vanishing quark chemical potential ($\mu_q = 0$) and analytic continuations of numerical calculations performed with an imaginary chemical potential [4]. The complementary approach at fixed baryon number [5,6] has also been used recently in simulations of 4-flavor QCD with light quarks [7]. These calculations yield the additional contribution to the pressure that arises from the presence of a net excess of baryons over antibaryons in a strongly interacting medium. We will further explore here the approach based on a Taylor expansion of the pressure and make use of our earlier calculations for 2-flavor QCD [2] to determine also the energy and entropy densities at nonvanishing baryon chemical potential.

In a heavy ion collision a dense medium is created which after thermalization is expected to expand without further generation of entropy (S) and with fixed baryon number (N_B) or, equivalently, with fixed quark number $N_q = 3N_B$. During the isentropic expansion the ratio S/N_B thus remains constant.[1] The cooling of the expanding system then is controlled by the equation of state on lines of constant S/N_B. From a knowledge of the energy density and pressure at nonvanishing quark chemical potential we can calculate trajectories in the μ_q-T phase diagram of QCD that correspond to constant S/N_B and can determine the *isentropic equation of state* on these trajectories.

This paper is organized as follows. In the next section we summarize the basic setup for calculating Taylor expan-

sions for bulk thermodynamic observables and present a calculation of the Taylor expansion coefficients for energy and entropy densities at nonvanishing quark chemical potential up to $\mathcal{O}(\mu_q^6)$. We use these results in Sec. III to determine lines of constant S/N_B and study the temperature dependence of pressure and energy density along these lines. Our conclusions are given in Sec. IV.

II. TAYLOR EXPANSION OF PRESSURE, ENERGY AND ENTROPY DENSITY

The analysis we are going to present in this paper is based on numerical results previously obtained in simulations of 2-flavor QCD on lattices of size $16^3 \times 4$ with an improved staggered fermion action [1,2]. In these calculations Taylor expansion coefficients for the pressure have been obtained up to $\mathcal{O}(\mu^6)$ for a fixed bare quark mass value ($\hat{m} = 0.1$) which at temperatures close to the transition temperature (T_0) corresponds to a still quite large pion mass of about 770 MeV. However, in particular, at temperatures above the QCD transition temperature the remaining quark mass dependence of thermodynamic observables is nonetheless small as deviations from the massless limit are controlled by the quark mass in units of the temperature, which is a small number. In fact, recent studies of energy density and pressure in $(2 + 1)$-flavor QCD [8] with quark masses that are about a factor 4 smaller than those used here show quite similar temperature dependence and deviation from ideal gas behavior. Moreover, in the high temperature ideal gas limit the influence of a nonvanishing quark mass is known to be small, it starts at $\mathcal{O}((m/T)^2)$ and these corrections have a small coefficient, e.g. $\epsilon_{SB}(0)/\epsilon_{SB}(m/T) = 1 - 0.0766(m/T)^2$.

We closely follow here the approach and notation used in Ref. [2]. We start with a Taylor expansion for the

---

[1]Here N_B denotes the net baryon number, i.e. the excess of baryons over antibaryons.

pressure in 2-flavor QCD for nonvanishing quark chemical potential μ_q (and vanishing isospin chemical potential, $\mu_I \equiv 0$),

$$\frac{p}{T^4} \equiv \frac{1}{VT^3} \ln Z(T, \mu_q) = \sum_{n=0}^{\infty} c_n(T, m_q)\left(\frac{\mu_q}{T}\right)^n, \quad (1)$$

with

$$c_n(T, m_q) = \frac{1}{n!} \frac{1}{VT^3} \frac{\partial^n \ln Z(T, \mu_q)}{\partial(\mu_q/T)^n}\bigg|_{\mu_q=0}. \quad (2)$$

Here we explicitly indicated that the Taylor expansion coefficients c_n depend on temperature as well as the quark mass.[2] In the context of lattice calculations it is more customary to think of this quark mass dependence in terms of a dependence on an appropriately chosen ratio of hadron masses characterizing *lines of constant physics*. We thus may replace m_q by a ratio of pseudoscalar (pion) and vector (rho) meson masses, $m_q \equiv (m_{PS}/m_V)^2$.

Because of the invariance of the partition function under exchange of particle and antiparticle the Taylor expansion is a series in even powers of μ_q/T; expansion coefficients c_n vanish for odd values of n. From the pressure we immediately obtain the quark number density,

$$\frac{n_q}{T^3} = \frac{1}{VT^3} \frac{\partial \ln Z(T, \mu_q)}{\partial \mu_q/T}\bigg|_T = \sum_{n=2}^{\infty} n c_n(T, m_q)\left(\frac{\mu_q}{T}\right)^{n-1}. \quad (3)$$

Using standard thermodynamic relations we also can calculate the difference between energy density (ϵ) and 3 times the pressure,

$$\frac{\epsilon - 3p}{T^4} = \sum_{n=0}^{\infty} c_n'(T, m_q)\left(\frac{\mu_q}{T}\right)^n, \quad (4)$$

where

$$c_n'(T, m_q) = T \frac{dc_n(T, m_q)}{dT}. \quad (5)$$

Combining Eqs. (1), (3), and (4) we then obtain Taylor expansions for the energy and entropy densities,

$$\frac{\epsilon}{T^4} = \sum_{n=0}^{\infty} (3c_n(T, m_q) + c_n'(T, m_q))\left(\frac{\mu_q}{T}\right)^n,$$

$$\frac{s}{T^3} \equiv \frac{\epsilon + p - \mu_q n_q}{T^4} = \sum_{n=0}^{\infty} ((4 - n)c_n(T, m_q) + c_n'(T, m_q))$$
$$\times \left(\frac{\mu_q}{T}\right)^n. \quad (6)$$

---

[2]We assume that the thermodynamic limit has been taken so that we can ignore any volume dependence of thermodynamic quantities. For an aspect ratio $TV^{1/3} = 4$ and the quark mass value used in our calculation finite volume effects are expected to be small.

The expansion coefficients $c_n(T, m_q)$ have been calculated in 2-flavor QCD at several values of the temperature and for a fixed value of the bare quark mass, $\hat{m} = 0.1$ [2]. We note that the bare coupling $\hat{m}$ introduces an implicit temperature dependence if $\hat{m}$ is kept fixed while the lattice cut-off is varied. The latter controls the temperature of the system on lattices of temporal extent N_τ, *i.e.* $T^{-1} = N_\tau a$. The difference between derivatives taken at fixed $\hat{m}$ and fixed m_q, however, is negligible at high temperature, where the quark mass dependence of bulk thermodynamic quantities is $\mathcal{O}((m_q/T)^2)$ as we have discussed above. Of course, the difference will vanish at all temperatures in the chiral limit as this also defines a line of constant physics. A more quantitative analysis of the remaining quark mass effects thus will become possible through studies of the equation of state with lighter quark masses. We thus approximate the derivative, Eq. (5), by a derivative taken at fixed $\hat{m}$ which becomes exact in the chiral limit. These derivatives are evaluated using finite difference approximants. Alternatively we may express derivatives of Taylor expansion coefficients with respect to T in terms of derivatives with respect to the bare lattice couplings $\beta \equiv 6/g^2$ and $\hat{m}$ and rewrite Eq. (5) as,

$$c_n'(T, m_q) = -a \frac{d\beta}{da} \frac{\partial c_n(T, m_q)}{\partial \beta} - a \frac{d\hat{m}}{da} \frac{\partial c_n(T, m_q)}{\partial \hat{m}}. \quad (7)$$

Here the two lattice β-functions, $ad\beta/da$ and $ad\hat{m}/da$, have to be determined by demanding that the temperature derivative is taken along lines of constant physics [9,10]. The β-function controlling the variation of the bare quark mass with the lattice cut-off, $ad\hat{m}/da$, is proportional to the bare quark mass $\hat{m}$ and thus vanishes in the chiral limit [9]. For small quark masses the first term in Eq. (7) thus gives the dominant contribution to $c_n'(T, m_q)$. We have evaluated this first term in Eq. (7) for $n = 2$ and find that this approximation for c_2' agrees within errors with the calculation of c_2' from finite difference approximants. Only close to T_0 we find differences of the order of 10%.

The expansion coefficients $c_n(T, m_q)$ have been calculated in [2] for $\hat{m} = 0.1$ at a set of 16 temperature values in the interval $T/T_0 \in [0.81, 1.98]$. From this set of expansion coefficients we have calculated the partial derivatives c_n' at temperature T by averaging over finite difference approximants for left and right derivatives at T. With this we have constructed the expansion coefficients for the energy and entropy densities,

$$\epsilon_n \equiv 3c_n(T, m_q) + c_n'(T, m_q),$$
$$s_n \equiv (4 - n)c_n(T, m_q) + c_n'(T, m_q). \quad (8)$$

Results for the 2nd, 4th and 6th order expansion coefficients of energy and entropy densities obtained in the approximation discussed above are shown in Fig. 1. Also shown in this figure are the corresponding expansion coefficients for the pressure ($p_n \equiv c_n$). For temperatures
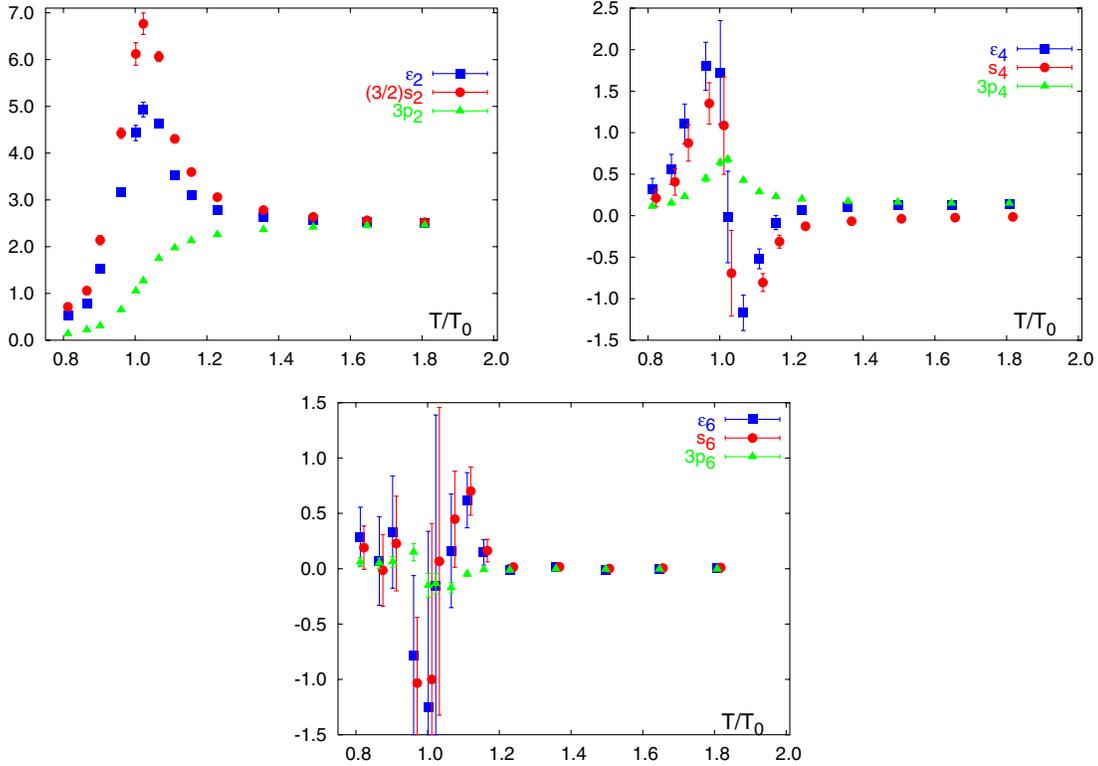
FIG. 1 (color online).   The 2nd, 4th and 6th order Taylor expansion coefficients for pressure ($p_n \equiv c_n$), energy density (ϵ_n) and entropy density (s_n) as functions of T/T_0.

larger than $T/T_0 \simeq 1.5$ all expansion coefficients satisfy quite well the ideal gas relations, $\epsilon_n^{SB} = 3p_n^{SB}$ and $\epsilon_2^{SB} = 3s_2^{SB}/2$. As noted already in Ref. [2] results for the 2nd and 4th order expansion coefficients are close to those of an ideal Fermi gas which describes the high temperature limit for the energy density of 2-flavor QCD,

$$\frac{\epsilon_F^{SB}}{T^4} = \frac{7\pi^2}{10} + 3\left(\frac{\mu_q}{T}\right)^2 + \frac{3}{2\pi^2}\left(\frac{\mu_q}{T}\right)^4. \qquad (9)$$

Nonetheless, quantitative statements on the approach to ideal gas behavior may be at present premature. Deviations from the continuum result clearly arise in our current analysis which has been performed on lattices with a lattice spacing in units of the inverse temperature given by $aT \equiv 1/N_\tau = 1/4$. Although cut-off effects are reduced due the usage of an improved action the approach to the continuum limit has to be analyzed further through calculations on lattices with larger temporal extent N_τ.

Although errors on the 6th order expansion coefficients are large it is apparent that subsequent orders in the expansion decrease in magnitude thus making the expansions for energy and entropy density well behaved for $\mu_q/T \lesssim 1$. In fact, the 6th order contribution to ϵ/T^4 and s/T^3 never exceeds 10% for $\mu_q/T < 0.9$. We will see below that this is the regime of interest for a discussion of the thermodynamics of matter formed in heavy ion collisions at the

SPS and RHIC and even is appropriate for the energy range to be covered by the AGS and the future FAIR facility at GSI/Darmstadt. Results for the μ_q-dependent contribution to the energy and entropy densities, $\Delta\epsilon/T^4 \equiv \epsilon/T^4 - \epsilon_0$ and $\Delta s/T^3 = s/T^3 - s_0$, obtained in 4th order Taylor expansion are shown in Fig. 2 for some values of μ_q/T.
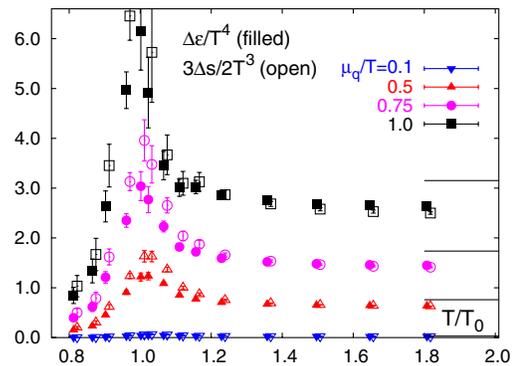

FIG. 2 (color online).   The μ_q dependent contribution to energy density (filled symbols) and entropy density (open symbols) calculated in 4th order Taylor expansion. Data points for the entropy density have been shifted slightly for better visibility. Horizontal lines show the corresponding ideal gas values for the energy density.
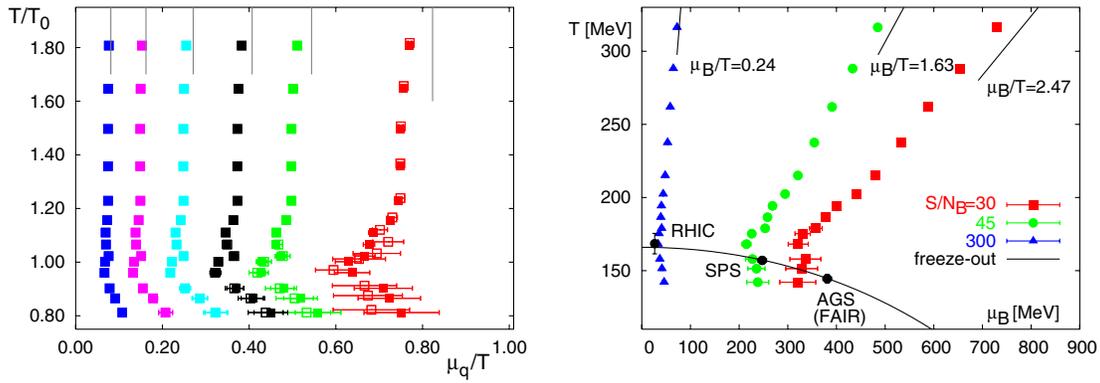
FIG. 3 (color online). Lines of constant entropy per quark number versus μ_q/T (left) and in physical units using $T_0 = 175$ MeV to set the scales (right). In the left hand figure we show results obtained using a 4th (full symbols) and 6th (open symbols) order Taylor expansion of the pressure, respectively. Data points correspond to $S/N_B = 300, 150, 90, 60, 45, 30$ (from left to right). The vertical lines indicate the corresponding ideal gas results, $\mu_q/T = 0.08, 0.16, 0.27, 0.41, 0.54$ and 0.82 in decreasing order of values for S/N_B. For a detailed description of the right hand figure see the discussion given in the text.

III. LINES OF CONSTANT ENTROPY PER BARYON NUMBER

When discussing the equation of state at finite baryon number density we have to eliminate the quark chemical potential in the thermodynamic quantities introduced in the previous section. We may do this by demanding that a thermodynamic quantity, for instance the quark number density, stays constant. We will use here a prescription more appropriate for discussing the thermodynamics of matter created in relativistic heavy ion collisions. After equilibration the dense medium created in such a collision will expand along lines of constant entropy per baryon. It then is of interest to calculate thermodynamic quantities along such isentropic lines.

We thus should first analyze how the quark chemical potential changes along lines of constant S/N_B. It is readily seen that in an ideal quark-gluon gas these lines are defined by $\mu_q/T = $ const,

$$\frac{S}{N_B} = 3\frac{\frac{37\pi^2}{45} + (\frac{\mu_q}{T})^2}{\frac{\mu_q}{T} + \frac{1}{\pi^2}(\frac{\mu_q}{T})^3}. \tag{10}$$

In the zero temperature limit, however, the resonance gas reduces to a degenerate Fermi gas of nucleons and the chemical potential approaches a finite value to obtain finite baryon number and entropy, i.e. $\mu_q/T \sim 1/T$.

In Fig. 3 we show the values for μ_q needed to keep S/N_B fixed. These lines of constant S/N_B in the $T - \mu_q$ plane have been obtained by calculating the total entropy density for 2-flavor QCD from Eq. (6) using results for the pressure and energy density calculated at $\mu_q = 0$ [11,12] and the corresponding μ_q dependent contributions shown in Fig. 1. The ratio of s/T^3 and n_q/T^3 obtained in this way is then solved numerically for μ_q/T.

We find that isentropic expansion at high temperature indeed is well represented by lines of constant μ_q/T down to temperatures close to the transition, $T \simeq 1.2T_0$. In the low temperature regime we observe a bending of the isentropic lines in accordance with the expected asymptotic low temperature behavior. The isentropic expansion lines for matter created at SPS correspond to $S/N_B \simeq 45$ while the isentropes at RHIC correspond to $S/N_B \simeq 300$. The energy range of the AGS which also corresponds to an energy range relevant for future experiments at FAIR/ Darmstadt is well described by $S/N_B \simeq 30$. These lines are shown in Fig. 3 (right) together with points characterizing the chemical freeze-out of hadrons measured at AGS, SPS and RHIC energies.[3] These points have been obtained by comparing experimental results for yields of various hadron species with hadron abundances in a resonance gas [13,14]. The solid curve shows a phenomenological parametrization of these *freeze-out data* [14]. In general our findings for lines of constant S/N_B are in good agreement with phenomenological model calculations that are based on combinations of ideal gas and resonance gas equations of state at high and low temperature, respectively [15,16].

Our current analysis yields stable results for lines of constant S/N_B also for temperatures $T \lesssim T_0$ and values of the chemical potential $\mu_B \simeq 400$ MeV. This value of μ_B

─────

[3]We note that the presentation of experimental results with lattice results in Fig. 3 (right) is done to indicate the relevant range of parameters S/N_B appropriate for the discussion of physics studied with different beam energies in heavy ion experiments. A more quantitative comparison, in particular a discussion of the question to what extent the experimentally observed freeze-out parameters are related to the "critical" parameter for the transition to the quark-gluon plasma phase, will require also an extension of the current lattice calculations for 2-flavor QCD to the physical case of $(2 + 1)$-flavor QCD with a realistic light quark mass spectrum.
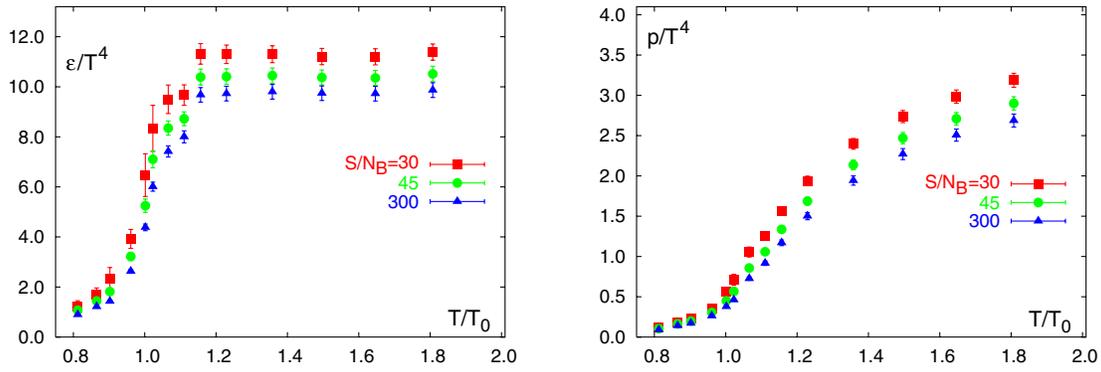
FIG. 4 (color online).   The complete energy density (left) and pressure (right) evaluated on lines of constant S/N_B using Taylor expansions up to 6th order in the quark chemical potential. Shown are results for $S/N_B = 30, 45, 300$.

is larger than recent estimates for the location of the chiral critical point in 2-flavor [17] and $(2 + 1)$-flavor [18] QCD. We thus may expect modifications to our current analysis at temperatures below T_0 once we include higher orders in the Taylor expansion and/or perform this analysis at smaller values of the quark mass.

We now can proceed and calculate energy density and pressure on lines of constant entropy per baryon number using our Taylor expansion results up to $\mathcal{O}(\mu_q^6)$. In Fig. 4 we show both quantities for the parameters relevant for AGS (FAIR), SPS and RHIC energies. At high temperatures the relevant values of the quark chemical potential in an ideal quark gas are $\mu_q/T = 0.82, 0.54$ and 0.08, respectively. However, as can be seen from Fig. 3 at temperatures $T \lesssim 1.8T_0$ the required values for the chemical potentials are significantly smaller. In particular, for the AGS (FAIR) energies ($S/N_B = 30$) we find $\mu_q/T = 0.77$ at $T \simeq 1.8T_0$. The 6th order Taylor expansion thus is still well behaved at these values of the quark chemical potential. The dependence of ϵ and p on S/N_B cancels to a large extent in the ratio p/ϵ, which is most relevant for the analysis of the hydrodynamic expansion of dense matter. This may be seen by considering the leading $\mathcal{O}(\mu_q^2)$ cor-

rection,

$$\frac{p}{\epsilon} = \frac{1}{3} - \frac{1}{3}\frac{\epsilon_0 - 3p_0}{\epsilon_0}\left(1 + \left[\frac{c_2'}{\epsilon_0 - 3p_0} - \frac{\epsilon_2}{\epsilon_0}\right]\left(\frac{\mu_q}{T}\right)^2\right).$$
(11)

In Fig. 5 we show p/ϵ as function of T/T_0 (left) as well as function of the energy density (right). In the latter case we again used $T_0 = 175$ MeV to set the scale. The small dependence on S/N_B visible in Fig. 5 (left) gets further reduced when considering the ratio p/ϵ at fixed energy density (Fig. 5 (right)). The softest point of the equation of state is found to be $(p/\epsilon)_{\min} \simeq 0.075$ and within our current numerical accuracy it is independent of S/N_B. However, the analysis of the temperature regime $T \lesssim T_0$ at present clearly suffers from poor statistics and still needs a more detailed analysis. We also find that the equation of state for $T_0 < T < 2T_0$ is well parametrized by

$$\frac{p}{\epsilon} = \frac{1}{3}\left(1 - \frac{1.2}{1 + 0.5\epsilon \text{ fm}^3/\text{GeV}}\right).$$
(12)

We note, however, that this phenomenological parametrization is not correct at asymptotically large temperatures as it is obvious that Eq. (12) would lead to corrections to the
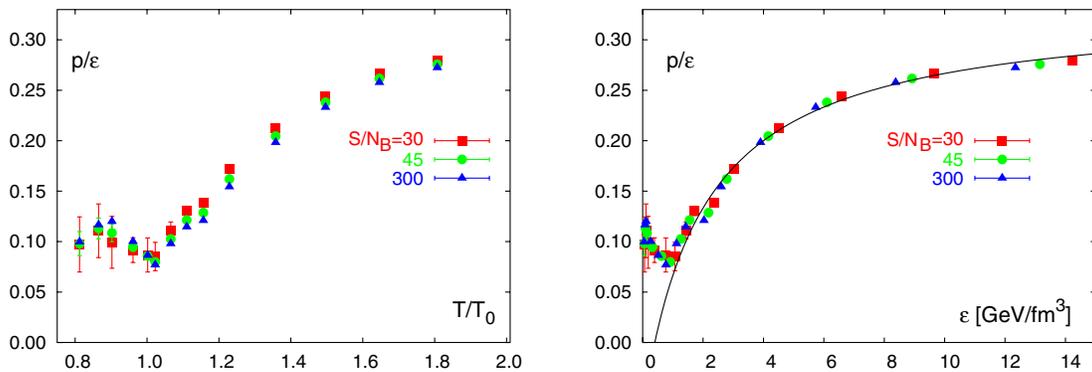

FIG. 5 (color online).   Equation of state on lines of constant entropy per quark number versus T/T_0 (left) and in physical units using $T_0 = 175$ MeV to set the scale (right). The solid curve in the right hand figure is the parametrization of the high temperature part of the equation of state given in Eq. (12).

ideal gas result $p/\epsilon = 1/3$ that are proportional to T^{-4} while perturbative corrections vanish only logarithmically as function of T.

IV. CONCLUSIONS

We have determined Taylor expansion coefficients for the energy and entropy densities in 2-flavor QCD at non-zero quark chemical potential. At present these expansion coefficients have been determined from calculations performed at one value of the bare quark mass which gives a good description of QCD thermodynamics at high temperature. In this temperature regime lines of constant entropy per baryon are well described by lines of constant μ_q/T. On these isentropic lines we have determined the equation of state and find that the ratio of pressure and energy density shows remarkably little dependence on the ratio S/N_B.

The regime close to and below the transition temperature T_0 is not yet well controlled in our current analysis. Smaller quark masses and the introduction of a nonvanishing strange quark mass, kept fixed in physical units, will be needed to explore this regime as well as the approach to the 3-flavor high temperature limit in more detail.

ACKNOWLEDGMENTS

[1] C. R. Allton, S. Ejiri, S. J. Hands, O. Kaczmarek, F. Karsch, E. Laermann, and C. Schmidt, Phys. Rev. D **68**, 014507 (2003).

[2] C. R. Allton, M. Döring, S. Ejiri, S. J. Hands, O. Kaczmarek, F. Karsch, E. Laermann, and K. Redlich, Phys. Rev. D **71**, 054508 (2005).

[3] Z. Fodor, S. D. Katz, and K. K. Szabo, Phys. Lett. B **568**, 73 (2003); F. Csikor, G. I. Egri, Z. Fodor, S. D. Katz, K. K. Szabo, and A. I. Tóth, J. High Energy Phys. 05 (2004) 046.

[4] M. D'Elia and M. P. Lombardo, Phys. Rev. D **70**, 074509 (2004).

[5] D. E. Miller and K. Redlich, Phys. Rev. D **35**, 2524 (1987).

[6] J. Engels, O. Kaczmarek, F. Karsch, and E. Laermann, Nucl. Phys. **B558**, 307 (1999).

[7] S. Kratochvila and Ph. de Forcrand, Proc. Sci., LAT2005 (2005) 167.

[8] C. Bernard *et al.*, Proc. Sci. **LAT2005**, 156 (2005).

[9] T. Blum, L. Kärkkäinen, D. Toussaint, and S. Gottlieb, Phys. Rev. D **51**, 5153 (1995).

[10] A. AliKhan *et al.* (CP-PACS), Phys. Rev. D **64**, 074510 (2001).

[11] F. Karsch, E. Laermann, and A. Peikert, Nucl. Phys. **B605**, 579 (2001).

[12] A. Peikert, Ph.D. thesis, Bielefeld, 2000.

[13] J. Cleymans and K. Redlich, Phys. Rev. Lett. **81**, 5284 (1998); Phys. Rev. C **60**, 054908 (1999).

[14] J. Cleymans, H. Oeschler, K. Redlich, and S. Wheaton, hep-ph/0511094.

[15] C. M. Hung and E. Shuryak, Phys. Rev. C **57**, 1891 (1998).

[16] V. D. Toneev, J. Cleymans, E. G. Nikonov, K. Redlich, and A. A. Shanenko, J. Phys. G **27**, 827 (2001); J. Phys. G **27**, 827 (2001).

[17] R. V. Gavai and S. Gupta, Phys. Rev. D **71**, 114014 (2005).

[18] Z. Fodor and S. D. Katz, J. High Energy Phys. 04 (2004) 050.