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Power-law eigenstates of a regular Vicsek fractal with hierarchical interactions

Zhifang Lin*
Graduate School of Science and Technology, Niigata University, Niigata 950-21, Japan

Masaki Goda†
Graduate School of Science and Technology, Niigata University, Niigata 950-21, Japan
and Faculty of Engineering, Niigata University, Niigata 950-21, Japan

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The nature of transverse vibrational eigenmodes of a regular Vicsek fractal with hierarchical nearest-neighbor interactions has been analyzed using a real-space renormalization-group approach. Different types of spatial scaling behavior for the nondegenerate transverse vibrational modes are obtained depending on a positive hierarchical parameter R: for $R > R_c$, the nondegenerate modes are all extended in nature, while for $R < R_c$, they exhibit a type of power-law behavior, with the exponent of power depending explicitly on R.

In recent years, studies concerning the nature of the eigenvalue spectrum and the eigenstates of fractal systems have revealed a wealth of interesting and exotic features. For instance, on random fractals, numerical results suggested the existence of anomalous density of states[1,2] and superlocalization[3,4] of eigenmodes. In the case of deterministic fractals, detailed analysis by Rammal has shown that the eigenfrequency spectrum of a Sierpinski gasket is the superposition of two distinct pure point spectra.[5] There are consequently two types of localized modes. The first type of mode is termed as molecular state, since it has nonvanishing amplitudes only at a finite set of sites. The second type of mode, referred as hierarchical state, is localized around the empty region of the fractal lattice. More recently, Jayanthi and Wu (JW) carried out an investigation of the transverse vibration modes of a regular Vicsek fractal (VF).[6,7] Among other things, two unusual and interesting features have been found: (i) the frequency spectrum of the regular VF is a pure point spectrum consisting of nondegenerate as well as degenerate modes, while all the nondegenerate modes (NDM's) are extended, and (ii) the extended states of the NDM's exist side by side with the superlocalized states of the persistent degenerate modes, so that there is no frequency demarcation which separates the extended states from the localized ones. In the present study, we investigate the nature of the transverse vibrational modes of particles arranged on a regular VF with a hierarchical distribution of nearest-neighbor interactions. It is found that for a positive hierarchical parameter R $(0<R\le 1)$ greater than a critical value $R > R_c = \frac{2}{3}$, the NDM's are extended, while for $R < R_c$, the NDM's exhibit a type of power-law decay behavior.

The model treated here is similar to that of JW.[6,7] The first-stage VF consists of five particles with four of them located at the corners and the last one at the center of a square. The force constant describing the interaction between the nearest-neighboring particles is γ and the mass of the particle m. With the use of the reduced unit of $\gamma/m = 1$, the transverse equations of motion for the first-stage VF with nearest-neighbor interactions and with its outer particles anchored by force constant γ to a rigid boundary are

$$(4-\omega^2)u_1 - \sum_\alpha v_{1\alpha} = 0 \,, \tag{1}$$

$$(2-\omega^2)v_{1\alpha} - u_1 = 0 \quad (\alpha = a,b,c,d) \,, \tag{2}$$

where u_1 denotes the displacement of the central site and $v_{1\alpha}$, with $\alpha = a$, b, c, and d, the displacements of the four outer sites on the first-stage fractal.

The hierarchy of the interactions (force constants) for the model under consideration is introduced in the second- and higher-stage fractals. In the present model, a second-stage VF is constructed by assembling five copies of the first-stage fractal clusters with one at the center and the other four at the corners of a square. However, different from the model studied by JW,[6,7] in the present model, the force constants between the particles in the central first-stage fractal cluster and the nearest-neighboring particles in the outer first-stage clusters are set to be $R\gamma$ (with $0<R\le 1$) instead of γ, see the schematic illustration given in Fig. 1(a). The higher-stage fractals are built by a similar construction pattern. That is, an $(n+1)$th-stage fractal is made by assembling five copies of the nth-stage fractal clusters with one located at the center and the other four at the corners of a square, whereas every two adjoining nth-stage clusters are connected by force constant $R^n\gamma$, rather than γ. Notice that if one sets $R=1$, the present model reduces to that studied by JW,[6,7] while if we remain only a linear chain in one direction and remove all branches in the other direction, we are left with a one-dimensional hierarchical model, which has attracted a lot of attention in recent years.[8,9]

The present hierarchical VF model preserves both the local symmetry and the self-similar nature. The heuristic analysis by JW on the nature of the frequency spectrum and eigenmodes remains tenable. So the present model preserves persistent degenerate modes which are confined to a finite region so that they do not "feel" the effect due to the hierarchy of the interactions and thus remain superlocalized.[6] On the other hand, as the NDM's are all

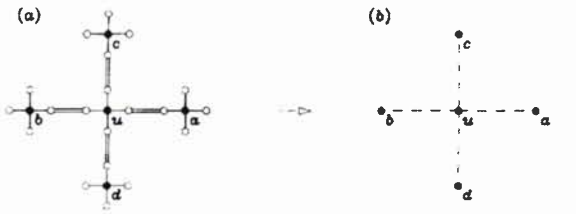

FIG. 1. Reduction of a second-stage fractal (a) to a "first-stage fractal," (b) consisting of five central sites (solid circles) with renormalized interactions (dashed lines). In (a), the single lines between the circles stand for the interactions with force constant γ, while the double lines represent the interactions with force constant $R\gamma$, i.e., every two adjoining first-stage fractal clusters are connected by force constant $R\gamma$, instead of γ, for the second-stage fractal.

extended throughout the entire fractal lattice when $R=1$,[7] one may expect the change of the nature of these modes when the hierarchy of the force constants is introduced. In this work, we shall focus on the investigation of the NDM's.

As was found by JW,[6] for a NDM of the nth-stage VF, the displacement at the central site of the entire fractal must be nonzero. Furthermore, the central sites of the four outer $(n-1)$th-stage clusters must have equal and finite (nonvanishing) displacements. To take advantage of this property, it is preferable to reduce the equations of motion describing any stage of fractal to a set of equations connecting only the five central sites. To do so, the decimation method proves to be useful.

For the second-stage VF composed of 25 particles, let u_2 denote the displacement of the central site of the central cluster, whereas $v_{2\alpha}$, with $\alpha=a$, b, c, and d, denote the displacements of the central sites of the four outer component clusters. By eliminating all other displacement coordinates except u_2 and $v_{2\alpha}$ (see the illustration in Fig. 1), we are led to the following renormalized equations of motion connecting the desired five central sites

$$(4-\alpha_2)u_2-\kappa_2\sum_{\alpha}v_{2\alpha}=0\ , \tag{3}$$

$$(4-\beta_2)v_{2\alpha}-\kappa_2u_2=0\quad(\alpha=a,b,c,d)\ , \tag{4}$$

with

$$\alpha_2=\omega^2+\frac{4(1+R-\omega^2)}{(1+R-\omega^2)^2-R^2}\ ,$$
$$\beta_2=\omega^2+\frac{3}{2-\omega^2}+\frac{1+R-\omega^2}{(1+R-\omega^2)^2-R^2}\ , \tag{5}$$
$$\kappa_2=\frac{R}{(1+R-\omega^2)^2-R^2}\ .$$

For a general nth-stage fractal with $n\geq3$, let u_n be the displacement at the central site of the whole fractal and $v_{n\alpha}$, with $\alpha=a$, b, c, and d, the displacements at the central sites of the four outer $(n-1)$th-stage clusters. By decimation, we can obtain a set of renormalized equations of motion for the five central sites. They are

$$(4-\alpha_n)u_n-\kappa_n\sum_{\alpha}v_{n\alpha}=0\ , \tag{6}$$

$$(4-\beta_n)v_{n\alpha}-\kappa_nu_n=0\quad(\alpha=a,b,c,d)\ , \tag{7}$$

where the renormalized parameters are given by

$$\alpha_n=\alpha_{n-1}+\frac{4\kappa_{n-1}^2[4-\beta_{n-1}^{(1)}]}{[4-\beta_{n-1}^{(1)}]^2-[\kappa_{n-1}^{(1)}]^2}\ , \tag{8}$$

$$\beta_n=\alpha_{n-1}+\frac{3\kappa_{n-1}}{4-\beta_{n-1}}+\frac{\kappa_{n-1}^2[4-\beta_{n-1}^{(1)}]}{[4-\beta_{n-1}^{(1)}]^2-[\kappa_{n-1}^{(1)}]^2}\ , \tag{9}$$

$$\kappa_n=\frac{\kappa_{n-1}^2\kappa_{n-1}^{(1)}}{[4-\beta_{n-1}^{(1)}]^2-[\kappa_{n-1}^{(1)}]^2}\ , \tag{10}$$

$$\beta_n^{(i)}=\alpha_{n-1}+\frac{2\kappa_{n-1}}{4-\beta_{n-1}}+\frac{\kappa_{n-1}^2[4-\beta_{n-1}^{(1)}]}{[4-\beta_{n-1}^{(1)}]^2-[\kappa_{n-1}^{(1)}]^2}+\frac{\kappa_{n-1}^2[4-\beta_{n-1}^{(i+1)}]}{[4-\beta_{n-1}^{(i+1)}]^2-[\kappa_{n-1}^{(i+1)}]^2}\ , \tag{11}$$

$$\kappa_n^{(i)}=\frac{\kappa_{n-1}^2\kappa_{n-1}^{(i+1)}}{[4-\beta_{n-1}^{(i+1)}]^2-[\kappa_{n-1}^{(i+1)}]^2}\ , \tag{12}$$

$$\beta_2^{(i)}=\omega^2+\frac{2}{2-\omega^2}+\frac{1+R-\omega^2}{(1+R-\omega^2)^2-R^2}+\frac{1+R^{i+1}-\omega^2}{(1+R^{i+1}-\omega^2)^2-R^{2(i+1)}}\ , \tag{13}$$

$$\kappa_2^{(i)}=\frac{R^{i+1}}{(1+R^{i+1}-\omega^2)^2-R^{2(i+1)}}\ , \tag{14}$$

with $i=1,2,3,\ldots$.

For NDM's of the nth-stage fractal, with the use of the property $u_n\neq0$ and $v_{na}=v_{nb}=v_{nc}=v_{nd}$,[6] Eqs. (6) and (7) give rise to

$$\left[4-\alpha_n-\frac{4\kappa_n^2}{4-\beta_n}\right]u_n=0\ . \tag{15}$$

Thus, the roots of the equation

$$f_n(\omega^2)=4-\alpha_n(\omega^2)-\frac{4\kappa_n^2(\omega^2)}{4-\beta_n(\omega^2)}=0 \tag{16}$$

yield the eigenfrequencies of the NDM's of the nth-stage fractal.

Now we are ready to examine the effect on the nature of the NDM's due to the hierarchy of the force constants. To this end, we have carried out an extensive numerical calculation using quadruple precision. The outline of the calculation is as follows: Given a fractal of certain stage n, the eigenfrequencies of the fundamental NDM and the highest excited NDM are first calculated by using Eq. (16). The renormalized parameters α_n, $\beta_n^{(i)}$, and $\kappa_n^{(i)}$ are next calculated through Eqs. (5) and (8)–(14). The ratio $v_{n\alpha}/u_n$, which has been used as a criterion for the extended states,[7] is then given by $\kappa_n(\omega^2)/[4-\beta_n(\omega^2)]$. Finally, by using $v_{n\alpha}/u_n$ and setting $u_n=1$, the vibrational amplitudes at any site of the fractal can be computed with the help of the renormalized parameters.

Figures 2 and 3 show the log-log plots of the vibration-

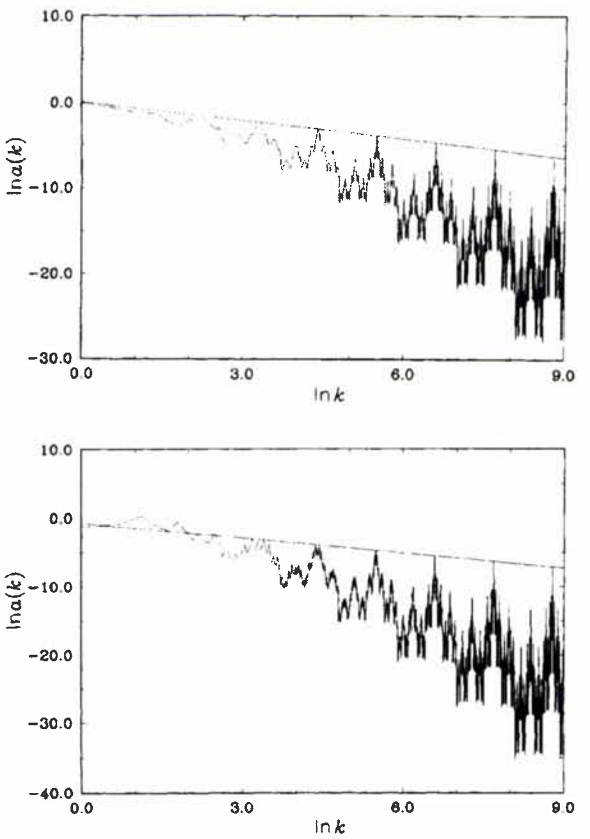

FIG. 2. log-log plot of the transverse vibrational amplitude $a_n(k)$ vs k for (a) the fundamental nondegenerate mode and (b) the highest excited nondegenerate mode with $R=0.3$. Here $a_n(k)$ denotes the vibrational amplitude at site k along one of the four symmetric linear chains starting from the central site of an nth-stage fractal, so that $a_n(0)=u_n$ and $a_n(3^{n-1})=v_{n\alpha}$. The straight line in the figure is of slope $(\ln 3R - \ln 2)/\ln 3$ and serves as a guide to eye showing the power-law decay behavior of the maximum amplitudes along the linear chain. The data for the figure are obtained on a 10th-stage fractal.

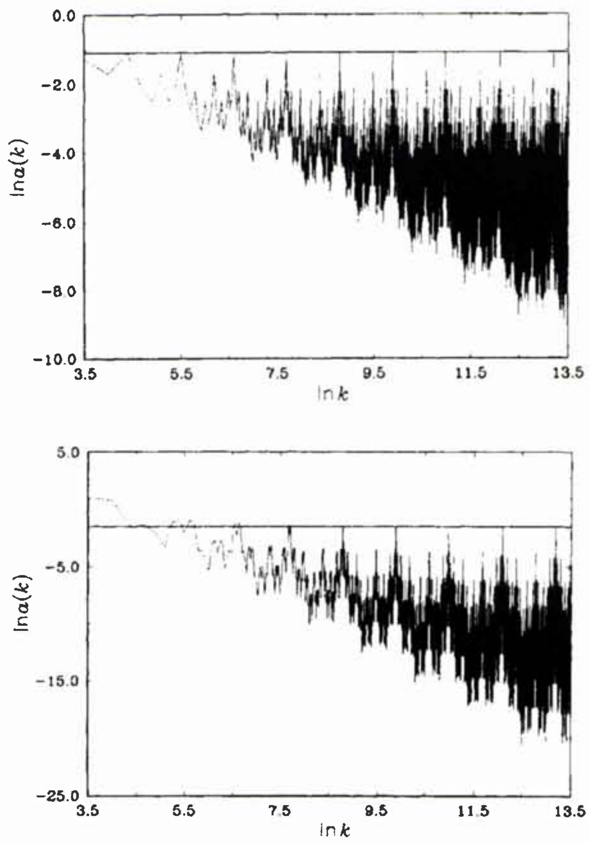

FIG. 3. The same as in Fig. 2 except that $R=0.72$ and the data for the figure are obtained on a 14th-stage fractal. The horizontal straight line serves to show that the maximum amplitudes do not decay.

al amplitude $a_n(k)$ vs k for the cases with $R=0.3$ and 0.72, respectively. Here $a_n(k)$ denotes the amplitudes at site k along one of the four symmetric linear chains starting from the central site of the nth-stage fractal, so that $u_n=a_n(0)$ and $v_{n\alpha}=a_n(3^{n-1})$. It can be seen from Figs. 2 and 3 that different types of spatial behavior appear depending on the value of R. For $R=0.3$, the maximum amplitudes decay by a power law with the increasing distance from the central site. While for $R=0.72$, the maximum amplitudes do not decay, suggesting that the NDM's remain extended throughout the whole fractal, as in the case with $R=1$.[7] To be more specific, we have examined the positions of the maximum amplitudes along the linear chain. Our extensive numerical results indicate that the maximum vibrational amplitudes occur at site $k=3^l$, with $l<n$, for the fundamental mode. While for the highest excited mode, the maximum vibrational amplitudes appear at $k=3^l \pm 3^{l^*}$, with l^* dependent on the value of R but independent of l. In addition, the ratio of $a_n(3^l)$ to $a_n(3^l \pm 3^{l^*})$ is independent of l for the highest excited mode. So for both the fundamental and the highest modes, the scaling behavior of the vibrational amplitudes can be determined by the values of amplitudes at the sites with $k=3^l$. We have studied the ratios $A_n(l) = a_n(3^{l+1})/a_n(3^l)$ for various values of R. Figure 4 shows a plot of $A_n(l)$ vs l for several typical values of R. It can be seen that $A_n(l)$ approaches to different asymptotic values depending on R. A careful analysis of the numerical results indicates that

$$\lim_{l\to\infty} A_n(l) = \begin{cases} 3R/2 & \text{for } R < \frac{2}{3} , \\ 1 & \text{for } R \geq 2/3 . \end{cases} \tag{17}$$

So, the crossover for a transition from the extended state to the power-law state is observed at $R_c = \frac{2}{3}$. Although we have only presented the data for the fundamental mode and the highest excited mode, the scaling behavior is believed to be the same for other NDM's.[7] It is, therefore, concluded that when the positive hierarchical parameter $R > R_c = \frac{2}{3}$, the nondegenerate transverse vibra-

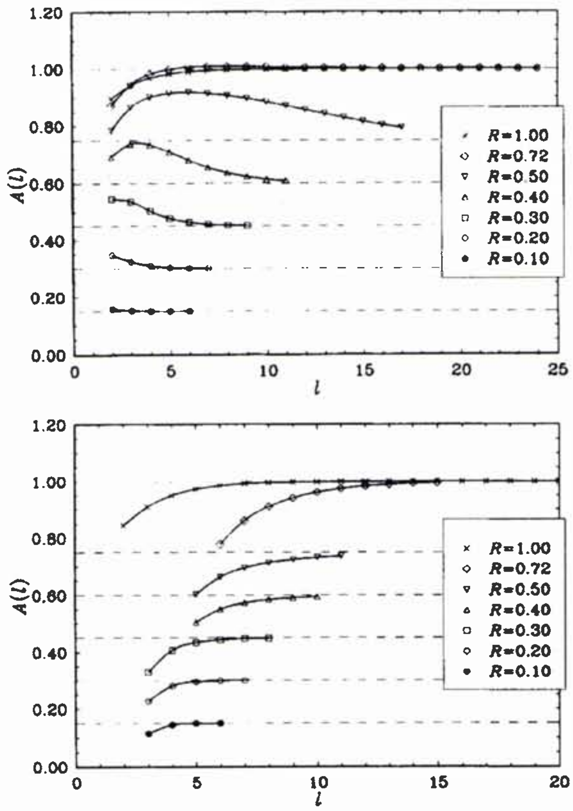

FIG. 4. Plot of the ratio of the vibrational amplitudes $A_n(l)=a_n(3^{l+1})/a_n(3^l)$ vs l along one of the four symmetric linear chains for (a) the fundamental nondegenerate mode and (b) the highest excited nondegenerate mode.

tional modes of the VF with hierarchical interactions are all extended, while for $R<R_c$, all NDM's exhibit a type of power-law spatial scaling behavior, with the exponent of power given by

$$\chi=\frac{\ln 3R-\ln 2}{\ln 3} \quad \text{for } R<\tfrac{2}{3} . \tag{18}$$

It should be emphasized that the numerical results show that the value of $A_n(l)$ does not depend on the stage number n for fairly large n, i.e., any two sets of the ratios, $A_{n_1}(l)$ and $A_{n_2}(l)$ with $n_1 \neq n_2$, agree with each other rather well up to $l=\min(n_1-2,n_2-2)$ for both the fundamental and the highest modes. As a result, although we cannot carry out the numerical calculation for extremely large n, especially for small R, due to the limit of quadruple precision, the asymptotic behavior of $A_n(l)$, as was shown in Fig. 4, is believable enough to draw the conclusion (17). In fact, the data for Fig. 4 are obtained on the fractals with the largest value of n which can be achieved within the limit of quadruple precision.

Finally, it is interesting to note that when $R<\frac{2}{9}$, the exponent of power χ is smaller than -1, which implies that the NDM's become power-law localized. For $\frac{2}{9}<R<\frac{2}{3}$, our numerical results suggest that there is a region for R, in which the NDM's are power-law critical, like the critical states of one-dimensional quasicrystals.[10] In this sense, there seem to be two transitions when the value of R is decreased from unity to zero, although we are unable to find the crossover behavior for the transition from the power-law critical state to the power-law localized state on the basis of the numerical results in quadruple precision.

To summarize, by a real-space renormalization-group approach, we have examined the spatial behavior of the nondegenerate eigenmodes for the transverse vibration of particles arranged on a regular VF with a hierarchical distribution of the nearest-neighbor interactions. It is found that the NDM's display different types of spatial behavior depending on the positive hierarchical parameter R. When $R>R_c$, all the NDM's are extended, while for $R<R_c$, the NDM's exhibit a type of power-law spatial scaling behavior, with the exponent of power depending explicitly on R.

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*Permanent address: Department of Physics, Fudan University, Shanghai 200433, China.

†Author to whom all correspondence should be addressed.

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