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Surface magnetization and critical behavior of a hierarchical quantum Ising chain

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We study the surface magnetization and critical behavior of a hierarchical quantum Ising chain. The surface magnetization exponent β_s is calculated exactly by using scaling arguments. It is found that, although the size of the fluctuations in the exchange couplings may be bounded or unbounded depending on the value of the hierarchical parameter r, the surface critical behavior stays nonuniversal with β_s varying continuously with the value of r. This result is rather different from the cases where the aperiodicity of the couplings is generated by substitution rules and different types of critical behavior are observed for bounded and unbounded fluctuations in the couplings.

The discovery of quasicrystals has stimulated a growing interest in understanding their structural and physical properties (for a recent review, see Ref. 1). Among others, the study of the critical behavior of the quasiperiodic and, more generally, the deterministic aperiodic systems is also an active field of theoretical research, since these systems can be regarded as a state of matter intermediate between the periodic and random extremes. The question to be addressed is how the imperfection (aperiodicity) of the underlying lattice affects the critical properties of a system. Exact results in this field can be obtained on two-dimensional layered Ising models with constant intralayer interactions and an aperiodic modulation of the interlayer couplings.[2] In this aspect, one usually works in the extreme anisotropic limit, in which the system can be described by a one-dimensional quantum Ising model (QIM) governed by the Hamiltonian

$$H=-\frac{1}{2}\sum_{k=1}^{\infty}(\lambda_k\sigma_k^x\sigma_{k+1}^x+\sigma_k^z)\ , \tag{1}$$

where the σ's are Pauli matrices and the exchange couplings λ_k are modulated in some aperiodic way.

On the bulk critical properties of system (1), a systematic study by Luck[2] showed that the effect of the aperiodicity can be relevant or irrelevant, depending on the size of the fluctuations in the couplings λ_k. For bounded fluctuations, the system undergoes an Ising-type phase transition like the homogeneous one; e.g., the specific heat of the system diverges logarithmically at critical point. For unbounded fluctuations, on the other hand, the critical behavior is anomalous; e.g., the specific heat displays an essential singularity like in the system with random couplings.[3] The marginal case, which has the fluctuations growing on a logarithmic scale, may exhibit nonuniversal critical behavior with critical exponents dependent on the strength of the modulation.

For the surface critical behavior, based on scaling considerations, it was recently claimed[4] that the relevance or irrelevance of the aperiodicity is related to the wandering exponent ω, which is defined by the deviation from the averaged coupling as follows:

$$\Delta(L)=\sum_{k=1}^{L}(\lambda_k-\bar{\lambda})\sim L^{\omega}\ , \tag{2}$$

where L is the length of the chain and $\bar{\lambda}$ is the averaged coupling given by

$$\bar{\lambda}=\lim_{L\to\infty}\frac{1}{L}\sum_{k=1}^{L}\lambda_k\ . \tag{3}$$

For $\omega<0$, the aperiodic modulation is an irrelevant perturbation so that the surface magnetization of the corresponding QIM vanishes with a square-root singularity, i.e., the surface critical exponent $\beta_s=\frac{1}{2}$, like in the homogeneous Ising system. For $\omega>0$, which corresponds to the relevant case, the system exhibits an anomalous surface critical behavior. And finally, in the marginal case with $\omega=0$, nonuniversal critical behavior is expected with the critical exponent β_s dependent on the strength of the aperiodicity. The relevance-irrelevance criterion has been verified on three QIM's with the couplings arranged in specific aperiodic sequences, i.e., the Thue-Morse, the period-doubling, and the Rudin-Shapiro sequences,[4] which are all generated by substitution rules and were chosen to represent three typical types of aperiodic modulation with $\omega<0$, $\omega=0$ and $\omega>0$, respectively.

Now an interesting question is whether or not the surface magnetization of QIM on a hierarchical lattice, which is another type of deterministic aperiodic system and has attracted much attention recently (see, e.g., Ref. 5 for a review), will subject to this relevance-irrelevance criterion. In this report, we examine the surface critical behavior of a hierarchical quantum Ising chain. The critical exponent β_s is calculated exactly by using scaling arguments. Our result shows that the hierarchical QIM display a nonuniversal surface critical behavior, even when the value of the wandering exponent ω is smaller

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than 0.

The model treated here is described by the Hamiltonian (1) with the couplings λ_k given by

$$\lambda_k=\begin{cases}\lambda, & k=2l+1\\ \lambda r^n, & k=2^n(2l+1)\end{cases}$$

$$l=0,1,2,\ldots, \text{ and } n=1,2,3,\ldots, \quad (4)$$

where r, satisfying $0<r<2$,[6] is a parameter characterizing the hierarchy. For a QIM with any distribution of the couplings λ_k, the surface magnetization m_s can be expressed as[7]

$$m_s=\langle 1|\sigma_1^x|0\rangle , \quad (5)$$

where $|0\rangle$ is the ground state of H and $|1\rangle$ its first excited state. Using a Jordan-Wigner transformation of the spin operator and by a canonical transformation, one can rewrite the surface magnetization in the form simply involving sums of products of the couplings[8]

$$m_s=[S(\lambda,r)]^{-1/2}, \quad S(\lambda,r)=1+\sum_{j=1}^{\infty}\prod_{k=1}^{j}\lambda_k^{-2} . \quad (6)$$

The bulk critical coupling λ_c is determined by the condition[9] $\lim_{L\to\infty}\prod_{k=1}^{L}(\lambda_k)_c=1$, so that

$$\lambda_c=1/r \quad (7)$$

for the hierarchical QIM described by (1) and (4).

To evaluate the infinite sum $S(\lambda,r)$ for the surface magnetization m_s, let us define

$$S_N(\lambda,r)\equiv 1+\sum_{j=1}^{2^N-1}\prod_{k=1}^{j}\lambda_k^{-2} . \quad (8)$$

Due to the peculiar structure of the exchange couplings (4), it is not difficult to derive the following recursion relation:

$$S_{N+1}(\lambda,r)=(1+u^{2^N-1}/\lambda^2)S_N(\lambda,r), \quad N\geq 1 , \quad (9)$$

where $u=(\lambda/\lambda_c)^2$ and the initial condition reads $S_1(\lambda,r)=1+1/\lambda^2$. The above recursion relation, together with (6), yields the surface magnetization shown in Fig. 1.

To analyze the critical behavior of the surface magnetization, we take advantage of the scaling method proposed by Iglói.[10] Denote by $S(u)$ the series expansion of $S(\lambda,r)$ in terms of the powers of $u=(\lambda/\lambda_c)^2$. It follows from Eq. (6) that $S(u)$ should show a power-law singularity near the critical point $u=1$:

$$S(u)\sim(1-u)^{-2\beta_s} , \quad (10)$$

where β_s is the surface critical exponent describing the singularity of the surface magnetization at the critical point[4]

$$m_s\sim(1-u)^{\beta_s} . \quad (11)$$

On the other hand, if one truncates the series expansion $S(u)$ by retaining the terms up to u^{L-1} and discarding terms of higher powers of u, it can be shown that the truncated series, $S_L(u)$, should behave as $L^{2\beta_s}$ at the critical point $u=1$ for large values of L.[4,10] Paying attention to the recursion relation (9) and its initial condition, one observes that

$$S_L(u)=S_N(\lambda,r)+(1+1/\lambda^2)u^{L-1}/\lambda^2 , \quad \text{with } L=2^N . \quad (12)$$

As a result, one has

$$S_{L=2^N}(u=1)=(1+r^2)^N+(1+r^2)r^2\sim(2^N)^{2\beta_s} , \quad (13)$$

where we have made use of the critical condition (7). From (13), it follows that the surface magnetization critical exponent

$$\beta_s=\frac{\ln(1+r^2)}{2\ln 2} . \quad (14)$$

The surface magnetization critical exponent depends on the value of the hierarchical parameter r, showing a nonuniversal surface critical behavior as in the marginal case with aperiodic sequence generated by substitution rules.[4] The variation of β_s vs r is shown in Fig. 2. It can be seen that β_s is no longer greater than $\frac{1}{2}$ when the hierarchical parameter $r<1$, which suggests that the singularity may be stronger than in the homogeneous case.

It should be emphasized that the deviation from the averaged coupling given by (2) now reads, for $0<r<2$,

$$\Delta(2^N)=\sum_{k=1}^{2^N}(\lambda_k-\bar{\lambda})=\lambda(1-r)r^N/(2-r) , \quad (15)$$

which leads to

$$\omega=\frac{\ln r}{\ln 2} \quad \text{for } r\neq 1 . \quad (16)$$

The wandering exponent ω is positive and negative when $r>1$ and $r<1$, respectively, while the surface critical exponent β_s keeps dependent on the amplitude of the aperiodicity through the hierarchical parameter r. This result is rather different from the cases where the

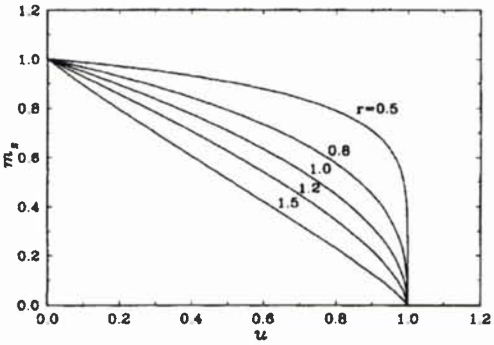

FIG. 1. The surface magnetization m_s as a function of $u=(\lambda_c/\lambda)^2$ for different values of r. The surface magnetization exponent β_s varies continuously with the value of r.

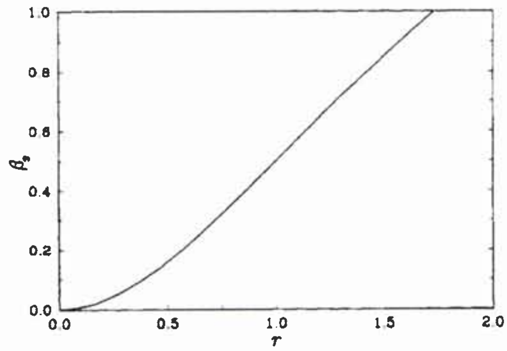

FIG. 2. The surface magnetization exponent β_s as a function of r. Notice that the system displays a stronger singularity than in the homogeneous case when $r < 1$.

aperiodic sequence of the couplings in QIM is generated by substitution rules.[4] For the latter cases, as the perturbation caused by the aperiodicity is finite, the critical behavior subjects to the relevance-irrelevance criterion. As a result, the surface critical exponent β_s is nonuniversal only when ω vanishes, whereas for positive ω and negative ω, the corresponding systems display, respectively, an anomalous and an Ising-type surface critical behavior. For the model treated here, on the other hand, because the exchange couplings can be infinitely strong (for $r > 1$) or infinitely weak (for $r < 1$), one cannot expect that the relevance-irrelevance criterion remains valid. Our result confirms this belief.

To summarize, we have studied the surface magnetization and the surface critical behavior of a hierarchical QIM. The surface magnetization exponent β_s has been calculated exactly using scaling arguments. It has been shown that although the system may have bounded or unbounded fluctuations in the exchange couplings λ_k, i.e., with the wandering exponent $\omega > 0$ or $\omega < 0$ depending on the hierarchical parameter r, the surface critical behavior keeps to be nonuniversal with the surface critical exponent β_s being a continuous function of the hierarchical parameter r.

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