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Unique Continuation Theorem for CR-hyperfunctions

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§0. Introduction

Let $\overline{\partial_b}$ be the tangential Cauchy-Riemann system induced on a real analytic hypersurface of a complex manifold. A hyperfunction solution of the system $\overline{\partial_b}$ is called a CR-hyperfunction.

We consider the following unique continuation problem for CR-hyperfunctions : What geometric conditions on the hypersurface ensure that every CR-hyperfunction which is zero on an arbitrary open subset has the property that it is identically equal to zero.

In the section one we show that the classical Kneser's edge theorem implies a local unique continuation theorem for CR-hyperfunctions.

In the section two we give a global unique continuation theorem. Note that our proof, which is based on the study of H. Lewy's phenomena and the results obtained in the first section of this paper, is different to the one given by Hunt-Polking-Strauss [4].

In the appendix we examine the problem from the microlocal point of view and we give an improvement to the local unique continuation theorem.

§1. Local results

Let N be a real analytic real hypersurface of a m-dimensional complex manifold X. Let $\overline{\partial}_b$ denotes the tangential Cauchy-Riemann system induced on N. Let U be an open subset of N. By a CR-hyperfunction on U we mean a hyperfunction h which satisfies the tangential Cauchy-Riemann equations $\overline{\partial}_b h = 0$ on U.

Let S be a real hypersurface of N. For $p \in S$, we denote by $H_p(S)$ the holomorphic tangent space to S at p. It is easy to see that $m-2 \leqq \dim_{\mathbf{C}} H_p(S) \leqq m-1$. A hypersurface S is said to be generic at p if it satisfies the condition $\dim_{\mathbf{C}} H_p(S) = m-2$.

We have the following theorem.

Theorem 1.

Let S be a real analytic real hypersurface of N. Let h be a CR-hyperfunction on a neighborhood $U \subset N$ of a point $p \in S$. Suppose that the CR-hyperfunction h vanishes on one side of S. If the hypersurface S is generic at p, then $h \equiv 0$ in a neighborhood of p.

Proof. Let r be a real-valued real analytic local defining function for N. Since the submanifold S is generic at p, there exists a real-valued real analytic function s which satisfies following conditions:

(i) $S \cap \Omega = \{ z \in \Omega \mid r(z, \overline{z}) = s(z, \overline{z}) = 0 \}$,

(ii) $\overline{\partial} r \wedge \overline{\partial} s \neq 0$ in Ω,

for some neighborhood Ω in X of p.

We set :

$$\Omega_{\pm} = \{ z \in \Omega \mid r(z, \overline{z}) \gtrless 0 \},$$

$$\omega_{\pm} = \{ z \in \Omega \mid r(z, \overline{z}) = 0, \ s(z, \overline{z}) \gtrless 0 \}.$$

By shrinking Ω if necessary, we may assume that the CR-hyperfunction h is the difference of two boundary values (see Stormark [10]) :

$$h = \mathrm{b}(h_{+}) - \mathrm{b}(h_{-}) \qquad \text{on} \qquad \Omega \cap N,$$

where $\mathrm{b}(h_{+})$ denotes the boundary value of holomorphic function $h_{\pm} \in \Gamma(\Omega_{\pm}, \mathcal{O}_{X})$.

Without loss of generality we may asumme that $h \equiv 0$ on ω_{-}. Hence

$$\mathrm{b}(h_{+}) = \mathrm{b}(h_{-}) \qquad \text{holds on} \quad \omega_{-}.$$

We set:

$$\phi(z) = \begin{cases} h_{+}(z) & z \in \Omega_{+} \\ h_{-}(z) & z \in \Omega_{-}. \end{cases}$$

Then the function ϕ has a holomorphic continuation to $\Omega - \omega_{-}$.

Since the submanifold S is generic, we can apply the classical Kneser's edge theorem [8](see [1],[2]) to conclude that the function ϕ can be continued analytically to a neighborhood of p. Hence we have

$$h = \mathrm{b}(h_{+}) - \mathrm{b}(h_{-}) = \mathrm{b}(\phi) - \mathrm{b}(\phi) \equiv 0.$$

Q.E.D.

Corollary 2 (see [13]).

Let h be a non-zero CR-hyperfunction defined in an open subset U of N. Assume that the support of h, we denote it by V, is a real analytic hypersurface in U. Then V is a complex $m-1$ dimensional complex manifold.

Proof. Theorem 1 implies that the submanifold V is not generic at any point of V. Hence the following lemma concludes the result.

Lemma (see Theorem 17.1 of Fuks [2]).

Let V be a real hypersurface in N. Assume that $\dim_{\mathbf{C}} H_p(V) = m-1$ holds at every point p of V. Then the submanifold V is a complex hypersurface in X. Q.E.D.

We also have the following result.

Corollary 3.

Let h be a CR-hyperfunction on an open subset U of N. Assume that the support of h is a real analytic subset of dimension less than $2m-2$. Then we have $h \equiv 0$.

Note that Corollary 2 and Corollary 3 follow from the classical result of Hartogs concerning removable singularities of holomorphic functions.

§2. global result.

Let N be a connected real analytic hypersurface in a m-dimensional complex manifold X such that $X-N$ has two connected components X_+ and X_-.

The aim of this section is to prove the next theorem.

Theorem 4.

Assume that the manifold N does not contain a complex hypersurface. Then every CR-hyperfunction h on N which vanishes on an arbitrary open subset of N vanishes identically.

Remark. Hunt-Polking-Strauss [4] obtained an analogous result for CR-distributions. Let us emphasize that our proof, based on the study of H. Lewy's phenomena [12], is quite different from the one given in [4].

Let $L_p(N)$ be the Levi form of N at a point $p \in N$.

Lemma.

Let U be an open connected subset of N. Assume that the Levi form $L_p(N)$ has eigenvalues of opposite sign at every point $p \in U$. Then every CR-hyperfunction on U which vanishes on an open subset of U vanishes identically.

Proof. Let h be a CR-hyperfunction on U. Since the Levi form has at least one positive eigenvalue and one negative eigenvalue, the CR-hyperfunction h is a real analytic function which satisfies the tangential Cauchy-Riemann equations on U. Hence the Cauchy-Kowalewskaja theorem implies that the CR-function h is equal to a restriction to U of a holomorphic function (see Theorem 1.2.2 of [11]). Therefore every CR-hyperfunction h possesses the unique continuation property. Q.E.D.

Proposition 5.

Let U be an open connected subset of N. Assume that the Levi form $L_p(N)$ is non-degenerate at any point of U. Then every CR-hyperfunction on U which vanishes on an open subset of U vanishes identically.

Proof. By the lemma above it suffices to consider the case when the Levi form is strictly positive (or negative) definite at every point of U. In this case, H. Lewy's extension theorem (cf. Theorem 3.1 of [12]) implies that every CR-hyperfunction on U can be uniquely extended to a holomorphic function in one sided neighborhood of U (see Theorem 5.8 of [9]).

Hence the principle of unique analytic continuation for holomorphic functions yields the result. Q.E.D.

Proof of Theorem 4.

Let h be a CR-hyperfunction on N which vanishes on an open subset $U' \subseteq N$. Let us denote by $\operatorname{supp}(h)$ the support of h. We set :

$$W = \{ \ p \in N \mid \text{the Levi form } L_p(N) \text{ is degenerate at } p \ \}.$$

To prove the theorem it is enough to consider the following two cases, for N can not be a Levi-flat manifold.

(i) the case when $\dim_{\mathbf{R}} W \leqq 2m-3$.

In this case $N - W$ is an open connected subset of N. Therefore Proposition 5 immediately implies that $h \equiv 0$ on $N - W$. We hence get $\dim_{\mathbf{R}} \operatorname{supp}(h) \leqq 2m-3$. Corollary 3 yields the result.

(ii) the case when $\dim_{\mathbf{R}} W = 2m-2$.

As we have pointed out before (Proposition 6), each connected component of $N - W$ has the global unique continuation property. Hence we have the following direct sum decomposition to the connected components :

$$N - W = (\bigcup_{i \in \mathrm{I}} U_i) \cup (\bigcup_{j \in \mathrm{J}} U_j).$$

Here $h \equiv 0$ on U_i $(i \in \mathrm{I})$ and $h \neq 0$ on U_j $(j \in \mathrm{J})$. Since $U' \neq \phi$, We have $\mathrm{I} \neq \phi$. If we assume $\mathrm{J} \neq \phi$, then there exists at least one pair of components U_{i_0} $(i_0 \in \mathrm{I})$ and U_{j_0} $(j_0 \in \mathrm{J})$ and a generic point p_0 of W such that

$$p_0 \in \overline{U}_{i_0} \cap \overline{U}_{j_0}.$$

Since the hypersurface W is generic at the point p_0, we can apply Theorem 1 to this situation. Consequently, the CR-hyperfunction h vanishes in a neighborhood of p_0. Hence $j_0 \notin J$ which is a contradiction. Therefore we get $J = \phi$ and $\dim_{\mathbf{R}} \mathrm{supp}(h) \leq 2m-3$. Corollary 3 completes the proof.

Q.E.D.

Appendix. microlocal view point.

Let N be a real analytic hypersurface in a complex manifold X. Let $\overline{\partial_b}$ be the tangential Cauchy-Riemann system induced on N. Let M be a real analytic submanifold of N. Recall that the following fact.

Proposition 6 (cf. Theoreme 1.1.1 of [11]).

The submanifold M of N is non-characteristic with respect to the system $\overline{\partial_b}$ at a point $p \in M$ if and only if the submanifold M is generic at p.

Therefore, by using Sato's fundamental theorem, we conclude that if a hypersurface S in N is generic at a point p, then every CR-hyperfunction h defined in a neighborhood of p satisfies the following condition :

$$(\mathrm{S.S.}(h) \cap \sqrt{-1}\, S_S^* N)_p = \phi .$$

Here $\mathrm{S.S.}(h)$ denotes the singular spectrum of the CR-hyperfunction h.

Hence Holmgren type theorem (Proposition 3.5.2 of [7]), due to Kashiwara-Kawai, and the argument above imply Theorem 1.

We also have the following result which is an improvement of of Theorem 1.

Theorem 7 (cf. Theorem 3.5.1 of [7]).

Let N *be a real analytic real hypersurface in a complex manifold* X*. Let* S *be a hypersurface in* N*. Let* h *be a CR-hyperfunction defined on a neighborhood of a point* $p\in S$ *in* N*. If the hypersurface* S *is generic at* p*, then the restriction* $h|_S$ *is a well defined hyperfunction. Furthermore if* $h|_S=0$ *holds near* p*, then* $h\equiv 0$ *in a neighborhood of* p*.*

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