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On vector equilibrium problems: Remarks on a general existence theorem and applications

El Mostafa Kalmoun, Hassan Riahi and Tamaki Tanaka

Abstract

We focus our attention on generalized vector equilibrium problems. In particular, we formulate a general and unified existence theorem, present an analysis for the assumptions used in this result, and give some applications to vector variational inequalities, vector complementarity problems and vector optimization.

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1 Introduction and preliminaries

In the present paper, we investigate the following form of generalized vector equilibrium problems:

$$\text{Find } \bar{x} \in C \text{ such as to satisfy } \varphi(\bar{x}, y) \not\subseteq K(\bar{x}) \text{ for all } y \in C, \qquad \text{(GVEP)}$$

where

- X and E are topological vector spaces,
- C is a nonempty closed convex subset of X,
- $\varphi : C \times C \to 2^E$ is a set-valued map, and

- K is a set-valued map from C to E.

In particular, we formulate a general existence result, present an analysis for the assumptions used, and give some applications.

Let us first recall and introduce some concepts about convexity and semi-continuity for set-valued vector maps. We denote by $\mathcal{F}(C)$ the set of all finite subsets of C.

1.1 Generalized set-valued convexity

Definition 1.1. *We say that $\psi : C \times X \to 2^E$ is diagonally quasi convex in its first argument relatively to L, in short L-diagonally quasi convex in x, if for any A in $\mathcal{F}(C)$ and any y in $co(A)$, we have $\psi(A, y) \not\subseteq L(y)$.*

Let us consider the scalar case, that is when $E = \mathbb{R}$ and ψ is a numeric valued function. If we set $L :=]-\infty, \gamma[$ (resp. $L :=]\gamma, +\infty[$) for all $x \in C$, where $\gamma \in \overline{\mathbb{R}}$, then the L-diagonal quasi convexity of ψ in x collapses to the following:

$$\forall A \in \mathcal{F}(C)\ \forall y \in co(A) : \min_{x \in A} \psi(x, y) \leq \gamma \text{ (resp. } \max_{x \in A} \psi(x, y) \geq \gamma),$$

which is the γ-diagonal quasi convexity (resp. concavity) considered by Zhou and Chen in [20].

Let us also recall the definition of some concepts of convexity for set-valued maps related to moving cones in E. Let $(P(y))_{y \in C}$ be a family of proper cones in E with $int\, P(y) \neq \emptyset$ for all $y \in C$.

Definition 1.2. *[14] $\psi : C \times C \to 2$ is said to be right P_y-convex if, for all $x_1, x_2, y \in C$ and all $\lambda \in [0, 1]$, one has*

$$\psi(\lambda x_1 + (1 - \lambda)x_2, y) \subseteq \lambda\psi(x_1, y) + (1 - \lambda)\psi(x_2, y) - P(y).$$

Definition 1.3. *[3] ψ is said to be right P_y-quasiconvex[1] if, for all $x_1, x_2, y \in C$ and all $\lambda \in [0, 1]$, one has either*

$$\psi(\lambda x_1 + (1 - \lambda)x_2, y) \subseteq \psi(x_1, y) - P(y)$$

or

$$\psi(\lambda x_1 + (1 - \lambda)x_2, y) \subseteq \psi(x_2, y) - P(y).$$

[1] In [3], it is called P_y-quasiconvex-like, and the left P_y-quasiconvexity below is called P_y-quasiconvex in [13]. When P is constant, it is called type-(v) P-properly quasiconvex in [9].

It should be mentioned that a right P_y-convex set-valued map is not necessarily right P_y-quasiconvex. When F is single-valued and P is constant, we recover the P-convexity and the properly quasi-P-convexity of [8], respectively. All these concepts reduce to the ordinary convexity and quasiconvexity when considering the scalar case.

Furthermore, Konnov and Yao [13] work with following set-valued concept.

Definition 1.4. *[13] ψ is said to be left P_y-quasiconvex if, for all $x_1, x_2, y \in C$ and all $\lambda \in [0,1]$, one has either*

$$\psi(x_1, y) \subseteq \psi(\lambda x_1 + (1-\lambda)x_2, y) + P(y)$$
$$\text{or}$$
$$\psi(x_2, y) \subseteq \psi(\lambda x_1 + (1-\lambda)x_2, y) + P(y).$$

1.2 Transfer semicontinuity

Definition 1.5. *We say that φ is K-transfer semicontinuous in y if for any $(x,y) \in C \times X$ with $\varphi(x,y) \subset K(y)$, there exist an element $x' \in C$ and an open $V \subset X$ containing y such that $\varphi(x', v) \subset K(v)$ for all $v \in V$.*

An equivalent statement is as follows: for each $y \in X$ satisfying

$$\forall x \in C\, \exists y_\alpha \to y \text{ in } X \ : \ \varphi(x, y_\alpha) \not\subseteq K(y_\alpha),$$

we have

$$\varphi(x,y) \not\subseteq K(y) \quad \forall x \in C.$$

Definition 1.6. *φ is said to be transfer u.s.c. (resp. l.s.c.) in y if, for any $(x,y) \in C \times X$ and any open $N \subset E$ with $\varphi(x,y) \subset N$ (resp. $\varphi(x,y) \cap N \neq \emptyset$) there exist an element $x' \in C$ and an open $V \subset X$ such that $\varphi(x', v) \subset N$ (resp. $\varphi(x', v) \cap N \neq \emptyset$) for all $v \in V$.*

Note here that an u.s.c. (resp. l.s.c.) set-valued bi-map in one of its arguments is obviously transfer u.s.c. (resp. l.s.c.). Moreover, the relationship between the two concepts of transfer continuity, introduced above, is the following.

Proposition 1.1. *If φ is transfer u.s.c. in y with compact values and if K has an open graph, then φ is K-transfer semicontinuous in y.*

Proof. Let $y \in X$ such that for any $x \in C$ there is a net (y_α) converging to y in X with $\varphi(x, y_\alpha) \not\subset K(y_\alpha)$. Therefore, for every $x \in C$, there exists $z_\alpha \in \varphi(x, y_\alpha)$ with $z_\alpha \notin K(y_\alpha)$. Passing to a subnet, if necessarily, we can certainly assume that, for every $x \in C$ there exists $z \in \varphi(x, y)$ such that $z_\alpha \to z$. Otherwise, there exists $x' \in C$ such that, for every $z \in \varphi(x', y)$, there is an open $N_z \subset E$ containing z and satisfying, for every $y'_\alpha \to y$ in X and for every $z'_\alpha \in \varphi(x', y'_\alpha)$, $z'_\alpha \notin N_z$ for α sufficiently large. Since $\varphi(x', y) \subset \bigcup_{z \in \varphi(x',y)} N_z$ and $\varphi(x', y)$ is compact, $\varphi(x', y) \subset N = \bigcup_{i=1}^{i=n} N_{z_i}$ for some finite subset $\{z_1, \ldots, z_n\}$ in $\varphi(x', y)$. Hence, there exists $x'' \in C$ such that, for every $y'_\alpha \to y$ in X and for α large enough, $\varphi(x'', y'_\alpha) \subset N$ because of the transfer upper semicontinuity of φ. But, we can also write, for some $y_\alpha \to y$ in X and some $z_\alpha \in \varphi(x'', y_\alpha)$, $z_\alpha \notin N$ for α large enough. It follows that $\varphi(x'', y_\alpha) \not\subset N$ for some $y_\alpha \to y$ in X, which is a contradiction.
The same argument is still applicable to show the assertion on compact subsets of X. This completes our proof. ∎

2 The existence theorem

For the proof of the main result, we need the following KKM lemma, which is a particular case of of the extended version of Fan-KKM theorem [11, Theorem 2.1].

Lemma 2.1. *Assume that $F, G : C \to 2^X$ satisfy*

(0i) $G(x) \subset F(x)$ for all $x \in C$,

(i) G is a KKM map,

(ii) F is transfer closed-valued,

(iii) there is a compact subset B in X such that for each $A \in \mathcal{F}(C)$ there exists a compact convex B_A in X containing A such that

$$\bigcap_{x \in B_A \cap C} \overline{G(x)} \cap B_A \subset B.$$

Then $\displaystyle\bigcap_{x \in C} B \cap F(x) \neq \emptyset.$

Recall that G is a KKM map when $co(A) \subset \bigcup_{x\in A} G(x)$ for any subset $A \in \mathcal{F}(C)$, and that F is transfer closed-valued whenever $y \in X$ and $x \in C$ such that $y \notin F(x)$ there exists $x' \in C$ such that $y \notin \overline{F(x')}$.

Now we are in position to state the following existence theorem for $(GVEP)$ in topological vector spaces.

Theorem 2.1. *Suppose that*

(A0) $\psi(x,y) \not\subseteq L(y) \implies \varphi(x,y) \not\subseteq K(y)\ \forall x, y \in C$;

(A1) ψ *is* L*-diagonally quasi-convex in* x;

(A2) φ *is* K*-transfer semicontinuous in* y;

(A3) there is a nonempty compact subset B *in* X *such that for each* $A \in \mathcal{F}(C)$ *there is a compact convex* $B_A \subset X$ *containing* A *such that, for every* $y \in B_A \setminus B$*, there exists* $x \in B_A \cap C$ *with*

$$y \in int_X \{v \in X : \psi(x,v) \subseteq L(v)\}.$$

Then there exists $\overline{y} \in B$ *such that* $\varphi(x,\overline{y}) \not\subseteq K(\overline{y})$ *for all* $x \in C$.

Proof. Define two set-valued maps $F, G : C \rightrightarrows X$ as follows:

$$F(x) = \{y \in X : \varphi(x,y) \not\subseteq K(y)\}$$

and

$$G(x) = \{y \in X : \psi(x,y) \not\subseteq L(y)\}.$$

The existence of a generalized vector equilibrium for φ in B with respect to K is now equivalent to

$$\bigcap_{x\in C} B \cap F(x) \neq \emptyset$$

Hence, we need only to check the assumptions of Lemma 2.1 for F and G.

(0i) Let $x \in C$ and let $y \in G(x)$; then $\psi(x,y) \not\subseteq L(y)$; it follows, by assumption (A0), that $\varphi(x,y) \not\subseteq K(y)$; thus $y \in F(x)$.

(i) Let $A \in \mathcal{F}(C)$ and let $y \in coA$. By assumption (A1), we have $\psi(A,y) \not\subseteq L(y)$. Hence there exists $x \in A$ such that $\psi(x,y) \not\subseteq L(y)$; that is $y \in G(x)$. Hence $co(A) \subset \bigcup_{x\in A} G(x)$. We conclude that G is a KKM map.

(ii) Let us consider $(x, y) \in C \times X$ with $y \notin F(x)$. Suppose, contrary to our claim, that $y \in cl_X F(x')$ for all $x' \in C$. Therefore, for every $x' \in C$, there is a net (y_α) in X converging to y and satisfying $\varphi(x', y_\alpha) \not\subseteq K(y_\alpha)$. It follows that $\varphi(x', y) \not\subseteq K(y)$ for all $x' \in C$ since φ is K-transfer semicontinuous in y, and hence that $y \in F(x')$ for all $x' \in C$, a contradiction. We conclude that F is transfer closed-valued.

(iii) It suffices to see that assumption (A3) leads to

$$B_A \setminus B \subset \bigcup_{x \in B_A \cap C} int_X G(x);$$

hence $B_A \cap \bigcap_{x \in B_A \cap C} cl_X G(x) \neq \emptyset$.

The proof is complete. ∎

Theorem 2.1 generalizes [3, Theorem 2.1], which is proved by means of a Fan-Browder fixed point theorem - an immediate consequence of the Fan-KKM theorem. As we will mention in the 'Assumptions analysis' subsection, our hypotheses are more general than those used in [3]. Besides, the scalar version of this result extends [17, Theorem 4] (we take $C_A = co(A \cup R) \cap X$ where R is the convex compact which contains C in [17, Theorem 4, (4iii)]). Other particular cases are [1, Theorem 2], [19, Theorem 2.1], [20, Theorem 2.11], [18, Theorem 1], [15, Corollary 2.4], [16, Lemma 2.1] and [4, Theorem 2]. The origin of this kind of results goes back to Ky Fan [7]. His classical minimax inequality can be deduced from our result by setting $E = \mathbb{R}$, $K(x) = \mathbb{R}_+^*$ and $\varphi(x, y) = \psi(x, y) = f(x, y) - \sup_{x \in C} f(x, x)$ for all $x, y \in C$.

3 Assumptions analysis

In this section, we analyze the requirements of Theorem 2.1 by presenting different situations where assumptions (A0)-(A3) hold true. Let $(P(y))_{y \in C}$ a family of proper convex closed cones on E with $int\, P(y) \neq \emptyset$ for all $y \in C$.

- **Pseudomonotonicity**

Remark 3.1. *(A0) holds provided one of the following statements is satisfied.*

(a) $\varphi = \psi$ *and* $K = L$.

(b) $X = C$, $K(y) = -L(y) = -int\, P(y)$, $\psi(x,y) = \varphi(y,x)$ *for all* $x, y \in C$, *and* φ *is* P_x*-pseudomonotone, that is,*

$$\varphi(x,y) \not\subseteq int\, P(x) \Longrightarrow \varphi(y,x) \not\subseteq -int\, P(x)\ \forall x, y \in C.$$

- **Convexity**. Suppose that $X = C$.

Remark 3.2. *(A1) holds provided that, for every* $y \in C$, *one has either*

(a) $\psi(y,y) \not\subseteq L(y)$, *and*

(b) the set $\{x \in C : \psi(x,y) \subseteq L(y)\}$ *is convex,*

or

(i) $L(y) = -int\, P(y)$ *and* $P(y)$ *is w-pointed* [2],

(ii) $\psi(y,y) \subseteq P(y)$, *and*

(iii) ψ *is left* P_y*-quasiconvex.*

Indeed, if we suppose in contrary that there exist $x_1, \dots, x_n \in C$, $\lambda_1, \dots, \lambda_n \in [0,1]$, and $y = \sum_{j=1}^n \lambda_j x_j$ such that $\sum_{i=1}^n \lambda_i = 1$ and

$$\psi(x_i, y) \subseteq L(y) \tag{1}$$

for each $i = 1, \dots, n$. For the first assertion, assumption (b) shows that $\psi(y,y) \subseteq L(y)$, which contradicts (a).
For the second one, (iii) implies

$$\psi(x_{i_0}, y) \subseteq \psi(y,y) + P(y) \subseteq P(y) + P(y) \subseteq P(y)$$

for some $i_0 \in \{1, \dots, n\}$; this contradicts (1) since $P(y)$ is w-pointed. The assertion is proved.

Moreover condition (b) in Remark 3.2 is satisfied provided that $L(y)$ is convex, and for all $x_1, x_2 \in C$ and all $\lambda \in [0,1]$,

$$\psi(\lambda x_1 + (1-\lambda)x_2, y) \subseteq \lambda\psi(x_1,y) + (1-\lambda)\psi(x_2,y),$$

In case of $L(y) = -int\, P(y)$, (b) holds if either

[2] A cone P is w-pointed if $P \cap -int\, P = \emptyset$.

- ψ is right P-convex, or
- ψ is right P-quasiconvex.

- **Continuity**

Remark 3.3. *(A2) holds provided that one of the following statement is satisfied.*

(a) φ *is (transfer) u.s.c. in* y *with compact values and if* K *has an open graph.*

(b) φ *is (transfer) u.s.c. in* y *and* $K(x) = O$ *for all* $x \in C$*, where* O *is an open subset of* E*.*

(c) For each $x \in C$*, the set* $\{y \in X : \varphi(x,y) \not\subseteq K(y)\}$ *is closed in* C*.*

While the assertion "(c) $\Rightarrow$ (A2)" is obvious, the other assertions follow from Proposition 1.1.

- **Coercivity**. Recall first the following definition.

Definition 3.1. *We will say that a set-valued map* $\phi : D \to 2^E$ *is* K*-compact if the set* $cl\{y \in D : \phi(y) \not\subseteq K(y)\}$ *is compact in* Y*.*

It has to be observed that this definition is equivalent to the following fact: there exists a nonempty compact subset B in D such that $\phi(y) \subseteq K(y)$ for all $y \in D \setminus B$.

Remark 3.4. *(A3) holds if one of the following statements is satisfied.*

(a) X *is compact.*

(b) There is $x_0 \in C$ *such that* $\psi(x_0, \cdot)$ *is* K*-compact.*

(c) There is a nonempty compact subset B *in* X *such that for each* $y \in X \setminus B$ *there exists* $x \in B \cap C$ *such that* $\psi(x,y) \subseteq L(y)$*.*

(d) There is a nonempty compact subset B *of* X *and a compact convex subset* $B' \in X$ *such that for each* $y \in X \setminus B$ *there exists* $x \in B' \cap C$ *with*

$$y \in int_X \{v \in X : \psi(x,v) \subseteq L(v)\}.$$

Besides, when the classical assumption (c) of Remark 3.3 is satisfied, (A3) holds provided that

(e) there is a nonempty compact subset B in X such that for each $A \in \mathcal{F}(C)$ there is a compact convex $B_A \subset X$ containing A such that, for every $y \in C \setminus B$, there exists $x \in B_A \cap C$ with $\varphi(x,y) \subseteq K(y)$.

4 Applications

Throughout this section, and otherwise stated, C is supposed to be a nonempty closed convex subset in a t.v.s X, and E to be a real topological vector space. Assume also we are given a family $\{P(x) : x \in C\}$ of convex cones in E with $int\, P(x) \neq \emptyset$ for all $x \in C$. Also, we denote by $L(X,E)$ the space of all linear bounded applications from X to E.

4.1 Generalized vector variational like-inequalities

Let us consider a set-valued operator T from C into $L(X,E)$, and a bifunction η from C to itself. We write for $\Pi \subset L(X,E)$ and $x \in C$, $\langle \Pi, x\rangle = \{\langle \pi, x\rangle : \pi \in \Pi\}$, where $\langle \pi, x\rangle$ denotes the evaluation of the linear mapping π at x which is supposed to be continuous on $L(X,E) \times X$[3].

The generalized vector variational inequality problem $(GVVLIP)$ takes the following form:

$$\text{Find } \overline{x} \in C \text{ such that }, \langle T\overline{x}, \eta(y,\overline{x})\rangle \nsubseteq -int\, P(\overline{x})\ \forall y \in C.$$

Thus $(GVVLIP)$ is a particular case of $(GVEP)$ if we take

$$\varphi(x,y) = \{\langle t, \eta(y,x)\rangle : t \in Tx\}.$$

For the reader's convenience, we recall the following definitions.

Definition 4.1. *(see [2])*
1) T is said to be η-pseudomonotone if, for all $x, y \in C$,

$$\langle Tx, \eta(y,x)\rangle \nsubseteq -int\, P(x) \Rightarrow \langle Ty, \eta(y,x)\rangle \nsubseteq -int\, P(x).$$

[3]A typical situation when X is a reflexive Banach and E is a Banach

2) T is said to be V-hemicontinuous if for any $x, y \in C$ and $t \in]0,1[$ $T(tx + (1-t)y) \to T(y)$ as $t \to 0_+$ (that is, for any $z_t \in T(tx + (1-t)y)$ there exists $z \in Ty$ such that for any $a \in C$, $\langle z_t, a\rangle \to \langle z, a\rangle$ as $t \to 0_+$).

It has to be observed that when T is single-valued, we recover the hemicontinuity used in [5]. If $\eta(x,y) = x - y$ for all $x, y \in C$, η is dropped from the definition of pseudomonotonicity.

The linearization lemma plays a significant role in variational inequalities. Chen [5] extended this lemma to the single-valued vector case. For our need in this subsection, we state it in the set-valued case by using standard Minty's argument. Consider the following problem, which may be seen as a dual problem of $(GVVLIP)$,

$$\text{Find } \overline{x} \in C \text{ such that } \langle Ty, \eta(y, \overline{x})\rangle \not\subseteq -int\, P(\overline{x}) \ \forall y \in C. \qquad \text{(GVVLIP*)}$$

Lemma 4.1. *Suppose that $\eta(\cdot, x)$ is affine and $\eta(x,x) = 0$ for each $x \in C$. If T is η-pseudomonotone and V-hemicontinuous then $(GVVLIP)$ and $(GVVLIP*)$ are equivalent.*

Proof. We only need to show that every solution of $(GVVLIP*)$ is a solution of $(GVVLIP)$; the reverse assertion follows clearly from the η-pseudomonotonicity of T. To this end, let $\overline{x} \in C$ such that

$$\langle Ty, \eta(y, \overline{x})\rangle \not\subseteq -int\, P(\overline{x}) \ \forall y \in C.$$

Let $y \in C$ be fixed and set $y_t = ty + (1-t)\overline{x}$ for $t \in [0,1]$, which is in the convex subset C. Then

$$\langle Ty_t, \eta(y_t, \overline{x})\rangle \not\subseteq -int\, P(\overline{x}).$$

Hence

$$\langle Ty_t, \eta(y, \overline{x})\rangle \not\subseteq -int\, P(\overline{x}).$$

By the V-hemicontinuity of T and the closedness of $E \setminus (-int\, P(\overline{x}))$, it follows that

$$\langle T\overline{x}, \eta(y, \overline{x})\rangle \not\subseteq -int\, P(\overline{x}).$$

The last assertion holds for an arbitrary $y \in C$. The lemma is proved. ■

As an application of Theorem 2.1, we are now in position to formulate the following existence result for $(GVVLIP)$.

Theorem 4.1. *Suppose that*

(i) the mapping $int\, P(\cdot)$ has an open graph in $C \times L(X, E)$;

(ii) for each $x \in C$, $\eta(\cdot, x)$ is affine, $\eta(x, \cdot)$ is continuous and $\eta(x, x) = 0$;

(iii) T is compact valued, η-pseudomonotone and V-hemicontinuous;

(iv) there is a nonempty compact subset B in C such that for each $A \in \mathcal{F}(C)$ there is a compact convex $B_A \subset C$ containing A such that, for every $y \in B_A \setminus B$, there exists $x \in B_A \cap C$ with

$$y \in int_C \{v \in C : \langle Tv, \eta(x, v)\rangle \subseteq -int\, P(v)\}.$$

Then $(GVVLIP)$ has at least one solution, which is in B.

Proof. Set $\varphi(x, y) = \langle Tx, \eta(x, y)\rangle$, $\psi(x, y) = \langle Ty, \eta(x, y)\rangle$ and $K(x) = -int\, P(x)$ for all $x, y \in C$. Let us show that the assumptions of Theorem 2.1 are satisfied:
(A0) follows clearly from the η-pseudomonotonicity of T.
(A1) First we have,

- $\psi(x, x) = \langle Tx, \eta(x, x)\rangle = 0 \notin -int\, P(x)$ for all $x \in C$, and
- for y being fixed in C, the set

$$\{x \in C : \langle Ty, \eta(x, y)\rangle \subseteq -int\, P(y)\}$$

is convex since $\eta(\cdot, y)$ is affine.
Thus conditions (a) and (b) of Remark 3.2 hold, which lead to (A1).
(A2) Fix x in C and let (y_α) be a net on C converging to $y \in C$ such that

$$\langle Tx, \eta(x, y_\alpha)\rangle \not\subseteq -int\, P(y_\alpha).$$

Therefore there exists $z_\alpha \in Tx$ such that

$$\langle z_\alpha, \eta(x, y_\alpha)\rangle \notin -int\, P(y_\alpha).$$

Since Tx is compact then, passing to a subnet, if necessarily, we may assume that z_α converges to $z \in Tx$. By the continuity of the map $\eta(x, \cdot)$ and $\langle ., .\rangle$, we get $\langle z_\alpha, \eta(x, y_\alpha)\rangle \to \langle z, \eta(x, y)\rangle$. Hence, according to (i), we obtain

$$\langle z, \eta(x, y)\rangle \notin -int\, P(y);$$

thus

$$\langle Tx, \eta(x,y)\rangle \not\subseteq -int\, P(y).$$

(A3) follows easily from (iv). Therefore, from Theorem 2.1, there exists $\overline{x} \in B$ such that

$$\langle Ty, \eta(y,\overline{x})\rangle \not\subseteq -int\, P(\overline{x})\ \forall y \in C.$$

Hence $(GVVLIP*)$ has a solution in B, which completes the proof of the theorem according to Lemma 4.1. ■

The coercivity condition (iv) is better than that formulated in [12], namely, there is a compact B of X and $x_o \in C \cap B$ such that

$$\langle Tx_o, \eta(x_o,y)\rangle \subset -int\, P(y)\ \ \forall y \in C \setminus B.$$

4.2 Vector complementarity problems

A natural extension of the classical nonlinear complementarity problem, (CP) for short, is considered as follows. Let T be a single-valued operator from C, which is supposed to be a convex closed cone, to $L(X,E)$. The vector complementarity problem considered in this subsequent, (VCP) for short, is to find $\overline{x} \in C$ such that

$$\langle T(\overline{x}), \overline{x}\rangle \notin int\, P(\overline{x}), \text{ and } \langle T(\overline{x}), y\rangle \notin -int\, P(\overline{x}) \text{ for all } y \in C.$$

This problem collapses to (CP) when $E = \mathbb{R}$ and $P(x) = \mathbb{R}_+$ for all $x \in C$.

By means of vector variational inequalities, we can formulate the following existence theorem for (VCP).

Theorem 4.2. *Suppose that*

(i) the set-valued map $int\, P(\cdot)$ has an open graph in $C \times L(X,E)$;

(ii) T is pseudomonotone and hemicontinuous;

(iv) there is a nonempty compact subset B in C such that for each $A \in \mathcal{F}(C)$ there is a compact convex $B_A \subset C$ containing A such that, for every $y \in B_A \setminus B$, there exists $x \in B_A \cap C$ with

$$y \in int_C \{v \in C : \langle Tv, x - v\rangle \in -int\, P(v)\}.$$

Then (VCP) has at least one solution, which is in B.

Proof. It is clear that all the assumptions of Theorem 4.1 are satisfied with $\eta(x,y) = x - y$ for all $x, y \in C$. Therefore there exists $\overline{x} \in B$ such that

$$\langle T\overline{x}, z - \overline{x}\rangle \notin -int\, P(\overline{x})\ \forall z \in C. \tag{2}$$

Since C is a convex cone, then setting in (2), $z = 0$ and $z = y + \overline{x}$ for an arbitrary $y \in C$, we get respectively

$$\langle T\overline{x}, \overline{x}\rangle \notin int\, P(\overline{x}) \text{ and } \langle T\overline{x}, y\rangle \notin -int\, P(\overline{x}).$$

Hence we conclude that $\overline{x}$ is also a solution to (VCP). ∎

4.3 Vector optimization

Here, to convey an idea about the use of vector variational-like inequalities in vector optimization which involves smooth vector mappings, we prove the existence of solutions to weak vector optimization problems, $(WVOP)$ for short, by considering the concept of invexity. Let us state the problem as follows.

$$\text{Find } \overline{x} \in C \text{ such that } \phi(y) - \phi(\overline{x}) \notin -int\, P \text{ for all } y \in C, \qquad \text{(WVOP)}$$

where $\phi : C \to E$ be a given vector-valued function and P is a given convex cone in E.

Let $\eta : C \times C \to X$ be a given function, and denote by $\nabla\phi$ the Fréchet derivative of ϕ once the latter is assumed to be Fréchet differentiable.

Definition 4.2. *Suppose that ϕ is Fréchet differentiable. ϕ is said to be P-invex with respect to η if*

$$\phi(y) - \phi(x) - \langle \nabla\phi(x), \eta(y,x)\rangle \in P \quad \forall x, y \in C.$$

If $E = \mathbb{R}$ and $P = \mathbb{R}^+$ then we recover the real invexity introduced by Hanson [10] and later labeled so by Craven [6] due to its "invariance" under "convex" transformations.

Theorem 4.3. *Suppose that P is a convex cone in E with $int\, P \neq \emptyset$, and let $\phi : C \to E$ be a Fréchet differentiable mapping. Assume that* *(i)* $\langle \nabla\phi(x), \eta(y,x)\rangle \notin -int\,P$ *implies* $\langle \nabla\phi(y), \eta(y,x)\rangle \notin -int\,P$ *for all* $x, y \in C$;

(ii) ϕ *is* P*-invex with respect to* η;

(iii) $\nabla\phi$ *is hemicontinuous;*

(iv) for each $x \in C$, $\eta(\cdot, x)$ *is affine,* $\eta(x, \cdot)$ *is continuous and* $\eta(x,x) = 0$;

(v) there is a compact subset B *in* X *such that for every finite subset* A *in* C *there is a compact convex* $C_A \subset X$ *containing* A *such as to satisfy, for every* $y \in C \backslash B$, *there exists* $x \in C_A \cap C$ *with* $\langle \nabla\phi(x), \eta(x,y)\rangle \in -int\,P$.

Then (WVOP) has at least one solution.

Proof. First, by virtue of Theorem 4.1 with $T := \nabla\phi$, we get

$$\langle \nabla\phi(\overline{x}), \eta(y,\overline{x})\rangle \notin -int\,P \;\; \forall y \in C.$$

Then the P-invexity of ϕ allows us to conclude. ∎

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E. M. Kalmoun and H. Riahi
Cadi Ayyad University, Faculty of Science Semlalia, Department of Mathematics, B.P. 2390, Marrakech-40000, Morocco.

T. Tanaka
Graduate School of Science and Technology, Niigata University, Niigata 950-2181, Japan.

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