Anomalous flux quantization in the spin-imbalanced attractive Hubbard ring

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We investigate the one-dimensional Hubbard ring with attractive interaction in the presence of imbalanced spin populations by using the exact diagonalization method. The singlet pairing correlation function is found to show spatial oscillations with power-law decays as expected in the Fulde-Ferrell-Larkin-Ovchinnikov state of the Tomonaga-Luttinger liquid. In the strong coupling regime, the system shows an anomalous flux quantization with a half of the superconducting flux quantum $h/4e$ as recently predicted by the mean-field analysis, together with various flux quanta smaller than $h/4e$. Remarkably, the observed flux quanta are determined by the difference between the system size N_L and the electron number N_e as $h/(N_L - N_e)e$.

KEYWORDS: FFLO state, flux quantization, spin-imbalance, attractive Hubbard model, exact diagonalization

1. Introduction

The development of the ultra cold neutral atoms system in optical trap has stimulated interest in fundamental problems of superfluid states.[1–4] In particular, two component Fermi particle systems with different populations attract much attention due to the possibility of exotic superfluid phase which is the so-called Fulde-Ferrell-Larkin-Ovchinnikov (FFLO) state.[5,6] The FFLO state is characterized by the formation of Cooper pairs with finite center-of-mass momentum caused by the imbalance of the Fermi surfaces of two component fermions and exhibits inhomogeneous superconducting phases with a spatially oscillating order parameter.

Recent theoretical works have revealed that the FFLO state accompanies a variety of spatial structure of the pairing order parameter which is dominated by the difference of the Fermi wave vectors of two component fermions.[7–14] Based on the bosonization approach and the conformal field theory, Lüscher *et al.* have discussed the correlation exponents of the FFLO state in the one-dimensional (1D) attractive Hubbard model and have obtained the phase diagram of the electronic states on the parameter plane of the electron density vs. the spin-imbalance.[15] It indicates that the FFLO region expands with increase in strength of the attractive interaction.

More recently, Yoshida and Yanase[16] have pointed out that the FFLO state indicates an anomalous flux quantization of the period "$h/4e$" in mesoscopic rings. However, this result is restricted within the mean-field approximation such as the Bogoliubov-de Gennes equation. At this stage, exact and/or numerical results for the anomalous flux quantization of the FFLO state has not been obtained. To achieve a certain understanding without ambiguity from the approximation, we believe that nonperturbative and reliable approaches would be required.

In the present paper, we investigate the 1D Hubbard model with the attractive interaction by using the exact diagonalization method. To confirm the anomalous flux quantization such as the period "$h/4e$" without any approximation, we calculate the periodicity of the ground state energy $E(\Phi)$ respect to the magnetic flux Φ numerically. Although our calculation is limited to small systems, we expect that the reliable result of the FFLO state can be obtained.

2. Model and Formulation

We consider the 1D Hubbard ring given by the following Hamiltonian:

$$H = -t \sum_{i,\sigma} (e^{i2\pi\Phi/N_L} c^\dagger_{i,\sigma} c_{i+1,\sigma} + h.c.) - |U| \sum_i n_{i,\uparrow} n_{i,\downarrow} \quad (1)$$

where $c^\dagger_{i,\sigma}$ stands for the creation operator of an electron with spin σ $(=\uparrow,\downarrow)$ at site i and $n_{i,\sigma} = c^\dagger_{i,\sigma} c_{i,\sigma}$. Here, t represents the hopping integral between nearest-neighbor sites and we set $t = 1$ in this study. Φ corresponds to the magnetic flux through the ring measured in units of h/e, and N_L is the system size, respectively. The interaction parameter $|U|$ stands for the attractive interaction on the site.

We numerically diagonalize the model Hamiltonian eq. (1) up to 20 sites using the standard Lanczos algorithm. To carry out a systematic calculation, we use the periodic boundary condition for $N_\uparrow$ and $N_\downarrow$ being odd number and the antiperiodic boundary condition for these being even, where $N_\uparrow$ and $N_\downarrow$ are the total number of up- and down-spin electrons, respectively.[17] The filling n of electrons is defined by $n = N_e/N_L$, where $N_e(= N_\uparrow + N_\downarrow)$ is the total number of electron, and the spin-imbalance is given as $p = \frac{N_\uparrow - N_\downarrow}{N_e}$. We also calculate the correlation function $C(r)$ of singlet superconducting pairing on the same site as

$$C(r) = \frac{1}{N} \sum_i \langle c^\dagger_{i,\uparrow} c^\dagger_{i,\downarrow} c_{i+r,\downarrow} c_{i+r,\uparrow} \rangle. \quad (2)$$

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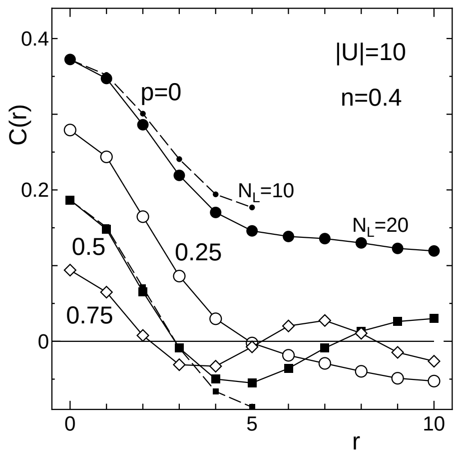

Fig. 1. The singlet pairing correlation functions $C(r)$ as functions of r for several values of p at n=0.4 for $|U| = 10$. The solid lines represent the result for $N_L = 20$ and the dashed line for $N_L = 10$.

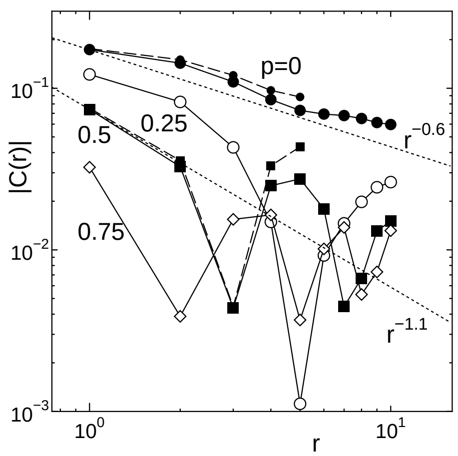

Fig. 2. The logarithmic plot of $|C(r)|$ as functions of r for several values of p at n=0.4 for $|U| = 10$. The solid lines represent the result for $N_L = 20$ and the dashed line for $N_L = 10$. The dotted lines are guides for eyes.

3. Pairing Correlation

To confirm the FFLO state and estimate the finite size effect of the small system, we examine the pairing correlation of two different size systems as $N_L = 10$ and $N_L = 20$. In Fig.1, we show $C(r)$ for p =0 and 0.5 at $N_L = 10$, and p =0, 0.25, 0.5 and 0.75 at $N_L = 20$. Compared with both systems of $N_L = 10$ and 20, we find the difference of the result to be not so large. It suggests that the size effect of $C(r)$ is small.[18]

We find that $C(r)$ for $p = 0$ decays monotonically as a function of r, however, the others are oscillate. The period of oscillation seems to be shorter with increasing the value of p. This result is related to a spatial oscillation of superconducting order parameter $\Delta(r)$ which has been already pointed out in the previous works.[7–13,15,16] The periodicity of $\Delta(r)$ is determined by the difference of Fermi wave numbers of $k_{F\uparrow}$ and $k_{F\downarrow}$ as $\Delta(r) \propto \exp(iQr)$ for Fulde-Ferrell state or $\Delta(r) \propto \cos(Qr)$ for Larkin-Ovchinnikov state, where $Q = |k_{F\uparrow} - k_{F\downarrow}| = n\pi p$.[15] In Fig.1, we see that the spatial period of $C(r)$ is 20, 10, and 20/3 at n=0.4 for $p = 0.25$, 0.5, and 0.75 respectively. In this case, Q is yielded as $\pi/10$, $\pi/5$, and $3\pi/10$, respectively, and the behavior of $C(r)$ is consistent with the result of $\Delta(r)$ obtained by the previous works.[7–13,15,16]

It is noted that $\Delta(r)$ obtained from the mean-field theory has long range order, but our result is obtained with the pure 1D model and the correlations should show power low decay at large r as a Tommonga-Luttinger liquid.[19–21] As shown in Fig.2, the logarithmic plot of the $|C(r)|$ indicates that the pair correlation function for $p = 0$ decays roughly as $\sim r^{-0.6}$, and $\sim r^{-1.1}$ for $p > 0$. The result[22] seems to be consistent with that in Ref.15. It suggests that the characteristic feature of the FFLO state is well described even by the small size Hubbard ring such as the system of $N_L = 10$.

4. Flux Quantization

Next, we consider the flux quantization of the FFLO state. It is well known that the ordinary superconductivity indicates the flux quantization of the cooper pair in the unit of $h/2e$. However, Yoshida and Yanase has shown that an anomalous flux quantization with the period of $h/4e$ occurs for the FFLO state in mesoscopic rings. They have claimed that the FFLO state has multi-component order parameter and shows rich phases distinguished by the order parameter in the presence of the mass flow which corresponds to the magnetic flux through the ring in our model. When the mass flow increases, a FFLO state transitions to another FFLO state with different order parameter. Therefore, the ground state of our system is expected to alternate between several FFLO states with the increase of the flux and it may lead the anomalous flux quantization such as a half period. To ensure this prediction without any approximation, we calculate the periodicity of the ground state energy $E(\Phi)$ with respect to the magnetic flux Φ by using the exact diagonalization method.[25] Although the system is limited to small one, it would give a direct evidence of the anomalous flux quantization of the FFLO state.

In Figs.3 (a)-(c), we show the deference of the ground state energy $\Delta E = E(\Phi) - E(0)$ as a function of Φ for n=0.5 (4 electrons/8 sites) at $|U| = 10$, 200 and 1000, respectively. We see that the energy levels cross at $\Phi \sim 0.5$ in the case of $p = 0.5$, and $\Phi \sim 0.25$ and 0.75 for $p = 0$ as shown in Fig.3(a). It means that the usual flux quantization of the period $h/2e$ appears in the case of $p = 0$, while the other is not so. On the other hand, the energy levels cross at $\Phi \sim 0.2, 0.4, 0.6$ and 0.8 for $p = 0.5$ in the case of more large attractive interaction as shown in Figs.3(b) and (c). They indicate that the period of the ground state with respect to Φ in the case of $p = 0.5$ is clearly a half of that in the case of $p = 0$ and the anomalous flux quantization occurs in the period of $h/4e$ for large $|U|$. These results seem to agree with that of Yoshida and Yanase[16] except for the strength of the attractive interaction $|U|$ being fairly large. This discrepancy will be discussed later.

In Fig. 4, we show the comparison of ΔE between two different size systems, $N_L = 8$ and 16 with $p = 0.5$, $n =$

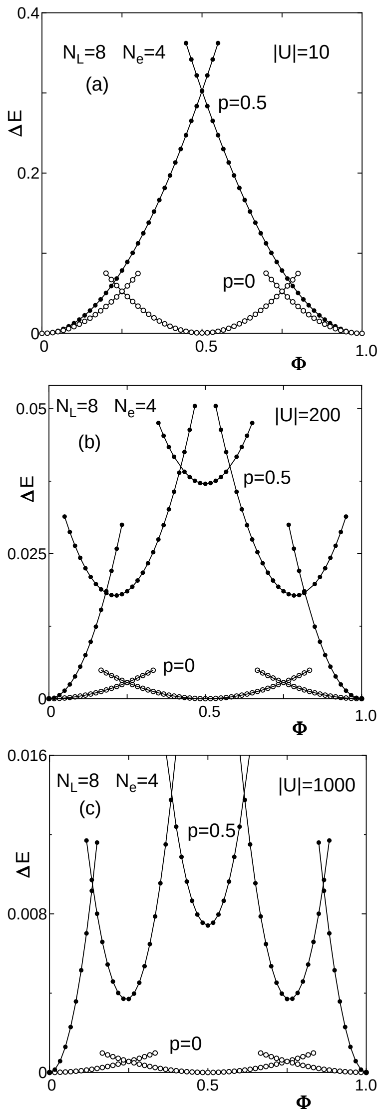

Fig. 3. The deference of the ground state energy $\Delta E = E(\Phi) - E(0)$ as a function of the magnetic flux Φ for $p = 0$ and 0.5 at n=0.5 (4 electrons/8 sites) for (a) $|U| = 10$, (b) $|U| = 200$ and (c) $|U| = 1000$, respectively.

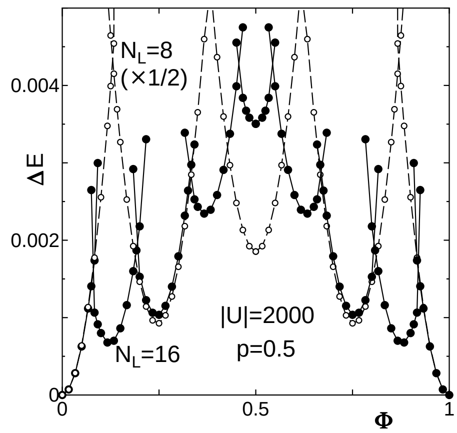

Fig. 4. The deference of the ground state energy $\Delta E = E(\Phi) - E(0)$ as a function of the magnetic flux Φ for n=0.5 (4 electrons/8 sites and 8 electrons/16 sites) at $|U| = 1000$ with $p = 0.5$.

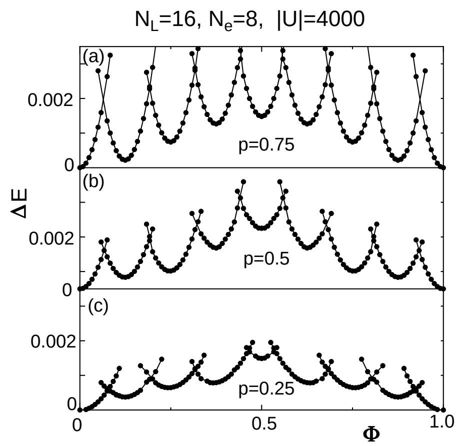

Fig. 5. The deference of the ground state energy $\Delta E = E(\Phi) - E(0)$ as a function of the magnetic flux Φ with n=0.5 (8 electrons/16 sites) at $|U| = 4000$ for (a) $p = 0.75$, (b) $p = 0.5$, and (c) $p = 0.25$, respectively.

0.5 and $U = 2000$. Here, the value of ΔE for $N_L = 8$ is set a half to compare both systems easily. The result suggests that the size dependence of ΔE is approximately given by $1/N_L$ for small Φ. The relation is easily understood as follows. If we assume $E(\Phi)$ to be analytic function with respect to Φ, we can expand it to the power series of Φ,

$$E(\Phi) = E(0) + \frac{1}{2!}\frac{\partial^2 E(\Phi)}{\partial \Phi^2}\bigg|_{\phi=0}\Phi^2 + \frac{1}{4!}\frac{\partial^4 E(\Phi)}{\partial \Phi^4}\bigg|_{\phi=0}\Phi^4 + ..., \quad (3)$$

where the odd powers of Φ vanish because of the inversion symmetry of $E(\Phi)$ for Φ. Using the equation $D = \frac{N_L}{4\pi}\left.\frac{\partial^2 E(\Phi)}{\partial \Phi^2}\right|_{\Phi=0}$, where D is Drude weight,[23,24] we obtain

$$\Delta E \simeq 2\pi D \Phi^2/N_L, \quad (4)$$

for small Φ. It indicates that the size dependence of ΔE is dominated by $1/N_L$. It also suggests and the flux quantization disappears in the limit $N_L \to \infty$, since the values of ΔE become zero in this limit. In fact, Yoshida and Yanase[16] have also claimed that the anomalous flux quantization of $h/4e$ will be observed in not the infinite system but the mesoscopic system.

In Figs. 4, we also find a shorter period in the system of $N_L = 16$, that is, the flux quantum is given by $h/8e$, although that of $N_L = 8$ is $h/4e$. It seems to be very curious that both systems have same n and p except the system size, however, their periods are different from each other. To clarify this puzzling behavior, we perform detailed calculation for many finite systems. For example, Fig. 5 shows the period of the system with $N_L = 16$ and $N_e = 8$ for $p = 0.25$, 0.5, and 0.75. It clearly indicates that the periods of these systems are same and given by $h/8e$. It suggests that the period is independent of p. Figure 6 shows that the period is yield as $h/4e$ for $N_e = 12$, $h/8e$ for $N_e = 8$, and $h/12e$ for $N_e = 4$ in the system of $N_L = 16$. The result suggests that the

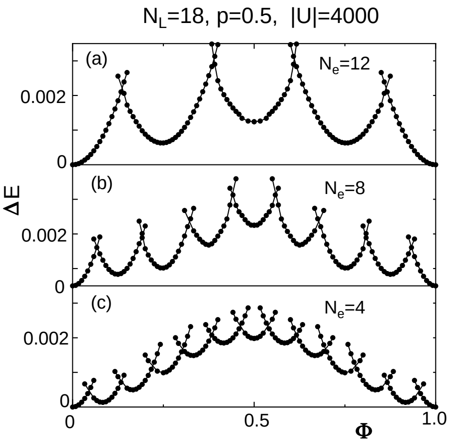

Fig. 6. The deference of the ground state energy $\Delta E = E(\Phi) - E(0)$ as a function of the magnetic flux Φ at $N_L = 18$, $p = 0.5$, and $|U| = 4000$ for (a) $N_e = 12$, (b) $N_e = 8$, and (c) $N_4 = 4$, respectively.

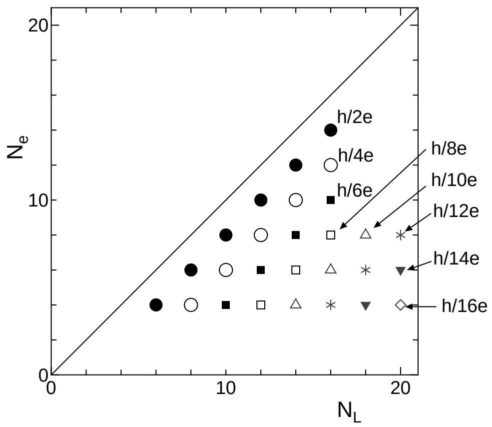

Fig. 7. The phase diagram of the observed flux quanta on the N_L-N_e plane.

period depends on the number of the difference between the system size N_L and the electron number N_e.

By more systematic calculation, we confirm that the periodicity of the flux quantization of the finite systems is presented by the relation $h/(N_L - N_e)e$ and is independent of p and n at $0 < p < 1$ for sufficiently large $|U|$ as shown in Fig. 7. The result suggests that the period of the flux quantization could be very short and fine in the case of large $N_L - N_e$ in contrast with the result of the mean field analysis.[16)]

The result reminds us the period of the anomalous flux quantization of the positive U Hubbard ring.[26,27)] It is known that the period is given by h/N_e at $N_L|U| >> N_e|t|$, in contrast to the attractive case. We also note that the well known transformation of the on-site attraction into the repulsion for the Hubbard model[15,28,29)] can not be simply adapted to the case with the flux. The effect of the flux in the original Hamiltonian is not equivalent to that of the flux in the transformed system.[30)] Although the 1D Hubbard model is one of the simplest models in the correlated electron systems, it shows the very complex behaviors upon the flux quantization.

5. Summary and Discussion

We have investigated the FFLO state of the 1D Hubbard model with the attractive interaction in the case of spin-imbalance electron. To obtain reliable results including strong correlation regime, we have used the exact diagonalization method beyond the mean-field approximation. The correlation function of the singlet pairing indicates spatial oscillating and power low decay as a pure 1D electronic system. The result suggests that the FFLO state is surely realized in the attractive 1D Hubbard model.

We have also present the direct evidence of the anomalous flux quantization in the case of very strong attraction ($|U| \gtrsim 200$). Of cause, the period predicted by Yoshida and Yanase[16)] is $h/4e$ and they does not show the shorter period less than $h/4e$. However, our systematic calculation of the finite systems suggests that the flux quanta are generally described by $h/(N_L - N_e)e$. When $N_L - N_e$ is large, the period becomes very short and the result seems to be very strange at a glance. We think that the appearance of the shorter periods relates to the exactly one-dimensionality of the system. If three dimensionality, which is implicitly contained in the mean-field theory, is added into the system, the short periods may almost vanish except $h/4e$. Thus, only the period $h/4e$ in the observed periods might correspond to that obtained by the mean field analysis.[16)]

Their work[16)] also indicates that the anomalous flux quantization occurs even for weak interaction ($|U| \sim 1.5$), however, we could not find it in the case of $|U| \lesssim 100$. This discrepancy might be caused by the finite size effect. It is plausible that the finite size effect of the FFLO state described by the complex order parameter is larger than that of the usual BCS state with simple order parameter. We also point out that large fluctuation due to the one-dimensionality of the system, which is not treated in the mean-field theory, may prevent us to detect the anomalous flux quantization for small $|U|$. When the attractive interaction is very large, the pairing order of FFLO state is expected to be large and the anomalous flux quantization can be more easily observed even if the system is small.

At this stage, it is difficult to prove the curious behavior of the anomalous flux quantization exactly, however, it may be possible to give an intuitive explanation. To find out the origin of the periodicity, we consider the rotation of electrons around the ring through the flux. It would bring us an qualitative understanding of the period of the flux quantization. We assume $|U| \to \infty$, because the allowed state are dramatically limited. For example, one of an electron pair on the same site can not hop to the adjacent site, unless an electron with the opposite spin is there.

We treat the three electrons system in the N_L sites, which consists an electron pair(doublon) and an up-spin electron, as shown in Fig. 8. Here, the initial state is set to the doublon to be on the first site and the up-spin electron on the second site. We move the up-spin electron from the second site to the N_L-th site through many counterclockwise hoppings, and the down-spin electron from the first site to the N_L-th site. Then, the accomplished state consists of the doublon on the N_L-th site and the up-spin electron on the first site. It becomes the rotated state from the initial state by one site.

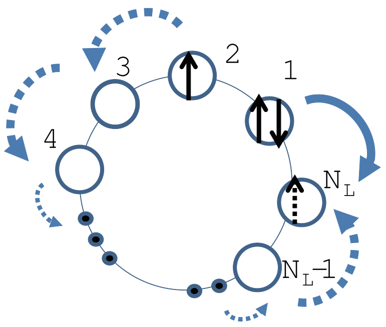

Fig. 8. Possible hopping process of the electrons in the attractive 1D Hubbard ring at $|U| \to \infty$.

The process contains $N_L - 2$ times counterclockwise hopping and one time clockwise hopping. Each electron hopping under the flux may originate the oscillation into the flux quantization. Since the hopping of opposite direction counteracts the effect to the oscillation each other, the number of the oscillation for the flux quantization is estimated as $N_L - 3$ and the period may be given by $h/(N_L - 3)e$. It is easily shown that the situation is same for any configuration of the doublon and the up-spin electron.

We can not give a exact proof for the general case with any N_e and p, however, the above consideration seems to stand even in the general case and the period of the flux quantization might be given by $h/(N_L - N_e)e$, except the case of $p = 0$ and 1. Of cause, our argument is limited to an intuitive one and the relationship between our result and that of Yoshida and Yanase[16] has not been clear yet. More detailed and exact analysis would be needed to clarify the nature of the anomalous flux quantization of the attractive 1D Hubbard model. We will address it in future.

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