Anomalous oscillations due to Aharonov-Bohm and Aharonov-Casher effects of the one-dimensional Hubbard ring in the strong coupling limit

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We investigate anomalous oscillations due to the Aharonov-Bohm(AB) and Aharonov-Casher(AC) effects of the one-dimensional Hubbard ring with flux in the strong coupling limit. By using the exact diagonalization method and the Shiba transformation, we examine the energies of the ground state and few excited states in the presence of the flux producing the AB or AC effect, where the transformation not only reverses the sign of the interaction U, but also exchanges the role between the AB and AC effects in the model Hamiltonian. We classify the number of oscillation of the ground state as a function of the flux systematically and clarify the close relationship between the AB and AC effects. For example, it is shown that the number of the periods are given by $N_L - N_e(N_L - N_\uparrow + N_\downarrow)$ for the AB(AC) effect in the very strong attraction, where N_L, N_e, $N_\uparrow$, and $N_\downarrow$ are the system size, the number of the total electrons, the number of electrons with up-spin, and the number of electrons with down-spin, respectively, in the condition of $N_L > N_e > N_\downarrow > N_\uparrow$. More special case, such as the half-filled band and spin balanced case ($N_L = N_e$ and $N_\downarrow = N_\uparrow$), we find the periodicity of the system to be 0(2) for the AB(AC) effect in the very strong repulsion. These results show us the nature of interesting phenomena originated by the interplay between the strong correlation and the quantum interference effect in the mesoscopic ring.

1. Introduction

The Aharonov-Bohm(AB) and Aharonov-Casher(AC) effects have stimulated interest in fundamental problems of quantum physics.[1–3] Here, the AB effect is characterized by the quantum phase shift of the charged particle $\Delta\phi_{AB}$ through some trajectory of P as, $\Delta\phi_{AB} = \frac{2\pi e}{h}\int_P \boldsymbol{A}\cdot d\boldsymbol{x}$, where $\boldsymbol{A}$ is the vector potential that represents magnetic fields.[1] On the other hand, the AC effect is known as an electromagnetic dual phenomenon to the AB effect.[2] It originates in a kind of the spin-orbital coupling between the particle with a magnetic moment $\boldsymbol{\mu}$ and the electric field $\boldsymbol{E}$. In this case, the phase shift is given as $\Delta\phi_{AC} = \frac{2\pi}{hc^2}\int_P(\boldsymbol{E}\times\boldsymbol{\mu})\cdot d\boldsymbol{x}$, and its sign depends on the direction of $\boldsymbol{\mu}$.

The AB and AC effects(ABC effects) cause the interference due to the phases deference between two different paths of the particle. Recent progress in microstructuring technology allows a direct detection of the AB effect of mesoscopic rings which make a closed circuit around a tube of magnetic flux.[4,5] For the AC effect, neutron interferometry is tried to observer the phase shift, although the value of it is very small.[6] More sophisticated measurement of the AC phase shift is attempted by using the traveling atoms which consist a coherent superposition of opposite spin in a uniform electric field.[7–9]

Ultracold atomic gases in optical traps have also concerned the AB effect through an exotic superfluid phase, which is the so-called Fulde-Ferrell-Larkin-Ovchinnikov (FFLO) state.[10–14] Here, the FFLO state is characterized by the formation of Cooper pairs with finite center-of-mass momentum caused by the imbalance of the Fermi surfaces of two-component fermions. Because the mass flow of atoms corresponds to the magnetic flux through the ring, the interference effect of atoms is expected. Recently, Yoshida and Yanase[15] have shown that the FFLO state in an attractive Hubbard ring with 200 sites exhibits an anomalous flux quantization of period $h/4e$, which is half of the superconducting flux quantum $h/2e$ in the presence of the mass flow. They claimed that the FFLO state has a multicomponent order parameter and shows rich phases distinguished by the order parameter. When the mass flow increases, the ground state is expected to alternate between several types of FFLO states with a different order parameter and shows anomalous flux quantization such as the half period $h/4e$.

We note that its fractional flux quantization can be interpreted as a kind of the AB effect related to the particle correlations. Since strong correlation has be known to produce interesting phenomena, such as high-T_c superconductivity and/or metal-insulator transition,[16] we expect strange properties originated from the interplay of ABC effects and strong electron correlation.

In fact, several analyses for the one-dimensional(1D) Hubbard model predict curious fractional periods of the AB effect in the case of the interaction U being larger.[17–27] Kusmartsev et al. indicate that a fractional period of the AB effect is observed as h/N_e by the Bethe ansatz method for $U/|N_L t| >> 1$, where N_e and N_L are the number of the electron and the system size, respectively, and t is the hopping integral of electron.[18,19] Zvyagin et al. also study the periods of ABC effects on the Hubbard model with repulsion and the attraction by using the Bethe ansatz method.[21–23] Recently, we show that fractional periods of the AB effect as $h/(N_L - N_e)e$ appear in the spin-imbalanced Hubbard ring with the very strong attractive interaction by using the exact diagonalization (ED) method.[26,27]

We expect that there is any relation or connection be-

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tween the periods of the attractive and repulsive case, because of the similarity of these periodicities. In this work, we pursue the fractional periodicity of the oscillation induced by ABC effects and the very strong correlation based on the Shiba transformation. It will give us a clear relationship between the AB and AC effects, and systematic results of the periodicity of the attractive and repulsive Hubbard models. We also present the detailed comparison between the result of the Bethe ansatz method obtained by Zvyagin et al .,[22,23] and the ED result for the attractive Hubbard model.

2. Model and Formulation

We consider the following Hamiltonian of the 1D Hubbard ring with the AB or AC phase shift,

$$\begin{aligned} H_{AB/AC} &= -t\sum_{i,\sigma}(e^{i2\pi\Gamma_{AB/AC}\phi/N_L}c^{\dagger}_{i,\sigma}c_{i+1,\sigma} \\ &+ e^{-i2\pi\Gamma_{AB/AC}\phi/N_L}c^{\dagger}_{i+1,\sigma}c_{i,\sigma} \\ &+ U\sum_i n_{i,\uparrow}n_{i,\downarrow} \end{aligned} \quad (1)$$

where $c^{\dagger}_{i,\sigma}$ stands for the creation operator of an electron with spin σ ($=\uparrow,\downarrow$) at site i and $n_{i,\sigma} = c^{\dagger}_{i,\sigma}c_{i,\sigma}$. Here, $\Gamma_{AB/AC}$ is a parameter which represents the sign of the phase shift of ABC effects, that is, a unity for the AB effect, and ±1 depends on the spin direction for the AC effect.

ϕ is the normalized phase shift given by $\phi = \Phi^{AB}/(\Phi_0^{AB})$, where Φ^{AB} corresponds to the magnetic flux through the ring measured in units of the flux quantum $\Phi_0^{AB} = h/e$ for the AB effect, on the other hand, it stands $\phi = \Phi^{AC}/\Phi_0^{AC}$, where Φ^{AC} is the electrical flux determined by the spin-orbital coupling measured in units of the flux quantum $\Phi_0^{AC} = h/\mu$ for the AC effect. t represents the hopping integral between the sites and we set $t = 1$ in this study. The interaction parameter U is set as positive(negative) for the repulsive(attractive) interaction at very strong correlation($|U| >> t$). The filling n of electrons is defined by $n = N_e/N_L$, where $N_e(= N_\uparrow + N_\downarrow)$ is the total number of electron, and N_L is the system size.

To analyze the Hamiltonian 1 and find the relation between the AB and the AC effects, we introduce the Shiba transformation,[28–30] which is known that the attractive interaction in the Hubbard model changes to the repulsive one by using a kind of the particle-hole transformation;

$$\begin{aligned} &c^{\dagger}_{j,\downarrow} = (-1)^j\tilde{c}_{j,\downarrow}, \\ &c_{j,\downarrow} = (-1)^j\tilde{c}^{\dagger}_{j,\downarrow} \\ &c^{\dagger}_{j,\uparrow} = \tilde{c}^{\dagger}_{j,\uparrow}, \\ &c_{j,\uparrow} = \tilde{c}_{j,\uparrow}. \end{aligned} \quad (2)$$

It replaces the annihilation operator with down spin at the site j to the creation operator with alternate sign depending on j, although the creation operator with up spin is left being. We note that the transformation requires N_L to be even, because the transformed operators with down spin have the alternate sign.[29]

Using this transformation, the Hamiltonian is changed to

$$\begin{aligned} \tilde{H}_{AC/AB} &= -t\sum_{i,\sigma}(e^{i2\pi\Gamma_{AC/AB}\phi/N_L}\tilde{c}^{\dagger}_{i,\sigma}\tilde{c}_{i+1,\sigma} \\ &+ e^{-i2\pi\Gamma_{AC/AB}\phi/N_L}\tilde{c}^{\dagger}_{i+1,\sigma}\tilde{c}_{i,\sigma}) \\ &- U\sum_i \tilde{n}_{i,\uparrow}\tilde{n}_{i,\downarrow}, \end{aligned} \quad (3)$$

where $\tilde{n}_{i,\sigma} = \tilde{c}^{\dagger}_{i,\sigma}\tilde{c}_{i,\sigma}$. Through this transformation, the sign of the interaction U is reversed and the form of the Hamiltonian is exchanged between the AB and AC effects, where we represent it as $\tilde{H}_{AC/AB}$. Further, the number of down-spin electrons $N_\downarrow$ is changed into $N_L - N_\downarrow$, although the number of up-spin electrons $N_\uparrow$ is not changed. Any eigenstate of the system for $U < 0(> 0)$ with N_e electrons and a magnetization $M_z = 2S_z = N_\uparrow - N_\downarrow$, is transformed into the eigenstate of the system for $U > 0(< 0)$ with $\tilde{N}_e = N_L + M_z$ electrons with a magnetization $\tilde{M}_z = 2\tilde{S}_z = N_e - N_L$, and vice versa.[28,30] The energies of the corresponding states are represented by the equation

$$\begin{aligned} &E_{AB/AC}(N_L, N_e, N_\uparrow, N_\downarrow, \pm|U|, \phi) \\ &= E_{AC/AB}(N_L, N_L + M_z, N_\uparrow, N_L - N_\downarrow, \mp|U|, \phi) \pm |U|N_\uparrow, \end{aligned} \quad (4)$$

where we can limit $M_z \leq 0$ and $N_e(\tilde{N}_e) \leq N_L$ without loss of generality.

To analyze flux dependence of the systems concretely, we obtain lower energies of the Hamiltonian (1) as a function of normalized flux ϕ, where these eigenvalues are numerically calculated by the standard Householder or Lanczos algorithm. Using systematic ED calculation with the help of the Shiba transformation, we revel whole aspect of an anomalous periodicity of ABC effects of the Hubbard model in the strong coupling limit.

3. Numerical results

3.1 Periodicity of the ground state

In this subsection, we limit the analysis in the case of N_L being even, because the Shiba transformation is not applicable to the odd case.[29] At first, we consider the half-filling($n = 1$) and spin balanced($M_z = 0$) case, as the simplest one. In this case, the highest symmetry is realized and the transformation does not change the number of the electron N_e and the magnetization M_z, that is, $\tilde{N}_e = N_e$, and $\tilde{M}_z = M_z = 0$. Figure 1(a) shows the the energy spectrum $E_{AB/AC}(\phi)$ as a function of ϕ of the $N_L = 8$ site system. The grid-line of left side axis is corresponding to the energies of the AB effect $E_{AB}(\phi)$ with $U = -50$, and that of right side is the AC effect $E_{AC}(\phi)$ with $U = 50$. By calculating energies of both systems independently, we have numerically reconfirmed that eigenvalues $E_{AB}(\phi)$ and $E_{AC}(\phi)$ completely agree with each other except the energy shift of $|U|N_\uparrow$.

In the side of the AB effect, the attraction is very strong($U = -50$) and the period of $E_{AB}(\phi)$ is determined by the superconducting flux quantization relevant

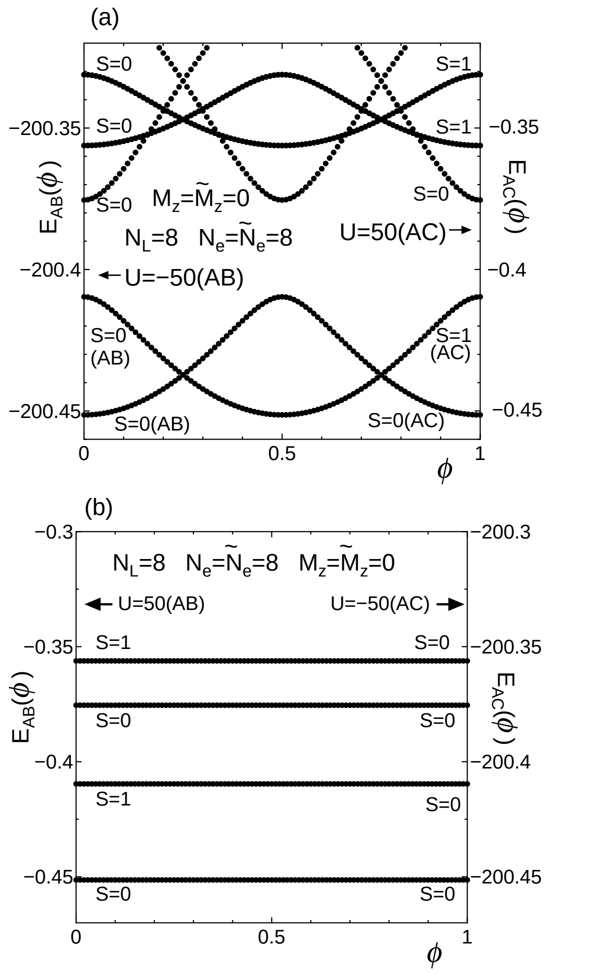

Fig. 1. The ground state energy and few lower energies (a) $E_{AB}(\phi)(E_{AC}(\phi))$ as a function of the renormalized flux ϕ with $M_z = 0(0)$ and $U = -50(50)$ for the AB(AC) effect, (b)$E_{AB}(\phi)(E_{AC}(\phi))$ with $M_z = 0(0)$ and $U = 50(-50)$ for the AB(AC) effect.

to so called Cooper pairs on the same site. On the other hand, the period of $E_{AC}(\phi)$ transformed from $E_{AB}(\phi)$ should not be interpreted as the superconducting flux quantum, because the strong repulsion($U = 50$) brakes Cooper pairs and the system becomes insulator with large charge gap in the side of the AC effect. The value of the charge gap is considered to be corresponding to that of the spin gap relate to a bound energy of the Cooper pair in the side of the AB effect. In Fig. 1(a), both levels of alternating the ground and the first excited states have the total spin angular momentum $S = 0$, but one of the transformed states is $S = 1$. It is noted that corresponding state does not necessarily conserve total spin S through the transformation.

When the repulsive interaction U is sufficiently large, the Hubbard model can be mapped to the Heisenberg model with the exchange interaction $J \simeq 4t^2/U$ at half-filling. In this case, the first excited state is corresponding to the spin wave state with $S = 1$. Since the phase shift of the AC effect is a kind of the spin twist, it affects the spin state. As shown in Fig.1(a) , the phase shift pushes down the $S = 1$ first excited state, and raises up the $S = 0$ ground state. Then, the alternation of the ground state may be interpreted as a spin fluctuation effect produced by the AC effect. Using the ED method, we confirm that same behavior stands other size systems at half-filling ($n = 1$) and $M_z = 0$. These results suggest that the characteristic number of the periodicity of the system is given by 2.

Figure 1(b) also shows the result at $n = 1$ and $M_z = 0$, but the interaction U is positive for the AB effect and negative for the AC effect. It indicates that the flux dependence of the $E_{AB/AC}(\phi)$ becomes very small and it seems to be nearly flat. It means that the Drude weight $D_c(\propto \frac{\partial^2 E_{AB}(\phi)}{\partial \phi^2}|_{\phi=0})$ closes to zero. In this case, large charge gap[28] is opened as a insulator in the side of the AB effect, and few lower excited states should be regarded as a spin excitation. On the other hand, in the side of the AC effect, attractive interaction($U = -50$) produces large spin gap and suppresses the spin fluctuation. In this case, the spin stiffness $D_s(\propto \frac{\partial^2 E_{AC}(\phi)}{\partial \phi^2}|_{\phi=0})$ becomes zero.[31] Then, lower excited states are originated by the freedom of the charge as an electron pair (its effective transfer may be given as $\sim 4t^2/U$). Since the variation of $E_{AB/AC}(\phi)$ as a function of ϕ is almost zero as shown in Fig1(b), we define the number of the period as zero. These results are also consistent with the Bethe ansatz results of the half-filled Hubbard model with repulsive interaction obtained by Zvyagin and Krive.[21]

Next, we present the result of $n = 1$ and spin-imbalanced($|M_z| > 0$) case in Fig. 2. It is noted that the system of $n = 1$ and $|M_z| > 0$ is transformed to the system of $\tilde{N}_e = N_L - |M_z| < N_L$ with $\tilde{M}_z = 0$. Figure 2(a) indicates that the periodicity is given as 1 for the AB(AC) effect at $U < 0(U > 0)$. Figure 2(b) shows the flat ground state. In this case, the charge gap state appears as well as the $n = 1$ and $|M_z| = 0$ case. Further, the transformed case is corresponding to the spin gap state, because $\tilde{M}_z = 0$ and $U < 0$.

In Fig.3, we indicates the the ground state energy and few lower excited energies at at $N_L = 8$ with $n < 1$ and $M_z = 0$. In this case, the transformed system is corresponding to the case of $n = 1$ and $|M_z| > 0$. As shown in Fig.3(a), the period is 2 and it can be interpreted by the flux quantization of usual cooper pairs for the AB effect, because the on-site interaction is negative as $U = -100$ at $n < 1$. Fig.3(b) indicates the $U = 300$ case of the AB effect. In this case, the Bethe ansatz method[18,19] gives the periodicity of the AB effect as N_e, which is consistent with the ED calculation.

Finally, we consider the non-half-filling($n < 1$) and $|M_z| > 0$ case in Figs. 4. In this case, the AB(AC) effect with $U < 0$ is transformed to the the AC(AB) effect with $U > 0$. For the attractive Hubbard model, we have shown that the periodicity is given by $N_L - N_e$ in the previous ED work.[26] On the other hand, in the repulsive case, the Bethe ansatz method also gives the same periodicity of the AB effect as N_e as well as the case of $M_z = 0$.[18,19] When we adapt the transformation to the above results, we can immediately find the number of the periodicity of the AC effect to be $|M_z|$ for $U > 0$, and $N_L - |M_z|$ for $U < 0$, where we have reconfirmed these periodicity by the ED calculation.

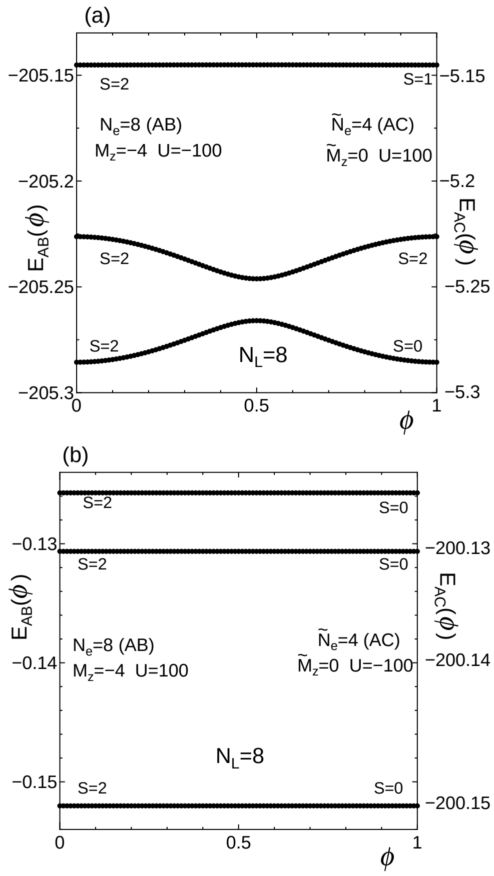

Fig. 2. The ground state energy and few lower excited energies (a) $E_{AB}(\phi)(E_{AC}(\phi))$ as a function of the renormalized flux ϕ with $N_e = 8(4)$, $M_z = -4(0)$, and $U = -100(100)$ for the AB(AC) effect, (b) $E_{AB}(\phi)(E_{AC}(\phi))$ at $U = 100(-100)$ for the AB(AC) effect.

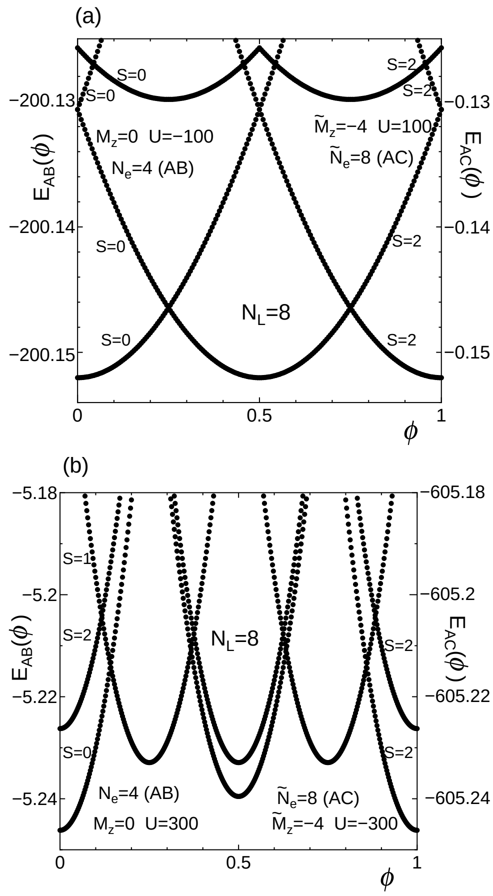

Fig. 3. The ground state energy and few lower excited energies (a) $E_{AB}(\phi)(E_{AC}(\phi))$ as a function of the renormalized flux ϕ with $N_e = 4(8)$, $M_z = 0(-4)$, and $U = -100(100)$ for the AB(AC) effect, (b) $E_{AB}(\phi)(E_{AC}(\phi))$ with $N_e = 4(8)$, $M_z = 0(-4)$, and $U = 300(-300)$ for the AB(AC) effect.

In this subsection, we give only the results of $N_L = 8$ as typical examples, but we confirmed that same results are obtained for all different systems in the scope of the systematical ED calculation. To show these results at once conveniently, we present the number of the period in following tables. These give the periodicity of oscillations classified for all possible case. Here, we add the superscript index (Roman figure) to the number of the period in the columns, which represents the corresponding periodicity connected by the Shiba transformation between the AB and AC effects.[32)]

Table I. Number of the period for $n = 1$.

	AB effect	AC effect
$M_z = 0$		
Attractive ($U < 0$)	$2^{(\mathrm{i})}$	$0^{(\mathrm{ii})}$
Repulsive ($U > 0$)	$0^{(\mathrm{ii})}$	$2^{(\mathrm{i})}$
$\lvert M_z\rvert > 0$		
Attractive ($U < 0$)	$1^{(\mathrm{iii})}$	$(N_L - \lvert M_z\rvert)^{(\mathrm{vi})}$
Repulsive ($U > 0$)	$0^{(\mathrm{iv})}$	$2^{(\mathrm{v})}$

Table II. Number of the period for $n < 1$.

	AB effect	AC effect
$M_z = 0$		
Attractive ($U < 0$)	$2^{(\mathrm{v})}$	$0^{(\mathrm{iv})}$
Repulsive ($U > 0$)	$N_e^{(\mathrm{vi})}$	$1^{(\mathrm{iii})}$
$\lvert M_z\rvert > 0$		
Attractive ($U < 0$)	$(N_L - N_e)^{(\mathrm{vii})}$	$(N_L - \lvert M_z\rvert)^{(\mathrm{viii})}$
Repulsive ($U > 0$)	$N_e^{(\mathrm{viii})}$	$\lvert M_z\rvert^{(\mathrm{vii})}$

3.2 The periodicity for N_L =odd

When N_L is odd, the symmetry of the system decreases and the Shiba transformation can not be used. It may reduce the regularity of the period of ABC effects, however, we find almost similar results within our ED calculation. When $n < 1$, the periodicity does not change and the table of II just stand. Further, the case of $n = 1$ and $U > 0$ gives also the same result. However, we find the exceptional cases for the AB effect at $n = 1(|M_z| > 0)$ and $U < 0$ as follows, where it is noted that the case of $M_z = 0$ is does not exist at $n = 1$.

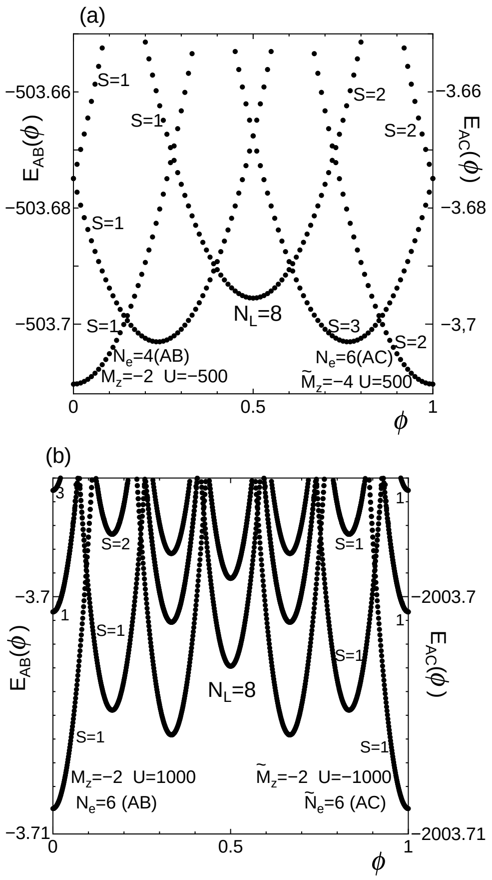

Fig. 4. The ground state energy and few lower excited energies (a) $E_{AB}(\phi)(E_{AC}(\phi))$ as a function of the renormalized flux ϕ with $N_e = 4(6)$, $M_z = -2(-4)$, and $U = -500(500)$ for the AB(AC) effect, (b) $E_{AB}(\phi)(E_{AC}(\phi))$ with $N_e = 6(6)$, $M_z = -2(-2)$, and $U = 1000(-1000)$ for the AB(AC) effect.

(I) When $N_\uparrow$(or $N_\downarrow$)=2, the period becomes 0.

(II) When $N_\uparrow$(and $N_\downarrow$) $\neq 2$, the period is 2.

In Fig.5, we show the behavior of the ground state as a function of ϕ as a typical example of the case (I). It indicate that the ground state is almost independent of ϕ(insulator like state), although the corresponding period is indicated as 1 in the table of I. We confirm that this insulator like ground state is observed in the system of N_L =5,7,9,11,13,15,17, and 19 by the ED calculation. It may expect that the same result is obtained for more large systems. The state descends to the ground state from the excited state, when the attractive interaction $|U|$ becomes larger than some critical value of $|U_c|$. We find $|U_c| \sim 40$ for $N_L = 7$, and ~ 20 for $N_L = 19$. It suggests that $|U_c|$ decreases with the increase of the system size N_L. It is much interesting, but we can not clarify the origin of this ground state at this stage.

3.3 *Comment on the Bethe ansatz result for $U < 0$*

As mentioned in the previous section, there are several works of the Bethe ansatz method for the repulsive and/or attractive Hubbard model with the AB and/or AC flux.[18,19,31,33] In the repulsive model, the periodicity of the AB effect[18,19] agrees with our result. The AC

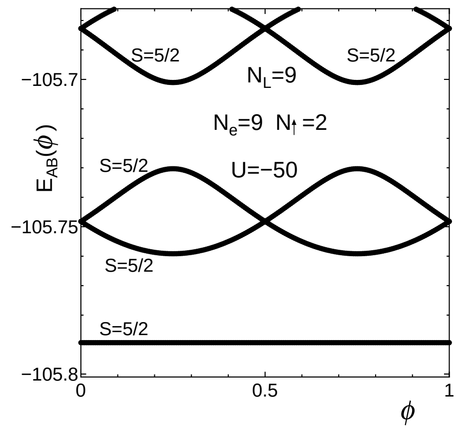

Fig. 5. The ground state and few lower excited energies $E_{AB}(\phi)$ as a function of ϕ obtained by the ED method for the attractive Hubbard model at $n = 1$.

effect of the half-filling ($n = 1$) and spin balanced($M_z = 0$) case,[21] also consists with the ED result. However, we find the disagreement between our ED result and the Bethe ansatz result obtained by Zvyagin et al .,[22,23] which addresses ABC effects of the attractive Hubbard model($U < 0$). They claim that the period of the system is N_e for the AB effect, and $N_e - M_b$ for AC effect, where M_b is the number of electrons bound in local pairs, that is $N_\uparrow$ for $M_z < 0$.[34] It indicates that the period is independent of the system size N_L and does not agree with our ED results of the attractive case and the transformed result from the the Bethe ansatz result($U > 0$) obtained by Kusmartsev et al.,[18,19] as shown in the table II.

Therefore, we examine this discrepancy in the following. At first, we consider the validity of the equations constructed by their Bethe ansatz method, since their formulation seems to be not compatible to the result of the Bethe ansatz method for the attractive Hubbard model obtained by Takahashi.[35] Although all pairs as possible should been considered in the wave function, their formulation seems to limit only local pairs on the same site, and neglect pair fluctuations. We think that it might be corresponding to a kind of approximation to the fully exact Bethe ansatz result.[35] To examine our interpretation, we numerically solve the nonlinear simultaneous equations derived in their work,[22,23]

$$N_L k_j = 2\pi(I_j + \phi_{AB} + \phi_{AC}) + 2\sum_{\alpha=1}^{M_b} \tan^{-1}(\frac{\sin k_j - \lambda_\alpha}{U}),$$

$$2N_L \mathrm{Re}\{\sin^{-1}(\lambda_\alpha - iU)\} = 2\pi(J_\alpha + 2\phi_{AB})$$

$$+2\sum_{j=1}^{N_e-2M_b} \tan^{-1}(\frac{\lambda_\alpha - \sin k_j}{U}) + 2\sum_{\beta=1,\beta\neq\alpha}^{M_b} \tan^{-1}(\frac{\lambda_\alpha - \lambda_\beta}{2U}),$$

where ϕ_{AB}, and ϕ_{AC} are the AB-, and AC-flux, respectively. The total energy of the system $E(\phi_{AB}, \phi_{AC})$ is given as

$$E(\phi_{AB}, \phi_{AC})/N_L = -\sum_{j=1}^{N_e-2M_b} 2\cos k_j$$

$$-\sum_{\alpha=1}^{M_b} 4\mathrm{Re}\sqrt{1-(\lambda_\alpha - iU)^2}. \qquad (5)$$

We calculate the ground state energy from the above formulation numerically and compare it with the ED result for the same system as a function of $|U|$. In Fig.6, we show the normalized energy difference $|\Delta E/E(0,0)_{U=0}|$ between the ground state energies obtained by both method, where $E(0,0)_{U=0}$ is the ground state energy at $U = 0$. It indicates that the difference is not zero especially in the weak coupling region, but exponentially close to zero with increasing $|U|$. In Fig.7, we also give the ϕ_{AC} dependence of the ground state energy obtained by the above formulation with the result of the ED method, where we set $N_L = 5$, $N_e = 3$, and $M_b = |M_z| = 1$ at $U = -300$. Both results well agree with each other for any ϕ_{AC}, and it is difficult to distinguish these results numerically. We also confirm that the same result is obtained for the ϕ_{AB} dependence.

Generally speaking, the energy calculated by the exact Bethe ansatz method is completely consistent with the ED result for any strength of U. In fact, we confirm that the ground state energy of the repulsive Hubbard model exactly agrees with the ED result for even small U, within the numerical error.[19,33] These results suggest that their formulation is valid as a kind of an approximation for $|U| >> 1$, since local pairs dominate in the presence of the strong attractive on-site interaction and other pair fluctuations can be neglected.

However, more serious problem is that their determination of the lowest-energy state may be wrong in the case of finite ϕ and the obtained periodicity is not correct. In fact, if we adopt their claim of the periodicity, it should be $3 - 2 = 1$ in the system of $N_e = 3$ and $M_b = 1$. On the other hand, the result of Fig.7 clearly indicates that the period is given 4, which is consistent with our ED result, but inconsistent with their claim of the periodicity. It indicates that their formulation is valid as an approximation, but they may mistake in the periodicity extracted from Eqs.5 and 5. We think that it is hard to find out analytically which state becomes the lowest-energy state at given ϕ_{AB} and/or ϕ_{AC}.

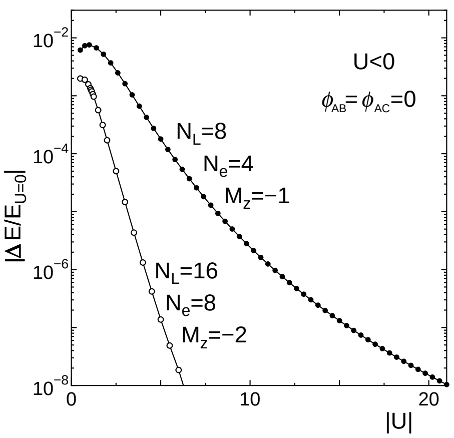

Fig. 6. The normalized energy difference $|\Delta E/E(0,0)_{U=0}|$ between the ground state obtained by the ED and Zvyagin et al. as a function of $|U|$ for the the attractive Hubbard model.

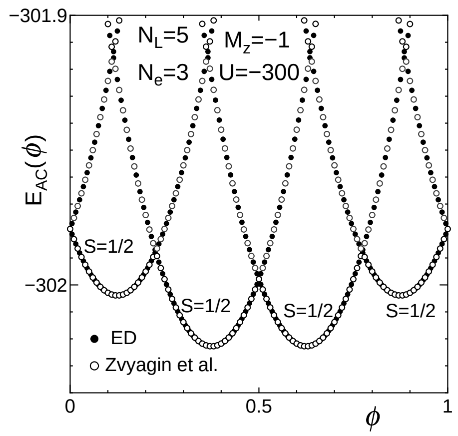

Fig. 7. The ground states energy of the attractive Hubbard model as a function of the AC-flux ϕ_{AC} obtained by Zvyagin et al. with the result of the ED method . The difference between both results is very small.

4. Discussion

In this work, we present anomalous oscillations of the ground state due to ABC effects of the one-dimensional Hubbard ring with flux in the strong coupling limit. By using the exact diagonalization method and the Shiba transformation, we classify the periodicity of anomalous oscillations as a function of the flux. As an example, the number of the periods are given by $N_L - N_e(N_L - |M_z|)$ for the AB(AC) effect in the very strong attraction at $n < 1$. The result suggests that the interference effect relates to the whole site of the system, since the system size N_L appears in the the number of the period. We think that the strong correlation may be crucial to produce this anomalous phenomena.

When we consider the attractive interaction case, the origin of the periodicity of that ABC effects can be easily understood intuitively. In the previous work,[26] we have already give an intuitive explanation for the fractional periods of the AB effect in the attractive Hubbard model($U \to -\infty$) by considering the hopping of an electron pair(doublon) binding concretely at the same site. The consideration is easily extended to the case the AC effect by taking account of the change of the sign of the phase shift of the electron with down spin. If we reverse the contribution of the phase shift from $N_\downarrow$ in the period $N_L - N_e$ of the AB effect, we can easily transform it as $N_L - N_\uparrow + N_\downarrow$. It shows that the number of the flux quantization is given by $N_L - |M_z|$ for the AC effect, where we noted that the result stands even if the N_L is odd.[36] The above intuitive argument may be useful to understand the physics of the anomalous periodicities in the case of $U > 0$ by combining the Shiba transformation.

Finally, we discuss the scope of our result of the anomalous periodicity. Our calculation are limited to the single chain Hubbard model and seems to be not capable of the two-band Hubbard models, such as ladder or zigzag models. However, same anomalous periodicity is observed in these models by using the ED method.[25,27] It may suggest that the main mechanism producing the anomalous ABC effects survives even in more general attractive Hubbard models.

Even though main amplitude of the period is such as the superconducting flux quantum $h/2e$ or the half period $h/4e$ derived by Yoshida and Yanase,[15] it may not break the fine period such as $N_L - N_e$. If the period consists of the superposition of two different oscillations, we may recognize a fine oscillation to be a representative period. We expect that the anomalous periodicity is relatively robust and experimental measurements might be not so difficult, as long as the system is not so large.

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