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# Convolution powers of singular-symmetric measures

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## 1. Introduction.

Let $G$ be a L.C.A. group and $\hat{G}$ be its dual group. Let $M(G)$ be the measure algebra on $G$ and $L^1(G)$ be the group algebra on $G$. In [7], Taylor showed that: There are a compact topological abelian semigroup $S$ and an isometric isomorphism $\theta$ of $M(G)$ into $M(S)$ such that;

(a) $\theta(M(G))$ is a weak-*dense subalgebra of $M(S)$;

(b) $\hat{S}$, the set of all continuous semicharacters on $S$, separates the points of $S$;

(c) for $f\in\hat{S}$, $\mu\to\int_S fd\theta\mu$ $(\mu\in M(G))$ is a non-zero complex homomorphism of $M(G)$;

(d) for a non-zero complex homomorphism $F$ of $M(G)$, there is an $f\in\hat{S}$ such that $F(\mu)=\int_S fd\theta\mu$ for $\mu\in M(G)$.

We can consider that $\hat{S}$ is the maximal ideal space of $M(G)$, $\hat{G}\subset\hat{S}$, and the Gelfand transform of $\mu\in M(G)$ is given by $\hat{\mu}(f)=\int_S fd\theta\mu$ $(f\in\hat{S})$. A closed subspace (ideal, subalgebra) $N\subset M(G)$ is called an $L$-subspace ($L$-ideal, $L$-subalgebra) if $L^1(\mu)\subset N$ for every $\mu\in N$, where $L^1(\mu)=\{\lambda\in M(G);\ \lambda$ is absolutely continuous with respect to $\mu$ $(\lambda\ll\mu)\}$. We denote by $\mathrm{Rad}\,L^1(G)$ the radical of $L^1(G)$ in $M(G)$, that is, $\mathrm{Rad}\,L^1(G)=\{\mu\in M(G);\ \hat{\mu}(f)=0$, for all $f\in\hat{S}\backslash\hat{G}\}$. We put $\mathfrak{L}(G)=\sum_\tau \mathrm{Rad}\,L^1(G_\tau)$, where $\tau$ runs through over L.C.A. group topologies on $G$ which are stronger than the original one. Then $\mathfrak{L}(G)\subset M(G)$ and $\mathfrak{L}(G)$ is an $L$-subalgebra ([2]). For $\mu\in M(G)$, we put $\mu^*(E)=\overline{\mu(-E)}$ for every Borel subset $E$ of $G$. We denote by $\mathfrak{M}$ the set of all symmetric measures of $M(G)$, that is, $\mathfrak{M}=\{\mu\in M(G);\ \hat{\mu}^*(f)=\overline{\hat{\mu}(f)}$ for every $f\in\hat{S}\}$. Then it is easy to show that $\mathfrak{L}(G)\subset\mathfrak{M}$. A measure $\mu\in\mathfrak{M}$ is called singular-symmetric if $\mu$ is singular with

$\mathfrak{L}(G)$ $(\mu\perp\mathfrak{L}(G))$. In [4], the author shows that if $\bar{R}$ is the Bohr compactification of the real line $R$, then there exists a singular-symmetric measure $\mu$ on $\bar{R}$. Moreover it is easy to show that $\mu$ (constructed in [4]) has the property $\mu*\mu\in\mathfrak{L}(\bar{R})$. By the same method as in [4], we can construct $\mu$ on an infinite compact abelian group $G$ whose dual group has an infinite independent subset, such that $\mu$ is singular-symmetric with $\mu*\mu\in\mathfrak{L}(G)$. In this paper, we show

THEOREM. *Let $G$ be an infinite compact abelian group. If $\hat{G}$ has an infinite independent subset, then there exists a singular-symmetric measure $\mu$ on $G$ such that $\mu^n$ is singular-symmetric for every positive integer $n$, where $\mu^n=\mu^{n-1}*\mu$ $(n\geqq 2)$ and $\mu^1=\mu$.*

## 2. Proof of theorem.

Let $G$ be an infinite compact abelian group such that $\hat{G}$ has an infinite independent subset $E$ which we may suppose to generate $\hat{G}$ without loss of generality. Then there is a family of infinite subsets of $E$, $\{E_{n,i}; n=1, 2, \cdots, i=1, 2, \cdots, 2^n\}$, which satisfies the following properties:

1) For $n\geqq 1$, $\cup\{E_{n,i}; 1\leqq i\leqq 2^n\}=E$;

2) for $1\leqq i<j\leqq 2^n$, $E_{n,i}\subsetneqq E_{n,j}$ and $E_{n,j}\backslash E_{n,i}$ is an infinite set;

3) $E_{n+1,k}\subset E_{n,i}$ for $k<2i$ and $E_{n+1,2i}=E_{n,i}$ $(1\leqq i\leqq 2^n)$.

Let $H_{n,i}$ be the subgroup of $\hat{G}$ generated by $E_{n,i}$, then $\{H_{n,i}\}_{n,i}$ has the following properties by 1), 2) and 3):

4) For $n\geqq 1$ and $1\leqq i<j\leqq 2^n$, $H_{n,i}\subsetneqq H_{n,j}$, $H_{n,j}/H_{n,i}$ is an infinite group, and $H_{n,2^n}=\hat{G}$;

5) $H_{n,i}\supsetneqq H_{n+1,k}$ and $H_{n,i}/H_{n+1,k}$ is an infinite group for $k<2i$, and $H_{n,i}=H_{n+1,2i}$ for $1\leqq i\leqq 2^n$.

By the above facts 4) and 5), we have:

6) For $n\leqq s$ and $1\leqq i\leqq 2^{s-n}j$, $H_{n,j}\supset H_{s,i}$ and $H_{n,j}/H_{s,i}$ is an infinite group if $i\neq 2^{s-n}j$, and $H_{n,j}=H_{s,2^{s-n}j}$.

Let $G_{n,i}$ be the annihilator of $H_{n,i}$ in $G$ $(G_{n,i}=H_{n,i}^{\perp}\subset G)$, then $G_{n,i}$ is a compact subgroup of $G$ and $\{G_{n,i}\}_{n,i}$ satisfies the following by 4), 5) and 6):

7) For $n\geqq 1$ and $1\leqq i<j\leqq 2^n$, $G_{n,i}\supsetneqq G_{n,j}$, $G_{n,i}/G_{n,j}$ is an infinite compact group, and $G_{n,2^n}=\{0\}$, where 0 is the unit element of $G$;

8) $G_{n,i}\subsetneqq G_{n+1,k}$ and $G_{n+1,k}/G_{n,i}$ is an infinite compact group for $k<2i$, and $G_{n,i}=G_{n+1,2i}$ for $1\leqq i\leqq 2^n$;

9) for $n\leqq s$ and $1\leqq i\leqq 2^{s-n}j$, $G_{n,j}\subset G_{s,i}$ and $G_{s,i}/G_{n,i}$ is an infinite compact group if $i\neq 2^{s-n}j$, and $G_{n,j}=G_{s,2^{s-n}j}$.

For a compact subgroup $G_0 \subset G$, we denote by $m(G_0)$ the normalized Haar measure on $G_0$. We put

$$\mu_n = \sum\{(1/2)^n m(G_{n,i}) ; 1 \leqq i \leqq 2^n\} \quad (n \geqq 1),$$

then $\mu_n \geqq 0$, $\|\mu_n\| = 1$, $\mu_n^* = \mu_n$. For a fixed $\gamma \in \hat{G}$, there is a non-negative integer $p_n (0 \leqq p_n < 2^n)$ such that $\gamma \notin H_{n,p_n}$ and $\gamma \in H_{n,p_n+1}$, where $H_{n,0} = \emptyset$. Then we have $\hat{\mu}_n(\gamma) = (1/2)^n (2^n - p_n) > 0$. Also there is $p_{n+1}$ $(0 \leqq p_{n+1} < 2^{n+1})$ such that $\hat{\mu}_{n+1}(\gamma) = (1/2)^{n+1}(2^{n+1} - p_{n+1})$. Since $p_{n+1} = 2p_n$ or $p_{n+1} = 2p_n + 1$ by 4) and 5), we have

$$\hat{\mu}_{n+1}(\gamma) = (1/2)^{n+1}(2^{n+1} - 2p_n) = (1/2)^n (2^n - p_n) = \hat{\mu}_n(\gamma)$$

or

$$\hat{\mu}_{n+1}(\gamma) = (1/2)^{n+1}(2^{n+1} - 2p_n - 1) = \hat{\mu}_n(\gamma) - (1/2)^{n+1}.$$

So that $\hat{\mu}_n(\gamma) \geqq \hat{\mu}_{n+1}(\gamma)$ for every $n \geqq 1$. This implies that $\{\mu_n\}_{n=1}^{\infty}$ has only one weak-*cluster point $\mu$ in $M(G)$ and $\mu$ has the following properties:

10) $\mu \geqq 0$, $\|\mu\| = 1$, $\mu^* = \mu$ and $\{\hat{\mu}(\gamma) ; \gamma \in \hat{G}\}$ is dense in $\{x \in R ; 0 \leqq x \leqq 1\}$;

11) $\hat{\mu}(\gamma) = \lim_{n \to \infty} \hat{\mu}_n(\gamma)$ for every $\gamma \in \hat{G}$.

We will show that $\mu$ satisfies the conditions of our theorem. At first, we show that $\mu \in \mathfrak{M}$. For $1 \leqq n$, $1 \leqq i \leqq 2^n$ and $n \leqq k$, we put

$$\mu_{n,k,i} = \sum\{(1/2)^k m(G_{k,j}) ; 2^{k-n}(i-1) < j \leqq 2^{k-n} i\}.$$

Then

$$\mu_{n,k,i} \geqq 0, \ \|\mu_{n,k,i}\| = (1/2)^k 2^{k-n} = (1/2)^n$$

and

12) $\mu_k = \sum\{\mu_{n,k,i} ; 1 \leqq i \leqq 2^n\}$.

By the same way as in the previous part, $\{\mu_{n,k,i}\}_{k=n}^{\infty}$ has only one weak-*cluster point $\lambda_{n,i}$ in $M(G)$ and which satisfies

13) $\lambda_{n,i} \geqq 0$, $\|\lambda_{n,i}\| = (1/2)^n$, $\hat{\lambda}_{n,i}(\gamma) = \lim_{k \to \infty} \hat{\mu}_{n,k,i}(\gamma)$ for $\gamma \in \hat{G}$, and $\lambda_{n,i} \in M(G_{n,i})$ by 8) and 9).

Since $\hat{\mu}_{n,k,i}(\gamma) = 0$ for $\gamma \notin H_{n,i}$ by the definition of $\mu_{n,k,i}$, we have

14) $\hat{\lambda}_{n,i}(\gamma) = 0$ if $\gamma \notin H_{n,i}$.

By 11), 12) and 13), we have

$$\hat{\mu}(\gamma) = \lim_{k \to \infty} \hat{\mu}_k(\gamma) = \sum\{\lim_{k \to \infty} \hat{\mu}_{n,k,i}(\gamma) ; 1 \leqq i \leqq 2^n\}$$

$$= \sum\{\hat{\lambda}_{n,i}(\gamma) ; 1 \leqq i \leqq 2^n\} \qquad \text{for } \gamma \in \hat{G}.$$

This implies

15) $\mu=\sum\{\lambda_{n,i}\,;\,1\leqq i\leqq 2^n\}$ for $n=1, 2, \cdots$.

Let $f\in\hat{S}(f\geqq 0)$ and $n\geqq 1$. Since $m(G_{n,i})*m(G_{n,j})=m(G_{n,i})$ for $1\leqq i\leqq j\leqq 2^n$ by 7), there exists $j_n(1\leqq j_n\leqq 2^n)$ such that

16) $m(G_{n,k})\hat{}(f)=1$ if $j_n\leqq k\leqq 2^n$ and

$m(G_{n,k})\hat{}(f)=0$ if $1\leqq k<j_n$.

Then we have the following:

17) For $1\leqq k<j_n$, $\hat{\lambda}_{n,k}(f)=0$;

18) for $j_n<k\leqq 2^n$, $\hat{\lambda}_{n,k}(f)=\|\lambda_{n,k}\|$.

Because, let $1\leqq k<j_n$, then we have $\lambda_{n,k}*m(G_{n,j_n-1})=\lambda_{n,k}$ by 4) and 14). By 16), we have $\hat{\lambda}_{n,k}(f)=\hat{\lambda}_{n,k}(f)\,m(G_{n,j_n-1})\hat{}(f)=0$. This implies 17). Let $j_n<k\leqq 2^n$. Since $\mu_{n,q,k}\in M(G_{n,j_n})$ for $n\leqq q$ by 9) and the definition of $\mu_{n,q,k}$, we have

19) $\lambda_{n,k}\in M(G_{n,j_n})$.

Since $m(G_{n,j_n})\hat{}(f)=1$ by 16), we have that $\hat{\lambda}_{n,k}(f)=\hat{\lambda}_{n,k}(1)=\|\lambda_{n,k}\|$. This shows 18).

Let $M$ be a prime $L$-subalgebra generated by $\{m(G_{n,j_n})\}_{n=1}^{\infty}$, where $M\subset M(G)$ is called a prime $L$-subalgebra if $M$ is an $L$-subalgebra and $M^{\perp}=\{\lambda\in M(G)\,;\,\lambda\perp M\}$ is an $L$-ideal. Then there is a $\pi_f\in\hat{S}$ such that $\pi_f^2=\pi_f$ and $M=\{\lambda\in M(G)\,;\,\theta\lambda$ is concentrated on $O(\pi_f)\}$, where $O(\pi_f)=\{x\in S\,;\,\pi_f(x)=1\}$ (see [7]). By Dunkl and Ramirez [1], we have $\pi_f\in cl(\hat{G})\backslash\hat{G}$, where $cl(\hat{G})$ is the closure of $\hat{G}$ in $\hat{S}$. Since $m(G_{n,j_n})\hat{}(\pi_f)=1$, we have

20) $\hat{\lambda}_{n,k}(\pi_f)=\|\lambda_{n,k}\|$ $\quad(j_n<k\leqq 2^n)$ by 19).

Since $f\geqq\pi_f$, we have

21) $\hat{\lambda}_{n,k}(\pi_f)=0$ $\quad$ for $1\leqq k<j_n$ by 17).

Then we have that for $n\geqq 1$,

$$\begin{aligned}|\hat{\mu}(f)-\hat{\mu}(\pi_f)|&=|\sum\{\hat{\lambda}_{n,i}(f)\,;\,1\leqq i\leqq 2^n\}-\sum\{\hat{\lambda}_{n,i}(\pi_f)\,;\,1\leqq i\leqq 2^n\}|\\&=|\hat{\lambda}_{n,j_n}(f)-\hat{\lambda}_{n,j_n}(\pi_f)|\leqq\|\lambda_{n,j_n}\|=(1/2)^n,\end{aligned}$$

by 13), 15), 17), 18), 20) and 21). This implies

22) $\hat{\mu}(f)=\hat{\mu}(\pi_f)$ for every $f\in\hat{S}$ $(f\geqq 0)$.

Here we note that

23) $\hat{\mu}(f)=\lim\limits_{n\to\infty}\sum\{\hat{\lambda}_{n,k}(f)\,;\,j_n<k\leqq 2^n\}$ for $f\in\hat{S}$ $(f\geqq 0)$.

We put $J(f)=\{x\in S; f(x)\neq 0\}$ and $\mu=\eta_1+\eta_2$, where $\theta\eta_1$ is concentrated on $S\backslash J(f)$ and $\theta\eta_2$ is concentrated on $J(f)$. Then $\theta\eta_2$ is concentrated on $O(\pi_f)$ by 22). This implies that $\hat{\mu}(g)=\hat{\mu}(g\cdot\pi_{|g|})$ for every $g\in\hat{S}$. Since $0\leqq\hat{\mu}(\gamma)\leqq 1$ for $\gamma\in\hat{G}$ and $g\cdot\pi_{|g|}\in cl(\hat{G})\backslash\hat{G}$ (this fact is proved easily by [1]), we have $0\leqq\hat{\mu}(g\cdot\pi_{|g|})\leqq 1$. This shows

24) $\hat{\mu}(g)\geqq 0$ for every $g\in\hat{S}$.

Since $\mu^*=\mu$ and $\mu\geqq 0$ by 10), we have $\mu\in\mathfrak{M}$.

In the rest of this paper, we will show that $\mu^n\perp\mathfrak{L}(G)$ for every positive integer $n$.

Suppose that $\mu^{n_0}\not\perp L^1(G_\tau)$ for a positive integer $n_0$ and a L.C.A. group topology $\tau$ on $G$ which is stronger than the original one. Since $M(G_\tau)$ is a prime $L$-subalgebra of $M(G)$, there exists $f(\tau)\in\hat{S}$ such that $f(\tau)^2=f(\tau)$ and $M(G_\tau)=\{\lambda\in M(G); \theta\lambda$ is concentrated on $O(f(\tau))\}$. We put $\mu=\nu_1+\nu_2$ and $a_1=\|\nu_1\|$, where $\nu_1\in M(G_\tau)$ and $\nu_2\perp M(G_\tau)$, then $\hat{\mu}(f(\tau))=a_1$. Since $M(G_\tau)$ is a prime $L$-subalgebra and $L^1(G_\tau)\subset M(G_\tau)$, we have $\|\nu_1\|=a_1>0$. Since $\|\mu\|=1$, we have $0<a_1\leqq 1$. Let $\nu_1^{n_0}=\lambda_1+\lambda_2$, where $\lambda_1\in L^1(G_\tau)$ and $\lambda_2\perp L^1(G_\tau)$. Then $\lambda_1$ is the part of $\mu^{n_0}$ which is contained in $L^1(G_\tau)$, and put $a_2=\|\lambda_1\|$. Then we have $a_1\geqq a_2>0$. By 16), there is $1\leqq j_n\leqq 2^n$ (depending on $f(\tau)$ and $n$) such that

25) $M(G_{n,j_n})\subset M(G_\tau)$ and $M(G_{n,k})\not\subset M(G_\tau)$ for $k<j_n$.

Since $\hat{\mu}(f(\tau))\neq 0$, we have that $j_n<2^n$ for sufficient large positive integers $n$ by 23). Since $\lambda_{n,p}\in M(G_{n,q})$ and $M(_{n,p})\subsetneqq M(G_{n,q})$ for $1\leqq q<p\leqq 2^n$ by 7) and 13), we have that by 25)

26) $\lambda_{n,k}\in M(G_{n,j_n+1})$, $M(G_{n,j_n+1})\perp L^1(G_{n,j_n})$, $M(G_{n,j_n+1})\perp L^1(G_\tau)$ and $\lambda_{n,k}\perp L^1(G_\tau)$ for $j_n+1<k\leqq 2^n$.

Since $\hat{\lambda}_{n,j_n+1}(f(\tau))=\|\lambda_{n,j_n+1}\|=(1/2)^n\to 0$ $(n\to\infty)$ by 13) and 18), we have

27) $\lim_{n\to\infty}\sum\{\hat{\lambda}_{n,k}(f(\tau)); j_n+1<k\leqq 2^n\}=\hat{\mu}(f(\tau))=a_1$ by 23).

Since $\hat{\lambda}_{n,k}(f(\tau))=\|\lambda_{n,k}\|$ $(k>j_n)$ and $a_1\geqq\sum\{\|\lambda_{n,k}\|; j_n+1<k\leqq 2^n\}$ by 7), 25) and the definition of $a_1$, there is a positive integer $n_1$ such that

28) $0\leqq a_1^{n_0}-(\sum\{\|\lambda_{n_1,k}\|; j_{n_1}+1<k\leqq 2^{n_1})^{n_0}<a_2$ by 27).

Since $\sum\{\lambda_{n,k}; j_{n_1}+1<k\leqq 2^{n_1}\}\in M(G_{n_1,j_{n_1}+1})$ by 26) and $M(G_{n_1,j_{n_1}+1})$ is an $L$-subalgebra, we have that

$$\|\lambda_1\|\leqq\|\nu_1^{n_0}-(\sum\{\lambda_{n_1,k}; j_{n_1}+1<k\leqq 2^{n_1}\})^{n_0}\|$$
$$=a_1^{n_0}-(\sum\{\|\lambda_{n_1,k}\|; j_{n_1}+1<k\leqq 2^{n_1}\})^{n_0}<a_2,$$

because $\nu_1-\sum\{\lambda_{n_1,k}\,;\,j_{n_1}+1<k\leqq 2^{n_1}\}$ is a positive measure, and by 25) and 28). This contradicts $\|\lambda_1\|=a_2$. Thus we have that $\mu^n\perp L^1(G_\tau)$ for every positive integer $n$ and L.C.A. group topology $\tau$ on $G$. Moreover we have $\mu^n\perp\mathrm{Rad}\,L^1(G_\tau)$ by [8]. This shows that $\mu^n\perp\mathfrak{L}(G)$ for every positive integer $n$. This completes the proof.

REMARK 1. We denote by $\sigma(\lambda)$ the spectrum of $\lambda\in M(G)$, that is, $\sigma(\lambda)=\{\hat{\lambda}(f)\,;\,f\in\hat{S}\}$. By 10) and 24), we have

$$\sigma(\mu)=\{x\in R\,;\,0\leqq x\leqq 1\}.$$

REMARK 2. In [5], it is proved that for a positive integer $n$, there exists $\mu\in M(G)$ such that $\mu^k\perp\mathfrak{L}(G)$ for $k<n$ and $\mu^q\in\mathfrak{L}(G)$ for $q\geqq n$, under the same assumptions of $G$.

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