

belongs to some segment which joins two points of U , and we put $C^k U := C(C^{k-1}U)$ inductively, $k=1, 2, 3, \dots$, $C^0 U := U$. Clearly $HU = \bigcup_{k=0}^{\infty} C^k U$ holds for any $U \subset B_r(p)$. We may think that C^k corresponds to convex combinations in the linear space.

It is the nature of a space of constant curvature that the convex hull of sufficiently close $n+1$ points x_0, x_1, \dots, x_n can be obtained by $C^k\{x_0, x_1, \dots, x_n\}$, where the integer k satisfies $2^{k-1} \leq n < 2^k$. And if M satisfies the axiom of n -planes with $2 \leq n < \dim M$, then the set of $n+1$ points x_0, x_1, \dots, x_n , which are sufficiently close to each other, has the property that $C^k\{x_0, x_1, \dots, x_n\} = H\{x_0, x_1, \dots, x_n\}$, where the integer k satisfies $2^{k-1} \leq n < 2^k$.

However it is not easy to verify the converse. This is because $C^k\{ \}$ does not in general carry the structure of a smooth submanifold, and because the dimension of $H\{ \}$ is in general greater than n .

Thus our main result is

THEOREM 1. *Let $\dim M$ be greater than 3. If for each point p in M there exists a convex neighborhood V of p in M such that $H\{x_0, x_1, x_2, x_3\} = C^2\{x_0, x_1, x_2, x_3\}$ for any points x_0, x_1, x_2, x_3 in V , then M is a space of constant curvature.*

The author does not know whether the above theorem for convex combinations of three points is true. On this problem the following holds.

THEOREM 2. *Let $\dim M$ be greater than 2. If for each point p in M there exists a convex neighborhood V of p in M such that $H\{x_0, x_1, x_2\} = C^2\{x_0, x_1, m(x_1, x_2)\} \cup C^2\{x_0, x_2, m(x_1, x_2)\}$ for any points x_0, x_1, x_2 in V , where $m(x_1, x_2)$ is the midpoint of the segment $T(x_1, x_2)$ which joins x_1 and x_2 , then M is a space of constant curvature.*

In the proofs of our theorems we shall need to estimate the dimensions (defined in [8] p. 24) of convex hulls. For this purpose we will often use the a -measure $m_a(X)$, $0 \leq a < \infty$, of a (separable) metric space X which is defined in [8] p. 102 as follows. Given $\varepsilon > 0$, let $m_a^\varepsilon(X) := \inf \sum_{i=1}^{\infty} [\delta(A_i)]^a$, where $X = \bigcup_{i=1}^{\infty} A_i$ is any decomposition of X in a countable number of subsets such that for every i the diameter $\delta(A_i)$ of A_i is less than ε , and the superscript a denotes the exponentiation. Let $m_a(X) := \sup_{\varepsilon > 0} m_a^\varepsilon(X)$.

Concerning this measure it is well known ([8] p. 104) that if X is a metric space such that $m_{n+1}(X) = 0$, $0 \leq n < \infty$, then $\dim X \leq n$, and this fact is used in the proof of Lemma 2 in § 2.

In § 2 we shall give lemmas which are used in the proofs of our theorems and we will prove theorems in § 3. In § 4 we give remarks of the theorems.

The author would like to express his thanks to Professor K. Shiohama for his valuable suggestions.

2. Lemmas.

In [5] Cheeger-Gromoll showed that if S is a connected locally convex set in M , then there is a smooth totally geodesic imbedded submanifold N of M such that $N \subset S \subset \bar{N}$, where \bar{N} is the closure of N .

This fact and the axiom of 2-planes furnish the following.

LEMMA 1. *Let $m := \dim M$ be greater than 2. If for each point p in M and for some $n, 2 \leq n < m$, there is a convex neighborhood V of p in M such that $\dim H\{x_0, x_1, \dots, x_n\} \leq n$ for any points x_0, x_1, \dots, x_n in V , then M is a space of constant curvature.*

PROOF. We first claim that $\dim H\{x_0, x_1, \dots, x_k\} \leq k$ holds for every $k, 2 \leq k \leq n$, and for any points x_0, x_1, \dots, x_k in V . Suppose $\dim H\{x_0, x_1, \dots, x_{n-1}\} > n-1$ for some points x_0, x_1, \dots, x_{n-1} in V , i. e., $\dim H\{x_0, x_1, \dots, x_{n-1}\} = n$. Then there exists a smooth totally geodesic n -dimensional imbedded submanifold N such that $N \subset H\{x_0, x_1, \dots, x_{n-1}\} \subset \bar{N}$. Take a point q in N and a normal vector v of N at q such that $\exp_N v \in V$. Then $\dim H\{x_0, x_1, \dots, x_{n-1}, \exp_N v\} > n$, a contradiction. Thus we obtain $\dim H\{x_0, x_1, \dots, x_{n-1}\} \leq n-1$ for any points x_0, x_1, \dots, x_{n-1} in V . By the same argument inductively we have our claim. In particular, $\dim H\{x_0, x_1, x_2\} \leq 2$ for any points x_0, x_1, x_2 in V .

Now we show that M satisfies the axiom of 2-planes. Let T'_p be an arbitrary 2-dimensional subspace of $T_p M$ and let v_1 and v_2 be vectors in T'_p such that $x_1 := \exp_p v_1$ and $x_2 := \exp_p v_2$ belong to V , and p, x_1 and x_2 are non-collinear. Take q in the interior of the segment $T(x_1, x_2)$ joining x_1 and x_2 , and take x_0 in V on the other side of q with respect to p on the extension of $T(p, q)$. Let N_0 be the set of all points each of which belongs to a certain segment from x_0 to a point of $T(x_1, x_2)$. Then N_0 is a smooth surface except at x_0 . We need to prove that $T_p N_0 = T'_p$ and $N_0 - \{x_0\}$ is totally geodesic in M . Let N be a smooth totally geodesic submanifold in M such that $N \subset H\{x_0, x_1, x_2\} \subset \bar{N}$. Since $p \in N_0 \subset H\{x_0, x_1, x_2\}$ and $\dim H\{x_0, x_1, x_2\} \leq 2$, it follows that $N_0 \subset N$ and $\dim N_0 = \dim N = 2$. Thus $T_p N_0 = T'_p$ and $N_0 - \{x_0\}$ is totally geodesic in M .

We know from this lemma that in order to prove our theorems we have only to estimate the dimension of $H\{\}$. We then need the following lemma.

LEMMA 2. *Let T_1 and T_2 be two segments contained in a convex set V in M . Let A be the set of all points each of which belongs to some segment joining a point of T_1 and a point of T_2 . Then $\dim A \leq 3$ and A is closed.*

PROOF. Let $x(\tau), 0 \leq \tau \leq \alpha$, and $y(\nu), 0 \leq \nu \leq \beta$, represent segments T_1 and T_2 respectively, and let W_q , for each q in V , be a subset of $T_q M$ where $\exp_q|W_q$ is diffeomorphic onto V . Define a map G of $[0, 1] \times [0, \alpha] \times [0, \beta]$ into $T_p M$ by $G(\mu, \tau, \nu) := (\exp_p|W_p)^{-1}[\exp_{x(\tau)}\{\mu(\exp_{x(\tau)}|W_{x(\tau)})^{-1}(y(\nu))\}]$ for $(\mu, \tau, \nu) \in [0, 1] \times [0, \alpha] \times [0, \beta]$, where p is a fixed point in V . Then G is differentiable, and

hence G is Lipschitz continuous. Therefore it follows from the definition of 4-measure that the 4-measure of the image of G is zero since the 4-measure of $[0, 1] \times [0, \alpha] \times [0, \beta]$ is zero. Note that A is the image of $\exp_p \circ G$ and that the property of having at most dimension n is topologically invariant. Thus we conclude $\dim A \leq 3$ by the fact in § 1.

Closedness of A is evident.

LEMMA 3. Let p be a fixed point in M . For an arbitrary $\alpha > 0$, there exists an $r > 0$ such that for any points x, y and z in $B_r(p)$,

$$\mu(1-\alpha)yz \leq w_y(\beta\mu)w_z(\gamma\mu) \leq \mu(1+\alpha)yz$$

for any $\mu \in [0, 1]$, where yz is the distance between y and z , and $w_v(\tau)$, $0 \leq \tau \leq \beta$, and $w_z(\nu)$, $0 \leq \nu \leq \gamma$, represent segments $T(x, y)$ and $T(x, z)$ respectively.

PROOF. By a straightforward generalization of Proposition 9.10 in [7] p. 54 we obtain that for given $0 < \varepsilon < 1$ there is an $r > 0$ such that for any non-collinear points x, y and z in $B_r(p)$

$$1 - \varepsilon < \|(\exp_x|B_r)^{-1}(y) - (\exp_x|B_r)^{-1}(z)\|_x / yz < 1 + \varepsilon,$$

where $\|\cdot\|_x$ is the norm in T_xM , and B_r is the r -ball in T_xM centered at the origin.

We then have

$$1 - \varepsilon < \|(\exp_x|B_r)^{-1}(w_y(\beta\mu)) - (\exp_x|B_r)^{-1}(w_z(\gamma\mu))\|_x / w_y(\beta\mu)w_z(\gamma\mu) < 1 + \varepsilon$$

for $\mu \neq 0$. Therefore

$$(1 - \varepsilon)/(1 + \varepsilon) < (1/\mu)(w_y(\beta\mu)w_z(\gamma\mu)/yz) < (1 + \varepsilon)/(1 - \varepsilon).$$

If we choose an $\varepsilon > 0$ which satisfies

$$1 - \alpha < (1 - \varepsilon)/(1 + \varepsilon) < (1 + \varepsilon)/(1 - \varepsilon) < 1 + \alpha,$$

then it follows that $\mu(1-\alpha)yz \leq w_y(\beta\mu)w_z(\gamma\mu) \leq \mu(1+\alpha)yz$ for any $\mu \in [0, 1]$.

3. Proofs of Theorems.

3.1. PROOF OF THEOREM 1. We denote six segments each of which joins x_i and x_j , $0 \leq i < j \leq 3$ by T_k , $k=1, 2, \dots, 6$. Then from the assumption $H\{x_0, x_1, x_2, x_3\} = \bigcup_{1 \leq i < j \leq 3} \{x \in V; x \text{ belongs to some segment which connects a point of } T_i \text{ and a point of } T_j\}$. Therefore it follows from Lemma 2 and the sum theorem ([8] p. 30), i.e., a separable metric space which is the countable sum of closed subsets of dimension $\leq n$ has dimension $\leq n$, that $\dim H\{x_0, x_1, x_2, x_3\} \leq 3$. Hence we obtain our theorem by Lemma 1.

3.2. PROOF OF THEOREM 2. Let $\alpha > 0$ satisfy that $6((1+\alpha)/2)^3 < 1$. And

for this α we choose an $r > 0$ such that $B_r(p) \subset V$ satisfies the conclusion of Lemma 3 and the $4r$ -ball with center p is strongly convex.

By Lemma 1 it suffices to show that $\dim H\{x_0, x_1, x_2\} \leq 2$. If $\dim C^2\{y_0, y_1, y_2\} \leq 2$ for any points y_0, y_1 and y_2 in $B_r(p)$, then $\dim H\{x_0, x_1, x_2\} \leq 2$ because of the sum theorem.

In fact, $\dim C^2\{y_0, y_1, y_2\} \leq 2$ is established as follows. From the definition of $C^2\{y_0, y_1, y_2\}$ the diameter of $C^2\{y_0, y_1, y_2\}$ is not greater than $y_0y_1 + y_1y_2 + y_2y_0$, because $C^2\{y_0, y_1, y_2\}$ is contained in $B_{(y_0y_1 + y_1y_2 + y_2y_0)^{1/2}}(y_0)$. Since $H\{y_0, y_1, y_2\} \supset C^2\{y_0, y_1, y_2\}$, it holds that

$$C^2\{y_0, y_1, y_2\} \subset C^2\{y_0, y_1, m(y_1, y_2)\} \cup C^2\{y_0, y_2, m(y_1, y_2)\}.$$

Hence if we put $y'_0 := m(y_1, y_2)$, $y'_1 := m(y_0, y_2)$, $y'_2 := m(y_0, y_1)$ and $y' := m(y_0, y'_0)$, then

$$C^2\{y_0, y_1, y_2\} \subset C^2\{y_0, y'_1, y'\} \cup C^2\{y_0, y', y'_2\} \cup C^2\{y'_1, y_2, y'_0\} \\ \cup C^2\{y'_1, y'_0, y'\} \cup C^2\{y', y'_0, y'_2\} \cup C^2\{y'_2, y'_0, y_1\},$$

and the diameter of each $C^2\{\}$ on the right hand side are not greater than $((1+\alpha)/2)(y_0y_1 + y_1y_2 + y_2y_0)$ (by Lemma 3). If we repeat this $(n-1)$ times for each $C^2\{\}$ of the right hand side, then we obtain $6^n C^2\{\}$'s and their diameters are not greater than $((1+\alpha)/2)^n (y_0y_1 + y_1y_2 + y_2y_0)$. Hence for given $\varepsilon > 0$ there is an n_0 such that $n \geq n_0$ implies $((1+\alpha)/2)^n (y_0y_1 + y_1y_2 + y_2y_0) < \varepsilon$. Since $m_s(C^2\{y_0, y_1, y_2\}) \leq \sum [\delta(C^2\{\})]^s \leq 6^n [((1+\alpha)/2)^n (y_0y_1 + y_1y_2 + y_2y_0)]^s$ for $n \geq n_0$, we get $m_s(C^2\{y_0, y_1, y_2\}) = 0$. By the fact introduced in § 1, $\dim C^2\{y_0, y_1, y_2\} \leq 2$. The proof is complete.

4. Remarks.

If we try to describe Theorem 1 with only convex combinations we have Corollary 1. This is because $C^{k+1}U = C^kU$ for every subset U of $B_r(p)$ in M means $HU = C^kU$.

COROLLARY 1. *Let $\dim M$ be greater than 3. If for each point p in M there is a convex neighborhood V of p in M such that $C^3\{x_0, x_1, x_2, x_3\} = C^2\{x_0, x_1, x_2, x_3\}$ for any points x_0, x_1, x_2, x_3 in V , then M is a space of constant curvature.*

The following corollary is evident by the fact that $H\{y_0, y_1, y_2\} \supset C^2\{y_0, y_1, y_2\}$ for any y_0, y_1 and y_2 in $B_r(p) \subset M$. Moreover it is directly proved by the same way as in the proof of Theorem 2.

COROLLARY 2. *Let $\dim M$ be greater than 2. If for each point p in M there is a convex neighborhood V of p in M such that $H\{x_0, x_1, x_2\} = H\{x_0, x_1, m(x_1, x_2)\} \cup H\{x_0, x_2, m(x_1, x_2)\}$ for any points x_0, x_1 and x_2 in V , then M is a*

space of constant curvature.

It is natural to ask whether $H\{x_0, x_1, x_2\}$ in the assumption of Theorem 2 could be replaced by $C^2\{x_0, x_1, x_2\}$. On this question we show the following.

THEOREM 3. *Let $\dim M$ be greater than 2. If for each p in M there is a convex neighborhood V of p in M such that $C^2\{x_0, x_1, x_2\} = C^2\{x_0, x_1, x\} \cup C^2\{x_0, x_2, x\}$ for any points x_0, x_1 and x_2 in V and for any point x in the segment $T(x_1, x_2)$, then M is a space of constant curvature.*

If it is possible to replace x in the assumption with $m(x_1, x_2)$, then this theorem is stronger than Theorem 2. However the author does not know the possibility.

We prepare a lemma.

LEMMA 4. *Let M satisfy the assumption in Theorem 3. Let $x(\tau)$, $0 \leq \tau \leq \alpha$, and $y(\nu)$, $0 \leq \nu \leq \beta$, represent segments $T_1 = T(y_0, y_1)$ and $T_2 = T(y_0, y_2)$ respectively in $B_r(p) \subset V$ and let $t := \max_{\mu \in [0, 1]} x(\alpha\mu)y(\beta\mu)$. If the $3r$ -ball with center p is strongly convex, then $C^2\{y_0, y_1, y_2\}$ is contained in the union of the closed t -neighborhood of T_1 and the closed t -neighborhood of T_2 in M .*

PROOF OF LEMMA 4. Choose a partition $0 = \mu_0 < \mu_1 < \dots < \mu_n = 1$ of $[0, 1]$ such that $\alpha(\mu_i - \mu_{i-1}) < t$ and $\beta(\mu_i - \mu_{i-1}) < t$ for $1 \leq i \leq n$. Then $C^2\{x(\alpha\mu_{i-1}), x(\alpha\mu_i), y(\beta\mu_{i-1})\} \subset H\{x(\alpha\mu_{i-1}), x(\alpha\mu_i), y(\beta\mu_{i-1})\} \subset \overline{B_t(x(\alpha\mu_{i-1}))}$ and $C^2\{x(\alpha\mu_i), y(\beta\mu_{i-1}), y(\beta\mu_i)\} \subset \overline{B_t(y(\beta\mu_i))}$ for every $1 \leq i \leq n$, because for every $1 \leq i \leq n$ $\overline{B_t(x(\alpha\mu_{i-1}))}$ and $\overline{B_t(y(\beta\mu_i))}$ are contained in $B_{3r}(p)$ and hence are convex. Since $C^2\{y_0, y_1, y_2\} \subset \bigcup_{i=1}^n C^2\{x(\alpha\mu_{i-1}), x(\alpha\mu_i), y(\beta\mu_{i-1})\} \cup C^2\{x(\alpha\mu_i), y(\beta\mu_{i-1}), y(\beta\mu_i)\}$, $C^2\{y_0, y_1, y_2\}$ is contained in the union of the closed t -neighborhood of T_1 and the closed t -neighborhood of T_2 in M .

PROOF OF THEOREM 3. Let x_0, x_1 and x_2 be any points in $B_r(p) \subset V$ where r is a positive such that the $3r$ -ball with center p is strongly convex. Let S be the set of all points each of which belongs to the segment $T(x_0, x)$ for some x in $T(x_1, x_2)$. $S \subset C^2\{x_0, x_1, x_2\}$ is clear. We claim $S = C^2\{x_0, x_1, x_2\}$. In fact, suppose there exists a point $z \in C^2\{x_0, x_1, x_2\} - S$. Let s be the distance between z and S . Since S is closed we have $s > 0$. Choose a partition $x_1 = z_1, z_2, \dots, z_n = x_2$ of $T(x_1, x_2)$ in this order such that if $z_i(\tau)$, $0 \leq \tau \leq \alpha_i$, represents the segment $T(x_0, z_i)$ for each $1 \leq i \leq n$, and if we put $t_i = \max_{\mu \in [0, 1]} z_i(\alpha_i\mu)z_{i+1}(\alpha_{i+1}\mu)$ for each $1 \leq i \leq n-1$, then $t_i < s$ for all $1 \leq i \leq n-1$. By Lemma 4 and the assumption $C^2\{x_0, x_1, x_2\}$ is contained in the open s -neighborhood of S in M , a contradiction.

Next we assert $H\{x_0, x_1, x_2\} = C^2\{x_0, x_1, x_2\}$. Let z and y be any points of $C^2\{x_0, x_1, x_2\}$. Then by the above argument there are points z' and y' in $T(x_1, x_2)$ such that $z \in T(x_0, z')$ and $y \in T(x_0, y')$. Since $C^2\{x_0, x_1, x_2\} = C^2\{x_0, x_1, z'\} \cup C^2\{x_0, z', y'\} \cup C^2\{x_0, x_2, y'\}$, where we assume without loss of generality that

x_1, z', y' and x_2 are in this order on $T(x_1, x_2)$, and since $T(z, y)$ is contained in $C^2\{x_0, z', y'\}$, $T(z, y)$ is contained in $C^2\{x_0, x_1, x_2\}$, which implies the convexity of $C^2\{x_0, x_1, x_2\}$.

Thus $H\{x_0, x_1, x_2\} = C^2\{x_0, x_1, x_2\} = S$. Then we conclude $\dim H\{x_0, x_1, x_2\} \cong 2$, and hence we obtain our theorem by Lemma 1.

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