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# Residues of complex analytic foliation singularities

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In [3], Baum and Bott defined the residues of complex analytic foliation singularities and proved a general residue formula using differential geometry based on the Bott vanishing theorem. Let $M$ be a complex manifold. We define a foliation (of complete intersection type) on $M$ to be a locally free subsheaf $F$ of the cotangent sheaf $\Omega_M$ which satisfies the Frobenius integrability condition outside of the singular set (=the singular set of the coherent sheaf $\Omega_F=\Omega_M/F$). In this note, we express ((3.4) Theorem) a certain class of residues of $F$ in terms of the Chern classes of $F$ and the local Chern classes of the sheaf $\mathcal{E}xt^1_{\mathcal{O}}(\Omega_F, \mathcal{O})$, which appeared in the unfolding theory ([7]). As a corollary, the rationality of these residues is shown (cf. [3] p. 287 Rationality Conjecture). In a number of cases, the Riemann-Roch theorem for analytic embeddings (Atiyah-Hirzebruch [2]) can be used to compute the residues. The results of this paper were announced in [9].

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## 1. Residues.

We briefly review how the residues are defined in Baum-Bott [3]. Let $M$ be an $n$-dimensional complex manifold. We denote by $\mathcal{O}_M$ (or simply by $\mathcal{O}$), $\Theta_M$ and $\Omega_M$, respectively, the structure sheaf, the tangent sheaf and the cotangent sheaf of $M$. In [3] pp. 281-282, a foliation is defined to be a full integrable coherent subsheaf $\xi$ of $\Theta_M$. Let $Q$ be the quotient sheaf $\Theta_M/\xi$;

$$(1.1) \qquad 0 \longrightarrow \xi \longrightarrow \Theta_M \longrightarrow Q \longrightarrow 0\,.$$

The singular set $S$ of the foliation is defined by

$$(1.2) \qquad S=\{z\in M \mid Q_z \text{ is not a free } \mathcal{O}_z\text{-module}\},$$

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where for a sheaf $\mathcal{S}$ on $M$, $\mathcal{S}_z$ denotes the stalk of $\mathcal{S}$ over $z$. The sheaf $\xi$ defines an ordinary foliation on $M-S$, whose codimension is denoted by $q$. Let $Z$ be a connected component of $S$ and assume that $Z$ is compact. Take an open neighborhood $U$ of $Z$ in $M$ such that $Z$ is a deformation retract of $U$. Let $\sigma_1, \cdots, \sigma_n$ be the elementary symmetric functions in $n$ variables $X_1, \cdots, X_n$. On $U-Z$, the sheaf $Q$ is locally free and it admits a basic connection $D_{-1}$, which determines a closed $2i$-form $\sigma_i(K_{-1})$ on $U-Z$ for each $i$, $1 \leqq i \leqq n$. There exists a closed $2i$-form $\omega_i$ on $U$ which coincides with $\sigma_i(K_{-1})$ outside of a compact set in $U$ containing $Z$ in its interior (cf. [3] p. 312 Proof of (0.23)).

If $\phi$ is a symmetric and homogeneous polynomial of degree $d$ in $X_1, \cdots, X_n$, there is a polynomial $\tilde{\phi}$ in $\sigma_1, \cdots, \sigma_n$ with $\phi = \tilde{\phi}(\sigma_1, \cdots, \sigma_n)$. We set $\phi(Q) = (\sqrt{-1}/2\pi)^d \tilde{\phi}(\omega_1, \cdots, \omega_n)$, which is a closed $2d$-form on $U$. Note that in [3], the cohomology class of $\phi(Q)$ is denoted by $\phi(Q)$, however here the form itself is denoted by $\phi(Q)$. If $d > q$, then by the Bott vanishing theorem ([3](3.27)), $\phi(Q)$ has compact support and defines a cohomology class $[\phi(Q)]$ in $H_c^{2d}(U; \boldsymbol{C})$ (cohomology with compact support). We denote by $L$ the composition of the two isomorphisms

$$(1.3) \qquad H_c^{2d}(U;\boldsymbol{C}) \xrightarrow[\sim]{D_U} H_{2n-2d}(U;\boldsymbol{C}) \xrightarrow[\sim]{i_*^{-1}} H_{2n-2d}(Z;\boldsymbol{C}),$$

where $D_U$ denotes the Poincaré duality map and $i$ is the embedding $Z \hookrightarrow U$. Then the residue is defined by

$$\mathrm{Res}_\phi(\xi, Z) = L([\phi(Q)]).$$

## 2. The sheaf $\mathcal{E}xt^1_{\mathcal{O}}(\Omega_F, \mathcal{O})$.

In [7](1.2), a (reduced) foliation is defined to be a full coherent subsheaf $F$ of $\Omega_M$ satisfying the integrability condition. Let $\Omega_F$ be the quotient sheaf $\Omega_M/F$;

$$(2.1) \qquad 0 \longrightarrow F \longrightarrow \Omega_M \longrightarrow \Omega_F \longrightarrow 0.$$

The two definitions are equivalent if we set ([7](1.5)) $\xi = F^a = \{\theta \in \Theta_M \mid \omega(\theta) = 0, \forall \omega \in F\}$ or $F = \xi^a = \{\omega \in \Omega_M \mid \omega(\theta) = 0, \forall \theta \in \xi\}$. Note that $F^a$ is identical with the dual sheaf $\mathcal{H}om_{\mathcal{O}}(\Omega_F, \mathcal{O})$ of $\Omega_F$. The singular set $S(F)$ of $F$ is defined by

$$(2.2) \qquad S(F) = \{z \in M \mid \Omega_{F,z} \text{ is not a free } \mathcal{O}_z\text{-module}\}$$

and is identical with $S$ in (1.2). By taking the duals of (2.1), we obtain the exact sequence

$$(2.3) \qquad 0 \to \underset{\substack{\| \\ F^a}}{\mathcal{H}om_{\mathcal{O}}(\Omega_F, \mathcal{O})} \to \underset{\substack{\| \\ \Theta_M}}{\mathcal{H}om_{\mathcal{O}}(\Omega_M, \mathcal{O})} \to \underset{\substack{\| \\ F^*}}{\mathcal{H}om_{\mathcal{O}}(F, \mathcal{O})} \to \mathcal{E}xt^1_{\mathcal{O}}(\Omega_F, \mathcal{O}) \to 0.$$

By (2.2), the support of the sheaf $\mathcal{E}xt^1_{\mathcal{O}}(\Omega_F, \mathcal{O})$ is in $S$. Comparing (1.1) and (2.3), we get the exact sequence

$$0 \longrightarrow Q \longrightarrow F^* \longrightarrow \mathcal{E}xt^1_{\mathcal{O}}(\Omega_F, \mathcal{O}) \longrightarrow 0 . \tag{2.4}$$

From now on we consider only foliations of complete intersection type ([7](1.10)), i.e., we assume that $F$ is a locally free $\mathcal{O}$-module (of rank $q$). We do not distinguish locally free sheaves from holomorphic vector bundles. Thus (2.4) can be viewed as a "decomposition" of the sheaf $Q$ into the vector bundle part $F^*$ and the singular part $\mathcal{E}xt^1_{\mathcal{O}}(\Omega_F, \mathcal{O})$.

### 3. Residues and the local Chern classes of $\mathcal{E}xt^1_{\mathcal{O}}(\Omega_F, \mathcal{O})$.

Let $F$ be a codim $q$ foliation (of complete intersection type) and let $Z$ be a compact connected component of the singular set $S$. In this section, analytic objects on $M$ are restricted to the open set $U$ considered in section 1. Since $\mathcal{E}xt^1_{\mathcal{O}}(\Omega_F, \mathcal{O})$ is a coherent sheaf on $U$ with support in $Z$, there is the associated "Grothendieck element" $\gamma_Z(\mathcal{E}xt^1_{\mathcal{O}}(\Omega_F, \mathcal{O}))$, which we simply denote by $\mathcal{E}$, in $K^0(U, U-Z)$ ([1] § 4, [4] Ch. I, cf. also [5]). The Chern character gives a mapping

$$\mathrm{ch}: K^0(U, U-Z) \longrightarrow H^*(U, U-Z\,;\, \boldsymbol{Q}) .$$

Since $Z$ is a deformation retract of $U$, there is a canonical isomorphism

$$H^*(U, U-Z\,;\, \boldsymbol{Q}) \xrightarrow{\sim} H^*_c(U\,;\, \boldsymbol{Q}) .$$

Also there is a canonical homomorphism

$$\kappa : H^*_c(U\,;\, \boldsymbol{Q}) \longrightarrow H^*(U\,;\, \boldsymbol{Q}) . \tag{3.1}$$

Thus $\mathrm{ch}(\mathcal{E})$ determines the local Chern classes $c_1(\mathcal{E}), \cdots, c_n(\mathcal{E})$ in $H^*(U, U-Z\,;\, \boldsymbol{Q}) = H^*_c(U\,;\, \boldsymbol{Q})$ such that $1+\kappa(c_1(\mathcal{E}))+\cdots+\kappa(c_n(\mathcal{E}))$ is the total Chern class of the coherent sheaf $\mathcal{E}xt^1_{\mathcal{O}}(\Omega_F, \mathcal{O})$ on $U$. For each integer $k$ with $1 \leqq k \leqq n$, we set

$$d_k(\mathcal{E}) = \sum_{r=1}^{k} (-1)^r \sum_{\substack{j_1+\cdots+j_r=k \\ j_\nu>0}} c_{j_1}(\mathcal{E}) \cdots c_{j_r}(\mathcal{E}) . \tag{3.2}$$

Let $c(F^*)=1+c_1(F^*)+\cdots+c_n(F^*)$ be the (rational) total Chern class in $H^*(U\,;\, \boldsymbol{Q})$ of $F^*$. Note that $c_i(F^*)=0$, $q+1 \leqq i \leqq n$, since $F^*$ is a locally free sheaf of rank $q$. Also note that there is a canonical pairing

$$H^*(U\,;\, \boldsymbol{Q}) \times H^*_c(U\,;\, \boldsymbol{Q}) \longrightarrow H^*_c(U\,;\, \boldsymbol{Q}) .$$

(3.3) DEFINITION. For each integer $j$ with $q<j \leqq n$, $c_j(F^*-\mathcal{E})$ denotes the element

$$c_q(F^*)d_{j-q}(\mathcal{E})+\cdots+c_1(F^*)d_{j-1}(\mathcal{E})+d_j(\mathcal{E})$$

in $H_c^{2j}(U; \boldsymbol{Q})$, and for each integer $j$ with $1 \leqq j \leqq q$, it denotes the element

$$c_j(F^*)+c_{j-1}(F^*)\kappa(d_1(\mathcal{E}))+\cdots+c_1(F^*)\kappa(d_{j-1}(\mathcal{E}))+\kappa(d_j(\mathcal{E}))$$

in $H^{2j}(U; \boldsymbol{Q})$.

(3.4) THEOREM. *Let $F$ be a foliation (of complete intersection type) of codim $q$ on $M$ and let $U$ and $Z$ be as above. If $\phi=\sigma_{j_1}\cdots\sigma_{j_r}$ with $j_\nu>q$ for some $\nu$, then*

$$\mathrm{Res}_\phi(F, Z)=L(c_{j_1}(F^*-\mathcal{E})\cdots c_{j_r}(F^*-\mathcal{E}))\,,$$

*where* $\mathrm{Res}_\phi(F, Z)=\mathrm{Res}_\phi(F^a, Z)$ *and $L$ is the composition of two isomorphisms in* (1.3).

PROOF. Let $D_{-1}$ be a basic connection for $Q$ on $U-Z$. Since $Q=F^*$ on $U-Z$ and $F^*$ is locally free on $U$, by [3] (4.41), the connection $D_{-1}$ can be modified to obtain a connection $\tilde{D}_{-1}$ for $F^*$ on $U$ such that

$$\tilde{D}_{-1}=D_{-1} \quad \text{on} \quad U-\Sigma\,, \tag{3.5}$$

where $\Sigma$ is a compact set in $U$ containing $Z$ in its interior. The connection $\tilde{D}_{-1}$ determines, for each $i$ with $1\leqq i\leqq q$, a closed $2i$-form $\sigma_i(F^*)$ on $U$ such that the class of $(\sqrt{-1}/2\pi)^i\sigma_i(F^*)$ in $H^*(U; \boldsymbol{C})$ is $c_i(F^*)$. The equation

$$\begin{aligned}(1+\sigma_1(Q)+\cdots+\sigma_n(Q))(1+\sigma_1(\mathcal{E})+\cdots+\sigma_n(\mathcal{E}))\\ =1+\sigma_1(F^*)+\cdots+\sigma_q(F^*)\end{aligned} \tag{3.6}$$

can be solved to find $\sigma_1(\mathcal{E}), \cdots, \sigma_n(\mathcal{E})$ such that, for each $j$, $1\leqq j\leqq n$, $\sigma_j(\mathcal{E})$ is a closed $2j$-form on $U$. By (3.5), $\sigma_j(Q)=\sigma_i(F)$, $1\leqq i\leqq q$ on $U-\Sigma$. Also by the Bott vanishing theorem, $\sigma_{q+1}(Q), \cdots, \sigma_n(Q)$ have compact support. Hence each $\sigma_j(\mathcal{E})$, $1\leqq j\leqq n$, has compact support. Moreover, the class of $(\sqrt{-1}/2\pi)^j\sigma_j(\mathcal{E})$ in $H_c^*(U; \boldsymbol{C})$ is $c_j(\mathcal{E})$. If we set, for $k=1, \cdots, n$,

$$\tau_k(\mathcal{E})=\sum_{r=1}^{k}(-1)^r\sum_{\substack{j_1+\cdots+j_r=k\\ j_\nu>0}}\sigma_{j_1}(\mathcal{E})\cdots\sigma_{j_r}(\mathcal{E})\,,$$

then we have $(1+\tau_1(\mathcal{E})+\cdots+\tau_n(\mathcal{E}))(1+\sigma_1(\mathcal{E})+\cdots+\sigma_n(\mathcal{E}))=1$. From (3.6), we have

$$\sigma_j(Q)=\sum_{\substack{i+k=j\\ i,k\geqq 0}}\sigma_i(F^*)\tau_k(\mathcal{E})\,, \qquad j=1, \cdots, n\,, \tag{3.7}$$

where we set $\sigma_0(F^*)=\tau_0(\mathcal{E})=1$ and $\sigma_{q+1}(F^*)=\cdots=\sigma_n(F^*)=0$. Thus if $j>q$, each term in the right hand side of (3.7) has compact support and the class of $\left(\frac{\sqrt{-1}}{2\pi}\right)^j\sigma_j(Q)$ in $H_c^{2j}(U; \boldsymbol{Q})$ is $c_j(F^*-\mathcal{E})$ (see (3.3) Definition). Therefore, if $\phi=\sigma_{j_1}\cdots\sigma_{j_r}$ with $j_\nu>q$ for some $\nu$, then $c_{j_1}(F^*-\mathcal{E})\cdots c_{j_r}(F^*-\mathcal{E})$ is in $H_c^{2j}(U; \boldsymbol{Q})$, $j=j_1+\cdots+j_r$, and is the class of $\phi(Q)$, Q.E.D.

(3.8) COROLLARY. *Let $F$ and $Z$ be as above and let $\phi$ be a symmetric and homogeneous polynomial of degree $d$ in $X_1, \cdots, X_n$. If each monomial in the expression $\phi=\tilde{\phi}(\sigma_1, \cdots, \sigma_n)$ contains $\sigma_j$ with $j>q$, then $\mathrm{Res}_\phi(F, Z)$ is rational, i.e. it is in $H_{2n-2d}(Z; \boldsymbol{Q})$* (cf. [3] p. 287 Rationality Conjecture).

Suppose now that $Z$ is non-singular and that there is a holomorphic vector bundle $E$ on $Z$ such that $\mathcal{E}xt^1_{\mathcal{O}}(\Omega_F, \mathcal{O})=i_!\mathcal{O}_Z(E)$ (=the sheaf $\mathcal{O}_Z(E)$ of germs of holomorphic sections of $E$ extended by zero on $U-Z$), where $i$ is the embedding $Z\hookrightarrow U$. Then (the finer version of) the Riemann-Roch theorem for analytic embeddings (Atiyah-Hirzebruch [2] Theorem (3.1), see also the proof of Theorem (3.3)) gives the local Chern classes of $\mathcal{E}=\gamma_Z(\mathcal{E}xt^1_{\mathcal{O}}(\Omega_F, \mathcal{O}))$;

(3.9)
$$\mathrm{ch}(\mathcal{E})=i_*(\mathrm{td}(N)^{-1}\mathrm{ch}(E))$$

or

(3.10)
$$c_1(\mathcal{E})+\cdots+c_n(\mathcal{E})=i_*\left(\frac{c(\lambda_{-1}(N^*))*c(E)-1}{c_r(N)}\right),$$

where $N$ is the normal bundle of $Z$ in $U$, $r=\mathrm{rank}\, N=\mathrm{codim}\, Z$ in $U$, $\lambda_{-1}(N^*) =\sum_{i=0}^{r}(-1)^i\lambda^i(N^*)$ ($\lambda^i(N^*)=i$-th exterior power of $N^*$), $c(\lambda_{-1}(N^*))*c(E)$ is the total Chern class of the tensor product $\lambda_{-1}(N^*)\otimes E$ and $i_*$ is the Thom-Gysin homomorphism

(3.11)
$$i_*: H^*(Z; \boldsymbol{Q}) \longrightarrow H^*(U, U-Z; \boldsymbol{Q})=H^*_c(U; \boldsymbol{Q}).$$

By our assumption, $Z$ is non-singular. Thus we have a commutative diagram

$$\begin{array}{ccc} H^p(Z; \boldsymbol{Q}) & \xrightarrow{\ i_*\ } & H^{p+2r}_c(U; \boldsymbol{Q}) \\ D_Z \downarrow \wr & \swarrow L & \wr \downarrow D_U \\ H_{2n-2r-p}(Z; \boldsymbol{Q}) & \xrightarrow[\sim]{\ i_*\ } & H_{2n-2r-p}(U; \boldsymbol{Q}), \end{array}$$

where $D_Z$ is the Poincaré duality map, and $i_*$ in (3.11) is an isomorphism.

In particular, if the singularity is isolated, we have

(3.12) PROPOSITION. *Let $U$ be a polydisk about the origin $0$ in $\boldsymbol{C}^n$ and let $F=(\omega)$ be a codim 1 foliation on $U$ with an isolated singularity at $0$. We denote the stalks $\mathcal{O}_{\boldsymbol{C}^n,0}$ and $\Omega_{F,0}$ simply by $\mathcal{O}$ and $\Omega_F$, respectively. Then we have*

$$\mathrm{Res}_{\sigma_n}(F, \{0\})=(-1)^n(n-1)!\dim_{\boldsymbol{C}}\mathrm{Ext}^1_{\mathcal{O}}(\Omega_F, \mathcal{O}) \quad \text{in} \quad H_0(\{0\}; \boldsymbol{Q})=\boldsymbol{Q}.$$

PROOF. Since $H^{2j}_c(U; \boldsymbol{Q})=0$ for $j\neq n$, we have $c_j(\mathcal{E})=0$ for $1\leqq j\leqq n$. Also $c_i(F^*)=0$ for $i>0$. Hence by (3.4) Theorem and (3.2), we have

$$\mathrm{Res}_{\sigma_n}(F, \{0\})=L(d_n(\mathcal{E}))=-L(c_n(\mathcal{E})).$$

On the other hand, for a point $z$ in $U$,

$$\mathcal{E}xt^1_{\mathcal{O}}(\Omega_F, \mathcal{O})_z = \begin{cases} \mathrm{Ext}^1_{\mathcal{O}}(\Omega_F, \mathcal{O}), & \text{if } z=0, \\ 0 \quad , & \text{if } z \neq 0. \end{cases}$$

We set $E=\mathrm{Ext}^1_{\mathcal{O}}(\Omega_F, \mathcal{O})$ and think of it as a vector bundle over $Z=\{0\}$ of rank $\mu=\dim_{\boldsymbol{C}} E$. Then we have $\mathcal{E}xt^1_{\mathcal{O}}(\Omega_F, \mathcal{O})=i_!\mathcal{O}_Z(E)$. In (3.9), we have $\mathrm{td}(N)=1$ and $\mathrm{ch}(E)=\mu$ in $H^0(\{0\}\,;\,\boldsymbol{Q})=\boldsymbol{Q}$. Thus denoting by $\theta$ the image of 1 by the isomorphism $i_*: H^0(\{0\}\,;\,\boldsymbol{Q}) \to H^{2n}_c(U\,;\,\boldsymbol{Q})$, we have

$$\mathrm{ch}(\mathcal{E})=\mu\theta. \tag{3.13}$$

Writing formally $1+c_1(\mathcal{E})+\cdots+c_n(\mathcal{E})=\prod_{i=1}^n(1+\gamma_i)$, we have $\mathrm{ch}(\mathcal{E})=\sum_{i=1}^n(e^{\gamma_i}-1)$. From (3.13),

$$\gamma_1^j+\cdots+\gamma_n^j=\begin{cases} 0, & \text{if } 1\leqq j\leqq n-1, \\ n!\,\mu\theta, & \text{if } j=n. \end{cases}$$

Thus we have $n\gamma_1\cdots\gamma_n+(-1)^n(\gamma_1^n+\cdots+\gamma_n^n)=0$. Hence $c_n(\mathcal{E})=\gamma_1\cdots\gamma_n=(-1)^{n+1}(n-1)!\,\mu\theta$. Q. E. D.

(3.14) REMARK. In the situation of (3.12), if we write $\omega=\sum_{i=1}^n f_i(z)dz_i$, then

$$\mathrm{Ext}^1_{\mathcal{O}}(\Omega_F, \mathcal{O})=\boldsymbol{C}\{z_1, \cdots, z_n\}/(f_1, \cdots, f_n),$$

where $\mathcal{O}=\boldsymbol{C}\{z_1, \cdots, z_n\}$ is the ring of convergent power series in $z_1, \cdots, z_n$ and $(f_1, \cdots, f_n)$ is the ideal generated by the germs of $f_1(z), \cdots, f_n(z)$ at 0 ([7] (4.5)). Especially, if $\omega=df$ for some $f$, then $f_i=\dfrac{\partial f}{\partial z_i}$. Thus (3.12) can be viewed as a formula for the "generalized" multiplicity (cf. [6]). For the significance of $\mathrm{Ext}^1_{\mathcal{O}}(\Omega_F, \mathcal{O})$, see also [8].

Here is an example with non-isolated singular set.

(3.15) EXAMPLE. Let $\boldsymbol{P}^1=\boldsymbol{P}^1(\boldsymbol{C})$ be the projective line with homogeneous coordinates $(\zeta_0:\zeta_1)$. It is covered by two coordinate neighborhoods $U_0$ and $U_1$ with coordinates $z_0=\zeta_1/\zeta_0$ and $z_1=\zeta_0/\zeta_1$, respectively. We denote by $H$ the hyperplane bundle over $\boldsymbol{P}^1$. Letting $l$ and $m$ be two integers, consider the vector bundle $N$ of rank 2 over $\boldsymbol{P}^1$ given by $N=H^l\oplus H^m$. Thus $N$ can be expressed as a union $N=\boldsymbol{C}^2\times U_0\cup\boldsymbol{C}^2\times U_1$, where a point $(x_0, y_0, z_0)$ in $\boldsymbol{C}^2\times U_0$ is identified with $(x_1, y_1, z_1)$ in $\boldsymbol{C}^2\times U_1$ if and only if

$$x_0=z_1^{-l}x_1, \quad y_0=z_1^{-m}y_1 \quad \text{and} \quad z_0=z_1^{-1}. \tag{3.16}$$

We identify $\boldsymbol{P}^1$ with the zero section $x_i=y_i=0$, $i=0, 1$, in $N$. Let $a$ and $b$ be positive integers satisfying $l(a-1)=m(b-1)$. We set $r=l(a-1)=m(b-1)$. C: each $W_i=\boldsymbol{C}^2\times U_i$, $i=0, 1$, we consider two holomorphic 1-forms $\tau_i$ and $\omega_i$ given by

$$\tau_i = dz_i \quad \text{and} \quad \omega_i = y_i^b dx_i - x_i^a dy_i .$$

In the intersection $W_0 \cap W_1$, we have

$$(3.17) \qquad \begin{pmatrix} \tau_0 \\ \omega_0 \end{pmatrix} = \begin{pmatrix} -z_1^{-2} & 0 \\ x_1 y_1 z_1^{-s-1}(m x_1^{a-1} - l y_1^{b-1}) & z_1^{-s} \end{pmatrix} \begin{pmatrix} \tau_1 \\ \omega_1 \end{pmatrix},$$

where $s=r+l+m$. Thus we may consider the locally free sub-$\mathcal{O}_N$-module $F$ of $\Omega_N$ generated by $\tau_i$ and $\omega_i$ on $W_i$. Clearly $F$ satisfies the integrability condition and defines a codim 2 foliation on $N$ with singular set the zero section $\boldsymbol{P}^1$. Now we find the sheaf $\mathcal{E}xt^1_{\mathcal{O}_N}(\Omega_F, \mathcal{O}_N)$. From (2.3), we have

$$\mathcal{E}xt^1_{\mathcal{O}_N}(\Omega_F, \mathcal{O}_N)|_{W_i} \cong \mathcal{O}^2_{W_i}/((0, y_i^b), (0, x_i^a), (1, 0)),$$

where the denominator in the right hand side denotes the sub-$\mathcal{O}_{W_i}$-module of $\mathcal{O}^2_{W_i}$ generated by $(0, y_i^b)$, $(0, x_i^a)$ and $(1, 0)$. Hence we have

$$(3.18) \qquad \mathcal{E}xt^1_{\mathcal{O}_N}(\Omega_F, \mathcal{O}_N)|_{W_i} \cong \mathcal{O}_{W_i}/(x_i^a, y_i^b),$$

where $(x_i^a, y_i^b)$ is the ideal generated by the sections $x_i^a$ and $y_i^b$. For an element $h$ in $\mathcal{O}_{W_i}$, we denote by $[h]$ its class in $\mathcal{O}_{W_i}/(x_i^a, y_i^b)$. The right hand side of (3.18) is a free $\mathcal{O}_{U_i}$-module generated by $[x_i^\alpha y_i^\beta]$, $0 \leqq \alpha \leqq a-1$, $0 \leqq \beta \leqq b-1$. Moreover, by (3.16), we have

$$x_0^\alpha y_0^\beta = z_1^{-(\alpha l + \beta m)} x_1^\alpha y_1^\beta .$$

Hence we may write $\mathcal{E}xt^1_{\mathcal{O}_N}(\Omega_F, \mathcal{O}_N) = i_! \mathcal{O}_{\boldsymbol{P}^1}(E)$, where $E$ is the vector bundle over $\boldsymbol{P}^1$ of rank $ab$ given by

$$E = \bigoplus_{\substack{0 \leqq \alpha \leqq a-1 \\ 0 \leqq \beta \leqq b-1}} H^{\alpha l + \beta m} .$$

We have

$$\mathrm{ch}(E) = \sum_{\substack{0 \leqq \alpha \leqq a-1 \\ 0 \leqq \beta \leqq b-1}} (1+\eta)^{\alpha l + \beta m} = ab(1 + r\eta),$$

where $\eta$ is the first Chern class of $H$ and is a generator of $H^2(\boldsymbol{P}^1; \boldsymbol{Q}) \cong \boldsymbol{Q}$. On the other hand, from $N = H^l \oplus H^m$, we have

$$\mathrm{td}(N)^{-1} = 1 - \frac{l+m}{2}\eta .$$

Hence we have

$$\mathrm{td}(N)^{-1}\mathrm{ch}(E) = \mathrm{ab}\left(1 + \left(r - \frac{l+m}{2}\right)\eta\right).$$

We have the Thom isomorphism $H^p(\boldsymbol{P}^1; \boldsymbol{Q}) \overset{i_*}{\cong} H^{p+4}(N, N-\boldsymbol{P}^1; \boldsymbol{Q})$. Setting $\theta_2 = i_*(1) \in H^4(N, N-\boldsymbol{P}^1; \boldsymbol{Q})$ and $\theta_3 = i_*(\eta) \in H^6(N, N-\boldsymbol{P}^1; \boldsymbol{Q})$, we have from (3.9),

$$\mathrm{ch}(\mathcal{E})=ab\left(\theta_2+\left(r-\frac{l+m}{2}\right)\theta_3\right).$$

Thus we have

$$c_1(\mathcal{E})=0, \quad c_2(\mathcal{E})=-ab\theta_2 \quad \text{and} \quad c_3(\mathcal{E})=ab(2r-(l+m))\theta_3 .$$

From (3.2), we have

$$d_1(\mathcal{E})=0, \quad d_2(\mathcal{E})=ab\theta_2 \quad \text{and} \quad d_3(\mathcal{E})=ab(l+m-2r)\theta_3 .$$

Next we find $c(F^*)$. Since $H^*(N; \boldsymbol{Q}) \overset{i^*}{\simeq} H^*(\boldsymbol{P}^1; \boldsymbol{Q})$, it suffices to find $c(i^*F^*)$. From (3.17), we have $i^*F^*=H^2\oplus H^s$. Thus $c(i^*F^*)=1+(s+2)\eta$. Therefore, $c(F^*)=1+(s+2)\sigma$, where $\sigma$ denotes $i^{*-1}\eta$ and is a generator of $H^2(N; \boldsymbol{Q})\cong\boldsymbol{Q}$. We have

$$\begin{aligned} c_3(F^*-\mathcal{E}) &= c_2(F^*)d_1(\mathcal{E})+c_1(F^*)d_2(\mathcal{E})+d_3(\mathcal{E}) \\ &= ab(s+2)\sigma\theta_2+ab(l+m-2r)\theta_3 \\ &= ab(2(l+m+1)-r)\theta_3 . \end{aligned}$$

Therefore,

$$\mathrm{Res}_{\sigma_3}(F, \boldsymbol{P}^1)=ab(2(l+m+1)-r) \quad \text{in} \quad H_0(\boldsymbol{P}^1; \boldsymbol{Q})=\boldsymbol{Q} .$$

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