

## Projective plane curves and the automorphism groups of their complements

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### 1. Introduction.

Let  $C$  be an irreducible algebraic curve of degree  $d$  on  $P^2 = P^2(C)$  and put  $V = P^2 \setminus C$ . Let  $\mathcal{G}$  be the automorphism group of the algebraic surface  $V$  and  $\mathcal{L}$  the linear part of  $\mathcal{G}$ , i. e.,  $\mathcal{L} = \{T \in \text{Aut}(P^2) \mid T(C) = C\}$ . If  $d=1$ , then  $\mathcal{G}$  is generated by linear transformations and de Jonquières transformations of  $V$  (Nagata [5]); if  $d=2$ , then generators of the similar kind have been found by Gizatullin and Danilov [2]. In this paper we shall study the structure of  $\mathcal{G}$  and at the same time the property of  $C$  in the case when  $d \geq 3$ .

We shall use the following notations in addition to the above ones. Let  $(X, Y, Z)$  be a set of homogeneous coordinates on  $P^2$  and put  $x = X/Z$  and  $y = Y/Z$ . Usually we do not treat the line  $Z=0$ , so we say that for an irreducible polynomial  $f$ , the curve  $Z^d f(X/Z, Y/Z) = 0$  is defined by  $f$ , where  $d = \deg f$ . Especially we denote by  $\Delta$  [resp.  $\Delta_e$ ] the curve defined by  $xy - x^3 - y^3$  [resp.  $y^e - x^d$ , where  $(e, d) = 1$  and  $1 \leq e \leq d - 2$ ]. Let  $M$  be the number of the singular points  $\{P_1, \dots, P_M\}$  of  $C$  and  $\mu: \tilde{C} \rightarrow C$  the normalization of  $C$ . Then let  $N$  denote the number of elements of  $\mu^{-1}(\{P_1, \dots, P_M\})$  and  $g$  the genus of  $\tilde{C}$ . In case  $N=1$ , let  $(e_1, \dots, e_p)$  be the sequence of the multiplicities of all successive infinitely near singular points of  $P_1$ , and put

$$R = d^2 - \sum_{i=1}^p e_i^2 - e_p + 1.$$

Let  $G_a$  and  $G_m$  be the additive and the multiplicative groups respectively.

First we shall prove the following with the help of the Plücker relations.

**PROPOSITION 1.** *Suppose that  $d \geq 3$ . Then the following three conditions are equivalent.*

- (1) *The order of  $\mathcal{L}$  is infinite.*
- (2) *The linear part  $\mathcal{L}$  is isomorphic to  $G_m$ .*
- (3) *The curve  $C$  is projectively equivalent to  $\Delta_e$ .*

Note that  $\mathcal{L}$  is a finite group if  $C$  is not projectively equivalent to  $\Delta_e$ .

Next we shall consider  $\mathcal{G}$ . Applying the Castelnuovo's criterion for contracting a curve, we shall give the condition that  $\mathcal{G}=\mathcal{L}$ . In case  $\mathcal{G}\neq\mathcal{L}$ , let  $\varphi$  be an element of  $\mathcal{G}\setminus\mathcal{L}$ . Then there is a composition  $\sigma$  of blow-ups such that the induced map  $\varphi\sigma$  is a morphism. Considering the total transform  $\sigma^{-1}(C)$  in detail, we shall prove the following main result.

**THEOREM A.** *The order of  $\mathcal{G}$  is finite if and only if  $C$  satisfies any one of the following conditions (1), (2) and (3).*

- (1)  $g\geq 1$ .
- (2)  $N\geq 2$  and  $C$  is projectively equivalent to neither  $\Delta$  nor  $\Delta_e$ .
- (3)  $N=1$  and  $R\leq -1$ .

On the contrary, for the remaining curves with  $d\geq 3$ ,  $\mathcal{G}$  has the following properties.

- (4) If  $C$  is projectively equivalent to  $\Delta_e$ , where  $e\geq 2$ , then  $\mathcal{G}=\mathcal{L}\cong G_m$ .
- (5) If  $C$  is projectively equivalent to  $\Delta$ , then the order of  $\mathcal{G}$  is countably infinite and  $\mathcal{L}$  is the dihedral group of order 6.
- (6) If  $C$  is a curve with  $g=0$ ,  $N=1$  and  $R\geq 0$ , then  $\mathcal{G}\supset(G_a)^n$  for every positive integer  $n$ . In this class of curves the order of  $\mathcal{L}$  is infinite if and only if  $C$  is projectively equivalent to  $\Delta_1$ .

**REMARK.** We do not know whether or not the curve with the properties  $g=0$ ,  $N=1$ ,  $R=0$  and  $e_{p-1}>e_p$  exists.

The structure of  $\mathcal{G}$  seems to be complicated for the curve  $g=0$ ,  $N=1$  and  $R\geq 0$ . According to Abhyankar-Moh [1], if  $C$  satisfies that  $C\setminus L\cong A^1$  for a line  $L$ , then there is an automorphism of  $P^2\setminus L\cong A^2$  by which  $C$  is transformed to a line  $L_1$ . Hence  $P^2\setminus(C\cup L)$  is isomorphic to  $P^2\setminus(L_1\cup L)$ , this implies that  $C$  is a curve with  $R\geq 2$  if  $d\geq 3$ . Moreover, if the logarithmic Kodaira dimension  $\bar{\kappa}(V)$  is  $-\infty$  and  $d\geq 3$ , then  $C$  is a curve with  $g=0$  and  $N=1$  (Iitaka [4]). By these facts it seems interesting to study the curves of this class.

Now these curves have similarly the following properties, which will be shown in the course of the proof of Theorem A.

**THEOREM B.** *If  $C$  is a curve with  $g=0$ ,  $N=1$  and  $R\geq 0$ , then there are one or two irreducible curves  $C'$  and  $C''$  and two or three lines  $L_i$ , where  $i=1, 2, 3$ , which have the following properties.*

- (1) In case  $R\neq 1$ , there is an isomorphism

$$P^2\setminus(C\cup C')\cong P^2\setminus(L_1\cup L_2).$$

- (2) In case  $R=1$ , there is an isomorphism

$$P^2\setminus(C\cup C'\cup C'')\cong P^2\setminus(L_1\cup L_2\cup L_3),$$

where  $L_1\cap L_2=L_1\cap L_3$ .

We observe that the above curves  $C'$  and  $C''$  have the same properties as the following  $C^*$ .

PROPOSITION 2. *If  $C$  satisfies the conditions  $C \setminus \{P\} \cong A^1$  and  $R \geq 0$ , then there is a curve  $C^*$  having the properties  $C \cap C^* = \{P\}$  and  $C^* \setminus \{P\} \cong A^1$ .*

Note that, in case  $d \geq 3$ , the condition  $C \setminus \{P\} \cong A^1$  is equivalent to the one  $g=0$  and  $N=1$ . Especially Theorem B implies the following

COROLLARY. *If  $C$  satisfies the conditions  $g=0$ ,  $N=1$  and  $R \geq 0$ , then  $\bar{\kappa}(V) = -\infty$ .*

This is a partial answer to the problem raised in [8]. Note that  $\bar{\kappa}(V)$  is not necessarily  $-\infty$  if  $R \leq -1$ . Indeed, there exist curves with  $g=0$ ,  $N=1$ ,  $R \leq -1$  and  $\bar{\kappa}(V)=1$  (Tsunoda [6] or Section 6).

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## 2. Structure of $\mathcal{L}$ .

Thanks to the theory of the logarithmic Kodaira dimension, we have that  $\mathcal{G}$  has a finite order if  $\bar{\kappa}(V)=2$  (Iitaka [4]). Further, Wakabayashi [7] has shown the following

LEMMA 2.1. *If  $C$  satisfies none of the following conditions, then  $\bar{\kappa}(V)=2$ .*

- (a)  $M=0$  and  $d \leq 3$ .
- (b)  $g=0$  and  $M=1$ .
- (c)  $g=0$  and  $M=N=2$ .

Later we shall make use of this lemma. Let  $\sim$  denote the projective equivalence. Then the curves  $\mathcal{A}$  and  $\mathcal{A}_e$  have the following properties.

- (1) If  $C \sim \mathcal{A}_1$ , then  $M=N=1$ ,  $R=d+1$  and  $\bar{\kappa}(V)=-\infty$ .
- (2) If  $C \sim \mathcal{A}_e$  and  $e \geq 2$ , then  $M=N=2$  and  $\bar{\kappa}(V)=1$ .
- (3) If  $C \sim \mathcal{A}$ , then  $M=1$ ,  $N=2$  and  $\bar{\kappa}(V)=0$ .

Let  $e(P, C)$  be the multiplicity of  $C$  at  $P$ , and  $(D_1 \cdot D_2)_P$  the intersection multiplicity of two curves  $D_1$  and  $D_2$  at  $P$ . If  $C_{ij}$  is an analytically irreducible branch of  $C$  at  $P_i$ , where  $1 \leq i \leq M$  and  $j=1, 2, \dots$ , then we put  $e_{ij}=e(P_i, C_{ij})$  and  $\lambda_{ij}=(C_{ij} \cdot L_{ij})_{P_i}$ , where  $L_{ij}$  is the tangent line to  $C_{ij}$  at  $P_i$ . If  $P$  is a flex, then we put  $W=\sum_{P \in \{\text{flexes}\}} \rho_P$ , where  $\rho_P=(C \cdot L_P)_P - 2$  and  $L_P$  is the tangent line to  $C$  at  $P$ . Then we have the following formula which is one of the Plücker relations (Iitaka [4]).

LEMMA 2.2. 
$$W=3d+6(g-1)-\sum_{i,j}(\lambda_{ij}+e_{ij}-3).$$

Applying this formula, we get the following

LEMMA 2.3. *Suppose that  $N=1$  and  $e(P_1, C)=d-1$ . Then  $C$  has one flex if*

and only if  $C \sim \Delta_1$ .

PROOF. If  $C$  has one flex  $Q$ , then by a suitable projective transformation we may assume that  $P_1=(0, 1, 0)$ ,  $Q=(0, 0, 1)$  and that  $C$  is defined by

$$f=y+\sum_{i=1}^d a_i x^i.$$

Since we have that  $W=d-2$  by Lemma 2.2, we get  $a_2=\dots=a_{d-1}=0$ , i.e.,  $f=y+a_d x^d+a_1 x$ , where  $a_d \neq 0$ . This implies that  $C$  is projectively equivalent to  $\Delta_1$ . The "if" part is proved easily by direct computation. Q.E.D.

Now, let us prove Proposition 1.

The implication (3) $\Rightarrow$ (2) is checked by direct computation and the implication (2) $\Rightarrow$ (1) is trivial. So let us prove the implication (1) $\Rightarrow$ (3). Suppose that  $C$  is not projectively equivalent to  $\Delta_e$ . Then, in case  $C$  is smooth, we have that  $W=3d(d-2)$ . Hence there are at least four non-collinear flexes, which implies the order of  $\mathcal{L}$  is finite. Next let us consider the non-smooth case. Then by Lemma 2.1 we have only to treat the curves in the cases (b) and (c). Thus we have that  $g=0$ . Let  $K$  be the number of flexes of  $C$ . Then we note that the order of  $\mathcal{L}$  is finite if  $N+K \geq 3$ . In fact, an element  $T$  of  $\mathcal{L}$  induces the automorphism  $\tilde{T}$  of  $\tilde{C} \cong P^1$  and this correspondence  $T \rightarrow \tilde{T}$  is injective. First let us consider the case (b). In this case, if  $N \leq 2$ , then  $W \geq 1$ , i.e.,  $K \geq 1$ . Suppose that  $N+K \leq 2$ . Then  $K \geq 1$ . Hence we have that  $N=K=1$ , thus  $W \geq d-2$ . Since  $K=1$ , it follows that  $W=d-2$ , which implies that  $e(P_1, C)=d-1$ . This is a contradiction by Lemma 2.3. Next let us consider the case (c). Let  $L_i$  be the tangent line to  $C$  at  $P_i$ , where  $i=1, 2$ . If  $L_1$  contains  $P_2$ , then  $e_2+\lambda_1 \leq d$ ,  $e_1 \leq d-1$  and  $\lambda_2 \leq d$ , hence  $W \geq 1$ . So that we may assume that  $L_1 \not\equiv P_2$  and  $L_2 \not\equiv P_1$ . Moreover we may assume that  $L_i$  intersects  $C$  only in  $P_i$ , and that  $W=0$ , otherwise there is a third fixed point for  $\mathcal{L}$ . Thus we have only to consider the case when  $\lambda_1=\lambda_2=d$  and  $e_1+e_2=d$ . In case  $e_1=e_2=e$ , seeing from that  $C$  is analytically irreducible at  $P_1$  and  $P_2$ , we infer that the multiplicities of infinitely near singular points to  $P_1$  and  $P_2$  of order one must be both  $e$ . Then we have that  $(d-1)(d-2) < 4e(e-1)$ , this contradicts the genus formula for plane curves. Therefore  $T$  fixes  $L_1$ ,  $L_2$  and the line passing through  $P_1$  and  $P_2$ . By a suitable projective transformation these lines are assumed to be  $Y=0$ ,  $Z=0$  and  $X=0$  respectively. Then the equation of  $C$  is

$$f=y^e+\sum^* c_{ij} x^i y^j - x^d,$$

where  $e=e_1$  and  $\sum^*$  denotes the summation for  $i$  and  $j$  satisfying  $d > i+j > e > j > 0$ . Suppose that  $(d-i)(e-j)=ij$  for all  $i, j$  in this equation. Then putting  $b=(e, d)$ ,  $e=be'$  and  $d=bd'$ , we have that  $bd'e'=ie'+jd'$ . Moreover putting  $i=d'i'$  and  $j=e'j'$ , where  $i'+j'=b$ , we see that  $f$  is a homogeneous polynomial with variables  $x^{d'}$  and  $y^{e'}$ . Hence  $f$  is factored as

$$\prod_{i=1}^b (y^{e'} + \alpha_i x^{d'})$$

for some  $\alpha_i$ , where  $1 \leq i \leq b$ . Since  $C$  is irreducible,  $b$  is 1, hence  $C \sim \Delta_e$ . This contradicts our hypothesis, so there are  $i$  and  $j$  such that  $k = (d-i)(e-j) - ij \neq 0$ . Since  $T$  is represented as a diagonal matrix

$$\begin{pmatrix} \alpha & & \\ & \beta & \\ & & 1 \end{pmatrix},$$

we have that  $\beta^e = \alpha^i \beta^j = \alpha^d$ , i. e.,  $\alpha^k = 1$ . Whence  $\mathcal{L}$  is a finite group. Q. E. D.

### 3. Relation between $\mathcal{G}$ and $\mathcal{L}$ .

Since an element  $\varphi$  of  $\mathcal{G}$  is the restricted mapping of a birational transformation of  $\mathbf{P}^2$ , let us denote by  $\varphi$  also the birational transformation. Let

$$S_r \xrightarrow{\sigma_r} S_{r-1} \xrightarrow{\sigma_{r-1}} \dots \xrightarrow{\sigma_2} S_1 \xrightarrow{\sigma_1} S_0 = \mathbf{P}^2$$

be a finite sequence of blow-ups  $\sigma_i$  with successive centers  $Q_i$  in  $S_{i-1}$ , where  $1 \leq i \leq r$  and  $S_0 = \mathbf{P}^2$ , and put  $\sigma = \sigma_1 \cdots \sigma_r$ . For a birational transformation  $\psi$  we denote by  $\psi(A)$  and  $\psi[A]$  the total and the proper transforms of  $A$  respectively.

LEMMA 3.1. *Let  $\varphi$  be an element of  $\mathcal{G}$ . Then the following assertions hold true.*

- (1) *Each birational transformation  $\varphi$  or  $\varphi \sigma_1 \cdots \sigma_i$  has at most one fundamental point, where  $1 \leq i \leq r$ .*
- (2) *The proper transform  $\varphi[C]$  is  $C$  [resp. one point] if and only if  $\varphi$  belongs to  $\mathcal{L}$  [resp.  $\mathcal{G} \setminus \mathcal{L}$ ].*

PROOF. Note that if  $\varphi$  has no fundamental points, then  $\varphi$  is a birational morphism from  $\mathbf{P}^2$  to  $\mathbf{P}^2$  and so is an isomorphism. Since  $C$  is irreducible and  $\varphi$  is an automorphism of  $\mathbf{P}^2 \setminus C$ , both assertions are proved readily. Q. E. D.

Put  $C_i = (\sigma_1 \cdots \sigma_i)^{-1}[C]$  and  $C_0 = C$ , where  $1 \leq i \leq r$ . Let  $D_1 \cdot D_2$  denote the intersection number of two curves  $D_1$  and  $D_2$  on some nonsingular complete surface. In case  $D_1 = D_2$ , let us write  $D_1^2$  instead of  $D_1 \cdot D_1$  and call it the *weight* of  $D_1$  for short. Then we have the following

LEMMA 3.2. *If  $C$  satisfies any one of the following conditions, then  $\mathcal{G} = \mathcal{L}$ .*

- (1)  $g \geq 1$ .
- (2) *There is some  $i$  ( $0 \leq i \leq r$ ) such that  $C_i$  has at least two singular points.*
- (3) *If  $C_i$  is smooth, then  $C_i^2 \leq -2$ , where  $1 \leq i \leq r$ .*
- (4)  $g = 0$ ,  $N = 1$  and  $R \leq -e_p - 1$ .

PROOF. From the above lemma and the Castelnuovo's criterion for con-

tracting a curve (Hartshorne [3]), the assertions (1), (2) and (3) follow easily. Note that in the case (4) the weight  $C_p^2$  is

$$d^2 - \sum_{i=1}^p e_i^2 = R + e_p - 1,$$

if the center  $Q_{i+1}$  of the blow-up  $\sigma_{i+1}$  coincides with the singular point of  $C_i$ , where  $0 \leq i \leq p-1$ . Hence this is a special case of (3). Q.E.D.

Combining Lemma 3.2 with Proposition 1, we get the following

**COROLLARY 3.3.** *If  $C$  satisfies any one of the following conditions, then  $\mathcal{G}$  is a finite group.*

- (1)  $g \geq 1$ .
- (2) *The curve  $C$  is not projectively equivalent to  $\Delta_e$  and there is some  $i$  ( $0 \leq i \leq r$ ) such that  $C_i$  has at least two singular points.*
- (3) *If  $C_i$  is smooth, then  $C_i^2 \leq -2$ , where  $1 \leq i \leq r$ .*
- (4)  $g=0$ ,  $N=1$  and  $R \leq -e_p - 1$ .

**PROOF.** It suffices to check (3) and (4). If  $C \sim \Delta_e$  and  $e \geq 2$ , then let  $(e_1, \dots, e_a)$  and  $(f_1, \dots, f_b)$ , where  $e_1 = e$  and  $f_1 = d - e$ , be the sequences of the multiplicities of all infinitely near singular points of  $(0, 0, 1)$  and  $(0, 1, 0)$  respectively. Since  $(e, d) = 1$ , we get

$$d^2 - \sum_{i=1}^a e_i^2 - \sum_{j=1}^b f_j^2 = e_a + f_b$$

by the Euclidean algorithm and the genus formula for plane curves. If  $e=1$ , then the multiplicity of the singular point is  $d-1$ , so  $R=d+1$ . Hence the curves in (3) and (4) are not projectively equivalent to  $\Delta_e$ . Q.E.D.

Now the assertions (1) and (4) in Theorem A follow from Lemma 3.2 and Proposition 1.

#### 4. Representation of automorphisms by graphs.

In this section we follow the notations fixed in the previous sections. Hereafter we shall study the curves that have not been treated in Corollary 3.3 and that are not projectively equivalent to  $\Delta_e$ , where  $e \geq 2$ . Therefore we assume that  $C$  satisfies all of the following conditions  $(A_1)$  and  $d \geq 3$ .

- $$(A_1) \left\{ \begin{array}{l} (1) \quad g=0. \\ (2) \quad \text{The proper transform } C_i \text{ has at most one singular point for all } i, \\ \quad \text{where } 0 \leq i \leq r. \\ (3) \quad \text{There is some } i \text{ (} 1 \leq i \leq r \text{) such that } C_i \text{ is smooth and } C_i^2 \geq -1. \\ (4) \quad \text{If } N=1, \text{ then } R \geq -e_p. \end{array} \right.$$

In particular  $M$  is 1. In view of the assertion (1) in Lemma 3.1 we may

assume moreover the following condition (A<sub>2</sub>).

(A<sub>2</sub>)  $\left\{ \begin{array}{l} \text{For an element } \varphi \text{ of } \mathcal{G} \setminus \mathcal{L} \text{ the birational morphism } \sigma \text{ is a composition} \\ \text{of } r \text{ blow-ups such that } r \text{ is minimal in order that } \varphi\sigma \text{ is a morphism.} \end{array} \right.$

DEFINITION 4.1. We denote by  $r(\varphi)$  the number of blow-ups defined in (A<sub>2</sub>) and call it the *rank* of  $\varphi$ . Of course  $r(\varphi)=0$  if and only if  $\varphi$  belongs to  $\mathcal{L}$ .

Put  $E_i=\sigma_i^{-1}(Q_i)$ , where  $1 \leq i \leq r$ . Then the following facts hold true, of which we shall make frequent use later.

LEMMA 4.2. *If  $\mathcal{G} \neq \mathcal{L}$  and  $\varphi$  belongs to  $\mathcal{G} \setminus \mathcal{L}$ , then we have the following.*

(i) *The center  $Q_{i+1}$  of  $\sigma_{i+1}$  coincides with the fundamental point of  $\varphi\sigma_1 \cdots \sigma_i$ , where  $0 \leq i \leq r-1$  and  $\varphi\sigma_0=\varphi$ .*

(ii) *The center  $Q_{i+1}$  of  $\sigma_{i+1}$  belongs to  $E_i$ , where  $1 \leq i \leq r-1$ .*

(iii) *If  $C_i$  is not smooth, then  $Q_{i+1}$  coincides with the singular point of  $C_i$ , where  $0 \leq i \leq r-1$ .*

(iv) *If  $C_i^2 \geq 0$ , then  $Q_{i+1}$  belongs to  $E_i \cap C_i$ , where  $1 \leq i \leq r-1$ .*

PROOF. First recall the assertion (1) in Lemma 3.1 and the conditions (A<sub>1</sub>) and (A<sub>2</sub>). Then (i) is clear. Similarly we get  $\sigma_i(Q_{i+1})=\sigma_i(E_i)$ . This proves (ii). Since  $\varphi$  has a fundamental point, the proper transform  $C_r$  must be contracted owing to the assertion (2) in Lemma 3.1. Then from the Castelnuovo's criterion we infer (iii) and (iv). Q.E.D.

Since there will be no danger of confusion, let  $E_i$  denote also the curve  $(\sigma_{i+1} \cdots \sigma_r)^{-1}[E_i]$  on  $S_r$ . When we treat more than one automorphism at a time, let us write  $\sigma_{i\varphi}$ ,  $E_{i\varphi}$  and  $S_{i\varphi}$  with the automorphism  $\varphi$  instead of  $\sigma_i$ ,  $E_i$  and  $S_i$  respectively. For a divisor we shall not consider the multiplicities of the components, hence *we identify the divisor with the reduced one obtained from it.*

DEFINITION 4.3. Let  $\varphi$  be an element of  $\mathcal{G} \setminus \mathcal{L}$  and put  $r=r(\varphi)$ . Since  $\sigma$  is determined by  $\varphi$ , let  $\Gamma(\varphi)=\sigma^{-1}(C)$ , i. e.,  $\Gamma(\varphi)=E_{1\varphi} + \cdots + E_{r\varphi} + C_\varphi$ , where  $C_\varphi = \sigma^{-1}[C]$ . The total transform  $\Gamma(\varphi)$  is called the *graph* of  $\varphi$ . Define the *orders* of  $E_{i\varphi}$  and  $C_\varphi$  to be  $i$  and  $r+1$  respectively, where  $1 \leq i \leq r$ . For an element  $\psi$  of  $\mathcal{G} \setminus \mathcal{L}$ , similarly  $\Gamma(\psi)$  has the decomposition  $E_{1\psi} + \cdots + E_{s\psi} + C_\psi$ . If  $r=s$  and there is an isomorphism  $\alpha: S_{r\varphi} \rightarrow S_{s\psi}$  such that  $\alpha(C_\varphi)=C_\psi$  and  $\alpha(E_{i\varphi})=E_{i\psi}$  for all  $i$ , where  $1 \leq i \leq r$ , then  $\Gamma(\varphi)$  is said to be *equivalent* to  $\Gamma(\psi)$ . Note that  $\alpha$  preserves the orders of the components. Let this equivalence be denoted by  $\approx$ . This equivalence satisfies the axiom of equivalence relation.

REMARK 4.4. If  $\varphi$  and  $\psi$  belong to  $\mathcal{G} \setminus \mathcal{L}$ , then the following hold true.

(1)  $E_{r\varphi}^2 = C_\varphi^2 = -1$  and  $E_{i\varphi}^2 \leq -2$  where  $1 \leq i \leq r-1$ .

(2) In order to check that the isomorphism  $\alpha$  in Definition 4.3 gives the equivalence, it suffices to verify  $\alpha(\Gamma(\varphi))=\Gamma(\psi)$  and  $\alpha(C_\varphi)=C_\psi$  [or  $\alpha(E_{r\varphi})=E_{r\psi}$ ].

PROOF. The first assertion is obtained from (ii) in Lemma 4.2 and the

Castelnuovo's criterion. In the second assertion we have that  $\alpha(C_\varphi)=C_\psi$  and  $\alpha(E_{r\varphi})=E_{r\psi}$ , since  $\alpha$  preserves the weights of the components. By the blow-down  $\sigma_{r\varphi}$  the isomorphism  $\alpha$  defines also the isomorphism  $\alpha_{r-1}:S_{r-1\varphi}\rightarrow S_{r-1\psi}$  such that  $\alpha_{r-1}\sigma_{r\varphi}=\sigma_{r\psi}\alpha$ . Since  $\sigma_{r\varphi}(E_{i\varphi})$  and  $\sigma_{r\psi}(E_{i\psi})$  have the weight  $-1$  if and only if  $i=r-1$ , we have that  $\alpha_{r-1}(\sigma_{r\varphi}(E_{r-1\varphi}))=\sigma_{r\psi}(E_{r-1\psi})$ , i. e.,  $\alpha(E_{r-1\varphi})=E_{r-1\psi}$ . In this way we complete the proof by induction. Q. E. D.

The following lemma is trivial, so its proof is omitted.

LEMMA 4.5. *We have that  $\Gamma(\varphi)\approx\Gamma(l\varphi)\approx\Gamma(\varphi l)$ , where  $\varphi\in\mathcal{G}\setminus\mathcal{L}$  and  $l\in\mathcal{L}$ .*

Since  $\varphi\sigma$  is a morphism, it can be expressed as  $\sigma'\tilde{\varphi}$ , applying the same factorization to  $\varphi^{-1}$ , where  $\sigma'=\sigma'_1\cdots\sigma'_r$  and  $\tilde{\varphi}:S_r\rightarrow S'_r$  is an isomorphism. In this expression  $\sigma'_i$  is a blow-up  $S'_i\rightarrow S'_{i-1}$ , where  $1\leq i\leq r$  and  $S'_0=\mathbf{P}^2$ . Note that  $S_r$ ,  $S'_r$  and  $\tilde{\varphi}$  are determined uniquely by  $\varphi$  owing to  $(A_2)$ .

DEFINITION 4.6. The above isomorphism  $\tilde{\varphi}$  is called the *resolution* of  $\varphi$ .

Here we have that  $\Gamma(\varphi^{-1})=\sigma'^{-1}(C)=E_{1\varphi^{-1}}+\cdots+E_{r\varphi^{-1}}+C_{\varphi^{-1}}$  and  $\tilde{\varphi}(\Gamma(\varphi))=\Gamma(\varphi^{-1})$ .

REMARK 4.7. (1) If  $\varphi$  belongs to  $\mathcal{G}\setminus\mathcal{L}$ , then  $\tilde{\varphi}(E_{r\varphi})=C_{\varphi^{-1}}$  and  $\tilde{\varphi}(C_\varphi)=E_{r\varphi^{-1}}$ .

(2) For two elements  $\varphi_1$  and  $\varphi_2$  of  $\mathcal{G}\setminus\mathcal{L}$  we have that  $\Gamma(\varphi_1)\approx\Gamma(\varphi_2)$  if and only if  $\Gamma(\varphi_1^{-1})\approx\Gamma(\varphi_2^{-1})$ .

PROOF. The first assertion follows from (2) in Remark 4.4. Let  $\tilde{\varphi}_i$  be the resolution of  $\varphi_i$ , where  $i=1, 2$ , and  $\alpha:S_{r\varphi_1}\rightarrow S_{r\varphi_2}$  give the equivalence between  $\Gamma(\varphi_1)$  and  $\Gamma(\varphi_2)$ . Then from the same assertion (2) in Remark 4.4 we see that  $\tilde{\varphi}_2\alpha\tilde{\varphi}_1^{-1}$  gives the equivalence between  $\Gamma(\varphi_1^{-1})$  and  $\Gamma(\varphi_2^{-1})$ . The converse is proved similarly. Q. E. D.

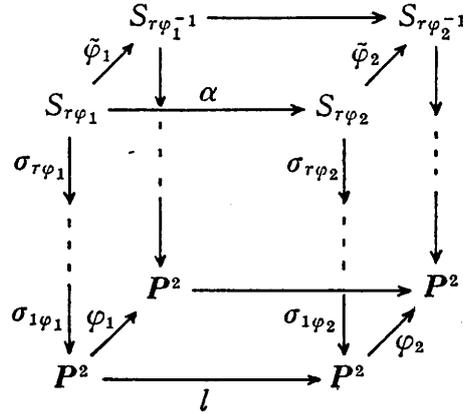
First contract  $\tilde{\varphi}^{-1}(E_{r\varphi^{-1}})=C_\varphi$ , secondly  $\tilde{\varphi}^{-1}(E_{r-1\varphi^{-1}})$ , and so on. In this way, by using  $\tilde{\varphi}$ , we have the composition  $\sigma''=\sigma''_1\cdots\sigma''_r$  of blow-downs  $\sigma''_i:S''_i\rightarrow S''_{i-1}$  such that  $S''_r=S_r$ ,  $S''_0=S''$  and  $\sigma''\tilde{\varphi}^{-1}=l\sigma'$ , where  $1\leq i\leq r$  and  $l$  is an isomorphism  $\mathbf{P}^2\rightarrow S''$ . Then we have that  $\varphi\sigma=l^{-1}\sigma''$ . Note that  $l$  is not necessarily the identity mapping and that the components of  $\Gamma(\varphi)$  define two morphisms  $\sigma$  and  $\sigma''$ .

DEFINITION 4.8. The above morphism  $\sigma''$  is called the *associate* of  $\sigma$ .

REMARK 4.9. Since  $\sigma''(E_r)=l(C)$ , the curve  $E_r$  is not contracted by the associate of  $\sigma$ .

LEMMA 4.10. *Let  $\varphi_1$  and  $\varphi_2$  be two elements of  $\mathcal{G}\setminus\mathcal{L}$ . Then there are two elements  $l_1$  and  $l_2$  of  $\mathcal{L}$  satisfying  $\varphi_2=l_2\varphi_1l_1$  if and only if  $\Gamma(\varphi_1)\approx\Gamma(\varphi_2)$ .*

PROOF. The "only if" part is an easy consequence of Lemma 4.5. Suppose that  $\Gamma(\varphi_1)\approx\Gamma(\varphi_2)$  and  $\alpha$  is the isomorphism  $S_{r\varphi_1}\rightarrow S_{r\varphi_2}$  defining its equivalence. Then  $\alpha$  induces an element  $l$  of  $\mathcal{L}$  which satisfies  $\sigma_{1\varphi_2}\cdots\sigma_{r\varphi_2}\alpha=l\sigma_{1\varphi_1}\cdots\sigma_{r\varphi_1}$ , where  $\sigma_{i\varphi_1}$  and  $\sigma_{i\varphi_2}$  are blow-ups,  $1\leq i\leq r$ . Let  $\tilde{\varphi}_i$  be the resolution of  $\varphi_i$ ,



where  $i=1, 2$ . Since  $\alpha(C_{\varphi_1})=C_{\varphi_2}$ ,  $\alpha(E_{r\varphi_1})=E_{r\varphi_2}$ ,  $\tilde{\varphi}_i(C_{\varphi_i})=E_{r\varphi_i^{-1}}$  and  $\tilde{\varphi}_i(E_{r\varphi_i})=C_{\varphi_i^{-1}}$ , where  $i=1, 2$ , it follows that  $\tilde{\varphi}_2\alpha\tilde{\varphi}_1^{-1}$  preserves the orders of the irreducible components of  $\Gamma(\varphi_1^{-1})$  and  $\Gamma(\varphi_2^{-1})$  by the assertion (2) in Remark 4.4. Hence we see that the isomorphism  $\tilde{\varphi}_2\alpha\tilde{\varphi}_1^{-1}$  induces an isomorphism  $\varphi_2\varphi_1^{-1}$  in  $\mathcal{L}$ . Q. E. D.

Let  $\mathcal{C}$  be the set consisting of the equivalence class of the [graphs of elements of  $\mathcal{G}\setminus\mathcal{L}$ , i. e.,

$$\mathcal{C} = \{\Gamma(\varphi) \mid \varphi \in \mathcal{G}\setminus\mathcal{L}\} / \approx.$$

Then the above lemma implies the following

COROLLARY 4.11.

$$\mathcal{G}\setminus\mathcal{L} = \bigcup_{\Gamma(\varphi) \in \mathcal{C}} \mathcal{L}\varphi\mathcal{L}.$$

Especially this means the following

LEMMA 4.12. *If the number of elements of  $\mathcal{L}$  and that of  $\mathcal{C}$  are both finite, then so is  $\mathcal{G}$ .*

Now, let us proceed to the proof of Theorem A. We shall find step by step the several conditions on which  $\mathcal{G}$  is a finite group. Recalling the condition (A<sub>1</sub>), we put

$$p = \min\{i \mid C_i \text{ is smooth}\}.$$

Then we have that  $C_p^2 \geq -1$ . If  $j \geq i$  and  $C_i^2 = -1$ , then  $Q_{j+1}$  does not lie on  $C_j$ , hence we put

$$q = \min\{i \mid C_i^2 = -1\}.$$

Note that the number  $q$  does not depend on  $\varphi$  and that  $C_q = C_r$ , where  $r = r(\varphi)$ . Since  $C_p^2 \geq -1$ , we have that  $C_{p-1}^2 \geq 3$ , i. e.,  $q \geq p$ .

LEMMA 4.13. *If the intersection number  $C_q \cdot E_q \geq 2$ , then  $p = q = r(\varphi)$  for all  $\varphi$  in  $\mathcal{G}\setminus\mathcal{L}$ .*

PROOF. Since  $C_q \cdot E_q \geq 2$ , the curve  $C_{q-1}$  is not smooth, i. e., we have that  $q = p$ . Suppose that  $r > q$  for some  $\varphi$ , where  $r = r(\varphi)$ . Then the fundamental point of  $\varphi\sigma_1 \cdots \sigma_q$  exists on  $E_q \setminus C_q$ , since  $C_q^2 = -1$  and  $C_q$  has to be contracted.

When  $C_q$  is contracted, the image of  $E_q$  is not smooth. By Remark 4.9 the curve  $E_r$  is not contracted, hence  $E_q$  has to be contracted, since  $q \neq r$ . This is a contradiction. Q. E. D.

LEMMA 4.14. *If  $q=p$ , then  $\mathcal{G}$  is a finite group.*

PROOF. Since  $q=p$ , we have that  $C_q \cdot E_q \geq 2$ , then by Lemma 4.13  $\varphi\sigma_1 \cdots \sigma_p$  is a morphism for all  $\varphi \in \mathcal{G} \setminus \mathcal{L}$ . Since  $Q_i$  coincides with the singular point of  $C_{i-1}$  by Lemma 4.2, the blow-up  $\sigma_i$  does not depend on  $\varphi$ , where  $1 \leq i \leq p$ . Hence we see that  $\mathcal{C}$  is a finite set (in fact it consists of at most one element). Since the order of  $\mathcal{L}$  is finite by Proposition 1, so is  $\mathcal{G}$  by Lemma 4.12. Q. E. D.

In view of the above lemma we may treat only the curves satisfying the following condition (A<sub>3</sub>) hereafter.

(A<sub>3</sub>) :  $q > p$ .

LEMMA 4.15. *If  $E_p$  and  $C_p$  satisfy either one of the following conditions, then  $\mathcal{G} = \mathcal{L}$  and it is a finite group.*

(i)  $E_p \cap C_p$  consists of at least three points.

(ii)  $E_p \cap C_p$  consists of two points such that  $E_p$  and  $C_p$  meet transversally at none of them.

PROOF. Suppose that  $\mathcal{G} \neq \mathcal{L}$  and take an element  $\varphi$  of  $\mathcal{G} \setminus \mathcal{L}$ . Then by the condition (A<sub>3</sub>)  $\varphi\sigma_1 \cdots \sigma_p$  has one fundamental point  $Q_{p+1}$  in  $E_p \cap C_p$ . After  $C_r$ ,  $r=r(\varphi)$ , is contracted, the image of  $E_p$  is not smooth on the condition of this lemma. Since  $r \geq q > p$  and  $E_r$  is not contracted, the curve  $E_p$  has to be contracted by the associate of  $\sigma$ . This is a contradiction. Of course  $\mathcal{L}$  is a finite group by Proposition 1. Q. E. D.

In view of the above lemma we may treat only the curves satisfying either one of the following conditions (1) and (2) hereafter.

(A<sub>4</sub>)  $\left\{ \begin{array}{l} (1) E_p \cap C_p \text{ consists of one point.} \\ (2) E_p \cap C_p \text{ consists of two points } \{Q', Q''\} \text{ such that } E_p \text{ and } C_p \text{ meet transversally at } Q' \text{ or } Q''. \end{array} \right.$

LEMMA 4.16. *If the total transform  $(\sigma_1 \cdots \sigma_q)^{-1}(C)$  has not normal crossings, then we have that  $r(\varphi)=q$  for every  $\varphi$  in  $\mathcal{G} \setminus \mathcal{L}$ , and hence  $\mathcal{G}$  is a finite group.*

PROOF. First of all, on the condition (A<sub>4</sub>) the possibilities for the centers of  $\sigma_{p+1}$  for every  $\varphi$  in  $\mathcal{G} \setminus \mathcal{L}$  are at most two, while  $Q_j$  coincides with  $E_{j-1} \cap C_{j-1}$  if  $p+2 \leq j \leq q$ . Thus  $Q_j$  depends on  $Q_{p+1}$ , where  $p+2 \leq j \leq q$ . Suppose that  $r(\varphi) > q$  for some  $\varphi$  in  $\mathcal{G} \setminus \mathcal{L}$ . Then  $Q_{q+1}$  lies in  $E_q \setminus C_q$ , hence there is some such that  $C_q \cdot E_i \geq 2$  or  $C_q$ ,  $E_i$  and  $E_j$  meet at one point, where  $i, j \leq q$ . After the contraction of  $C_q$  the image of  $E_i$  has a singular point or  $E_i$  and  $E_j$  do not meet transversal. Hence there is a curve not contracted besides  $E_r$ . This is a contradiction. Thus we have that  $r(\varphi)=q$  for every  $\varphi$  in  $\mathcal{G} \setminus \mathcal{L}$ . In view of the

first consideration we conclude that  $\mathcal{C}$  is a finite set (in fact it consists of at most two elements). Since, for the curve  $\Delta_1$ , we have that  $q=2d+1$  and that  $(\sigma_1 \cdots \sigma_q)^{-1}(\Delta_1)$  has normal crossings, the linear part  $\mathcal{L}$  has a finite order. Hence  $\mathcal{G}$  is a finite group by Lemma 4.12. Q. E. D.

**COROLLARY 4.17.** *If  $N=1$  and  $R \leq -1$ , then  $(\sigma_1 \cdots \sigma_q)^{-1}(\mathcal{C})$  has not normal crossings and hence  $\mathcal{G}$  is a finite group.*

**PROOF.** Note that  $C_{p+j}$ ,  $E_p$  and  $E_{p+j}$  meet at one point, where  $1 \leq j \leq e_p - 1$ . Since  $C_{p+j}^2 = R + e_p - 1 - j$ , we have that  $q = p + e_p + R$ , hence  $p + 1 \leq q \leq p + e_p - 1$ . This implies that  $(\sigma_1 \cdots \sigma_q)^{-1}(\mathcal{C})$  has not normal crossings. Q. E. D.

We have just proved the assertion (3) in Theorem A. In view of Lemma 4.16 we may treat only the curves satisfying the following condition (A<sub>6</sub>) hereafter.

(A<sub>6</sub>) :  $(\sigma_1 \cdots \sigma_q)^{-1}(\mathcal{C})$  has normal crossings.

Moreover the following lemma holds true.

**LEMMA 4.18.** *If  $N \geq 3$ , then  $\mathcal{G}$  is a finite group.*

**PROOF.** Suppose that  $N \geq 3$  and take an element  $\varphi$  of  $\mathcal{G} \setminus \mathcal{L}$ . Then  $C_r$  and  $E_1 + \cdots + E_r$  meet in at least 3 points, but  $E_r$  and  $E_1 + \cdots + E_{r-1}$  meet in at most 2 points, and  $E_r = \tilde{\varphi}^{-1}(C_{\varphi-1})$  meets at least three other components of  $\Gamma(\varphi)$ . Hence  $E_r$  must meet  $C_r$ . This means that  $r(\varphi) = q$ . As was shown in the proof of Lemma 4.16, the possibilities for the centers of blow-ups are at most two, so that the order of  $\mathcal{G}$  is finite. Q. E. D.

From the above results we may treat only the curves satisfying one of the following conditions hereafter.

(i)  $N=2$ .

(ii)  $N=1$  and  $R \geq 0$ .

In this section only the case (i) is considered. The other one will be treated in the next section.

In what follows the graphs will be represented as figures, where the following abbreviation will be used. The number  $r$  indicates the rank  $r(\varphi)$  and a positive integer  $i$  beside a component indicates the curve  $E_i$  and a non-positive integer  $j$  beside a component indicates the weight  $j$  of the component and a component without a non-positive integer has the weight  $-2$ . Since we treat only the case of normal crossings, we often adopt *dual graphs*, where the symbols  $\circ$  and  $\bullet$  indicate the components whose weights are  $-2$  and not  $-2$  respectively.

**DEFINITION 4.19.** If a divisor on some surface  $S$  admits two ways of contractions  $S \rightarrow \mathbf{P}^2$  such as  $\sigma$  and  $\sigma''$ , then it is said to be *contractible*. Of course the graph  $\Gamma(\varphi)$  for  $\varphi$  in  $\mathcal{G} \setminus \mathcal{L}$  is contractible.

**DEFINITION 4.20.** By using the resolution  $\tilde{\varphi}$  of  $\varphi$ , we define a permutation

$\varphi^*$  of the set  $\{E_{1\varphi}, \dots, E_{r\varphi}, C_\varphi\}$  as follows:

$$\varphi^*(C_\varphi) = E_{r\varphi}, \quad \varphi^*(E_{i\varphi}) = \bar{\varphi}^{-1}(E_{i\varphi-1}), \quad \text{where } 1 \leq i \leq r.$$

Let us call  $\varphi^*$  the *permutation* of  $\Gamma(\varphi)$  and denote by  $D(\varphi)$  the divisor consisting of all the components of  $\Gamma(\varphi)$  that are fixed by  $\varphi^*$ . In case all the components are moved, then we put  $D(\varphi) = 0$ . Let us call  $D(\varphi)$  the *center* of  $\Gamma(\varphi)$ . If  $\varphi^*$  has the order 2, then we may regard  $\Gamma(\varphi)$  as being symmetrical about  $D(\varphi)$ .

Now, let us study the curves with  $N=2$ .

DEFINITION 4.21. A divisor  $D = \sum_{i=1}^n D_i$  is said to contain a *loop* if a subset  $\bigcup_{j \in I} D_j$  forms a closed path for some  $I \subset \{1, 2, \dots, n\}$ . The divisor  $D$  is said to form a *simple loop* if the following two conditions are satisfied:

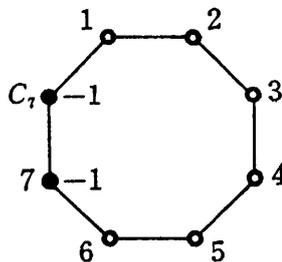
- (1)  $D$  has normal crossings.
- (2) Every irreducible component is nonsingular and rational and intersects others at two points.

Note that a divisor consists of just one loop if it forms a simple loop.

LEMMA 4.22. *The following three conditions are equivalent.*

- (1)  $\mathcal{G} \neq \mathcal{L}$  and  $\Gamma(\varphi)$  forms a simple loop for some  $\varphi$  in  $\mathcal{G} \setminus \mathcal{L}$ .
- (2) The singular point  $P_1$  of  $C$  is a node, i. e.,  $e(P_1, C) = N = 2$  and  $p = 1$ .
- (3)  $C \sim \Delta$ .

PROOF. If  $\Gamma(\varphi)$  forms a simple loop, then by contracting the components of  $\Gamma(\varphi)$  we see that  $P_1$  turns out to be a node. Next we assume that  $P_1$  is a node. Since  $g=0$ , we have that  $d=3$  by the genus formula for plane curves, which means that  $C \sim \Delta$ . Finally we assume that  $C \sim \Delta$ . Then by successive blow-ups we get the following figure.



In this figure, first contract  $C_1$ , secondly  $E_1$ , and so on. Eventually we get a curve which is projectively equivalent to  $\Delta$ . From these blow-ups and blow-downs we obtain an element  $\varphi$  of  $\mathcal{G} \setminus \mathcal{L}$ . Q. E. D.

Let us study the case (i) by examining the following cases separately.

- (i-1) The curve  $C$  is not projectively equivalent to  $\Delta$ .
- (i-2) The curve  $C$  is projectively equivalent to  $\Delta$ .

First let us consider the case (i-1).

Then by the above lemma we have that  $\mathcal{G} = \mathcal{L}$  or  $\Gamma(\varphi)$  does not form a

simple loop for all  $\varphi$  in  $\mathcal{G} \setminus \mathcal{L}$ . In the former case  $\mathcal{G}$  has a finite order owing to Proposition 1 and the assumption that  $M=1$  and  $N=2$ . Hence let us treat only the latter case. Put  $\Gamma_i = (\sigma_1 \cdots \sigma_i)^{-1}(C)$ , where  $i \geq q$ . By the condition (A<sub>5</sub>)  $\Gamma_i$  has normal crossings. The divisor  $E_1 + \cdots + E_i$  is connected and does not contain a loop, but it connects two points in  $C_i$ . Hence  $\Gamma_i$  contains just a simple loop, to which  $C_i$  belongs. We denote it by  $\Gamma_i^0$ .

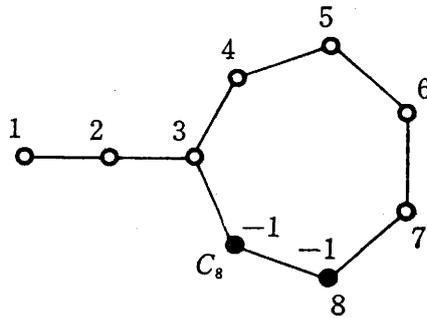
LEMMA 4.23. *If  $\varphi\sigma_1 \cdots \sigma_i$  has a fundamental point, where  $i \geq q$ , then the center  $Q_{i+1}$  of  $\sigma_{i+1}$  is not a free point on  $E_i$ , hence  $E_{i+1}$  is a component of  $\Gamma_r^0 = \Gamma(\varphi)^0$ . The curve  $E_q$  is a component of  $\Gamma_q^0$ , hence  $\Gamma_r^0$  is obtained from  $\Gamma_q^0$  by successive blow-ups.*

PROOF. Since  $\Gamma(\varphi)^0$  [resp.  $\Gamma(\varphi^{-1})^0$ ] is the unique simple loop contained in  $\Gamma(\varphi)$  [resp.  $\Gamma(\varphi^{-1})$ ], the resolution  $\tilde{\varphi}$  maps  $\Gamma(\varphi)^0$  onto  $\Gamma(\varphi^{-1})^0$ . As is mentioned above  $\Gamma(\varphi)^0$  [resp.  $\Gamma(\varphi^{-1})^0$ ] contains  $C_\varphi$  [resp.  $C_{\varphi^{-1}}$ ], hence  $\Gamma(\varphi)^0$  contains also  $\tilde{\varphi}^{-1}(C_{\varphi^{-1}}) = E_{r\varphi}$ . Since  $\Gamma_q^0$  has normal crossings, and  $E_q$  and  $C_q$  meets, and  $N=2$ , the curve  $E_q$  is a component of  $\Gamma_q^0$ . The above consideration proves the lemma. Q. E. D.

Since  $\Gamma(\varphi)$  does not form a simple loop but contains the one  $\Gamma_r^0$ , there is a component  $E_k$  of  $\Gamma_r^0$  which meets at least three other components. (For example, see the following figure, though it is not contractible, obtained from the curve

$$(y-x^2)^2 + tx^2y^2 + xy^3 = 0, \quad \text{where } t \neq 0.$$

In this case  $k=3$ .)



By Lemma 4.23 we have that  $k \leq q$ , in other words, for any  $i > q$ ,  $E_i$  is a component of  $\Gamma_r^0$ . Taking note of the position of  $E_k$ , we infer that the possibilities of graphs which represent automorphisms are finite. To explain this in detail, let us consider the one route in  $\Gamma_r^0$  that connects  $C_r$  and  $E_k$  and that does not pass through  $E_r$ . Then let  $h$  be the sum of the weights of the divisors in the route, i. e.,  $h = C_r^2 + \cdots + E_k^2$ . For the sake of simplicity, let us take  $E_k$  that gives the minimum value for  $-h$ . Of course  $h$  is determined uniquely by  $C$  and is independent of  $\varphi$ . Suppose that  $-h \ll r$ . Then there are a great many components in the other route, because the center  $Q_{q+1} \in E_q \setminus C_q$  and we have the facts

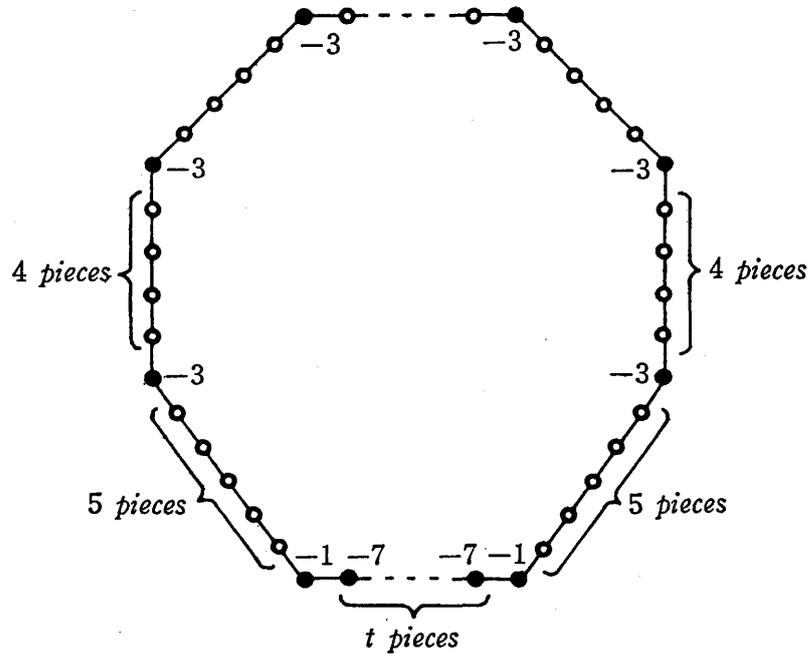
in Lemma 4.23. Now, let us contract the graph from  $C_r$ , then, after the contraction of  $E_h$ , we will have a divisor with not normal crossings, but the number of its components will be still more than  $q+1$ , since  $-h \ll r$ . This is a contradiction, hence we have the conclusion mentioned above. Thus  $\mathcal{C}$  consists of finitely many elements, i.e.,  $\mathcal{G}$  is a finite group.

Putting together all the results obtained above, we complete the proof of (2) in Theorem A.

Next let us consider the case (i-2).

Since  $C$  is projectively equivalent to  $\Delta$ , the linear part  $\mathcal{L}$  is the dihedral group of order 6. Let us consider non-linear elements.

LEMMA 4.24 (Wakabayashi). *For every element  $\varphi$  of  $\text{Aut}(\mathbb{P}^2 - \Delta) \setminus \mathcal{L}$ , the graph  $\Gamma(\varphi)$  forms a simple loop and its figure is as follows.*



In this figure  $t$  is a non-negative integer determined by  $\varphi$  and the number of components of  $\Gamma(\varphi)$  is  $8+6t$ , i.e.,  $r(\varphi)=6t+7$ . Conversely, such a figure yields an element of  $\text{Aut}(\mathbb{P}^2 - \Delta)$ .

PROOF. We perform successive blow-ups of  $\mathbb{P}^2$  in order that the total transform of  $\Delta$  may become contractible. In the proof of Lemma 4.22, we have obtained that  $q=7$  and the figure with  $t=0$ . Then, noting that Lemma 4.23 is also applicable to this case, we continue blow-ups if there is still a fundamental point. The center  $Q_8$  must coincide with  $E_6 \cap E_7$ , and  $Q_9$  with  $E_7 \cap E_8$ , and so on. When the weight of the proper transform of  $E_7$  becomes  $-7$ , we may stop the blow-ups and get the contractible divisor with  $t=1$  in the above figure. In

case there is still a fundamental point, the center  $Q_{14}$  must coincide with  $E_{12} \cap E_{13}$  and we proceed similarly. Finally we get all contractible divisors by such a manner, which are illustrated as above. Conversely, the above divisor is contractible and two curves which are images of the divisor are projectively equivalent. Hence from this divisor we get an element of  $\mathcal{G} \setminus \mathcal{L}$ . Q.E.D.

This lemma shows that the centers of blow-ups are always the intersection points of two curves, hence the order of  $\mathcal{G}$  is countably infinite. Thus the assertion (5) in Theorem A is proved.

Concerning the assertion (2) in Theorem A we raise the following conjecture, which is true for all examples we now have.

CONJECTURE. Suppose that  $N \geq 2$  and that  $C$  is not projectively equivalent to  $A$ . Then  $\mathcal{G} = \mathcal{L}$ .

### 5. Curves with $g=0$ , $N=1$ and $R \geq 0$ .

In this section also we follow the notations fixed in the previous sections and assume that  $C$  is a curve with  $g=0$ ,  $N=1$  and  $R \geq 0$ . Recall that  $(A_2)$  is always assumed, so  $C$  satisfies all the conditions  $(A_i)$ ,  $1 \leq i \leq 5$ .

DEFINITION 5.1. Let  $D_i$  be a component of a divisor  $D = \sum_{i=1}^n D_i$ . Then a point  $Q$  on  $D_i$  is called a *free point* on  $D_i$  if  $Q$  does not belong to any other component  $D_j$ , where  $j \neq i$ . The divisor  $D$  is called a *zigzag* (or a divisor of type  $A_n$ ) if its components are nonsingular and rational and it has the following expression:

$$D_i \cdot D_j = \begin{cases} -2 & \text{if } i=j. \\ 1 & \text{if } |i-j|=1, \quad \text{where } 1 \leq i, j \leq n. \\ 0 & \text{if } |i-j| \geq 2. \end{cases}$$

In the figures of graphs we sometimes use a dotted line which represents a zigzag. When we try to find the graphs of elements of  $\mathcal{G} \setminus \mathcal{L}$ , we shall make good use of the following facts freely.

REMARK 5.2. (1) The center  $Q_{i+1}$  of  $\sigma_{i+1}$  is contained in  $E_i$ , where  $1 \leq i \leq r-1$ .

(2) The graph  $\Gamma(\varphi)$  is symmetrical about  $D(\varphi)$  if  $\varphi^*$  has the order 2.

(3) There exists the associate  $\sigma''$  of  $\sigma$ , i.e., a contraction can be started from  $C_r$ . Moreover the following fact holds true, which will not be used explicitly. Put  $\Gamma_i'' = \sigma_{i+1}'' \cdots \sigma_r''(\Gamma(\varphi)) = (\sigma_1'' \cdots \sigma_i'')^{-1}(l(C))$ . Then  $\Gamma_i''$  contains one or two components with the weight  $-1$ . If  $\Gamma_i''$  contains two such components, then  $i \geq g$ , hence it has normal crossings.

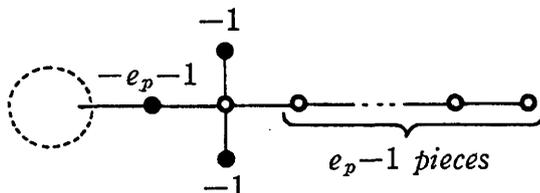
Then by considering blow-ups we get the following

PROPOSITION 5.3. Suppose that  $R=0$  and  $\mathcal{G} \neq \mathcal{L}$ . Then for every  $\varphi$  in  $\mathcal{G} \setminus \mathcal{L}$

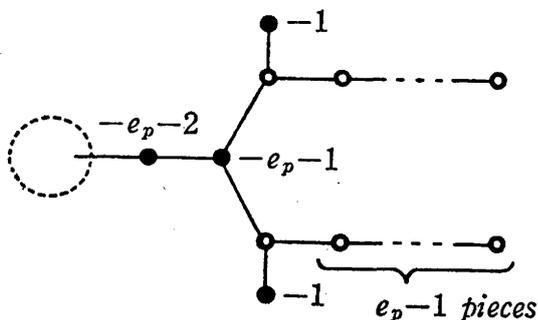
we have  $q=p+e_p$ . Suppose moreover that  $\varphi^*$  has the order 2 and that the center  $D(\varphi)$  of  $\Gamma(\varphi)$  contains  $E_1+\dots+E_{p-1}$ . Then the figure of  $\Gamma(\varphi)$  is (I) or (II) in the following according as  $e_{p-1}=e_p$  or  $e_{p-1}>e_p$ . In these figures the dotted circle represents the divisor  $E_1+\dots+E_{p-1}$ .

(I) The case  $e_{p-1}=e_p$  is separated into three subcases.

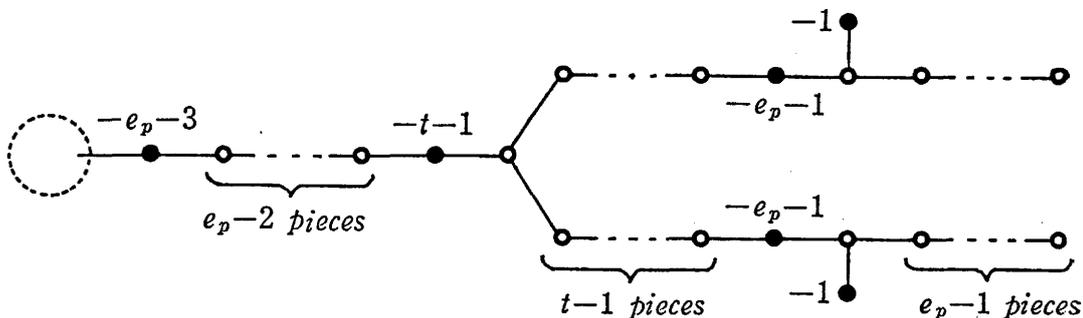
Subcase (a),  $r=q+1$ .



Subcase (b),  $r=q+e_p+2$ .

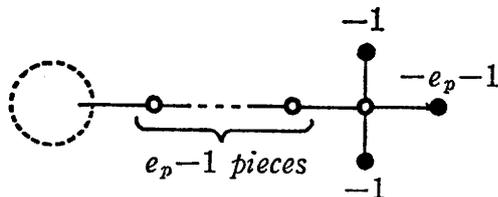


Subcase (c),  $r=q+2e_p+2t+1$ , where  $t$  is an integer  $\geq 1$ .

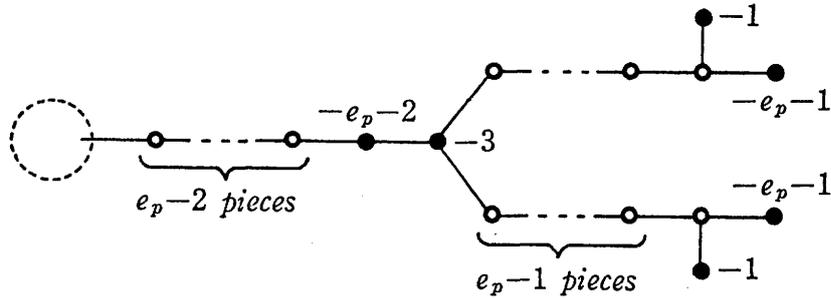


(II) The case  $e_{p-1}>e_p$  is also separated into three subcases.

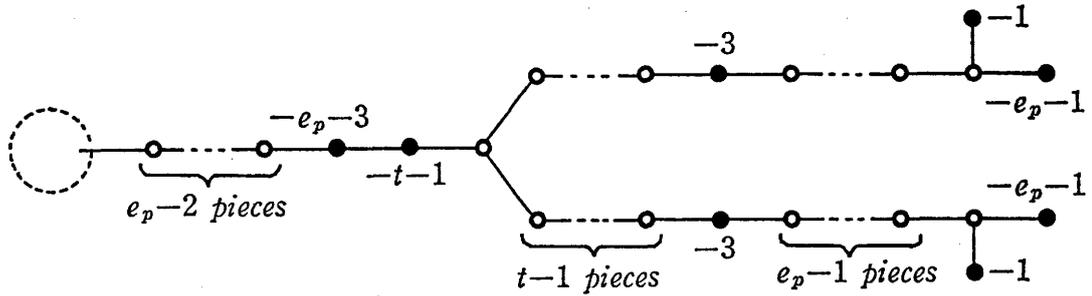
Subcase (a),  $r=q+1$ .



Subcase (b),  $r=q+2e_p+2$ .

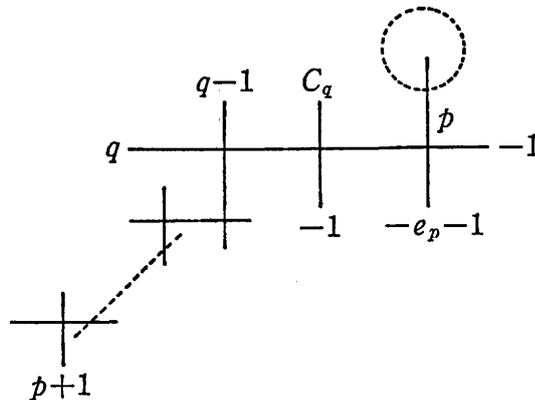


Subcase (c),  $r=q+2e_p+2t+3$ , where  $t$  is an integer  $\geq 1$ .



PROOF. Since  $N=1$  and  $d^2 - \sum_{i=1}^q e_i^2 - e_p = -1$ , we have that  $q=p+e_p$  and that the center  $Q_{i+1}$  of the blow-up  $\sigma_{i+1}$  coincides with the point  $E_i \cap C_i$  if  $i \leq q-1$ .

First let us take up the case (1). In this case the figure of  $\Gamma_q = (\sigma_1 \cdots \sigma_q)^{-1}(C)$  is as follows, since  $C_p$  meets only  $E_p$ .

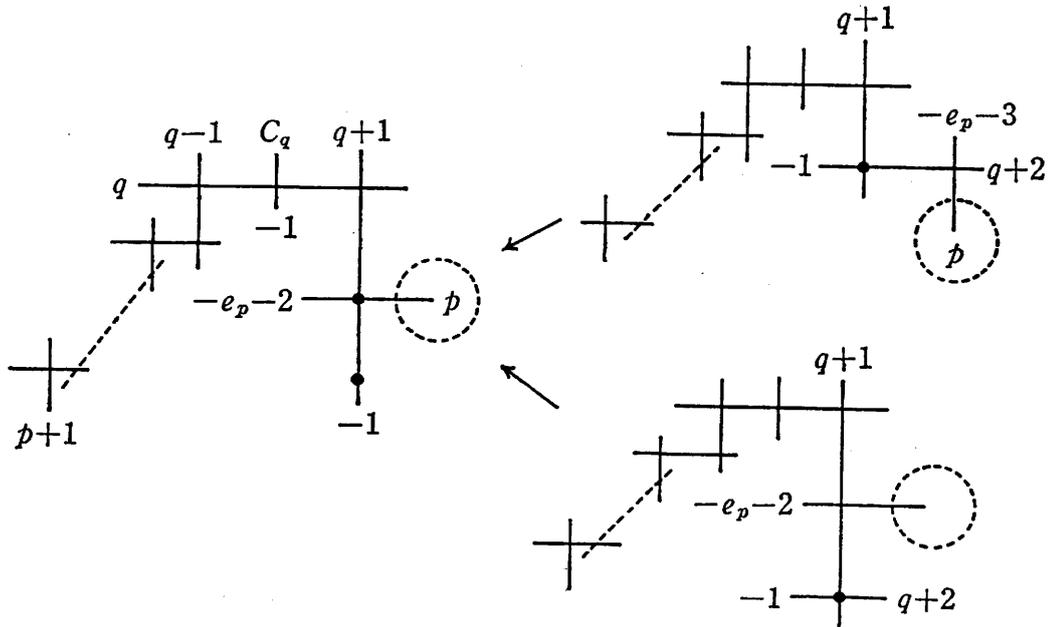


Note that  $\Gamma_q$  is independent of  $\varphi$  and  $\varphi^*$  is assumed to have the order 2. Hence by the hypothesis the divisor  $E_1 + \dots + E_p$  is contained in the center  $D(\varphi)$ . We prove first  $\{Q_{q+1}\} \neq E_q \cap E_{q-1}$ . Suppose the contrary. Then we have that  $\varphi^*(E_q) = E_q$ . Since  $\varphi^*(C_q) = E_r$ , the divisor  $E_r$  must meet  $E_q$ . Then  $\Gamma(\varphi)$  cannot be symmetrical about  $D(\varphi)$ . This is a contradiction. Thus we have that  $\{Q_{q+1}\} \neq E_q \cap E_{q-1}$ . Hence let us study this case (I) by examining the following cases separately.

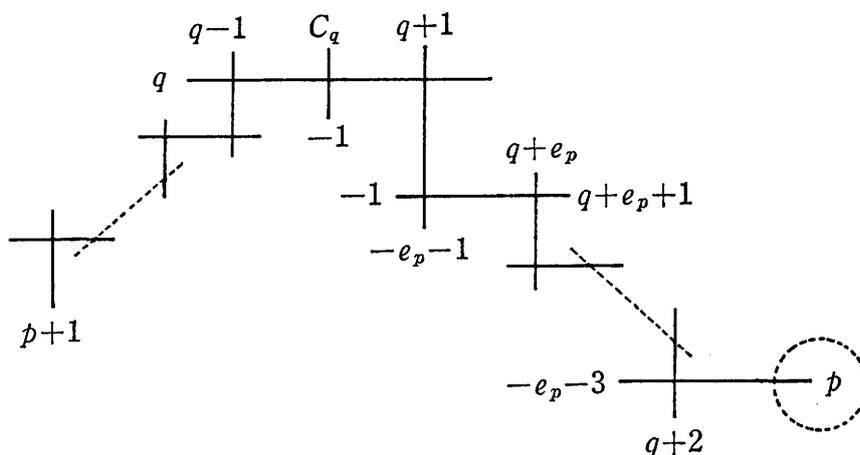
- (1)  $Q_{q+1}$  is a free point on  $E_q$ .
- (2)  $\{Q_{q+1}\} = E_p \cap E_q$ .

In the first case (1) the divisor  $E_1 + \dots + E_p + E_q$  is contained in the center  $D(\varphi)$ . Since  $\varphi^*(C_q) = E_r$ , we get  $r = q + 1$  and the figure of subcase (a).

In the second case (2) the center  $Q_{q+2}$  is a free point on  $E_{q+1}$  or coincides with the point  $E_p \cap E_{q+1}$ . Since the contraction can be started from  $C_q$ , the weight of the proper transform of  $E_{q+1}$  must be  $-e_p - 1$ . Now, in the former case, taking note that  $\Gamma(\varphi)$  is symmetrical about  $E_1 + \dots + E_p + E_{q+1}$ , we get the figure of subcase (b).

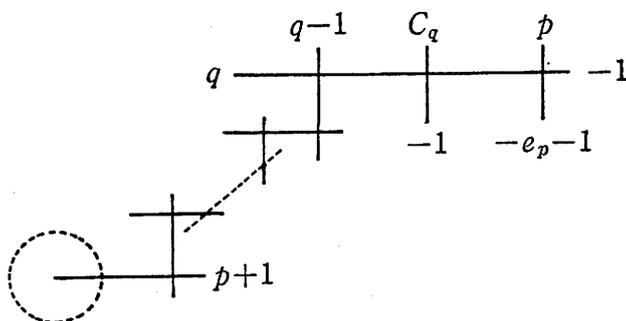


On the other hand, in the latter case, we get the following figure.



Then  $Q_{q+e_p+2}$  is a free point on  $E_{q+e_p+1}$  or coincides with the point  $E_{q+e_p+1} \cap E_{q+e_p}$ . In the former case, taking note that  $\Gamma(\varphi)$  is symmetrical about  $E_1 + \dots + E_p + E_{q+2} + \dots + E_{q+e_p+1}$ , we get the figure (c) with  $t=1$ . In the latter case,  $Q_{q+e_p+3}$  is a free point on  $E_{q+e_p+2}$  or coincides with the point  $E_{q+e_p+2} \cap E_{q+e_p+1}$ . Then we get the figure (c) with  $t=2$  by the same reason as above or we proceed similarly. In this way we get the figures of subcase (c). From the above procedure we see that every possible case is exhausted.

Next let us take up the case (II). Since  $C_{p-1} \cdot E_{p-1} = e_{p-1} > e_p$ , the curve  $E_{p-1}$  is a tangent to  $C_{p-1}$ , hence  $C_p$ ,  $E_p$  and  $E_{p-1}$  meet at  $Q_{p+1}$ . So that the figure of  $\Gamma_q = (\sigma_1 \dots \sigma_q)^{-1}(C)$  is as follows.

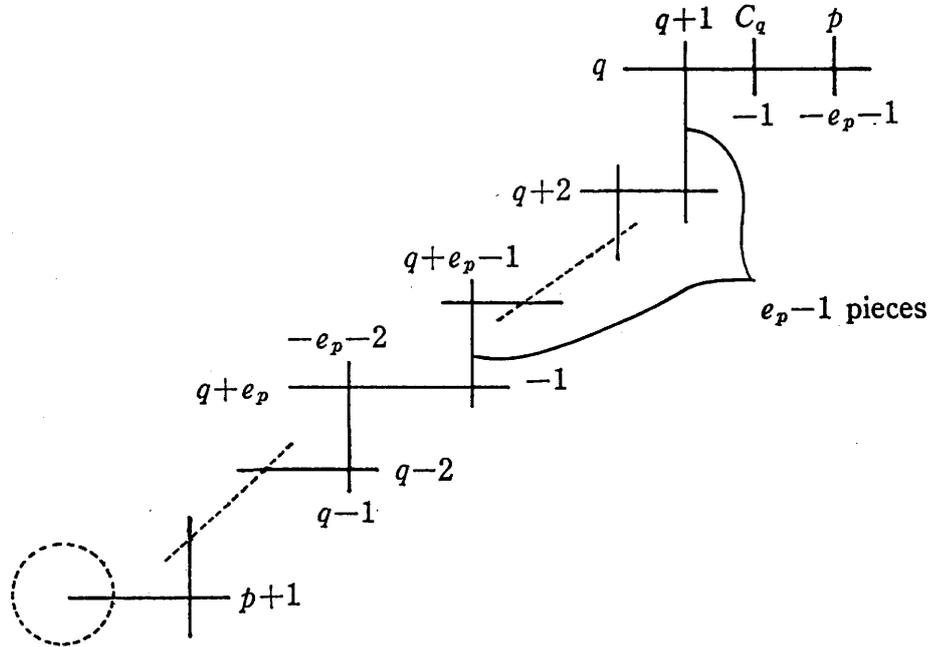


Note that  $\Gamma_q$  is independent of  $\varphi$  and  $\varphi^*$  is assumed to have the order 2. Hence by the hypothesis the divisor  $E_1 + \dots + E_{p-1} + E_{p+1} + \dots + E_{q-1}$  is contained in the center  $D(\varphi)$ . Then we conclude similarly that  $\{Q_{q+1}\} \neq E_p \cap E_q$ . Hence let us study this case (II) by examining the following cases separately.

- (1)  $Q_{q+1}$  is a free point on  $E_q$ .
- (2)  $\{Q_{q+1}\} = E_{q-1} \cap E_q$ .

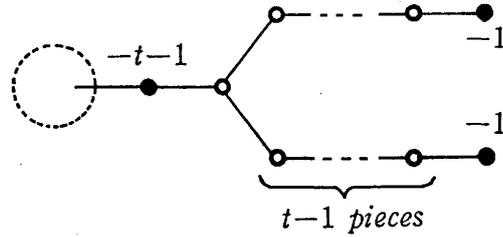
In the first case (1)  $E_q$  is also contained in the center  $D(\varphi)$ . Since  $\varphi^*(C_q) = E_r$ ,

we get  $r=q+1$  and the figure of subcase (a). In the second case (2) the weight of the proper transform of  $E_{q-1}$  must be  $-e_p-2$ , since the contraction can be started from  $C_q$  and  $E_p$  must be contracted. Then we get the following figure.

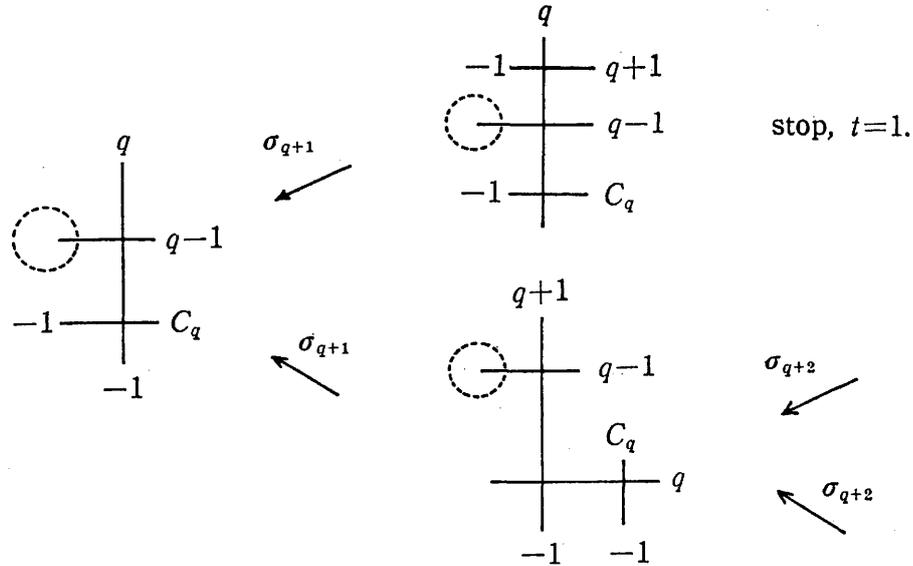


Similarly, since  $E_p$  must be contracted, we have that  $\{Q_{q+e_p+1}\} \neq E_{q+e_p} \cap E_{q+e_p-1}$ . Then  $Q_{q+e_p+1}$  is a free point on  $E_{q+e_p}$  or coincides with the point  $E_{q-1} \cap E_{q+e_p}$ . In the former case, taking note that  $\Gamma(\varphi)$  is symmetrical about  $E_1 + \dots + E_{p-1} + E_{p+1} + \dots + E_{q-1} + E_{q+e_p}$ , we get  $r=q+2e_p+2$  and the figure of subcase (b). In the latter case, since  $E_p$  must be contracted, we have that  $\{Q_{q+e_p+2}\} = E_{q+e_p} \cap E_{q+e_p+1}$ . Then similarly  $\{Q_{q+e_p+3}\} \neq E_{q+e_p} \cap E_{q+e_p+2}$ , i. e.,  $Q_{q+e_p+3}$  is a free point on  $E_{q+e_p+2}$  or coincides with  $E_{q+e_p+1} \cap E_{q+e_p+2}$ . In the former case, taking note that  $\Gamma(\varphi)$  is symmetrical, we get the figure (c) with  $t=1$ . In the latter case  $Q_{q+e_p+4}$  is a free point on  $E_{q+e_p+3}$  or coincides with  $E_{q+e_p+1} \cap E_{q+e_p+3}$ . Then we get the figure (c) with  $t=2$  by the same reason as above or proceed similarly. In this way we get the figures of subcase (c). From the above procedure we see that every possible case is exhausted. Q. E. D.

PROPOSITION 5.4. Suppose that  $R \geq 1$  and  $\mathcal{G} \neq \mathcal{L}$ . Then for every  $\varphi$  in  $\mathcal{G} \setminus \mathcal{L}$  we have that  $q=p+e_p+R$ . Suppose moreover that  $\varphi^*$  has the order 2 and the center  $D(\varphi)$  of  $\Gamma(\varphi)$  contains  $E_1 + \dots + E_{q-2}$ . Then  $r=q+2t-1$ , where  $t$  is a positive integer determined by  $\varphi$ , and the figure of  $\Gamma(\varphi)$  is as follows, where the dotted circle represents the divisor  $E_1 + \dots + E_{q-2}$ .



PROOF. By Lemma 4.2 we have that  $\{Q_i\} = E_{i-1} \cap C_{i-1}$  for  $i \leq q$ . Since  $C_q^2 = -1$ , the center  $Q_{q+1}$  lies in  $E_q \setminus C_q$ . First we consider the case when  $Q_{q+1}$  is a free point on  $E_q$ . Then  $E_1 + \dots + E_q$  is the center  $D(\varphi)$ , since  $E_1, \dots, E_{q-2}$  are contained in the center, hence we get  $r = q + 1$  and the figure with  $t = 1$ .



Next we consider the case when  $\{Q_{q+1}\} = E_{q-1} \cap E_q$ . Then we see that  $Q_{q+2}$  lies in  $E_{q+1} \setminus E_q$ . In case  $Q_{q+2}$  is a free point on  $E_{q+1}$ , then  $E_1 + \dots + E_{q-1} + E_{q+1}$  is the center  $D(\varphi)$ . Hence we get  $r = q + 3$  and the figure with  $t = 2$ . On the other hand in case  $\{Q_{q+2}\} = E_{q-1} \cap E_{q+1}$ , then we proceed similarly in order to get contractible graphs. Seeing from this procedure, we infer that every possible case is exhausted. Q. E. D.

REMARK 5.5. It seems that the above two propositions hold true without the assumptions about  $\varphi^*$  and  $D(\varphi)$ .

Let  $\Gamma$  denote a divisor on  $S_r$  with a figure in Proposition 5.3 or 5.4. As we see from the manner of the proof of them, we have the following result independently of the automorphisms.

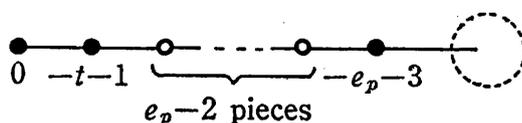
REMARK 5.6. For a curve with  $g=0$ ,  $N=1$  and  $R \geq 0$  we can perform blow-ups in order to get  $\Gamma$ . Conversely, for any divisor  $\Gamma$ , there is a curve  $C$  with  $g=0$ ,  $N=1$  and  $R \geq 0$  such that  $\Gamma$  is obtained from  $C$  by some blow-ups.

Now, let us begin the proof of (6) in Theorem A. It suffices to find a set of elements of  $\mathcal{G}$  which defines the group  $(G_a)^n$ . By the above remark we shall use the divisors  $\Gamma$  to find such elements. In the case when  $R=0$  we consider only the graphs of subcase (c), since they turn out to yield the wanted elements.

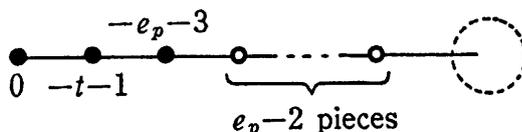
Since  $\Gamma$  is contractible, we have two curves on  $P^2$  which are images of  $\Gamma$ , but we do not know whether they are projectively equivalent. So we cannot conclude immediately that we can get an element of  $\mathcal{G}$  by the blow-ups and the blow-downs (compare this with the case of Lemmas 4.22 and 4.24).

Starting contractions from  $C_r$  and  $E_r$  at the same time and contracting components of  $\Gamma$  symmetrically, we get the following figures (here  $r$  is already irrelevant to the rank).

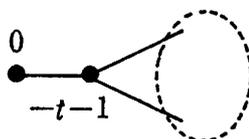
- (1) The case when  $R=0$  and  $e_{p-1}=e_p$ .



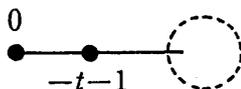
- (2) The case when  $R=0$  and  $e_{p-1} > e_p$ .



- (3) The case when  $R=1$ .



- (4) The case when  $R \geq 2$ .



Note that in case  $R=1$ , the component  $E_{q-1}$  meets  $E_p$ ,  $E_{q-2}$  and  $E_{q+t-1}$ , hence the figure is as above. Let  $F$  be the surface obtained from  $S_r$  by the above contractions and  $\tau_1$  be the morphism  $S_r \rightarrow F$ . In the above figures let us denote

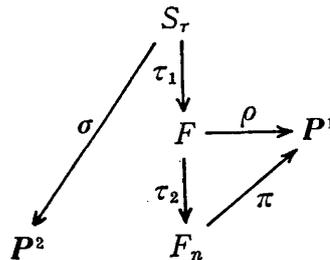
by  $E_f$  and  $E_s$  the components with the weight 0 and  $-t-1$  respectively. Since  $E_f^2=0$  and  $E_f \cdot K = -2$ , we have, by the Riemann-Roch theorem, that  $\dim|E_f| = 1$ . Hence we have a fiber space  $\rho: F \rightarrow P^1$  with a fiber  $E_f$  and a section  $E_s$ . Let  $\mathcal{E}$  be the divisor on  $F$  consisting of the components of  $\tau_1(\Gamma)$  except  $E_f$  and  $E_s$ , i.e.,  $\mathcal{E} = \tau_1(\Gamma)_{\text{red}} - E_f - E_s$ , where  $\tau_1(\Gamma)_{\text{red}}$  is the reduced divisor obtained from  $\tau_1(\Gamma)$ . Since  $\mathcal{E}$  does not meet  $E_f$ , the divisor  $\mathcal{E}$  is contained in fibers of  $\rho$ . Note that  $\mathcal{E}$  is disconnected if and only if  $R=1$ . In case  $R \neq 1$ , let  $\mathcal{F}$  be the unique fiber of  $\rho$  containing  $\mathcal{E}$ , on the other hand in case  $R=1$ , let  $\mathcal{E}_i$  be a connected component of  $\mathcal{E}$  and let  $\mathcal{F}_i$  be the fiber of  $\rho$  containing  $\mathcal{E}_i$ , where  $i=1, 2$ . In this case  $\mathcal{E} = \mathcal{E}_1 + \mathcal{E}_2$ , and put also  $\mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2$ .

Before proceeding further we fix some notations. Let  $F_n$  denote the rational ruled surface  $P(\mathcal{O}_{P^1} \oplus \mathcal{O}_{P^1}(n))$ , where  $n \geq 0$ , and  $\pi: F_n \rightarrow P^1$  the natural morphism. Let  $B_n$  be the base line, i.e., it is the unique irreducible curve on  $F_n$  with the negative weight  $-n$  in case  $n \neq 0$ .

Now, let us return to the proof.

LEMMA 5.7. *In case  $R \neq 1$ , there is only one singular fiber  $\mathcal{F}$ . On the other hand, in case  $R=1$ , there are just two singular fibers  $\mathcal{F}_1$  and  $\mathcal{F}_2$ .*

PROOF. Contracting components of singular fibers of  $\rho$ , we get a birational morphism  $\tau_2: F \rightarrow F_n$  for some  $n \geq 0$ . The number of components of  $\tau_2\{\tau_1(\Gamma) \cup \mathcal{F}\}$  is 3 [resp. 4], in case  $R \neq 1$  [resp.  $R=1$ ].



Comparing the number of blow-ups to obtain  $S_r$  from  $P^2$  with that of blow-downs to obtain  $F_n$  from  $S_r$ , we infer that the latter number is  $r-1$ . Since the number of components of  $\Gamma$  is  $r+1$ , there is one curve  $D$  [resp. two curves  $D_1$  and  $D_2$ ] not belonging to  $\mathcal{E}$  and contracted by  $\tau_2$ . Note that  $\mathcal{E}$  does not contain a curve with the weight  $-1$  by the assertion (1) in Remark 4.4. So the curve  $D$  has the weight  $-1$  and  $\mathcal{F}$  consists of  $\mathcal{E}$  and  $D$  in case  $R \neq 1$ . Similarly in the other case, since  $\mathcal{E}_1$  and  $\mathcal{E}_2$  do not contain curves with the weight  $-1$ , we infer that  $\mathcal{F}_i$  consists of  $\mathcal{E}_i$  and  $D_i$  such that  $D_i^2 = -1$ , where  $i=1$  and  $2$ . Whence the lemma is proved. Q.E.D.

Moreover we have the following

LEMMA 5.8. *There is a birational morphism  $\tau_2: F \rightarrow F_n$  such that*

- (1)  $\tau_2$  is a composition of blow-downs defined by contracting components of  $\mathcal{F}$ ,

(2)  $\tau_2$  does not contract the component(s) of  $\mathcal{F}$  which meet(s)  $E_s$ , hence  $n=t+1$ .

PROOF. Contracting components of  $\mathcal{F}$ , we get a birational morphism  $\tau'_2: F \rightarrow F_{n'}$ . Since  $E_f$  and  $E_s$  meet transversally at one point, the curve  $E_s$  is carried to a section of the natural morphism  $F_{n'} \rightarrow \mathbf{P}^1$ . If the weight of  $\tau'_2(E_s)$  is more than  $-t-1$ , then  $\tau'^{-1}_2$  has a fundamental point at  $\tau'_2(E_s \cap \mathcal{F})$ . Then by the elementary transformation with the center at this point we have a new surface which  $F$  dominates, too. We can repeat this procedure to get the surface  $F_n$  on which the weight of the image of  $E_s$  becomes equal to  $-t-1$ . This proves the whole parts of the lemma. Q.E.D.

As we have shown in the proof of Lemma 5.7, there is an irreducible curve  $D$  [resp. two irreducible curves  $D_1$  and  $D_2$ ] on  $F$  such that  $\mathcal{F} = \mathcal{E} + D$  [resp.  $\mathcal{F}_i = \mathcal{E}_i + D_i$ , where  $i=1, 2$ ]. Put  $C' = \sigma\tau_1^{-1}(D)$  [resp.  $C' = \sigma\tau_1^{-1}(D_1)$  and  $C'' = \sigma\tau_1^{-1}(D_2)$ ]. Then performing suitable successive elementary transformations starting from  $F_{t+1}$ , we complete the proof of Theorem B. Since  $\tau_1^{-1}(D)$ ,  $\tau_1^{-1}(D_1)$  and  $\tau_1^{-1}(D_2)$  do not meet  $C_r$ , Proposition 2 follows easily.

By definition,  $F$  and  $\Gamma$  depend on  $t$ , so hereafter let us write  $F(t)$  and  $\Gamma(t)$  instead of  $F$  and  $\Gamma$  respectively, when we emphasize the parameter  $t$ . Put

$$\mathcal{G}_1(t) = \{\varphi \in \text{Aut}(F(t)) \mid \varphi(\tau_1(\Gamma(t))) = \tau_1(\Gamma(t))\}.$$

Since  $\mathbf{P}^2 \setminus C \cong F(t) \setminus \tau_1(\Gamma(t))$ , we have the following

LEMMA 5.9. For any integer  $t \geq 1$ , the group  $\mathcal{G}_1(t)$  may be regarded as a subgroup of  $\mathcal{G}$ .

Let  $\tau_2: F(t) \rightarrow F_{t+1}$  be the morphism obtained in Lemma 5.8 and put  $\Sigma = E_f + E_s + \mathcal{F}$ .

First we consider the case when  $R \neq 1$ . Note that  $\tau_2(D) = A$  is a point on the fiber  $\tau_2(\mathcal{F})$  not lying on the base line. Then put

$$\mathcal{G}_2(t) = \{\psi \in \text{Aut}(F_{t+1}) \mid \psi(\tau_2(\Sigma)) = \tau_2(\Sigma) \text{ and } \psi(A) = A\}.$$

We shall find elements  $\psi$  of  $\mathcal{G}_2(t)$  such that  $\tau_2^{-1}\psi\tau_2 \in \mathcal{G}_1(t)$ . Let us call such  $\psi$  a *liftable* element. Before stating how to find liftable elements, we make some preparation.

Let  $B$  be a free point on  $D$ , i.e.,  $B$  does not lie on  $\tau_1(\Gamma)$ . Let  $T'$  be an irreducible curve intersecting  $D$  transversally at  $B$ , and  $U$  be a small neighbourhood of  $B$  such that  $U \cap \tau_1(\Gamma) = \emptyset$ . We assume that  $T = T' \cap U$  and  $D$  meet only at  $B$ . Let  $\tau_2 = \tau_{21} \cdots \tau_{2k}$  be the factorization of  $\tau_2$  into blow-downs  $\tau_{2i}$ . Recall that  $\tau_{2k}$  is defined by contracting  $D$ . (Here  $k = p + e_p - 2$  or  $q - 1$  according as  $R = 0$  or  $R \geq 2$ ). Let  $m_i$  be the multiplicity of  $\tau_{2i} \cdots \tau_{2k}(T)$  at  $\tau_{2i} \cdots \tau_{2k}(D)$ , where  $1 \leq i \leq k$ . Then we put  $m_0 = \sum_{i=1}^k m_i^2$ . Note that  $m_0$  is independent of  $t$ , since  $\mathcal{F}$  is independent of  $t$ , more precisely,  $F(t) \setminus (E_f \cup E_s)$  and  $F(1) \setminus (E_f \cup E_s)$  are isomorphic for each  $t$ .

Now, for an element  $\phi$  of  $\mathcal{G}_2(t)$  we define  $I(\phi)$  to be the intersection multiplicity  $(\tau_2(T) \cdot \phi(\tau_2(T)))_A$  [if  $\phi(\tau_2(T)) = \tau_2(T)$ , then we put  $I(\phi) = \infty$ ]. Then we have the following

LEMMA 5.10. *An element  $\phi$  of  $\mathcal{G}_2(t)$  is liftable if  $I(\phi) \geq m_0$ .*

PROOF. Since  $\phi(A) = A$ , we have an isomorphism  $\tau_{21}^{-1}\phi\tau_{21} = \phi_1$  such that  $\phi_1(\tau_{21}^{-1}(A)) = \tau_{21}^{-1}(A)$ . Since  $\phi_1$  also fixes the center of  $\tau_{22}$  by the hypothesis, we have an isomorphism  $\tau_{22}^{-1}\phi_1\tau_{22} = \phi_2$  such that  $\phi_2(\tau_{22}^{-1}\tau_{21}^{-1}(A)) = \tau_{22}^{-1}\tau_{21}^{-1}(A)$ . In view of the above preparation this procedure can be continued to get the lift of  $\phi$ .

Q. E. D.

By the way, we consider generally  $\text{Aut}(F_n)$  for a while. Let  $(u_0, u_1)$  be a set of homogeneous coordinates on  $P^1$  and  $(v_0, v_1)$  affine coordinates on  $A^2$ . The elements  $(u_0, u_1, v_0, v_1)$  and  $(u'_0, u'_1, v'_0, v'_1)$  of  $P^1 \times (A^2 \setminus \{(0, 0)\})$  determine the same point on  $F_n$  if and only if there are  $\alpha$  and  $\beta$  in  $C^* = C \setminus \{0\}$  such that

$$u'_0 = \alpha u_0, \quad u'_1 = \alpha \beta^n u_1, \quad v'_0 = \beta v_0, \quad v'_1 = \beta v_1.$$

For  $n > 0$  automorphisms of  $F_n$  are of the form

$$\begin{cases} u'_0 = u_0 \\ u'_1 = \gamma u_1 + u_0 \Phi(v_0, v_1), \end{cases} \quad \begin{cases} v'_0 = a v_0 + b v_1 \\ v'_1 = c v_0 + d v_1, \end{cases}$$

where  $\gamma \in C^*$  and  $\Phi$  is a form of degree  $n$ , and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PGL}_2.$$

Now, let us resume the proof. In the coordinates given above, the equation of  $\tau_2(E_s)$  is  $u_0 = 0$ . Let the equations of  $\tau_2(\mathcal{F})$  and  $\tau_2(E_f)$  be  $v_0 = 0$  and  $v_1 = 0$  respectively. Since  $A$  is a free point on  $\tau_2(\mathcal{F})$ , we may assume that  $A = (1, 0, 0, 1)$ . Let  $(\eta, \theta)$  be a set of local coordinates near  $A$ , where  $\eta = u_1/u_0$  and  $\theta = v_0/v_1$ . Since an element  $\phi$  of  $\mathcal{G}_2(t)$  fixes the curves  $\tau_2(\mathcal{F})$ ,  $\tau_2(E_f)$  and the point  $A$ , it is of the form

$$\begin{cases} \eta' = \gamma \eta + g(\theta), & \text{where } g(\theta) = v_1^{t+1} \Phi(\theta, 1) \text{ and } g(0) = 0. \\ \theta' = (a/d)\theta. \end{cases}$$

Note that  $g(\theta)$  is a polynomial of degree  $\leq t+1$ . Let the local equation of  $\tau_2(T)$  be  $h(\eta, \theta) = 0$ . Since the local equation of  $\tau_2(\mathcal{F})$  is  $\theta = 0$ , we have that  $h(\eta, 0) \neq 0$ . Then  $h(\eta, \theta)$  can be expressed as

$$\eta^m + w_1(\theta)\eta^{m-1} + \dots + w_m(\theta) = 0,$$

where  $w_i(\theta)$ ,  $1 \leq i \leq m$ , is a convergent power series with  $w_i(0) = 0$ . Let us take the automorphism  $\phi$  with  $a = d$  and  $\gamma = 1$ . Then the equation of  $\phi(\tau_2(T))$  is

$$(\eta - g(\theta))^m + w_1(\theta)(\eta - g(\theta))^{m-1} + \dots + w_m(\theta) = 0.$$

Whence we infer that  $I(\phi) \geq sm$  if  $g(\theta) = a_{t+1}\theta^{t+1} + \dots + a_s\theta^s$ . Thus by Lemma 5.10 we see that  $\phi$  is liftable to  $\mathcal{G}_1(t)$  if  $s \geq m_0$ . Taking such automorphisms, we have that  $\mathcal{G} \supset (\mathcal{G}_a)^{t+2-s}$  by Lemma 5.9. Recalling that  $m_0$  is independent of  $t$ , this relation holds good for every sufficiently large  $t$ , hence we complete the proof of (6) in Theorem A when  $R \neq 1$ .

Next we consider the case when  $R=1$ . The point  $A_i = \tau_2(D_i)$  lies on the fiber  $\tau_2(\mathcal{F}_i)$ , but not on the base line, where  $i=1, 2$ . Then put

$$\mathcal{G}_2(t) = \{\phi \in \text{Aut}(F_{t+1}) \mid \phi(\tau_2(\Sigma)) = \tau_2(\Sigma) \text{ and } \phi(A_i) = A_i \text{ for } i=1, 2\}.$$

Similarly let  $T_i$  be a part of a curve on  $F(t)$  intersecting  $D_i$  at a free point, where  $i=1, 2$ . For an element  $\phi$  of  $\mathcal{G}_2(t)$  we define  $I_i(\phi)$  to be  $(\tau_2(T_i) \cdot \phi(\tau_2(T_i)))_{A_i}$ . Then similarly there are integers  $m_{0i}$  such that  $\phi$  is liftable if  $I_i(\phi) \geq m_{0i}$  for  $i=1$  and  $2$ . Since  $\phi$  fixes three fibers, it is of the form  $a=d$  and  $b=c=0$ . Let the equations of three fibers be  $v_0=0$ ,  $v_0=v_1$  and  $v_1=0$  respectively. We may assume that  $A_1=(1, 0, 0, 1)$ ,  $A_2=(\alpha_1, \alpha_2, 1, 1)$  and  $\alpha_1 \neq 0$ . In case  $\gamma=1$  and  $a=1$ , the automorphism fixes  $A_1$  and  $A_2$  if and only if  $\Phi(0, 1) = \Phi(1, 1) = 0$ . Then take the automorphism of the form

$$\begin{cases} \eta' = \eta + g(\theta), & g(0) = g(1) = 0, \\ \theta' = \theta, \end{cases}$$

where the notations are the same as above.

Let the first condition  $I_1(\phi) \geq m_{01}$  be similarly described as above. The second condition  $I_2(\phi) \geq m_{02}$  decreases the number of the free coefficients of  $g(\theta)$ , i.e., the order of  $g(\theta+1) = a_{t+1}(\theta+1)^{t+1} + \dots + a_{m_{01}}(\theta+1)^{m_{01}}$  must be at least  $m_{02}$ , but as we have mentioned above  $t$  can take every sufficiently large integer. Hence we conclude similarly (6) in Theorem A. Thus we have finished the proof of the whole parts of Theorem A.

Lastly we prove the corollary, which is an immediate consequence of Theorem B. In fact, if  $R \neq 1$ , then we have that  $\bar{\kappa}(V) \leq \bar{\kappa}(\mathbf{P}^2 \setminus (C \cup C_1)) = \bar{\kappa}(\mathbf{P}^2 \setminus (L_1 \cup L_2)) = -\infty$ . If  $R=1$ , then similarly we have that  $\bar{\kappa}(V) \leq \bar{\kappa}(\mathbf{P}^2 \setminus (L_1 \cup L_2 \cup L_3)) = -\infty$ , since  $L_1 \cap L_2 = L_1 \cap L_3$ .

## 6. Examples and problems.

Also in this section we follow the notations in the previous sections. We present examples of curves with  $g=0$ ,  $N=1$  and  $R \leq 1$ . For the details, see [9]. Put  $F(X, Y, Z) = a^n(YZ^{n-1} - X^n)^{m_{n+1}} + \{aX(YZ^{n-1} - X^n)^m + \sum_{i=1}^m b_i Z^{ni+1}(YZ^{n-1} - X^n)^{m-i}\}^n$ , where  $a \neq 0$ ,  $m \geq 1$ ,  $n \geq 2$  and  $b_i$  are arbitrary for  $i=1, \dots, m$ . Let  $C$  be the curve defined by  $F/Z^{n-1}$ . Then  $C$  has the following properties.

- (1)  $C \setminus \{(0, 1, 0)\} \cong A^1$ .
- (2)  $d = mn^2 + 1$ ,  $p = 2m + 2n$ ,  $e_1 = mn^2 - mn$ ,  $e_2 = \dots = e_{2n} = mn$ ,  
 $e_{2n+1} = \dots = e_{2m+2n} = n$ .
- (3)  $R = 2 - n$ .

Moreover we have the following new one. Let  $\Theta_\lambda$  be the conic defined by

$$YZ - X^2 + \lambda Y^2 = 0.$$

Then  $\Theta_0$  and  $\Theta_{\lambda'}$ , where  $\lambda' \neq 0$ , meet at only one point  $(0, 0, 1)$ . Let  $\varphi$  be a non-linear automorphism of  $P^2 \setminus \Theta_0$  inducing an automorphism on the line  $Z=0$ . Then putting  $C = \varphi(\Theta_{\lambda'})$ , we have that  $C \setminus \{(0, 1, 0)\} \cong A^1$ ,  $e_p = 4$  and  $d^2 - \sum_{i=1}^p e_i^2 = 4$ , since the morphism  $\varphi\sigma$  contracts first the proper transform  $\sigma^{-1}[\Theta_0]$ . This shows that  $R=1$ .

Finally we raise problems concerning curves with  $g=0$  and  $N=1$ .

PROBLEM 1. Do there exist curves with  $R=0$  and  $e_{p-1} > e_p$ ?

PROBLEM 2. Find  $\bar{\kappa}(V)$  in the case when  $R \leq -1$ . Especially do there exist curves with  $\bar{\kappa}(V) = 2$ ?

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