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Functional calculus for certain Banach function algebras

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In this paper we study the symbolic calculus for a Banach function algebra with certain conditions. First we define a class of Banach function algebras which contains the class of all function algebras and the class of all ultraseparating Banach function algebras. Our purpose is to prove the theorem asserting that if A is a non-trivial Banach function algebra in the class, then only analytic functions can operate on A. It is a generalization of theorems of de Leeuw and Katznelson [**6**], Bernard [**2**] and Bernard and Dufresnoy [**3**].

1. Introduction.

Let A be a Banach function algebra on a compact Hausdorff space X, that is, a point separating unital subalgebra of $C(X)$ (the algebra of all complex valued continuous functions on X) with the Banach algebraic norm $N(\cdot)$. We say that A is a function algebra if $N(\cdot)$ is the supremum norm $\|\cdot\|_{\infty}$. Suppose that h is a complex valued continuous function on an open subset D of the complex plane. We say that h operates by composition on A if $h\circ f$ is in A whenever $f\in A$ has the range contained in D.

de Leeuw-Katznelson [**6**] proved that if A is a non-trivial function algebra (=non $C(X)$), the nany non-analytic function does not operate by composition on A. But it is not the case for Banach function algebras as many examples show. For example the algebra

$$A(\Gamma)=\{f\in C(\Gamma): \sum|\hat{f}(n)|<\infty\}$$

of all continuous functions on the unit circle Γ with absolutely convergent Fourier series with the norm

$$N(f)=\sum|\hat{f}(n)|$$

for f in $A(\Gamma)$, where $\hat{f}(n)$ denotes n-th Fourier coefficients, is conjugate closed non-trivial Banach function algebra. Bernard [**2**] defined ultraseparability for Banach function algebras and showed that $\bar{z}$ does not operate by composition on a non-trivial ultraseparating Banach function algebra. Bernard-Dufresnoy [**3**] also showed that if a non-analytic function h operates *boundedly* on an ultraseparating Banach function algebra, then it is trivial one. Our result is a generalization of above results by de Leeuw-Katznelson and Bernard and Bernard-Dufresnoy.

We denote the uniform closure of A in $C(X)$ by cl$A$ in this paper.

2. Definition and examples.

Let E be a normed space with the norm $n(\cdot)$. $\tilde{E}$ denotes the space of all bounded sequences in E. Suppose that A is a Banach function algebra on X with the norm $N(\cdot)$. We may consider that $\tilde{A}$ is a subalgebra of $C(\tilde{X})$, where $\tilde{X}$ is the Stone-Cech compactification of the direct product of the space of all positive integers $\boldsymbol{N}$ and the compact Hausdorff space X, with the Banach algebraic norm $\tilde{N}(\cdot)$ which is defined as follows:

$$\tilde{N}(\tilde{f})=\sup\{N(f_n):\ n=1,\ 2,\ \cdots\}$$

for $\tilde{f}=(f_n)$ in $\tilde{A}$. If $\tilde{A}$ separates the points of $\tilde{X}$, we say that A is ultraseparating on X. Suppose that Y is a compact subset of X. Put

$$A|Y=\{f\in C(Y):\ F|Y=f \text{ for } {}^{\exists}F\in A\},$$

where $F|Y$ is the restriction of F to Y. Then $A|Y$ is a Banach function algebra on Y with the norm

$$N_Y(f)=\inf\{N(F):\ F\in A \text{ and } F|Y=f\}$$

for f in $A|Y$. If A is ultraseparating on X, then $\tilde{A}$ is ultraseparating on $\tilde{X}$ with $\tilde{N}(\cdot)$ [1] and $A|Y$ is also ultraseparating on Y with $N_Y(\cdot)$. We may suppose that $(A|Y)^{\sim}=\tilde{A}|\tilde{Y}$ and $\tilde{N}_{\tilde{Y}}(\cdot)=(N_Y)^{\sim}(\cdot)$. We denote $(\tilde{A})^{\sim}$ by $\tilde{\tilde{A}}$ and $(\tilde{N})^{\sim}(\cdot)$ by $\tilde{\tilde{N}}(\cdot)$ and $(\tilde{X})^{\sim}$ by $\tilde{\tilde{X}}$ respectively. For further information for ultraseparability, see [1], [2] and [3].

DEFINITION. Let A and B be Banach function algebras on X. We say that A is B-ultraseparating if $\tilde{A}$ separates the points in $\tilde{X}$ which are separated by $\tilde{B}$.

Let A be an ultraseparating Banach function algebra on X with the norm $N(\cdot)$. Then A is $C(X)$-ultraseparating and cl$A$-ultraseparating on X. If A is a function algebra, then A is clearly cl$A$-ultraseparating by definition. If A is B-ultraseparating on X and Y is a closed subset of X, then $A|Y$ is $B|Y$-ultraseparating on Y.

Let H^∞ be the uniformly closed subalgebra of L^∞ (the algebra of all bounded measurable functions on the unit circle Γ in the complex plane) consisting of L^∞ functions whose negative Fourier coefficients vanish. Then H^∞ is ultraseparating on the maximal ideal space of L^∞ since H^∞ is a logmodular subalgebra of L^∞.

PROPOSITION 1. *Let B be a Douglas algebra on Γ (uniformly closed algebra lying between H^∞ and L^∞). We consider that H^∞ and B are function algebras on the maximal ideal space $M(B)$ of B. Then H^∞ is B-ultraseparating. Suppose that Y is a closed restriction set for B ($B|Y$ is uniformly closed); then $H^\infty|Y$ is* $\mathrm{cl}(H^\infty|Y)$*-ultraseparating.*

PROOF. Suppose that $\tilde{x}$ and $\tilde{y}$ are different points in $\tilde{M}(B)$ and $\tilde{f}=(f_n)\in\tilde{B}$ separates them. Let C_B be the C^*-algebra generated by invertible Blaschke products in B. In [**5**] it is shown that the linear span $H^\infty+C_B$ equals B. By the open mapping theorem there is a constant K which depends only on B with the property that for every f in B there are a $g\in H^\infty$ and an $h\in C_B$ such that $f=g+h$ and $\|g\|_\infty\leqq K\|f\|_\infty$ and $\|h\|_\infty\leqq K\|f\|_\infty$. Thus we have an expression $f_n=g_n+h_n$ with $\|g_n\|_\infty\leqq K\|f_n\|_\infty$ and $\|h_n\|_\infty\leqq K\|f_n\|_\infty$ for $g_n\in H^\infty$ and $h_n\in C_B$ for every positive integer n. So $\tilde{g}=(g_n)\in\tilde{H}^\infty$ or $\tilde{h}=(h_n)\in\tilde{C}_B$ separates $\tilde{x}$ and $\tilde{y}$. Suppose that $\tilde{g}$ does not separate them. Without loss of generality we may assume $\tilde{h}(\tilde{x})=0$ and $\tilde{h}(\tilde{y})=1$. Put

$$X_n^{(1)}=(\{n\}\times M(B))\cap\{z\in\tilde{M}(B):|\tilde{h}(z)|\leqq 1/3\}$$

and

$$X_n^{(2)}=(\{n\}\times M(B))\cap\{z\in\tilde{M}(B):|\tilde{h}(z)|\geqq 1/2\}.$$

Let us denote

$$PX_n^{(1)}=\{p\in M(C_B):\ ^\exists z\in X_n^{(1)} \text{ such that } f(z)=f(p) \text{ for } ^\forall f\in C_B\}$$

and

$$PX_n^{(2)}=\{p\in M(C_B):\ ^\exists z\in X_n^{(2)} \text{ such that } f(z)=f(p) \text{ for } ^\forall f\in C_B\}.$$

Since we may consider that the maximal ideal space $M(C_B)$ of C_B is the equivalent class of the points of $M(B)$ which is not separated by functions in C_B, it is easy to see that we may suppose that $PX_n^{(1)}$ and $PX_n^{(2)}$ are disjoint compact subsets of $M(C_B)$. Since $H^\infty\cap C_B$ is a logmodular subalgebra of C_B on $M(C_B)$ [**5**], we can take functions j_n in $H^\infty\cap C_B$ with $\|j_n\|_\infty\leqq 3$ and $|j_n|\leqq 1$ on $PX_n^{(1)}$ and $|j_n|\geqq 2$ on $PX_n^{(2)}$ for every n. It follows that $\tilde{j}=(j_n)\in(H^\infty\cap C_B)^\sim$ separates $\tilde{x}$ and $\tilde{y}$ as a function on $\tilde{M}(B)$. Thus we have concluded that H^∞ is B-ultraseparating.

Let $\tilde{x}$ and $\tilde{y}$ be a point in Y and suppose that $(\mathrm{cl}(H^\infty|Y))^\sim$ separates them. Then $(B|Y)^\sim$ separates $\tilde{x}$ and $\tilde{y}$ since $\tilde{Y}$ is a closed restriction set for $\tilde{B}$. Thus $(H^\infty|Y)^\sim$ separates $\tilde{x}$ and $\tilde{y}$ since H^∞ is B-ultraseparating.

Generally $H^\infty|Y$ is not uniformly closed nor ultraseparating, because Y may not be a closed restriction set for H^∞ and because Y may contain analytic disks.

Let A_i be B_i-ultraseparating Banach function algebra on X_i with the norm

$N_i(\cdot)$ for $i=1, 2$. Suppose that X is the formal union $X_1 \cup X_2$. Then we see that

$$A=\{f \in C(X): f|X_1 \in A_1 \text{ and } f|X_2 \in A_2\}$$

is a Banach function algebra on X with respect to the norm

$$N_A(f)=\max\{N_1(f|X_1), N_2(f|X_2)\}$$

and B-ultraseparating on X where B denotes the Banach algebra

$$\{f \in C(X): f|X_1 \in B_1 \text{ and } f|X_2 \in B_2\}$$

with the norm

$$N_B(f)=\max\{N_1'(f|X_1), N_2'(f|X_2)\}.$$

($N_i'(\cdot)$ denotes the norm of B_i for $i=1, 2$.)

There are many other examples of $B(\mathrm{cl}A)$-ultraseparating Banach function algebra A which is not uniformly closed nor ultraseparating.

PROPOSITION 2. *Let A and B be Banach function algebras on X. Then A is B-ultraseparating on X if and only if for every positive integer m, there exist a positive δ and a positive integer l with the following property* (*):

(*) *For every disjoint compact subsets X_1 and X_2 of X with the condition that there are $a_1, a_2, \cdots, a_m$ and $b_1, b_2, \cdots, b_m$ in the unit ball of B such that*

$$\sum_{j=1}^{m}(|a_j|-|b_j|) \geqq 1/2 \quad \text{on} \quad X_1$$

$$\sum_{j=1}^{m}(|a_j|-|b_j|) \leqq -1/2 \quad \text{on} \quad X_2,$$

there exist $f_1, f_2, \cdots, f_l$ and $g_1, g_2, \cdots, g_l$ in the unit ball of A such that

$$\sum_{j=1}^{l}(|f_j|-|g_j|) \geqq \delta \quad \text{on} \quad X_1$$

$$\sum_{j=1}^{l}(|f_j|-|g_j|) \leqq -\delta \quad \text{on} \quad X_2.$$

Proposition 2 is a generalization of the characterization for ultraseparating Banach function algebras [**1**] and we can prove it almost in the same way as Theorem in [**1**].

3. Main result.

In this section we show a generalization of the theorems of de Leeuw-Katznelson [**6**], Bernard [**2**] and Bernard-Dufresnoy [**3**].

THEOREM. *Let A be a* cl*A-ultraseparating Banach function algebra on X with the norm $N(\cdot)$ and h be a continuous but non-analytic function on an open subset of the complex plane. Suppose that h operates by composition on A; then* *we have* $A=C(X)$.

PROOF. It is trivial that h operates by composition on $\mathrm{cl}A$. It follows that $\mathrm{cl}A=C(X)$ by the theorem of de Leeuw-Katznelson. So we may consider only the case of ultraseparating Banach function algebras. Without loss of generality we may assume that h is defined on the closed unit disk and that h is not analytic at the origin. Let Δ_δ be a positive C^∞-function on the complex plane $\boldsymbol{C}$ supported in $\{w\in\boldsymbol{C}:|w|<\delta\}$ with a small positive δ such that $\iint\Delta_\delta(w)dxdy=1$, where $w=x+yi$. Put

$$H_\delta(z_1, z_2)=\iint h(z_1-z_2w)\Delta_\delta(w)dxdy \qquad |z_1|<1-\delta,\quad |z_2|<1.$$

Then $H_\delta(z_1, z_2)$ is a C^∞-function and tends to $h(z_1)$ uniformly on a compact subset of

$$\{(z_1, z_2)\in\boldsymbol{C}^2:\ |z_1|<1,\ 0<|z_2|<1\}$$

as δ tends to 0. We consider the following two cases:

(1) For some z_2 with $0<|z_2|<1$, there exists a $\eta>0$ such that $\partial/\partial\bar{z}_1H_\delta(z_1, z_2)=c_\delta$ for every small δ on $\{z_1\in\boldsymbol{C}:|z_1|<\eta\}$.

(2) For every z_2 with $0<|z_2|<1$ and for every small $\eta>0$, there exists a small δ_0 such that for every δ_1 with $0<\delta_1<\delta_0$ there exists δ with $0<\delta<\delta_1$ such that $\partial/\partial\bar{z}_1H_\delta(z_1, z_2)$ is not a constant on $\{z_1\in\boldsymbol{C}:|z_1|<\eta\}$ with respect to z_1.

Case (1). We may suppose that $H_\delta(z_1, z_2)-c_\delta\bar{z}_1=P_\delta(z_1)$ is analytic on $\{z_1\in\boldsymbol{C}:|z_1|<\eta/2\}$ and continuous on $\{z_1\in\boldsymbol{C}:|z_1|\leqq\eta/2\}$. Suppose that $\sup\|P_\delta\|_\infty=\infty$ (supremum takes for all small $\delta>0$). Then $\sup|c_\delta|=\infty$. So $\sup_\delta\inf\{\|c_\delta\bar{z}+f\|_\infty: f\in A_0\}=\infty$, where A_0 is a disk algebra on $\{z_1\in\boldsymbol{C}:|z_1|\leqq\eta/2\}$. It is a contradiction since $H_\delta(z_1, z_2)=c_\delta\bar{z}_1+P_\delta(z_1)$ and

$$\sup_\delta\sup\{|H_\delta(z_1, z_2)|:|z_1|\leqq\eta/2\}\leqq\sup\{|h(z)|:|z|\leqq\eta/2\}<\infty.$$

Thus we have $\sup\|P_\delta\|_\infty<\infty$ and it follows by the normal family argument that there is a sequence $\{\delta(n)\}$ of positive numbers converging to 0 which satisfies that $P_{\delta(n)}$ tends to an analytic function P as n tends to 0. So left hand side of

$$H_{\delta(n)}(z_1, z_2)-P_{\delta(n)}(z_1)=c_{\delta(n)}\bar{z}_1$$

tends to $h(z_1)-P(z_1)$ and thus right hand side to $c\bar{z}_1$ as n tends to ∞. We conclude that $h(z)=c\bar{z}+P(z)$ on $\{z\in\boldsymbol{C}:|z|\leqq\eta/3\}$ where $P(z)$ is analytic near $\{z\in\boldsymbol{C}:|z|\leqq\eta/3\}$ and we may suppose that $c\neq0$ since h is not analytic at the origin. Let f be a function in A. Put a sufficiently small positive real number α with

$$\alpha f(M_A)\subset\{z\in\boldsymbol{C}:\ |z|\leqq\eta/4\}$$

for the maximal ideal space M_A of A. Then

$$h(\alpha f)=c\cdot\overline{\alpha f}+P(\alpha f)$$

and

$$P(\alpha f)$$

are functions in A since $P(z)$ is analytic near $\{z\in \boldsymbol{C}: |z|\leqq\eta/3\}$. So we see that $\bar{f}$ is in A since $c\cdot\overline{\alpha f}$ is in A and $c\bar{\alpha}\neq 0$. Thus we conclude that $A=C(X)$ by a theorem of Bernard [**2**] since we have shown that A is conjugate closed and since A is ultraseparating.

Case (2). Put

$$A_x=\{f\in A: f(x)=0\}$$

and $A'_x=\{f\in A_x: \|f\|_\infty\leqq 1\}$ for a point x in X. Then we have

$$A'_x=\bigcup_n\{f\in A'_x: N(h\circ f)<n\}$$

and it follows that there is an integer n_0 such that the closure of $\{f\in A'_x: N(h\circ f)<n_0\}$ with respect to the topology on A_x induced by the norm $N(\cdot)$ contains an open subset by the Baire's category theorem. Therefore there exist an $\varepsilon>0$ and a g in A'_x and a dense subset U of

$$\{f+g\in A_x: N(f)<2\varepsilon\}$$

such that $N(h\circ F)<n_0$ for every F in U. Since A is ultraseparating, $\tilde{A}$ and $\tilde{\tilde{A}}$ are ultraseparating on $\tilde{X}$ and $\tilde{\tilde{X}}$ respectively. So we have

$$\inf\sup\{|f(y_1)|: f\in\tilde{\tilde{A}},\ f(y_2)=0,\ \tilde{N}(f)\leqq 1\}=2M>0,$$

where infimum takes for all different points y_1 and y_2 in $\tilde{\tilde{X}}$ [**2**]. Put a compact neighborhood

$$Y=\{y\in X: |g(y)|\leqq M^2\varepsilon/6\}$$

of x. We will show that Y is an interpolation set for A, that is, $A|Y=C(Y)$. Suppose that it follows that $A|Y=C(Y)$ for each point x in X. Then we have a finite number of interpolating compact subsets $Y_1, Y_2, \cdots, Y_n$ of X which cover X. Since $\tilde{X}=\bigcup\tilde{Y}_i$ and

$$\mathrm{cl}\tilde{A}|\tilde{Y}_i\supset\tilde{A}|\tilde{Y}_i=(A|Y_i)^{\sim}=C(\tilde{Y}_i)$$

for $i=1, 2, \cdots, n$, we see that $\mathrm{cl}\tilde{A}=C(\tilde{X})$. So we have $A=C(X)$ by Bernard's lemma.

Let $\tilde{Y}_x$ be the quotient space reduced by identifying the points of $\tilde{Y}$ which are not separated by $\tilde{A}_x|\tilde{Y}$. Let $\tilde{x}_0$ be the point in $\tilde{Y}_x$ which corresponds to

$$J=\{\tilde{x}\in\tilde{Y}: \tilde{f}(\tilde{x})=0 \text{ for } {}^\forall\tilde{f}\in\tilde{A}_x|\tilde{Y}\}.$$

In fact $\tilde{x}_0$ is the only point in $\tilde{Y}_x$ which is identified more than one points in

$\tilde{Y}$. Let $\tilde{x}_1$ and $\tilde{x}_2$ be points in $\tilde{Y}\setminus J$, then we see that an $\tilde{f}=(f_n)$ in $\tilde{A}$ separates $\tilde{x}_1$ and $\tilde{x}_2$ since A is ultraseparating on X and since we may suppose that $\tilde{x}_1$ and $\tilde{x}_2$ are points in $\tilde{X}$. Suppose that $(\tilde{f}-(f_n(x)))|\tilde{Y}$ separates $\tilde{x}_1$ and $\tilde{x}_2$. It shows that $\tilde{A}_x|\tilde{Y}$ separates $\tilde{x}_1$ and $\tilde{x}_2$ since $(\tilde{f}-(f_n(x)))|\tilde{Y}$ is in $\tilde{A}_x|\tilde{Y}$. Suppose that $(\tilde{f}-(f_n(x)))(\tilde{x}_1)=(\tilde{f}-(f_n(x)))(\tilde{x}_2)$, it follows that $(f_n(x))(\tilde{x}_1)\neq(f_n(x))(\tilde{x}_2)$. There is a $\tilde{k}$ in $\tilde{A}_x|\tilde{Y}$ such that $\tilde{k}(\tilde{x}_1)\neq 0$, because $\tilde{x}_1$ is in $\tilde{Y}\setminus J$. Thus $\tilde{k}$ or $(f_n(x))\tilde{k}$ separates $\tilde{x}_1$ and $\tilde{x}_2$. We have just shown that $\tilde{A}_x|\tilde{Y}$ separates the points of $\tilde{Y}\setminus J$. Let I be the uniform closed subalgebra of $C(\tilde{Y})$ which is generated by $\tilde{A}_x|\tilde{Y}$ and constant functions. Then we may suppose that I is a function algebra on $\tilde{Y}_x$. By the definition of J and $\tilde{x}_0$, for each $\tilde{y}$ in $\tilde{Y}_x\setminus\{\tilde{x}_0\}$ there exists an $\tilde{f}_0$ in $\tilde{A}_x|\tilde{Y}$ such that $\tilde{f}_0(\tilde{y})\neq 0$ and $\tilde{N}_{\tilde{Y}}(\tilde{f}_0-(g))<\varepsilon/3$ where (g) denotes $(g|Y, g|Y, \cdots)$ in $\tilde{A}_x|\tilde{Y}$. We put a compact neighborhood

$$G=\{\tilde{z}\in\tilde{Y}_x:\ |\tilde{f}_0(\tilde{z})|\geqq(1/2)|\tilde{f}_0(\tilde{y})|\}$$

of $\tilde{y}$ in $\tilde{Y}_x$ and we may suppose that G is also a compact subset of $\tilde{Y}$ since $\tilde{x}_0$ is the only point which is identified more than one point in $\tilde{Y}$ and $\tilde{f}_0(\tilde{x}_0)=0$. Put

$$V=\{f\in C(\tilde{G}):\ f\bar{F}\in \mathrm{cl}(\tilde{\tilde{A}}_x|\tilde{G})\ \text{for}\ {}^{\forall}F\in\mathrm{cl}(\tilde{\tilde{A}}_x|\tilde{G})\}$$

and let $[V\bar{V}+((c_{nk}))]$ be the uniformly closed subalgebra of $C(\tilde{G})$ which is generated by

$$\{F_1\bar{F}_2+((c_{nk})):\ F_1,\ F_2\in V,\ ((c_{nk}))\in\tilde{l}^{\infty}\}$$

where $\bar{F}$ denotes the complex conjugation of F. Let $F_1, F_2, \cdots, F_m$ and $G_1, G_2, \cdots, G_m$ be functions in V and let $((c_{nk}))$ be in $\tilde{l}^{\infty}$. Then $G_1\overline{((c_{nk}))(\tilde{f}_0)}$ is in $\mathrm{cl}(\tilde{\tilde{A}}_x|\tilde{G})$ by definition of V since $((c_{nk}))(\tilde{f}_0)$ is in $\tilde{\tilde{A}}_x|\tilde{G}$ where $(\tilde{f}_0)=(\tilde{f}_0|G, \tilde{f}_0|G, \cdots)$ in $\tilde{\tilde{A}}_x|\tilde{G}$. By the same way we see that $F_1\bar{G}_1((c_{nk}))(\tilde{f}_0)$ is in $\mathrm{cl}(\tilde{\tilde{A}}_x|\tilde{G})$, so we have

$$F_1F_2\cdots F_m\bar{G}_1\bar{G}_2\cdots\bar{G}_m((c_{nk}))(\tilde{f}_0)\in\mathrm{cl}(\tilde{\tilde{A}}_x|\tilde{G})$$

in general. Thus we have

$$[V\bar{V}+((c_{nk}))]\times(\tilde{f}_0)\subset\mathrm{cl}(\tilde{\tilde{A}}_x|\tilde{G}).$$

Since $[V\bar{V}+((c_{nk}))]$ is a self-adjoint closed unital subalgebra of $C(\tilde{G})$, if we prove that $[V\bar{V}+((c_{nk}))]$ separates the points of $\tilde{G}$, it follows that $[V\bar{V}+((c_{nk}))]=C(\tilde{G})$ by the Stone-Weierstrass theorem. Thus we have

$$[V\bar{V}+((c_{nk}))]\times(\tilde{f}_0)=C(\tilde{G})$$

since $(\tilde{f}_0)$ is bounded away from 0 on $\tilde{G}$. It follows that $\mathrm{cl}(\tilde{\tilde{A}}_x|\tilde{G})=C(\tilde{G})$ and then $\tilde{A}_x|G=C(G)$ by Bernard's lemma, so $I|G=C(G)$. Since I is a function algebra on $\tilde{Y}_x$ and since there is an interpolating compact neighborhood of every point in $\tilde{Y}_x\setminus\{\tilde{x}_0\}$, we have $I=C(\tilde{Y}_x)$ by Corollary 2.13 in [4]. So we have $A|Y=C(Y)$ by Bernard's lemma since $\mathrm{cl}(\tilde{A}|\tilde{Y})=[I, (c_n)]$ is conjugate closed and separates the points of $\tilde{Y}$, where $[I, (c_n)]$ is the function algebra generated by I and l^∞. Thus it remains only to prove that $[V\bar{V}+((c_{nk}))]$ separates the points of $\tilde{G}$.

LEMMA. $[V\bar{V}+((c_{nk}))]$ *separates the points of* $\tilde{G}$ *for case* (2).

PROOF. Suppose that $\tilde{\tilde{f}}_1=((f_{nk}^{(1)})_n)_k$ and $\tilde{\tilde{f}}_2=((f_{nk}^{(2)})_n)_k$ are in $\tilde{\tilde{A}}_x|\tilde{G}$ with $\tilde{\tilde{N}}_{\tilde{G}}(\tilde{\tilde{f}}_1)<\varepsilon$. Then we have

$$h(\tilde{\tilde{f}}_1+((g))-\tilde{\tilde{f}}_2 w)\in \mathrm{cl}(\tilde{\tilde{A}}_x|\tilde{G})$$

for a complex number w with the sufficiently small absolute value, where $((g))=((g), (g), \cdots)|\tilde{G}$. For, if $f_{nk}^{(1)}+g|Y-f_{nk}^{(2)}w$ is in U for every n and k, then

$$N_Y(h(f_{nk}^{(1)}+g|Y-f_{nk}^{(2)}w))<n_0$$

for each n and k and we see that

$$h(\tilde{\tilde{f}}_1+((g))-\tilde{\tilde{f}}_2 w)=(h(f_{nk}^{(1)}+g|Y-f_{nk}^{(2)}w)_n)_k$$

is in $\tilde{\tilde{A}}_x|\tilde{G}$, thus in $\mathrm{cl}(\tilde{\tilde{A}}_x|\tilde{G})$. For the general case, $\tilde{\tilde{f}}_1+((g))-\tilde{\tilde{f}}_2 w$ is the uniform limit of the functions in $\tilde{\tilde{U}}$, thus we have

$$h(\tilde{\tilde{f}}_1+((g))-\tilde{\tilde{f}}_2 w)\in \mathrm{cl}(\tilde{\tilde{A}}_x|\tilde{G}).$$

Since $H_\delta(\tilde{\tilde{f}}_1+((g)), \tilde{\tilde{f}}_2)$ is the uniform limit of linear combinations of $h(\tilde{\tilde{f}}_1+((g))-\tilde{\tilde{f}}_2 w)$, it follows that

$$H_\delta(\tilde{\tilde{f}}_1+((g)), \tilde{\tilde{f}}_2)\in \mathrm{cl}(\tilde{\tilde{A}}_x|\tilde{G})$$

for sufficiently small δ and for $\tilde{\tilde{f}}_1$ and $\tilde{\tilde{f}}_2$ in $\tilde{\tilde{A}}_x|\tilde{G}$ with $\tilde{\tilde{N}}_{\tilde{G}}(\tilde{\tilde{f}}_1)<\varepsilon$. For the same reason

$$\{H_\delta(\tilde{\tilde{f}}_1+((g))+\Delta\tilde{\tilde{f}}_3, \tilde{\tilde{f}}_2)-H_\delta(\tilde{\tilde{f}}_1+((g)), \tilde{\tilde{f}}_2)\}/\Delta$$

of sufficiently small δ is in $\mathrm{cl}(\tilde{\tilde{A}}_x|\tilde{G})$ for $\tilde{\tilde{f}}_1$ in $\{\tilde{\tilde{f}}\in\tilde{\tilde{A}}_x|\tilde{G}:\tilde{\tilde{N}}_{\tilde{G}}(\tilde{\tilde{f}})<\varepsilon\}$ and for $\tilde{\tilde{f}}_2$ and $\tilde{\tilde{f}}_3$ in $\tilde{\tilde{A}}_x|\tilde{G}$ such that $\tilde{\tilde{f}}_2$ is bounded away from 0 on $\tilde{G}$ and for a complex number Δ with the small absolute value. If Δ is real number and tends to 0, then

$$\{H_\delta(\tilde{\tilde{f}}_1+((g))+\Delta\tilde{\tilde{f}}_3, \tilde{\tilde{f}}_2)-H_\delta(\tilde{\tilde{f}}_1+((g)), \tilde{\tilde{f}}_2)\}/\Delta$$

tends to

$$(\mathrm{Re}\,\tilde{\tilde{f}}_3)\partial/\partial x_1 H_\delta(\tilde{\tilde{f}}_1+((g)), \tilde{\tilde{f}}_2)+(\mathrm{Im}\,\tilde{\tilde{f}}_3)\partial/\partial y_1 H_\delta(\tilde{\tilde{f}}_1+((g)), \tilde{\tilde{f}}_2)$$

in $\mathrm{cl}(\tilde{\tilde{A}}_x|\tilde{G})$, where $z_1=x_1+y_1 i$. By the same way, if Δ is purely imaginary number, then we see that

$$i(\mathrm{Im}\,\tilde{\tilde{f}}_3)\partial/\partial x_1 H_\delta(\tilde{\tilde{f}}_1+((g)), \tilde{\tilde{f}}_2)-i(\mathrm{Re}\,\tilde{\tilde{f}}_3)\partial/\partial y_1 H_\delta(\tilde{\tilde{f}}_1+((g)), \tilde{\tilde{f}}_2)$$

is in $\mathrm{cl}(\tilde{\tilde{A}}_x|\tilde{G})$. Thus we have

$$\bar{\partial}_1 H_\delta(\tilde{\tilde{f}}_1+((g)), \tilde{\tilde{f}}_2)\overline{\tilde{\tilde{f}}_3}\in \mathrm{cl}(\tilde{\tilde{A}}_x|\tilde{G})$$

for sufficiently small δ and for $\tilde{\tilde{f}}_1$, $\tilde{\tilde{f}}_2$ and $\tilde{\tilde{f}}_3$ in $\tilde{\tilde{A}}_x|\tilde{G}$ which satisfy that $\tilde{\tilde{N}}_{\tilde{G}}(\tilde{\tilde{f}}_1)<\varepsilon$ and that $\tilde{\tilde{f}}_2$ is bounded away from 0 on $\tilde{G}$, where we denote $(1/2)(\partial/\partial x_1+i\partial/\partial y_1)$ by $\bar{\partial}_1$. So

$$\bar{\partial}_1 H_\delta(\tilde{\tilde{f}}_1+((g)),\ \tilde{\tilde{f}}_2)\bar{\tilde{\tilde{f}}}_3\in \mathrm{cl}(\tilde{\tilde{A}}_x|\tilde{G})$$

for $\tilde{\tilde{f}}_1$ in $\{\tilde{\tilde{f}}\in\tilde{\tilde{A}}_x|\tilde{G}:\tilde{\tilde{N}}_{\tilde{G}}(\tilde{\tilde{f}})<\varepsilon\}$ and for $\tilde{\tilde{f}}_2$ in $\tilde{\tilde{A}}_x|\tilde{G}$ such that $\tilde{\tilde{f}}_2$ is bounded away from 0 on $\tilde{G}$ and for $\tilde{\tilde{f}}_3$ in $\mathrm{cl}(\tilde{\tilde{A}}_x|\tilde{G})$. Thus we conclude that

$$\bar{\partial}_1 H_\delta(\tilde{\tilde{f}}_1+((g)),\ \tilde{\tilde{f}}_2)\in V$$

for $\tilde{\tilde{f}}_1$ in $\{\tilde{\tilde{f}}\in\tilde{\tilde{A}}_x|\tilde{G}:\tilde{\tilde{N}}_{\tilde{G}}(\tilde{\tilde{f}})<\varepsilon\}$ and for $\tilde{\tilde{f}}_2$ in $\tilde{\tilde{A}}_x|\tilde{G}$ such that $\tilde{\tilde{f}}_2$ is bounded away from 0 on $\tilde{G}$.

Let a and b be different points in $\tilde{G}$ and suppose that $\tilde{l}^\infty$ does not separate them. By the definition of M, there is an $\tilde{\tilde{f}}=((f_{nk})_n)_k$ in $\tilde{\tilde{A}}$ such that

$$\tilde{\tilde{N}}(\tilde{\tilde{f}})\leqq 1/M,\qquad \tilde{\tilde{f}}(a)=1 \qquad\text{and}\qquad \tilde{\tilde{f}}(b)=0$$

since we may suppose that a and b are points in $\tilde{\tilde{X}}$. So $\tilde{\tilde{f}}'=\tilde{\tilde{f}}-((f_{nk})_n)_k$ is in $\tilde{\tilde{A}}_x$ and

$$\tilde{\tilde{f}}'(a)-\tilde{\tilde{f}}'(b)=1.$$

Without loss of generality we may assume $|\tilde{\tilde{f}}'(b)|\geqq 1/2$. (If not, take $\tilde{\tilde{f}}$ instead of $\tilde{\tilde{f}}''$ below and change a and b.) There is $\tilde{\tilde{f}}''$ in $\tilde{\tilde{A}}$ such that

$$\tilde{\tilde{N}}(\tilde{\tilde{f}}'')\leqq 1/M,\qquad \tilde{\tilde{f}}''(a)=0\qquad\text{and}\qquad \tilde{\tilde{f}}''(b)=1.$$

Thus we have $\tilde{\tilde{f}}'\tilde{\tilde{f}}''\in\tilde{\tilde{A}}_x$, $\tilde{\tilde{N}}(\tilde{\tilde{f}}'\tilde{\tilde{f}}'')\leqq 2/M^2$, $|\tilde{\tilde{f}}'\tilde{\tilde{f}}''(b)|\geqq 1/2$ and $\tilde{\tilde{f}}'\tilde{\tilde{f}}''(a)=0$. Since we may consider that b is in $\tilde{\tilde{Y}}$, we have $|((g))(b)|\leqq M^2\varepsilon/6$ by definition on Y.

There exists a complex number α with $|\alpha|<1$ such that

$$((g))(b)+\tilde{\tilde{f}}'\tilde{\tilde{f}}''(b)\times\alpha M^2\varepsilon/2=0.$$

Put $\eta=|((g))(b)+\tilde{\tilde{f}}'\tilde{\tilde{f}}''(b)\times\alpha M^2\varepsilon/(2|\alpha|)|$ if $\alpha\neq 0$ and put $\eta=|\tilde{\tilde{f}}'\tilde{\tilde{f}}''(b)\times M^2\varepsilon/2|$ if $\alpha=0$ and put $z_2=(\tilde{\tilde{f}}_0)(b)$. Since we consider only the case (2), there is a sufficiently small δ such that $\bar{\partial}_1 H_\delta(z_1, z_2)$ is not a constant function on $\{z_1\in\boldsymbol{C}:|z_1|<\eta\}$. Since $\tilde{\tilde{f}}'\tilde{\tilde{f}}''(a)=0$ we can choose appropriate complex numbers α_1 and α_2 with absolute values less than 1 which satisfy that

$$F(\alpha_1)(a)\overline{F(\alpha_2)}(a)\neq F(\alpha_1)(b)\overline{F(\alpha_2)}(b),$$

where we denote

$$F(\beta)(p)=\bar{\partial}_1 H_\delta(((g))+\tilde{\tilde{f}}'\tilde{\tilde{f}}''\times\beta M^2\varepsilon/2,\ (\tilde{\tilde{f}}_0))(p),$$

which shows that $V\bar{V}$ separates a and b. Thus we conclude that $[V\bar{V}+((c_{nk}))]$ separates the points of $\tilde{G}$.

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