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On intuitionistic many-valued logics

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Introduction.

G. Gentzen introduced the notion of sequent, which consists of the antecedent and of the succedent, each of which in turn is a sequence of finite formulas, and utilizing that notion he formulated the formal system LK for the classical logic. Then by restricting sequents to ones whose succedents are sequences of at most one formula, he obtained from LK the formal system LJ for the intuitionistic logic. Later, Takahashi in [3], and Rousseau in [1] independently, extended the notion of sequent to that of matrix, which consists of the 1st row, the 2nd row, $\cdots$, and of the M-th row, each of which in turn is a sequence of finite formulas, where M is a natural number greater than 1, and then utilizing that notion they formulated the formal system M-LK for each M-valued logic.

What is obtained from the system M-LK, when we restrict matrices to ones whose M-th rows or more rows are sequences of at most one formula? This paper is one answer to this problem.

Let U be a subset of the non-empty finite set T of truth-values. We take a formal system for a many-valued logic having T as the set of truth-values, and then restrict every inference rule by which a connective is introduced in some μ-th row where $\mu\notin U$ so that the ν-th rows where $\nu\notin U$ of the conclusion consist of one formula in all. We call by an *intuitionistic many-valued logic* what is represented by the above-obtained system. If $U=T$, then the intuitionistic many-valued logic is of course identical with the usual many-valued logic (cf. 3.43); if $T=\{\boldsymbol{t}, \boldsymbol{f}\}$ and $U=\{\boldsymbol{f}\}$, then the logic is identical with the intuitionistic logic as is expected (cf. 3.11). Though somewhat artificial, the intuitionistic many-valued logic can also be characterized semantically (cf. Theorem 1). If either $U=T$ or U contains at most one element, then the system enjoys the cut-elimination property (cf. Theorem 4). Moreover, if U contains one and only one element, then the logic enjoys the disjunction property (cf. Theorem 5). On the contrary, if U contains at least two elements (and if sufficiently many connectives are involved), then surprisingly the intuitionistic many-valued logic coincides with the

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usual many-valued logic (cf. 4.12).

In view of the above facts, the authors propose the intuitionistic many-valued logics with U containing one and only one element as a candidate for a many-valued analogue of the intuitionistic logic.

In this paper only propositional logics are studied. The intuitionistic many-valued logics studied in this paper differ from ones studied in Rousseau [2], each of which is determined by the help of a linear order on the set of truth-values instead of a subset.

§1. Preliminaries.

1.1. An *intuitionistic many-valued logic* is determined by choices of a non-empty finite set T, a set $\mathcal{F}$ of functions on T, and a subset U of T. Elements of T are denoted by $\lambda, \mu, \nu, \cdots$.

For μ in T, we put $\mu\hat{}=\{\lambda \mid \lambda\neq\mu\}$ following Takahashi [3].

1.2. *Primitive symbols* are countably many *propositional variables*, a *connective* C_f (abbreviated by 'f') for each f in $\mathcal{F}$, *parentheses* and a *comma*.

The connective f is *k-ary* iff f is a k-ary function on T.

1.3. *Formulas* are defined by the following recursion: a propositional variable standing alone is a formula; if f is a k-ary connective and $A_1, \cdots, A_k$ are formulas, then $f(A_1, \cdots, A_k)$ is also a formula. Formulas are denoted by $A, B, \cdots$.

1.4. A *signed formula* is an ordered pair $\langle\mu, A\rangle$ of μ in T and of a formula A. A *matrix* is a finite set of signed formulas. Matrices are denoted by K, $L, \cdots$.

The empty set $\emptyset$ is also called the *empty matrix*. For a subset S of T and a formula A, the direct product $S\times\{A\}$ denotes the matrix $\{\langle\mu, A\rangle \mid \mu\in S\}$ by one of set-theoretical conventions. For a matrix K, we put $K^U=\{\langle\mu, A\rangle\in K \mid \mu\in U\}$.

Expression of a matrix by a set is due to Takahashi [4].

1.41. $K^U\subseteq K$; $(K^U)^U=K^U$; $(K\cup L)^U=K^U\cup L^U$; $K^U\subseteq L^U$ *whenever* $K\subseteq L$. □

§2. A formal system for the intuitionistic many-valued logic.

2.1. A *proof-figure* is a finite tree $\mathfrak{P}$ of matrices such that every matrix in $\mathfrak{P}$ is either basic, where a matrix K is *basic* iff $T\times\{A\}\subseteq K$ for some A, or the conclusion of one of the following *inference rules* every premise of which is also in $\mathfrak{P}$.

Cut inference:

$$\frac{K\cup\{\langle\mu, A\rangle\} \qquad K\cup\{\langle\nu, A\rangle\}}{K},$$

where $\mu\neq\nu$.

Left inference (f, μ), where f is a k-ary connective and $\mu\in U$:

$$\frac{\begin{array}{l}K\cup(\mu_1{}^\wedge\times\{A_1\})\cup\cdots\cup(\mu_k{}^\wedge\times\{A_k\})\\ \quad\text{for every } \mu_1, \cdots, \mu_k \text{ falsifying } f(\mu_1, \cdots, \mu_k)=\mu\end{array}}{K},$$

where $\langle\mu, f(A_1, \cdots, A_k)\rangle\in K$. The pair $\langle\mu, f(A_1, \cdots, A_k)\rangle$ is called the *principal* signed formula of this inference.

Right inference (f, μ), where f is a k-ary connective and $\mu\notin U$:

$$\frac{\begin{array}{l}K^U\cup(\mu_1{}^\wedge\times\{A_1\})\cup\cdots\cup(\mu_k{}^\wedge\times\{A_k\})\\ \quad\text{for every } \mu_1, \cdots, \mu_k \text{ falsifying } f(\mu_1, \cdots, \mu_k)=\mu\end{array}}{K},$$

where $\langle\mu, f(A_1, \cdots, A_k)\rangle\in K$. The pair $\langle\mu, f(A_1, \cdots, A_k)\rangle$ is called the *principal* signed formula of this inference.

The *end-matrix* of a proof-figure is the lowest matrix in it.

The form of left and right inferences has come from Rousseau [**1**].

2.2. A matrix K is *provable* ($\vdash K$) iff it is the end-matrix of some proof-figure. A matrix K is *cut-free provable* ($\mapsto K$) iff it is the end-matrix of some proof-figure in which the cut inference is not applied.

More precisely, a matrix K is *provable with rank* n ($\vdash_n K$), where n is a natural number, iff it is the end-matrix of some proof-figure which is constructed from n matrices; the notion of K being *cut-free provable with rank* n ($\mapsto_n K$) is defined similarly.

The notation '$\vdash_{<n} K$ ($\mapsto_{<n} K$, resp.)' is an abbreviation for '$\vdash_m K$ ($\mapsto_m K$, resp.) for some m less than n'.

2.21. *$\vdash_n K$ ($\vdash K$, resp.) whenever $\mapsto_n K$ ($\mapsto K$, resp.); $\vdash_n L$ ($\mapsto_n L$, $\vdash L$ or $\mapsto L$, resp.) whenever $\vdash_n K$ ($\mapsto_n K$, $\vdash K$ or $\mapsto K$, resp.) and $K\subseteq L$.* □

§ 3. A semantical characterization of the intuitionistic many-valued logic.

3.1. A *model* is a triplet (X, R, v) of a non-empty set X, a reflexive, transitive relation R on X, and a function v which maps each pair of an element of X and of a formula into an element of T, satisfying the following conditions **M1**, **M2** and **M3**:

M1. $\Gamma R\Delta$ & $v(\Delta, A)\in U \implies v(\Gamma, A)=v(\Delta, A)$.

M2. $\mu \in U$ & $f(v(\Gamma, A_1), \cdots, v(\Gamma, A_k))=\mu$
$\Longrightarrow v(\Gamma, f(A_1, \cdots, A_k))=\mu.$

M3. $\mu \notin U$ & $\forall \Delta [\Gamma R \Delta \Rightarrow f(v(\Delta, A_1), \cdots, v(\Delta, A_k))=\mu]$
$\Longrightarrow v(\Gamma, f(A_1, \cdots, A_k))=\mu.$

3.11. In this paragraph, we assume that $T=\{\boldsymbol{t}, \boldsymbol{f}\}$, $\mathfrak{F}=\{\wedge, \vee, \supset, \neg\}$ and $U=\{\boldsymbol{f}\}$, where $\wedge$, $\vee$ and $\supset$ are binary functions while $\neg$ unary on T defined as follows: $\mu \wedge \nu=\boldsymbol{t}$ iff $\mu=\boldsymbol{t}$ and $\nu=\boldsymbol{t}$; $\mu \vee \nu=\boldsymbol{t}$ iff either $\mu=\boldsymbol{t}$ or $\nu=\boldsymbol{t}$; $\mu \supset \nu=\boldsymbol{t}$ iff either $\mu=\boldsymbol{f}$ or $\nu=\boldsymbol{t}$; $\neg \mu=\boldsymbol{t}$ iff $\mu=\boldsymbol{f}$.

Then the notion of model agrees with that of Kripke model for the intuitionistic logic, so that, in view of Theorem 1 stated in 3.3 below, the intuitionistic many-valued logic coincides with the intuitionistic logic.

3.2. Let (X, R, v) be a model and $\Gamma \in X$. A matrix K is *Γ-true* (*Γ-false*, resp.) *in* (X, R, v) iff $\langle v(\Gamma, A), A \rangle \in K$ for some A (for no A, resp.).

3.21. *If $\Gamma R \Delta$ and K is Γ-false, then K^U is Δ-false.* PROOF. Suppose that $\Gamma R \Delta$ and K^U is Δ-true. Then $\langle v(\Delta, A), A \rangle \in K^U$ for some A. Hence $v(\Delta, A) \in U$, so $\langle v(\Gamma, A), A \rangle = \langle v(\Delta, A), A \rangle \in K$ by **M1**, so K is Γ-true. □

3.3. A matrix is *valid* iff it is Γ-true in (X, R, v) for every model (X, R, v) and every Γ in X.

Then the intuitionistic many-valued logic is characterized semantically as follows.

THEOREM 1. *A matrix is provable if and only if it is valid.*

We shall prove the 'only if' part and the 'if' part in 3.5 and in 3.6-3.8, respectively.

3.4. A *valuation* is a function w which maps each formula into an element of T satisfying $w(f(A_1, \cdots, A_k))=f(w(A_1), \cdots, w(A_k))$ for every k-ary connective f and every formulas $A_1, \cdots, A_k$.

3.41. *Let w be a valuation. If we put $X=\{1\}, R=\{\langle 1, 1\rangle\}$ and $v(1, A)=w(A)$ for every A, then the triplet (X, R, v) forms a model.* □

3.42. *The empty matrix is not valid.* PROOF. Since valuations exist, so do models. □

3.43. In view of Theorem 2 below and of Theorem 1, when $U=T$, the intuitionistic many-valued logic coincides with the usual many-valued logic as is expected.

THEOREM 2. *Assume $U=T$. Then a matrix K is valid if and only if for* *every valuation w we obtain $\langle w(A), A\rangle \in K$ for some A.*

PROOF. To prove the 'only if' part, suppose that K is valid and w is a valuation. Since K is 1-true in (X, R, v), which is the model constructed by the method stated in 3.41, $\langle v(1, A), A\rangle \in K$ and so $\langle w(A), A\rangle \in K$ for some A.

Next, to prove the contraposition of the 'if' part, suppose that K is not valid. Then K is Γ-false in (X, R, v) for some model (X, R, v) and some Γ in X. We put $w(A)=v(\Gamma, A)$ for every A. Then w forms a valuation and $\langle w(A), A\rangle \in K$ for no A. □

3.5. PROOF of the 'only if' part of Theorem 1. It suffices to prove that K is valid whenever $\vdash_n K$, which we shall demonstrate by induction on n.

Suppose that $\vdash_n K$, (X, R, v) is a model and that $\Gamma \in X$. We must show that K is Γ-true.

Case 1. K is basic. Then $T\times\{A\}\subseteq K$ for some A. Hence $\langle v(\Gamma, A), A\rangle \in K$, so K is Γ-true.

Case 2. K is the conclusion of the cut inference. Suppose that $\vdash_{<n} K\cup\{\langle\mu, A\rangle\}$, $\vdash_{<n} K\cup\{\langle\nu, A\rangle\}$ and $\mu\neq\nu$. Suppose, on the contrary to the conclusion, that K is Γ-false. By the induction hypothesis both $K\cup\{\langle\mu, A\rangle\}$ and $K\cup\{\langle\nu, A\rangle\}$ are Γ-true, while either $v(\Gamma, A)\neq\mu$ or $v(\Gamma, A)\neq\nu$, which is a contradiction in either case. So K is Γ-true.

Case 3. K is the conclusion of a left inference. Suppose that $\vdash_{<n} K\cup(\mu_1{}^\wedge\times\{A_1\})\cup\cdots\cup(\mu_k{}^\wedge\times\{A_k\})$ for every $\mu_1, \cdots, \mu_k$ falsifying $f(\mu_1, \cdots, \mu_k)=\mu$, and that $\mu\in U$ and $\langle\mu, f(A_1, \cdots, A_k)\rangle\in K$. Suppose, on the contrary to the conclusion, that K is Γ-false. Then $v(\Gamma, f(A_1, \cdots, A_k))\neq\mu$, so $f(v(\Gamma, A_1), \cdots, v(\Gamma, A_k))\neq\mu$ by **M2**. Putting $\mu_j=v(\Gamma, A_j)$ for $j=1, \cdots, k$, we obtain $f(\mu_1, \cdots, \mu_k)\neq\mu$, so $K\cup(\mu_1{}^\wedge\times\{A_1\})\cup\cdots\cup(\mu_k{}^\wedge\times\{A_k\})$ is Γ-true by the induction hypothesis, which is a contradiction. Hence K is Γ-true.

Case 4. K is the conclusion of a right inference. Suppose that $\vdash_{<n} K^U\cup(\mu_1{}^\wedge\times\{A_1\})\cup\cdots\cup(\mu_k{}^\wedge\times\{A_k\})$ for every $\mu_1, \cdots, \mu_k$ falsifying $f(\mu_1, \cdots, \mu_k)=\mu$, and that $\mu\notin U$ and $\langle\mu, f(A_1, \cdots, A_k)\rangle\in K$. Suppose, on the contrary to the conclusion, that K is Γ-false. Then $v(\Gamma, f(A_1, \cdots, A_k))\neq\mu$, so $f(v(\Delta, A_1), \cdots, v(\Delta, A_k))\neq\mu$ for some Δ such that $\Gamma R\Delta$ by **M3**. Putting $\mu_j=v(\Delta, A_j)$ for $j=1, \cdots, k$, we obtain $f(\mu_1, \cdots, \mu_k)\neq\mu$, so $K^U\cup(\mu_1{}^\wedge\times\{A_1\})\cup\cdots\cup(\mu_k{}^\wedge\times\{A_k\})$ is Δ-true by the induction hypothesis, which contradicts 3.21. Hence K is Γ-true.

□

3.6. We shall devote the rest of this section to the proof of the 'if' part of Theorem 1.

A *generalized matrix* (abbreviated by '*g-matrix*') is a finite or infinite set of signed formulas. A g-matrix is *provable* iff it contains a provable matrix. A g-matrix is *maximal unprovable* iff it is unprovable and any proper extension of it is provable.

3.61. *Any matrix is a g-matrix. A matrix is provable iff it is provable as a g-matrix.* □

3.62. *Any unprovable g-matrix can be extended to a maximal unprovable one.* PROOF. Suppose that Π is an unprovable g-matrix, and let $\langle \mu_0, A_0 \rangle$, $\langle \mu_1, A_1 \rangle$, $\langle \mu_2, A_2 \rangle$, $\cdots$ be an enumeration of all the signed formulas. We define the g-matrix Π_n by the following recursion: $\Pi_0 = \Pi$; $\Pi_{n+1} = \Pi_n$ or $= \Pi_n \cup \{\langle \mu_n, A_n \rangle\}$ according as $\Pi_n \cup \{\langle \mu_n, A_n \rangle\}$ is provable or not. Then the g-matrix $\bigcup_{n=0}^{\infty} \Pi_n$ is the required one. □

3.63. *If Γ is a maximal unprovable g-matrix, then for every A there exists one and only one μ falsifying $\langle \mu, A \rangle \in \Gamma$.* PROOF. If $T \times \{A\} \subseteq \Gamma$, then Γ is provable, which is a contradiction. Hence $\langle \mu, A \rangle \notin \Gamma$ for some μ. Next, suppose that $\langle \mu_1, A \rangle \notin \Gamma$, $\langle \mu_2, A \rangle \notin \Gamma$ and $\mu_1 \neq \mu_2$. Then both $\Gamma \cup \{\langle \mu_1, A \rangle\}$ and $\Gamma \cup \{\langle \mu_2, A \rangle\}$ are provable since they are proper extensions of Γ. So in view of the cut inference, Γ is provable, which is a contradiction, too. Hence there is one and only one μ falsifying $\langle \mu, A \rangle \in \Gamma$. □

3.7. We introduce the model (X, R, v) as follows: X is the set of maximal unprovable g-matrices; $R = \{\langle \Gamma, \Delta \rangle \in X^2 \mid \langle \mu, A \rangle \in \Delta$ whenever $\langle \mu, A \rangle \in \Gamma$ and $\mu \in U\}$; for every Γ in X and every formula A, $v(\Gamma, A)$ is the unique μ falsifying $\langle \mu, A \rangle \in \Gamma$.

LEMMA. *The triplet (X, R, v) defined above certainly forms a model.*

PROOF. The empty matrix is unprovable by 3.5, so X is not empty; R is clearly reflexive and transitive.

To verify **M1**, suppose $\Gamma R \Delta$ and $v(\Delta, A) \in U$. Then $\langle v(\Delta, A), A \rangle \notin \Gamma$ since $\langle v(\Delta, A), A \rangle \notin \Delta$, so $v(\Gamma, A) = v(\Delta, A)$.

To verify **M2**, suppose $\mu \in U$ and $v(\Gamma, f(A_1, \cdots, A_k)) \neq \mu$. Since $\langle \mu, f(A_1, \cdots, A_k) \rangle \in \Gamma$, in view of the left inference (f, μ), the g-matrix $\Gamma \cup (\mu_1\hat{} \times \{A_1\}) \cup \cdots \cup (\mu_k\hat{} \times \{A_k\})$ is unprovable for some $\mu_1, \cdots, \mu_k$ falsifying $f(\mu_1, \cdots, \mu_k) = \mu$. Then $\langle \mu_1, A_1 \rangle \notin \Gamma, \cdots, \langle \mu_k, A_k \rangle \notin \Gamma$, so $v(\Gamma, A_1) = \mu_1, \cdots, v(\Gamma, A_k) = \mu_k$, so $f(v(\Gamma, A_1), \cdots, v(\Gamma, A_k)) \neq \mu$.

To verify **M3**, suppose $\mu \notin U$ and $v(\Gamma, f(A_1, \cdots, A_k)) \neq \mu$. Since $\langle \mu, f(A_1, \cdots, A_k) \rangle \in \Gamma$, in view of the right inference (f, μ), the g-matrix $\Gamma^U \cup (\mu_1\hat{} \times \{A_1\}) \cup \cdots \cup (\mu_k\hat{} \times \{A_k\})$ is unprovable for some $\mu_1, \cdots, \mu_k$ falsifying $f(\mu_1, \cdots, \mu_k) = \mu$, where $\Gamma^U = \{\langle \nu, B \rangle \in \Gamma \mid \nu \in U\}$. Then $\Gamma^U \cup (\mu_1\hat{} \times \{A_1\}) \cup \cdots \cup (\mu_k\hat{} \times \{A_k\}) \subseteq \Delta$ for some Δ in X. It follows $\Gamma R \Delta$ from $\Gamma^U \subseteq \Delta$. On the other hand, $\langle \mu_1, A_1 \rangle \notin \Delta, \cdots, \langle \mu_k, A_k \rangle \notin \Delta$, so $v(\Delta, A_1) = \mu_1, \cdots, v(\Delta, A_k) = \mu_k$, so $f(v(\Delta, A_1), \cdots, v(\Delta, A_k)) \neq \mu$. Hence it is not the case that $\forall \Delta\, [\Gamma R \Delta \Rightarrow f(v(\Delta, A_1), \cdots, v(\Delta, A_k)) = \mu]$.

□

3.8. PROOF of the 'if' part of Theorem 1. To prove the contraposition, suppose that K is unprovable. Then K is extended to a maximal unprovable g-matrix Γ. We claim that K is Γ-false in the model (X, R, v) introduced above. Suppose, on the contrary to the conclusion, that K is Γ-true. Then $\langle v(\Gamma, A), A\rangle \in K$ for some A. So $\langle v(\Gamma, A), A\rangle \in \Gamma$, which is a contradiction. Hence K is Γ-false, so it is not valid. □

§ 4. Syntactical properties of the formal system.

4.1. In this paragraph we wish to display the choice of U, so we denote $\vdash K$ and $\vdash_n K$ by $\vdash^U K$ and $\vdash_n^U K$, respectively.

4.11. *Suppose* $U \subseteq V \subseteq T$. *Then* $K^U \subseteq K^V$; $\vdash_n^V K$ $(\vdash^V K$, *resp.) whenever* $\vdash_n^U K$ $(\vdash^U K$, *resp.*). □

4.12. According to Theorem 3 below and to 3.43, if $\mathrm{Card}(U) \geqq 2$, where $\mathrm{Card}(U)$ denotes the cardinality of U, then the intuitionistic many-valued logic has no sense as an intuitionistic one.

THEOREM 3. *Assume that* $\mathrm{Card}(U) \geqq 2$ *and every unary function on* T *is contained in* $\mathfrak{F}$. *Then,* $\vdash^T K$ *if and only if* $\vdash^U K$.

PROOF. The 'if' part is a special case of 4.11. To show the 'only if' part, it suffices to prove, on the assumption of the theorem, that if $\vdash^U K \cup (\mu_1\hat{}\times\{A_1\}) \cup \cdots \cup (\mu_k\hat{}\times\{A_k\})$ for every $\mu_1, \cdots, \mu_k$ falsifying $f(\mu_1, \cdots, \mu_k) = \mu$, and if $\mu \notin U$ and $\langle \mu, f(A_1, \cdots, A_k)\rangle \in K$, then $\vdash^U K$.

Take λ and λ' such that $\lambda, \lambda' \in U$ and $\lambda \neq \lambda'$. Let $K = \{\langle \nu_1, B_1\rangle, \cdots, \langle \nu_n, B_n\rangle\}$ and let g_i be the unary function on T such that $g_i(\nu) = \lambda$ or $=\lambda'$ according as $\nu = \nu_i$ or not, for $i = 1, \cdots, n$.

First we remark the fact that for every matrix L and every formula B,

$$\vdash^U L \cup \{\langle \nu_i, B\rangle\} \quad \text{iff} \quad \vdash^U L \cup \{\langle \lambda, g_i(B)\rangle\}. \tag{1}$$

Suppose, first, that $\vdash^U L \cup \{\langle \nu_i, B\rangle\}$. If $g_i(\nu) \neq \lambda$, that is, if $\nu \neq \nu_i$, then $L \cup \{\langle \nu_i, B\rangle\} \subseteq L \cup \{\langle \lambda, g_i(B)\rangle\} \cup (\nu\hat{}\times\{B\})$, so $\vdash^U L \cup \{\langle \lambda, g_i(B)\rangle\} \cup (\nu\hat{}\times\{B\})$. Hence by the left inference (g_i, λ) we obtain $\vdash^U L \cup \{\langle \lambda, g_i(B)\rangle\}$. To show the converse, suppose $\vdash^U L \cup \{\langle \lambda, g_i(B)\rangle\}$. If $g_i(\nu) \neq \lambda'$, that is, if $\nu = \nu_i$, then $L \cup \{\langle \nu_i, B\rangle, \langle \lambda', g_i(B)\rangle\} \cup (\nu\hat{}\times\{B\})$ contains $T \times \{B\}$, so it is basic and so provable. Hence by the left inference (g_i, λ') we obtain $\vdash^U L \cup \{\langle \nu_i, B\rangle, \langle \lambda', g_i(B)\rangle\}$, from which together with $\vdash^U L \cup \{\langle \nu_i, B\rangle, \langle \lambda, g_i(B)\rangle\}$ by the cut inference we obtain $\vdash^U L \cup \{\langle \nu_i, B\rangle\}$. This completes the proof of (1).

Now suppose that $\vdash^U K \cup (\mu_1\hat{}\times\{A_1\}) \cup \cdots \cup (\mu_k\hat{}\times\{A_k\})$ for every $\mu_1, \cdots,$ μ_k falsifying $f(\mu_1, \cdots, \mu_k) = \mu$, and that $\mu \notin U$ and $\langle \mu, f(A_1, \cdots, A_k)\rangle \in K$. We must show $\vdash^U K$. If $f(\mu_1, \cdots, \mu_k) \neq \mu$, then

$$\vdash^{U} \{\langle \nu_1, B_1\rangle, \cdots, \langle \nu_n, B_n\rangle\} \cup (\mu_1{}^{\wedge} \times \{A_1\}) \cup \cdots \cup (\mu_k{}^{\wedge} \times \{A_k\}),$$

so by the repeated use of the 'only if' part of (1),

$$\vdash^{U} \{\langle \lambda, g_1(B_1)\rangle, \cdots, \langle \lambda, g_n(B_n)\rangle\} \cup (\mu_1{}^{\wedge} \times \{A_1\}) \cup \cdots \cup (\mu_k{}^{\wedge} \times \{A_k\}),$$

that is,

$$\vdash^{U} (\{\langle \mu, f(A_1, \cdots, A_k)\rangle, \langle \lambda, g_1(B_1)\rangle, \cdots, \langle \lambda, g_n(B_n)\rangle\})^{U} \cup (\mu_1{}^{\wedge} \times \{A_1\}) \cup \cdots \cup (\mu_k{}^{\wedge} \times \{A_k\}).$$

Hence by the right inference (f, μ),

$$\vdash^{U} \{\langle \mu, f(A_1, \cdots, A_k)\rangle, \langle \lambda, g_1(B_1)\rangle, \cdots, \langle \lambda, g_n(B_n)\rangle\},$$

so by the repeated use of the 'if' part of (1),

$$\vdash^{U} \{\langle \mu, f(A_1, \cdots, A_k)\rangle, \langle \nu_1, B_1\rangle, \cdots, \langle \nu_n, B_n\rangle\},$$

that is, $\vdash^{U} K$. □

4.2. Concerning the cut-elimination property the following theorem holds. Since the proof is rather long, we shall give it in 4.4.

THEOREM 4. *Assume that either* $U=T$ *or* $\mathrm{Card}(U) \leqq 1$. *Then every provable matrix is cut-free provable.*

4.3. With respect to the disjunction property, Theorem 5 below holds.

THEOREM 5. *Assume* $\mathrm{Card}(U)=1$. *If* $\vdash \{\langle \mu_1, A_1\rangle, \cdots, \langle \mu_n, A_n\rangle\}$ *and* $\mu_1, \cdots, \mu_n \notin U$, *then* $\vdash \{\langle \mu_i, A_i\rangle\}$ *for some* i $(i=1, \cdots, n)$.

PROOF. We put $K=\{\langle \mu_1, A_1\rangle, \cdots, \langle \mu_n, A_n\rangle\}$, and suppose $\vdash K$ and $\mu_1, \cdots, \mu_n \notin U$. Then $\mapsto K$ by Theorem 4 which is assumed to have been proved. Since $U \neq \emptyset$ the matrix K is not basic, so it is the conclusion of a left or right inference. Let $\langle \nu, f(B_1, \cdots, B_k)\rangle$ be the principal signed formula. Then $\langle \nu, f(B_1, \cdots, B_k)\rangle \in K$, so $\langle \nu, f(B_1, \cdots \cdots B_k)\rangle = \langle \mu_i, A_i\rangle$ for some i $(i=1, \cdots, n)$. Hence $\nu=\mu_i \notin U$, so K is the conclusion of the right inference (f, ν), so $\mapsto K^{U} \cup (\nu_1{}^{\wedge} \times \{B_1\}) \cup \cdots \cup (\nu_k{}^{\wedge} \times \{B_k\})$ for every $\nu_1, \cdots, \nu_k$ falsifying $f(\nu_1, \cdots, \nu_k)=\nu$. But $K^{U}=\emptyset=\{\langle \nu, f(B_1, \cdots, B_k)\rangle\}^{U}$, so

$$\mapsto \{\langle \nu, f(B_1, \cdots, B_k)\rangle\}^{U} \cup (\nu_1{}^{\wedge} \times \{B_1\}) \cup \cdots \cup (\nu_k{}^{\wedge} \times \{B_k\})$$

for every $\nu_1, \cdots, \nu_k$ falsifying $f(\nu_1, \cdots, \nu_k)=\nu$. Hence by the right inference (f, ν) we obtain $\mapsto \{\langle \nu, f(B_1, \cdots, B_k)\rangle\}$, so $\vdash \{\langle \mu_i, A_i\rangle\}$. □

4.4. PROOF of Theorem 4. It suffices to prove, on the assumption of the theorem, that

(2) if $\mapsto_{n_1} K\cup\{\langle\lambda_1, A\rangle\}$, $\mapsto_{n_2} K\cup\{\langle\lambda_2, A\rangle\}$ and $\lambda_1\neq\lambda_2$, then $\mapsto K$.

We shall prove (2) by induction on $\omega\cdot d(A)+n_1+n_2$, where $d(A)$ denotes the number of occurrences of connectives in A.

First, we remark the fact that for every matrix L and every formula B,

(3) if $\mapsto L\cup(\mu^\wedge\times\{B\})$ for every μ in T and if $d(B)<d(A)$,

then $\mapsto L\cup(S\times\{B\})$ for every subset S of T, in particular $\mapsto L$.

Suppose that $\mapsto L\cup(\mu^\wedge\times\{B\})$ for every μ in T and that $d(B)<d(A)$ and $S\subseteq T$. We shall prove $\mapsto L\cup(S\times\{B\})$ by induction on Card$(T-S)$. *Case* 1. Card$(T-S)=0$. The matrix $L\cup(S\times\{B\})$ is basic since it contains $T\times\{B\}$, so $\mapsto L\cup(S\times\{B\})$. *Case* 2. Card$(T-S)=1$. Since $S=\mu^\wedge$ for some μ in T, by the assumption $\mapsto L\cup(S\times\{B\})$. *Case* 3. Otherwise. Take λ_1 and λ_2 such that $\lambda_1, \lambda_2\in T-S$ and $\lambda_1\neq\lambda_2$, then by the hypothesis of induction on Card$(T-S)$ we have $\mapsto L\cup(S\times\{B\})\cup\{\langle\lambda_1, B\rangle\}$ and $\mapsto L\cup(S\times\{B\})\cup\{\langle\lambda_2, B\rangle\}$, so by the hypothesis of induction on $\omega\cdot d(A)+n_1+n_2$ we obtain $\mapsto L\cup(S\times\{B\})$. This completes the proof of (3).

Now, to prove (2) suppose that $\mapsto_{n_1} K\cup\{\langle\lambda_1, A\rangle\}$, $\mapsto_{n_2} K\cup\{\langle\lambda_2, A\rangle\}$ and $\lambda_1\neq\lambda_2$. We put $K_i=K\cup\{\langle\lambda_i, A\rangle\}$ for $i=1, 2$.

For the cut-free provable matrix K_i $(i=1, 2)$, one of the following five cases occurs:

I. K_i is basic.

II. K_i is the conclusion of a left inference, and the principal signed formula belongs to K.

III. K_i is the conclusion of a right inference, and the principal signed formula belongs to K.

IV. K_i is the conclusion of a left inference, and the principal signed formula is $\langle\lambda_i, A\rangle$.

V. K_i is the conclusion of a right inference, and the principal signed formula is $\langle\lambda_i, A\rangle$.

Remark that neither the case III nor the case V occurs when $U=T$.

We shall show $\mapsto K$ by cases.

Case 1. Either K_1 or K_2 is of case I. We suppose, without loss of generality, that K_1 is the case. Then $T\times\{B\}\subseteq K_1$ for some B. If B is distinct from A, then $T\times\{B\}\subseteq K$, so K is basic, so $\mapsto K$; if B is identical with A, then $\langle\lambda_2, A\rangle\in K$, so $K_2=K$, so $\mapsto K$.

Case 2. Either K_1 or K_2 is of case II. We suppose that K_1 is the case. Suppose that $\mapsto_{<n_1} K_1\cup(\nu_1^\wedge\times\{B_1\})\cup\cdots\cup(\nu_h^\wedge\times\{B_h\})$ for every $\nu_1, \cdots, \nu_h$ falsifying $g(\nu_1, \cdots, \nu_h)=\nu$, and that $\nu\in U$ and $\langle\nu, g(B_1, \cdots, B_h)\rangle\in K$. If $g(\nu_1, \cdots, \nu_h)\neq\nu$, then

$$\mapsto_{<n_1} K\cup(\nu_1{}^\wedge\times\{B_1\})\cup\cdots\cup(\nu_h{}^\wedge\times\{B_h\})\cup\{\langle\lambda_1, A\rangle\}$$

and

$$\mapsto_{n_2} K\cup(\nu_1{}^\wedge\times\{B_1\})\cup\cdots\cup(\nu_h{}^\wedge\times\{B_h\})\cup\{\langle\lambda_2, A\rangle\},$$

so by the induction hypothesis $\mapsto K\cup(\nu_1{}^\wedge\times\{B_1\})\cup\cdots\cup(\nu_h{}^\wedge\times\{B_h\})$. Hence $\mapsto K$ by the left inference (g, ν).

Case 3. Both K_1 and K_2 are of case III. In this case, $U\neq T$ and so Card(U) $\leqq 1$ by the assumption of the theorem, hence either $\lambda_1\notin U$ or $\lambda_2\notin U$. We suppose, without loss of generality, that $\lambda_1\notin U$. Suppose that $\mapsto_{<n_1}(K_1)^U\cup(\nu_1{}^\wedge\times\{B_1\})\cup\cdots\cup(\nu_h{}^\wedge\times\{B_h\})$ for every $\nu_1, \cdots, \nu_h$ falsifying $g(\nu_1, \cdots, \nu_h)=\nu$, and that $\nu\notin U$ and $\langle\nu, g(B_1, \cdots, B_h)\rangle\in K$. Then $\mapsto K^U\cup(\nu_1{}^\wedge\times\{B_1\})\cup\cdots\cup(\nu_h{}^\wedge\times\{B_h\})$ for every $\nu_1, \cdots, \nu_h$ falsifying $g(\nu_1, \cdots, \nu_h)=\nu$, since $(K_1)^U=K^U$. Hence $\mapsto K$ by the right inference (g, ν).

Case 4. One of K_1 and K_2 is of case III, while another of case IV. We suppose that K_1 is of case III, while K_2 of case IV. In this case also, $U\neq T$ and so either $\lambda_1\notin U$ or $\lambda_2\notin U$. Since K_2 is of case IV we have $\lambda_2\in U$, so $\lambda_1\notin U$. So $\mapsto K$ by the similar proof as Case 3.

Case 5. One of K_1 and K_2 is of case III, while another of case V. We suppose that K_1 is of case III, while K_2 of case V. Suppose that $\mapsto_{<n_1}(K_1)^U\cup(\nu_1{}^\wedge\times\{B_1\})\cup\cdots\cup(\nu_h{}^\wedge\times\{B_h\})$ for every $\nu_1, \cdots, \nu_h$ falsifying $g(\nu_1, \cdots, \nu_h)=\nu$, and that $\nu\notin U$ and $\langle\nu, g(B_1, \cdots, B_h)\rangle\in K$. Suppose further that $\mapsto_{<n_2}(K_2)^U\cup(\mu_1{}^\wedge\times\{A_1\})\cup\cdots\cup(\mu_k{}^\wedge\times\{A_k\})$ for every $\mu_1, \cdots, \mu_k$ falsifying $f(\mu_1, \cdots, \mu_k)=\lambda_2$, and that $\lambda_2\notin U$ and $f(A_1, \cdots, A_k)$ is identical with A. To show $\mapsto K$, it suffices to prove

(4) $\qquad \mapsto K^U\cup(\nu_1{}^\wedge\times\{B_1\})\cup\cdots\cup(\nu_h{}^\wedge\times\{B_h\})$
for every $\nu_1, \cdots, \nu_h$ falsifying $g(\nu_1, \cdots, \nu_h)=\nu$,

since from (4) it follows $\mapsto K$ by the right inference (g, ν). With a view to proving (4), suppose $g(\nu_1, \cdots, \nu_h)\neq\nu$. Then

$$\mapsto_{<n_1} K^U\cup(\nu_1{}^\wedge\times\{B_1\})\cup\cdots\cup(\nu_h{}^\wedge\times\{B_h\})\cup\{\langle\lambda_1, A\rangle\};$$

while

$$\mapsto_{<n_2} (K^U\cup(\nu_1{}^\wedge\times\{B_1\})\cup\cdots\cup(\nu_h{}^\wedge\times\{B_h\})\cup\{\langle\lambda_2, A\rangle\})^U \cup(\mu_1{}^\wedge\times\{A_1\})\cup\cdots\cup(\mu_k{}^\wedge\times\{A_k\})$$

for every $\mu_1, \cdots, \mu_k$ falsifying $f(\mu_1, \cdots, \mu_k)=\lambda_2$, so by the right inference (f, λ_2) we obtain

$$\mapsto_{n_2} K^U\cup(\nu_1{}^\wedge\times\{B_1\})\cup\cdots\cup(\nu_h{}^\wedge\times\{B_h\})\cup\{\langle\lambda_2, A\rangle\}.$$

Hence $\mapsto K^U\cup(\nu_1{}^\wedge\times\{B_1\})\cup\cdots\cup(\nu_h{}^\wedge\times\{B_h\})$ by the induction hypothesis. So (4) has been proved.

Case 6. Both K_1 and K_2 are either of case IV or of case V. Suppose that $\mapsto_{<n_1}(K_1)^* \cup (\mu_1{}^\wedge \times \{A_1\}) \cup \cdots \cup (\mu_k{}^\wedge \times \{A_k\})$ for every $\mu_1, \cdots, \mu_k$ falsifying $f(\mu_1, \cdots, \mu_k) = \lambda_1$, and that $f(A_1, \cdots, A_k)$ is identical with A, where $(K_1)^* = K_1$ or $= (K_1)^U$ according as K_1 is of case IV or of case V. Suppose further that $\mapsto_{<n_2}(K_2)^{**} \cup (\mu_1{}^\wedge \times \{A_1\}) \cup \cdots \cup (\mu_k{}^\wedge \times \{A_k\})$ for every $\mu_1, \cdots, \mu_k$ falsifying $f(\mu_1, \cdots, \mu_k) = \lambda_2$, where $(K_2)^{**} = K_2$ or $= (K_2)^U$ according as K_2 is of case IV or of case V. Since $\lambda_1 \neq \lambda_2$, either $f(\mu_1, \cdots, \mu_k) \neq \lambda_1$ or $f(\mu_1, \cdots, \mu_k) \neq \lambda_2$ for every $\mu_1, \cdots, \mu_k$. If $f(\mu_1, \cdots, \mu_k) \neq \lambda_1$, then

$$\mapsto_{<n_1} K \cup (\mu_1{}^\wedge \times \{A_1\}) \cup \cdots \cup (\mu_k{}^\wedge \times \{A_k\}) \cup \{\langle \lambda_1, A \rangle\}$$

and

$$\mapsto_{n_2} K \cup (\mu_1{}^\wedge \times \{A_1\}) \cup \cdots \cup (\mu_k{}^\wedge \times \{A_k\}) \cup \{\langle \lambda_2, A \rangle\},$$

so $\mapsto K \cup (\mu_1{}^\wedge \times \{A_1\}) \cup \cdots \cup (\mu_k{}^\wedge \times \{A_k\})$ by the induction hypothesis; if $f(\mu_1, \cdots, \mu_k) \neq \lambda_2$, we obtain the same result similarly. Hence for every $\mu_1, \cdots, \mu_k$ we have $\mapsto K \cup (\mu_1{}^\wedge \times \{A_1\}) \cup \cdots \cup (\mu_k{}^\wedge \times \{A_k\})$. So $\mapsto K$ by the repeated use of (3) in view of the fact that $d(A_1), \cdots, d(A_k) < d(A)$. □

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