J. Math. Soc. Japan
Vol. 39, No. 4, 1987

On some compact Einstein almost Kähler manifolds

By Kouei SEKIGAWA

(Received May 17, 1986)

§ 1. Introduction.

An almost Hermitian manifold $M=(M, J, <, >)$ is called an almost Kähler manifold if the corresponding Kähler form of M is closed (equivalently, $\langle(\nabla_X J)Y, Z\rangle+\langle(\nabla_Y J)Z, X\rangle+\langle(\nabla_Z J)X, Y\rangle=0$, for all $X, Y, Z\in\mathfrak{X}(M)$, where $\mathfrak{X}(M)$ denotes the Lie algebra of all differentiable vector fields on M). By the definition, a Kähler manifold ($\nabla J=0$) is necessarily an almost Kähler manifold. If the almost complex structure J of an almost Kähler manifold M is integrable, then M is a Kähler manifold [**10**]. A strictly almost Kähler manifold is an almost Kähler manifold whose almost complex structure is not integrable. Several examples of strictly almost Kähler manifolds are known [**1**], [**2**], [**3**], [**7**], [**9**]. By an Einstein almost Hermitian manifold we mean an almost Hermitian manifold which is Einstein in the Riemannian sense. The following conjecture is well-known [**4**], [**9**]:

CONJECTURE. *The almost complex structure of a compact Einstein almost Kähler manifold is integrable.*

Concerning this conjecture, some progress has been made under some curvature conditions ([**4**], [**6**], and etc.).

In this paper, we shall give a partial positive answer to the above conjecture. Namely, we shall prove the following

THEOREM. *Let $M=(M, J, <, >)$ be a compact Einstein almost Kähler manifold whose scalar curvature is non-negative. Then M is a Kähler manifold.*

§ 2. Preliminaries.

In this section, we prepare some elementary equalities which will be used in the proof of Theorem in § 1.

Let $M=(M, J, <, >)$ be a $2n$-dimensional almost Hermitian manifold with the almost Hermitian structure $(J, <, >)$ and Ω the Kähler form of M defined

This research was partially supported by Grant-in-Aid for Scientific Research (No. 441260018661), Ministry of Education, Science and Culture.

by $\Omega(X, Y)=\langle X, JY\rangle$, for $X, Y\in\mathfrak{X}(M)$. In the sequel, we assume that M is oriented by the volume form $\sigma=((-1)^n/n!)\Omega^n$. We denote by ∇, R, ρ and τ the Riemannian connection, the curvature tensor, the Ricci tensor and the scalar curvature of M, respectively. The curvature tensor R is defined by

$$R(X, Y)Z=[\nabla_X, \nabla_Y]Z-\nabla_{[X,Y]}Z\,, \tag{2.1}$$

for $X, Y, Z\in\mathfrak{X}(M)$. We introduce a tensor field ρ^* of type (0, 2) (the tensor field ρ^* is called the Ricci *-tensor [8]) defined by

$$\rho^*(x, y)=(1/2)\text{ trace of }(z\longmapsto R(x, Jy)Jz)\,, \tag{2.2}$$

for $x, y, z\in T_pM$ (the tangent space of M at p), $p\in M$. We denote by τ^* (τ^* is called the *-scalar curvature) the trace of the linear endomorphism Q^* defined by $\langle Q^*x, y\rangle=\rho^*(x, y)$, for $x, y\in T_pM$, $p\in M$. By (2.2), we get immediately

$$\rho^*(x, y)=\rho^*(Jy, Jx)\,, \tag{2.3}$$

for $x, y\in T_pM$, $p\in M$. We denote by TM and Λ^kM ($k\geqq 1$) the tangent bundle of M and the vector bundle of real exterior k-forms over M, respectively. Then we may regard Λ^kM as a Riemannian vector bundle over M in the natural way. The curvature operator (also denoted by R) is the symmetric endomorphism of the vector bundle Λ^2M of real exterior 2-forms defined by

$$\langle R(\iota(x)\wedge\iota(y)), \iota(z)\wedge\iota(w)\rangle=-\langle R(x, y)z, w\rangle\,, \tag{2.4}$$

for $x, y, z, w\in T_pM$, $p\in M$, where ι denotes the duality: $TM\rightarrow\Lambda^1M=T^*M$ (the cotangent bundle of M) defined by means of the metric $\langle\,,\rangle$. For 1-form ω, $J\omega$ is the 1-form defined by $J\omega(X)=-\omega(JX)$, for $X\in\mathfrak{X}(M)$. Then we have $J(\iota(x))=\iota(Jx)$, for $x\in T_pM$, $p\in M$. Let $\{e_i\}$ be an orthonormal basis of T_pM at any point $p\in M$. In this paper, we shall adopt the following notational convention:

$$\begin{aligned}
&R_{hijk}=\langle R(e_h, e_i)e_j, e_k\rangle\,,\\
&R_{\bar{h}ijk}=\langle R(Je_h, e_i)e_j, e_k\rangle\,,\\
&\dots\dots\dots\dots\dots\\
&R_{\bar{h}\bar{i}\bar{j}\bar{k}}=\langle R(Je_h, Je_i)Je_j, Je_k\rangle\,,\\
&\rho_{ij}=\rho(e_i, e_j),\ \cdots,\ \rho_{\bar{i}\bar{j}}=\rho(Je_i, Je_j)\,,\\
&\rho^*_{ij}=\rho^*(e_i, e_j),\ \cdots,\ \rho^*_{\bar{i}\bar{j}}=\rho^*(Je_i, Je_j)\,,\\
&J_{ij}=\langle Je_i, e_j\rangle\,,\qquad \nabla_iJ_{jk}=\langle(\nabla_{e_i}J)e_j, e_k\rangle\,,
\end{aligned} \tag{2.5}$$

and so on, where the Latin indices run over the range $1, 2, \cdots, 2n$. We get easily

$$\nabla_iJ_{\bar{j}\bar{k}}=-\nabla_iJ_{jk}\,. \tag{2.6}$$

Now, we shall define differentiable functions $f_1, \cdots, f_5$ on M respectively by

(2.7)
$$f_1(p)=\sum R_{abij}(R_{\bar a\bar bij}-R_{\bar a\bar b\bar i\bar j}),$$
$$f_2(p)=\sum R_{a\bar aij}(R_{b\bar bij}-R_{b\bar b\bar i\bar j}),$$
$$f_3(p)=\sum R_{a\bar aij}(\nabla_{\bar b}J_{ik})\nabla_b J_{jk},$$
$$f_4(p)=\sum R_{abij}(\nabla_{\bar b}J_{ik})\nabla_{\bar a}J_{jk},$$
$$f_5(p)=\sum(\langle R(e^i\wedge e^j-Je^i\wedge Je^j),\ e^a\wedge e^b-Je^a\wedge Je^b\rangle)^2,$$

at any point $p\in M$, where $e^i=\iota(e_i)$ $(1\leqq i\leqq 2n)$. We shall evaluate the values of the functions $f_1, \cdots, f_4$ at each point $p\in M$. By the definition of the function f_1, we have easily the following

LEMMA 2.1.
$$f_1(p)=\frac{1}{2}\sum\langle R(e^i\wedge e^j-Je^i\wedge Je^j),\ e^a\wedge e^b\rangle\langle R(e^i\wedge e^j-Je^i\wedge Je^j),\ Je^a\wedge Je^b\rangle.$$

Similarly, taking account of (2.2) and (2.3), we have the following

LEMMA 2.2. $f_2(p)=2\sum(\rho^*_{ij}-\rho^*_{ji})^2$.

In the rest of this section, we assume that $M=(M, J, \langle\ ,\ \rangle)$ is a $2n$-dimensional almost Kähler manifold. Then it is known that M is a quasi Kähler manifold [**10**], i.e.,

(2.8)
$$(\nabla_X J)Y+(\nabla_{JX}J)JY=0,$$

for $X, Y\in\mathfrak{X}(M)$.

LEMMA 2.3. $\sum(\nabla_b J_{ik})(\nabla_a J_{jk})(\nabla_{\bar a}J_{ih})\nabla_{\bar b}J_{jh}=0$.

PROOF. Taking account of (2.8), we get

(2.9)
$$\begin{aligned}\sum(\nabla_b J_{ik})(\nabla_a J_{jk})(\nabla_{\bar a}J_{ih})\nabla_{\bar b}J_{jh}&=\sum(\nabla_b J_{ik})(\nabla_a J_{\bar jk})(\nabla_{\bar a}J_{ih})\nabla_{\bar b}J_{\bar jh}\\&=-\sum(\nabla_b J_{ik})(\nabla_{\bar a}J_{jk})(\nabla_{\bar a}J_{ih})\nabla_b J_{jh}\\&=-\sum(\nabla_b J_{ik})(\nabla_a J_{jk})(\nabla_a J_{ih})\nabla_b J_{jh}.\end{aligned}$$

On one hand, we get also

(2.10)
$$\sum(\nabla_b J_{ik})(\nabla_a J_{jk})(\nabla_{\bar a}J_{ih})\nabla_{\bar b}J_{jh}=\sum(\nabla_b J_{ik})(\nabla_a J_{jk})(\nabla_a J_{ih})\nabla_b J_{jh}.$$

From (2.9) and (2.10), the lemma follows immediately. Q.E.D.

By (2.8), we get

(2.11)
$$\sum_{i,j}(\nabla_a J_{ij})\nabla_{\bar b}J_{ij}=-\sum_{i,j}(\nabla_{\bar a}J_{\bar ij})\nabla_b J_{\bar ij}=-\sum_{i,j}(\nabla_{\bar a}J_{ij})\nabla_b J_{ij}.$$

Similarly, by (2.6) and (2.8), we get

(2.12)
$$\sum_{i,j}(\nabla_j J_{ia})\nabla_j J_{i\bar b}=\sum_{i,j}(\nabla_{\bar j}J_{\bar ia})\nabla_{\bar j}J_{\bar i\bar b}=-\sum_{i,j}(\nabla_j J_{i\bar a})\nabla_j J_{ib}.$$

Since M is an almost Kähler manifold, we get

$$(2.13)\qquad \sum_{i,j,k}(\nabla_i J_{bk})J_{aj}\nabla_j J_{ki}=\frac{1}{2}\sum_{i,j,k}(\nabla_i J_{bk}-\nabla_k J_{bi})J_{aj}\nabla_j J_{ki}$$
$$=-\frac{1}{2}\sum_{i,k}(\nabla_b J_{ki})\nabla_{\bar{a}} J_{ki}.$$

Similarly, we get

$$(2.14)\qquad \sum_{i,j,k} J_{bk}(\nabla_i J_{aj})\nabla_j J_{ki}=-\sum_{i,j,k} J_{bk}(\nabla_a J_{ji})\nabla_j J_{ki}-\sum_{i,j,k} J_{bk}(\nabla_j J_{ia})\nabla_j J_{ki}$$
$$=-\frac{1}{2}\sum_{i,j,k} J_{bk}(\nabla_a J_{ji})(\nabla_j J_{ki}-\nabla_i J_{kj})+\sum_{i,j}(\nabla_j J_{ia})\nabla_j J_{i\bar{b}}$$
$$=-\frac{1}{2}\sum_{i,j}(\nabla_a J_{ij})\nabla_{\bar{b}} J_{ij}+\sum_{i,j}(\nabla_j J_{ia})\nabla_j J_{i\bar{b}}.$$

From (2.8), taking account of (2.11)~(2.14), we get

$$(2.15)\qquad \sum_i \nabla^2_{i\bar{a}} J_{\bar{b}i}=\sum_{i,j,k} J_{bk}J_{aj}\nabla^2_{ij}J_{ki}$$
$$=-\sum_i \nabla^2_{ia}J_{bi}-\sum_{i,j,k}(\nabla_i J_{bk})J_{aj}\nabla_j J_{ki}-\sum_{i,j,k} J_{bk}(\nabla_i J_{aj})\nabla_j J_{ki}$$
$$=-\sum_i \nabla^2_{ia}J_{bi}+\sum_{i,j}(\nabla_j J_{i\bar{a}})\nabla_j J_{ib}.$$

LEMMA 2.4. $\rho^*_{ab}+\rho^*_{ba}=\rho_{ab}+\rho_{\bar{a}\bar{b}}+\sum_{i,j}(\nabla_j J_{ia})\nabla_j J_{ib}.$

PROOF. By (2.2) and the first Bianchi identity, we get

$$(2.16)\qquad 2\rho^*_{a\bar{b}}-2\rho^*_{\bar{a}b}=\sum_i R_{i\bar{i}ab}+\sum_i R_{i\bar{i}\bar{a}\bar{b}}$$
$$=-\sum_i R_{iab\bar{i}}-\sum_i R_{ib\bar{i}a}-\sum_i R_{i\bar{a}\bar{b}\bar{i}}-\sum_i R_{i\bar{b}\bar{i}\bar{a}}.$$

On one hand, we get easily

$$(2.17)\qquad \sum_i \nabla^2_{ia}J_{bi}-\sum_i \nabla^2_{ai}J_{bi}=\rho_{a\bar{b}}+\sum_i R_{iab\bar{i}}.$$

From (2.17), taking account of (2.8), we get

$$(2.18)\qquad \sum_i R_{iab\bar{i}}=-\rho_{a\bar{b}}+\sum_i \nabla^2_{ia}J_{bi}.$$

By (2.12), (2.15), (2.16) and (2.18), we get

$$(2.19)\qquad 2\rho^*_{a\bar{b}}-2\rho^*_{\bar{a}b}=2\rho_{a\bar{b}}-2\rho_{\bar{a}b}-\sum_i \nabla^2_{ia}J_{bi}-\sum_i \nabla^2_{i\bar{a}}J_{\bar{b}i}+\sum_i \nabla^2_{ib}J_{ai}+\sum_i \nabla^2_{i\bar{b}}J_{\bar{a}i}$$
$$=2\rho_{a\bar{b}}-2\rho_{\bar{a}b}+2\sum_{i,j}(\nabla_j J_{ia})\nabla_j J_{i\bar{b}}.$$

From (2.19), the lemma follows immediately. Q. E. D.

Now, we evaluate the value $f_3(p)$ of the function f_3 at any point $p\in M$. We may choose an orthonormal basis $\{e_i\}=\{e_\alpha, e_{n+\alpha}=Je_\alpha\}$ $(1\leqq\alpha, \beta\leqq n)$ in such a way that

$$\sum_{j,k}(\nabla_j J_{ka})\nabla_j J_{kb}=\lambda_a\delta_{ab}, \tag{2.20}$$

where $\lambda_1=\lambda_{n+1}\leqq\cdots\leqq\lambda_n=\lambda_{2n}$. We denote by f the continuous function on M defined by

$$f(p)=\sum_{i,j}(\lambda_i-\lambda_j)^2. \tag{2.21}$$

By (2.21), we get

$$f(p)=4n\sum_i\lambda_i^2-2\sum_{i,j}\lambda_i\lambda_j=4n\sum_i\lambda_i^2-2\|\nabla J\|^4(p). \tag{2.22}$$

LEMMA 2.5.

$$f_3(p)=-2\sum\rho_{ij}(\nabla_b J_{ik})\nabla_b J_{jk}-\frac{1}{4n}f(p)-\frac{1}{2n}\|\nabla J\|^4(p),$$

at any point $p\in M$.

PROOF. By (2.7), (2.8), (2.20), (2.22) and Lemma 2.4, we get

$$\begin{aligned}f_3(p)&=\sum R_{a\bar{a}ij}(\nabla_{\bar{b}}J_{ik})\nabla_b J_{jk}\\&=\sum R_{a\bar{a}i\bar{j}}(\nabla_b J_{ik})\nabla_b J_{jk}\\&=-\sum(\rho^*_{ij}+\rho^*_{ji})(\nabla_b J_{ik})\nabla_b J_{jk}\\&=-2\sum\rho_{ij}(\nabla_b J_{ik})\nabla_b J_{jk}-\frac{1}{4n}f(p)-\frac{1}{2n}\|\nabla J\|^4(p).\end{aligned}$$

Q.E.D.

Lastly, we evaluate the value $f_4(p)$ of the function f_4 at any point $p\in M$. We denote by ξ the vector field on M defined by

$$\xi_p=\sum_a\Big(\sum_{b,i,j,k}R_{abij}(\nabla_b J_{ik})J_{jk}\Big)e_a, \quad \text{at } p\in M. \tag{2.23}$$

From (2.7) and (2.23), by the direct calculation, we have easily the following

LEMMA 2.6.

$$\begin{aligned}f_4(p)=&(\operatorname{div}\xi)(p)+\sum(\nabla_i\rho_{bj}-\nabla_j\rho_{bi})(\nabla_b J_{ik})J_{jk}\\&+\frac{1}{4}\sum(\langle R(e^i\wedge e^j-Je^i\wedge Je^j), e^a\wedge e^b\rangle)^2.\end{aligned}$$

By Lemmas 2.1, 2.6, and (2.7), we have the following immediately

LEMMA 2.7.

$$f_1(p)-2f_4(p)=-2(\operatorname{div}\xi)(p)-\frac{1}{4}f_5(p)-2\sum(\nabla_i\rho_{bj}-\nabla_j\rho_{bi})(\nabla_b J_{ik})J_{jk}.$$

§ 3. An integral formula.

In this section, we establish an integral formula on a compact almost Kähler manifold which plays an essential role in the proof of Theorem in § 1. First, we start with a general almost Hermitian manifold $M=(M, J, <, >)$. We assume that $\dim M=2n\geqq 4$. We denote by ∇' the linear connection on M defined by

$$\nabla'_X Y = \nabla_X Y - \frac{1}{2} J(\nabla_X J)Y, \tag{3.1}$$

for $X, Y \in \mathfrak{X}(M)$ [10]. Then we may easily check that both of the Riemannian metric $\langle , \rangle$ and the almost complex structure J are parallel with respect to the linear connection ∇'. Furthermore, by direct calculation, we have the following

LEMMA 3.1. *The curvature tensor R' of the linear connection ∇' is given by*

$$R'(X, Y)Z = \frac{1}{2}(R(X, Y)Z - JR(X, Y)JZ) - \frac{1}{4}((\nabla_X J)(\nabla_Y J)Z - (\nabla_Y J)(\nabla_X J)Z),$$

for $X, Y, Z \in \mathfrak{X}(M)$.

We denote by $\mu_1(\nabla)$ (resp. $\mu_1(\nabla')$) the first Pontrjagin form corresponding to the metric connection ∇ (resp. ∇'). Then, by the well-known Chern-Weil theorem, the first Pontrjagin class $p_1(M)$ of M is represented by the 4-form $\mu_1(\nabla)$ (resp. $\mu_1(\nabla')$) in the de Rham cohomology group. The 4-form $\mu_1(\nabla)$ (resp. $\mu_1(\nabla')$) is given by

$$\mu_1(\nabla)_p = \frac{1}{32\pi^2} \sum R_{abij} R_{cdij} e^a \wedge e^b \wedge e^c \wedge e^d \tag{3.2}$$

(resp. $\mu_1(\nabla')_p = \frac{1}{32\pi^2} \sum R'_{abij} R'_{cdij} e^a \wedge e^b \wedge e^c \wedge e^d$), at any point $p \in M$, [5]. Let $\{e_i\}$ be an orthonormal basis of the tangent space T_pM of the form $\{e_i\} = \{e_\alpha, Je_\alpha\}$. Then we get

$$\Omega = -\sum_\alpha e^\alpha \wedge Je^\alpha. \tag{3.3}$$

From (3.3), we get easily

$$\Omega^{n-2} = (-1)^{n-2}(n-2)! \sum_{\alpha<\beta} e^1 \wedge Je^1 \wedge \cdots \wedge \widehat{e^\alpha \wedge Je^\alpha} \wedge \cdots \wedge \widehat{e^\beta \wedge Je^\beta} \wedge \cdots \wedge e^n \wedge Je^n, \tag{3.4}$$

where $\wedge$ denotes the delation. We here assume $\Omega^0=1$. By (3.2) and (3.4), we get

(3.5) $$\mu_1(\nabla)\wedge\Omega^{n-2}=\frac{(-1)^{n-2}(n-2)!}{32\pi^2}(\sum R_{a\bar{a}ij}R_{b\bar{b}ij}-2\sum R_{abij}R_{\bar{a}\bar{b}ij})\sigma\ ,$$

$$\text{(resp. } \mu_1(\nabla')\wedge\Omega^{n-2}=\frac{(-1)^{n-2}(n-2)!}{32\pi^2}(\sum R'_{a\bar{a}ij}R'_{b\bar{b}ij}-2\sum R'_{abij}R'_{\bar{a}\bar{b}ij})\sigma).$$

In the rest of this section, we assume that M is a $2n(n\geqq 2)$-dimensional compact almost Kähler manifold. Then it follows that the $2n$-form $\mu_1(\nabla)\wedge\Omega^{n-2}-\mu_1(\nabla')\wedge\Omega^{n-2}$ is exact. Thus, by Stokes' theorem, we get

(3.6) $$\int_M(\mu_1(\nabla)-\mu_1(\nabla'))\wedge\Omega^{n-2}=0\,.$$

From (3.5) and (3.6), taking account of (2.7), (2.8) and Lemmas 2.3, 3.1, we have finally the following

PROPOSITION 3.2. *Let $M=(M, J, <, >)$ be a $2n(n\geqq 2)$-dimensional compact almost Kähler manifold. Then we have*

$$\int_M\Big(f_1-\frac{1}{2}f_2+f_3-2f_4\Big)\sigma=0\,.$$

§4. Proof of Theorem.

It is well-known that any 2-dimensional almost Hermitian manifold is a Kähler manifold. On one hand, the present author has proved that Theorem is true in the case $\dim M=4$ [**6**]. So, for the proof of Theorem, it suffices to consider the case $\dim M>4$. Let $M=(M, J, <, >)$ be a $2n(n>2)$-dimensional compact Einstein almost Kähler manifold. Then we have

(4.1) $$\rho(X, Y)=\frac{\tau}{2n}\langle X, Y\rangle,$$

for $X, Y\in\mathfrak{X}(M)$. By (4.1) and Lemma 2.7, we get

(4.2) $$\int_M(f_1-2f_4)\sigma=-\frac{1}{4}\int_M f_5\sigma\,.$$

Furthermore, by (2.20), (4.1) and Lemma 2.5, we get

(4.3) $$\int_M f_3\sigma=-\int_M\Big(\frac{\tau}{n}\|\nabla J\|^2+\frac{1}{4n}f+\frac{1}{2n}\|\nabla J\|^4\Big)\sigma\,.$$

Thus, from Proposition 3.2, taking account of (4.2) and (4.3), we have finally

(4.4) $$\int_M\Big(\frac{1}{4}f_5+\frac{1}{2}f_2\Big)\sigma=-\int_M\Big(\frac{\tau}{n}\|\nabla J\|^2+\frac{1}{4n}f+\frac{1}{2n}\|\nabla J\|^4\Big)\sigma\,.$$

From (4.4), taking account of (2.7), (2.21) and Lemma 2.2, we may easily show that if the scalar curvature τ of M is non-negative, then ∇J vanishes identically on M, that is, M is a Kähler manifold. This completes the proof of Theorem.

References

[1] E. Abbena, An example of an almost Kähler manifold which is not Kählerian, Boll. Un. Mat. Ital. (6) **3A** (1984), 383-392.
[2] L. A. Cordero, M. Fernandez and A. Gray, Symplectic manifolds with no Kaehler structure, preprint.
[3] L. A. Cordero, M. Fernandez and M. de Leon, Examples of compact non-Kähler almost Kähler manifolds, Proc. Amer. Math. Soc., **95** (1985), 280-286.
[4] S. I. Goldberg, Integrability of almost Kähler manifolds, Proc. Amer. Math. Soc., **21** (1969), 96-100.
[5] S. Kobayashi and K. Nomizu, Foundations of Differential Geometry, II, New York, Interscience, 1969.
[6] K. Sekigawa, On some 4-dimensional compact Einstein almost Kähler manifolds, Math. Ann., **271** (1985), 333-337.
[7] W. P. Thurston, Some simple examples of symplectic manifolds, Proc. Amer. Math. Soc., **55** (1976), 467-468.
[8] F. Tricerri and L. Vanhecke, Curvature tensors on almost Hermitian manifolds, Trans. Amer. Math. Soc., **267** (1981), 365-398.
[9] B. Watson, New examples of strictly almost Kähler manifolds, Proc. Amer. Math. Soc., **88** (1983), 541-544.
[10] K. Yano, Differential geometry on complex and almost complex spaces, New York, Pergamon, 1965.

Kouei SEKIGAWA
Department of Mathematics
Faculty of Science
Niigata University
Niigata 950-21
Japan