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Certain invariant subspace structure of analytic crossed products

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1. Introduction.

This paper extends some results of [**2**, **3**, **6**]. We have an interest in the invariant subspace structure of certain subalgebras of von Neumann algebras constructed as crossed products of finite von Neumann algebras by trace-preserving automorphisms. These subalgebras were studied systematically by McAsey, Muhly and the second author (and by others) [**2**, **3**, **4**, **5**, **6**, **7**, etc.] under the name "nonselfadjoint crossed products"; nowadays, for a variety of reasons, we call them "analytic crossed products".

In this paper, our setting is the following. Let (X, μ) be a σ-finite standard Borel space and let τ be an invertible measure-preserving ergodic transformation on X. Then τ induces uniquely a unitary operator u on $L^2(X, \mu)$ such that $(ux)(t)=x(\tau^{-1}t)$, $x\in L^\infty(X, \mu)\cap L^2(X, \mu)$. Form the Hilbert space $\boldsymbol{L}^2=l^2(\boldsymbol{Z})\otimes \boldsymbol{L}^2(X, \mu)$ and consider the operators L_x, $x\in L^\infty(X, \mu)$ and L_δ defined on $\boldsymbol{L}^2$ by the formulae $L_x=I\otimes x$ and $L_\delta=S\otimes u$ where S is the usual shift on $l^2(\boldsymbol{Z})$. Then the von Neumann crossed product determined by $L^\infty(X, \mu)$ $(=M)$ and τ is defined as the von Neumann algebra $\mathfrak{L}$ on $\boldsymbol{L}^2$ generated by $\{L_x : x\in L^\infty(X, \mu)\}$ $(=L(M))$ and L_δ, while the subalgebra which we call an analytic crossed product is the σ-weakly closed subalgebra $\mathfrak{L}_+$ generated by $L(M)$ and the positive powers of L_δ. Let $\boldsymbol{H}^2$ be the subspace $l^2(\boldsymbol{Z}_+)\otimes L^2(X, \mu)$ of $\boldsymbol{L}^2$, where $\boldsymbol{Z}_+=\{n\in\boldsymbol{Z} : n\geqq 0\}$. We shall denote by $\mathrm{Lat}(\mathfrak{L}_+)$ the set of all invariant subspaces $\mathfrak{M}$ under $\mathfrak{L}_+$ such that $\bigcap_{n\geq 0}L_\delta^n\mathfrak{M}=\{0\}$.

In [**2**, **3**], McAsey introduced the notion of canonical models for $\mathrm{Lat}(\mathfrak{L}_+)$. That is, a family of left-pure, left-full, left-invariant subspaces $\{\mathfrak{M}_i\}_{i\in I}$ in $\mathrm{Lat}(\mathfrak{L}_+)$ constitutes a complete set of canonical models for $\mathrm{Lat}(\mathfrak{L}_+)$ in case (a) for no two distinct indices i and j, $P_{\mathfrak{M}_i}$ is unitary equivalent to $P_{\mathfrak{M}_j}$ by a unitary operator in $\mathfrak{R}$ $(=\mathfrak{L}')$; and (b) for every $\mathfrak{M}$ in $\mathrm{Lat}(\mathfrak{L}_+)$, there is an i in I and a partial isometry V in $\mathfrak{R}$ such that $VP_{\mathfrak{M}_i}V^*=P_{\mathfrak{M}}$, so that $\mathfrak{M}=V\mathfrak{M}_i$. Let $M=l^\infty(X)$, where X is a finite set with elements $t_0, t_1, \cdots, t_{k-1}$ and let τ be the permuta-

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tion of X defined by $\tau(t_i)=t_{i+1}$ $(i\neq k-1)$ and $\tau(t_{k-1})=t_0$. Then McAsey [4] studied a complete set of canonical models for $\mathrm{Lat}(\mathfrak{L}_+)$ which consists of two-sided invariant subspaces of $\boldsymbol{L}^2$. Further, Solel [6] studied a complete set of canonical models for $\mathrm{Lat}(\mathfrak{L}_+)$ in case (X, μ) is a non-atomic standard Borel space with a finite measure μ. We refer the reader to [1, 5, 7, etc.] concerning invariant subspace structure in more general framework.

In this paper, we consider a complete set of canonical models for $\mathrm{Lat}(\mathfrak{L}_+)$ in the following setting. Let X be a standard Borel space with a σ-finite infinite positive measure μ, that is, $\mu(X)=\infty$. Let τ be an invertible measure-preserving ergodic transformation on X. First we shall prove that, for every $\boldsymbol{Z}_+\cup\{\infty\}$-valued measurable function m on X, there exists a left-pure, left-invariant subspace $\mathfrak{M}$ of $\boldsymbol{L}^2$ with the multiplicity function m. As a corollary, we can construct a left-pure, left-full, left-invariant subspace $\mathfrak{M}_\infty$ of $\boldsymbol{L}^2$ such that $m(t)=\infty$ for almost everywhere t in X where m is the multiplicity function of $\mathfrak{M}_\infty$. Therefore, we have that, for every non-zero $\mathfrak{M}\in\mathrm{Lat}(\mathfrak{L}_+)$, there exists a partial isometry V in $\mathfrak{R}$ such that $VP_{\mathfrak{M}_\infty}V^*=P_{\mathfrak{M}}$, so that $\mathfrak{M}=V\mathfrak{M}_\infty$. This implies that the complete set of canonical models is the singleton $\{\mathfrak{M}_\infty\}$ in this case. Finally we shall consider the structure of two-sided invariant subspaces of $\boldsymbol{L}^2$ and the case that (X, μ) is an atomic measure space.

2. Definitions and preliminaries.

Let (X, μ) be a σ-finite standard Borel space with $\mu(X)=\infty$. Let τ be an invertible measure-preserving ergodic transformation on X. Using the product of the counting measure on the integers $\boldsymbol{Z}$, and the measure μ on X, we may realize $\boldsymbol{Z}\times X$ as a measure space. The space $\boldsymbol{L}^2(\boldsymbol{Z}\times X)$ of all measurable functions on $\boldsymbol{Z}\times X$ satisfying

$$\sum_{n\in\boldsymbol{Z}}\int_X |f(n, t)|^2 d\mu(t) < \infty ,$$

is a Hilbert space with inner product

$$(f, g) = \sum_{n\in\boldsymbol{Z}}\int_X f(n, t)\overline{g(n, t)}d\mu(t) , \qquad f, g\in \boldsymbol{L}^2(\boldsymbol{Z}\times X) .$$

We shall denote it by $\boldsymbol{L}^2$. Define the following bounded linear operators on $\boldsymbol{L}^2$;

$$(L_\delta f)(n, t) = f(n-1, \tau^{-1}t) ,$$

$$(R_\delta f)(n, t) = f(n-1, t) ,$$

$$(L_\phi f)(n, t) = \phi(t)f(n, t) , \qquad \phi\in L^\infty(X)$$

and

$$(R_\phi f)(n, t) = \phi(\tau^{-n}t)f(n, t) , \qquad \phi\in L^\infty(X) .$$

Note that L_δ and R_δ are unitary operators on $\boldsymbol{L}^2$. Put $M=L^\infty(X)$. Let $L(M)$ (resp. $R(M)$) denote the algebra generated by $\{L_\phi : \phi\in M\}$ (resp. $\{R_\phi : \phi\in M\}$). Clearly $L(M)$ and $R(M)$ are abelian von Neumann algebras. The left (resp. right) von Neumann crossed product of $L^\infty(X)$ by τ is defined as the von Neumann algebra $\mathfrak{L}$ (resp. $\mathfrak{R}$) generated by $L(M)$ and L_δ (resp. $R(M)$ and R_δ). Define the left (resp. right) analytic crossed product as the σ-weakly closed subalgebra $\mathfrak{L}_+$ (resp. $\mathfrak{R}_+$) generated by $L(M)$ and L_δ (resp. $R(M)$ and R_δ). Furthermore, we define $\boldsymbol{H}^2=\{f\in\boldsymbol{L}^2 : f(n, \cdot)=0,\ n<0\}$.

DEFINITION 2.1. Let $\mathfrak{M}$ be a closed subspace of $\boldsymbol{L}^2$. We shall say that $\mathfrak{M}$ is left-invariant if $\mathfrak{L}_+\mathfrak{M}\subset\mathfrak{M}$, left-reducing if $\mathfrak{L}\mathfrak{M}\subset\mathfrak{M}$, left-pure if $\mathfrak{M}$ contains no non-trivial left-reducing subspace and left-full if the smallest left-reducing subspace containing $\mathfrak{M}$ is $\boldsymbol{L}^2$ itself. The right-hand versions of these concepts are defined similarly, and a closed subspace which is both left- and right-invariant will be said to be two-sided invariant.

In this paper, all results will be formulated in terms of left-invariant subspaces. We leave it to the reader to rephrase them to obtain "right-hand" statements.

An important tool for dealing with invariant subspaces is the notion of multiplicity function introduced in [2, 3]. To obtain it, note that the space $\boldsymbol{L}^2$ may be identified with the direct integral $\int_X^\oplus l^2(\boldsymbol{Z})d\mu(t)$, and the algebra $L(M)'$, acting on it, may be identified with $\int_X^\oplus B(l^2(\boldsymbol{Z}))d\mu(t)$, where $B(l^2(\boldsymbol{Z}))$ is the algebra of all bounded linear operators on $l^2(\boldsymbol{Z})$. Let $\mathfrak{M}$ be a left-invariant subspace of $\boldsymbol{L}^2$. Then the orthogonal projection $P_\mathfrak{F}$ on $\mathfrak{M}\ominus L_\delta\mathfrak{M}=\mathfrak{F}$ lies in $L(M)'$, so it is written as a direct integral $\int_X^\oplus P(t)d\mu(t)$, where $P(t)$ is a projection in $B(l^2(\boldsymbol{Z}))$ for almost everywhere $t\in X$. We define the multiplicity function m by letting $m(t)$ be the dimension of the range of $P(t)$. Then it is cleart hat m is a measurable function on X with values in $\boldsymbol{Z}_+\cup\{\infty\}$. By [3, Theorem 3.4], we have the following proposition.

PROPOSITION 2.2. *For $i=1, 2$, let $\mathfrak{M}_i$ be a left-pure, left-invariant subspace of $\boldsymbol{L}^2$. Let $\mathfrak{F}_i=\mathfrak{M}_i\ominus L_\delta\mathfrak{M}_i$ and m_i the multiplicity function of $\mathfrak{M}_i$. Then the following statements are equivalent:*

(1) $P_{\mathfrak{M}_1}=VP_{\mathfrak{M}_2}V^*$ *for a partial isometry V in $\mathfrak{R}$, so that* $\mathfrak{M}_1=V\mathfrak{M}_2$,

(2) $m_1(t)\leqq m_2(t)$ *a.e., and*

(3) $P_{\mathfrak{F}_1}\lesssim P_{\mathfrak{F}_2}$ *in* $L(M)'$.

Let $\mathfrak{M}$ be a left-pure, left-invariant subspace of $\boldsymbol{L}^2$. We shall denote the multiplicity function by $m[\mathfrak{M}](t)$ in this note.

3. Invariant subspace structure.

Keep the notations and the assumptions in §2. Our aim in this section is to construct a left-pure, left-full, left-invariant subspace of $\boldsymbol{L}^2$ such that the multiplicity function $m(t)=\infty$ for almost everywhere t in X. To do this, we need some lemmas.

LEMMA 3.1. *Let* $\{\mathfrak{M}_i\}_{i\in I}$ *is a finite or countable collection of left-pure, left-invariant subspaces of* $\boldsymbol{L}^2$ *such that* $\mathfrak{M}_i$ *is orthogonal to* $\mathfrak{M}_j$, *for* $i\neq j$. *Then* $\mathfrak{M}=\sum_{i\in I}\oplus\mathfrak{M}_i$ *is a left-pure, left-invariant subspace with the multiplicity function* $m[\mathfrak{M}](t)=\sum_{i\in I}m[\mathfrak{M}_i](t)$, *a.e.*

PROOF. See [**6**, Lemma 3.1].

Let χ_E be a characteristic function of a measurable subset E in X. We define a projection P in $L(M)'$ by

$$(Pf)(n,t)=\begin{cases}\chi_E(t)f(0,t)\,, & n=0\,,\\ 0\,, & n\neq 0\,.\end{cases}$$

Let E_n be the projection on $\boldsymbol{L}^2$ defined by the formula

$$(E_nf)(k,t)=\begin{cases}f(k,t)\,, & k=n\,,\\ 0\,, & k\neq n\,.\end{cases}$$

Since $P\leqq E_0$ and since $\{L_\delta^n E_0 L_\delta^{*n}\}_{n\in Z}$ is mutually orthogonal, $\{L_\delta^n PL_\delta^{*n}\}_{n\in Z}$ is mutually orthogonal. We define a subspace $\mathfrak{M}(E)$ of $\boldsymbol{L}^2$ by $\mathfrak{M}(E)=\sum_{n=0}^{\infty}\oplus(L_\delta^n PL_\delta^{*n})\boldsymbol{L}^2$. As in [**6**, Lemma 3.2] and [**5**, Lemma 5.1], we have

LEMMA 3.2. (i) $\mathfrak{M}(E)$ *is a left-pure left-invariant subspace of* $\boldsymbol{H}^2$ *with the multiplicity function* $\chi_E(t)$.

(ii) *If* $\mu(E)<\infty$, *then* $\mathfrak{M}(E)$ *is the closed linear span of* $\{L_\delta^n L_\phi e_0:\ \phi\in L^\infty(X,\mu),\ n\geqq 0\}$, *where* $e_0(n,t)=0$ *if* $n\neq 0$ *and* $e_0(0,t)=\chi_E(t)$.

Let E and F be measurable subsets of X such that there are measurable subsets $\{E_n\}_{n=0}^{\infty}$ and $\{F_n\}_{n=0}^{\infty}$ with the following properties:

(1) $E_n\subset E$ and $F_n\subset F$, $n\geqq 0$,

(2) $E_n\cap E_m=F_n\cap F_m=\emptyset$, $n\neq m$,

(3) $\mu(E\setminus\bigcup_{n=0}^{\infty}E_n)=\mu(F\setminus\bigcup_{n=0}^{\infty}F_n)=0$, and

(4) $F_n=\tau^n(E_n)$, $n\geqq 0$.

Then we have the following lemma.

LEMMA 3.3 ([**6**, Lemma 3.4]). *$U=\sum_{k=0}^{\infty}L_{\chi_{F_k}}L_\delta^k$ is a partial isometry in $\mathfrak{L}_+$ with the initial projection L_{χ_E} and the final projection L_{χ_F}.*

By the proof of [**6**, Lemma 3.5] and [**5**, Lemma 5.4], we have

LEMMA 3.4. *Let E, F, $\{E_n\}$, $\{F_n\}$ be as (1)~(4) in the above. Suppose that $\mu(E)=\mu(F)<\infty$. Then there exists a left-pure, left-invariant subspace $\mathfrak{M}$ of $\mathfrak{M}(E)$ such that $m[\mathfrak{M}](t)=\chi_F(t)$ a.e. and $\sum_{n\in Z}L_\delta^n P_{\mathfrak{F}} L_\delta^{*n}=R_{\chi_E}$ where $\mathfrak{F}=\mathfrak{M}\ominus L_\delta\mathfrak{M}$.*

Let $\mathfrak{M}$ be a left-pure, left-invariant subspace of $\boldsymbol{L}^2$. Then $m[\mathfrak{M}](t)$ is a measurable function with values in $\boldsymbol{Z}_+\cup\{\infty\}$. Conversely, we have the following

THEOREM 3.5. *Let m be a measuable function on X with values in $\boldsymbol{Z}_+\cup\{\infty\}$. Then there exists a left-pure, left-invariant subspace $\mathfrak{M}$ of $\boldsymbol{L}^2$ with the multiplicity function $m(t)$.*

PROOF. Put $E_n=\{t\in X: m(t)\geqq n\}$ for all $n\in\boldsymbol{Z}_+\cup\{\infty\}$. Then E_n is a measurable subset of X and $m(t)=\sum_{n=1}^{\infty}\chi_{E_n}(t)$. If $\mu(E_n)=\infty$, by the σ-finiteness of μ, there exists a family $\{E_{nk}\}_{k=1}^{\infty}$ of mutually disjoint measurable subsets of X such that $\mu(E_{nk})<\infty$, for all k, and such that $E_n=\sum_{k=1}^{\infty}E_{nk}$. Therefore we may rewrite

$$m(t)=\sum_{n=1}^{\infty}\chi_{E'_n}(t),\quad \mu(E'_n)<\infty,\quad n\geqq 1\,.$$

At first, put $F_1=E'_1$. Define the set $\{F_2^{(k)}\}_{k=0}^{\infty}$ and $\{G_2^{(k)}\}_{k=0}^{\infty}$, inductively as follows. For $k=0$, let $F_2^{(0)}=E'_2\cap(X\setminus F_1)$ and $G_2^{(0)}=F_2^{(0)}$. For $k\geqq 1$, put

$$F_2^{(k)}=\tau^{-k}\Big(E'_2\setminus\bigcup_{n=0}^{k-1}G_2^{(n)}\Big)\cap\Big(X\setminus\bigcup_{n=0}^{k-1}F_2^{(n)}\Big)\cap(X\setminus F_1)$$

and

$$G_2^{(k)}=\tau^k(F_2^{(k)})\,.$$

Then $\{F_2^{(k)}\}_{k=0}^{\infty}$ and $\{G_2^{(k)}\}_{k=0}^{\infty}$ are mutually disjoint respectively. Put $F_2=\bigcup_{k=0}^{\infty}F_2^{(k)}$ and $G_2=\bigcup_{k=0}^{\infty}G_2^{(k)}$. Then $F_1\cap F_2=\emptyset$ and $G_2\subset E'_2$. For $k\geqq 1$, we have

$$\begin{aligned}\emptyset &= F_2^{(k)}\cap(X\setminus F_2^{(k)})\\ &=\tau^{-k}\Big(E'_2\setminus\bigcup_{n=0}^{k-1}G_2^{(n)}\Big)\cap\Big(X\setminus\bigcup_{n=0}^{k-1}F_2^{(n)}\Big)\cap(X\setminus F_1)\cap(X\setminus F_2^{(k)})\\ &=\tau^{-k}\Big(E'_2\setminus\bigcup_{n=0}^{k-1}G_2^{(n)}\Big)\cap\Big(X\setminus\bigcup_{n=0}^{k}F_2^{(n)}\Big)\cap(X\setminus F_1)\\ &\supset\tau^{-k}(E'_2\setminus G_2)\cap(X\setminus F_2)\cap(X\setminus F_1)\\ &=\tau^{-k}(E'_2\setminus G_2)\cap(X\setminus(F_1\cup F_2))\,.\end{aligned}$$

Thus $\tau^{-k}(E'_2\setminus G_2)\subset F_1\cup F_2$ for all $k\geqq 1$. Put $K=\bigcup_{k=1}^{\infty}\tau^{-k}(E'_2\setminus G_2)$. Then $\tau^{-1}(K)\subset K\subset F_1\cup F_2$. Since τ is measure-preserving and $\mu(F_1\cup F_2)<\infty$, $\mu(K\setminus\tau^{-1}(K))=0$

and so $\tau^{-1}(K)=K$ a.e. Thus $\mu(K)=0$. This implies that $\mu(E_2'\setminus G_2)=0$. Thus $\{F_2^{(k)}\}_{k=0}^{\infty}$ and $\{G_2^{(k)}\}_{k=0}^{\infty}$ satisfy the following conditions:

(1) $F_2=\sum_{k=0}^{\infty}F_2^{(k)}$ and $E_2'=\sum_{k=0}^{\infty}G_2^{(k)}$ a.e., and

(2) $G_2^{(k)}=\tau^k(F_2^{(k)})$, $k\geqq 0$.

Inductively, we can define the measurable subsets $\{F_n\}_{n=1}^{\infty}$, $\{F_n^{(k)}\}_{k=1}^{\infty}$ and $\{G_n^{(k)}\}_{k=1}^{\infty}$ with the following properties: for $n\geqq 1$,

(1) $F_n=\sum_{k=0}^{\infty}F_n^{(k)}$, $F_n^{(k)}\cap F_n^{(k')}=\emptyset$ $(k\neq k')$ and $E_n'=\sum_{k=0}^{\infty}G_n^{(k)}$,

(2) $G_n^{(k)}=\tau^k(F_n^{(k)})$, $k\geqq 0$, $G_n^{(k)}\cap G_n^{(k')}=\emptyset$ $(k\neq k')$ and

(3) $F_n\cap F_m=\emptyset$, for $n\neq m$.

By Lemma 3.4, there exists a left-pure, left-invariant subspace $\mathfrak{M}_n$ of $\mathfrak{M}(F_n)$ such that $m[\mathfrak{M}_n](t)=\chi_{E_n'}(t)$. Since $\{F_n\}_{n=1}^{\infty}$ is mutually disjoint, the family $\{\mathfrak{M}(F_n)\}_{n=1}^{\infty}$ of left-pure, left-invariant subspaces of $\boldsymbol{L}^2$ is mutually orthogonal. Put $\mathfrak{M}=\sum_{n=1}^{\infty}\oplus\mathfrak{M}_n$. By Lemma 3.1, $\mathfrak{M}$ is a left-pure, left-invariant subspace of $\boldsymbol{L}^2$ and

$$m[\mathfrak{M}](t)=\sum_{n=1}^{\infty}m[\mathfrak{M}_n](t)=\sum_{n=1}^{\infty}\chi_{E_n'}(t)=m(t)\,.$$

Thus the multiplicity function of $\mathfrak{M}$ is m. This completes the proof.

COROLLARY 3.6. *Let m be a measurable function on X such that $m(t)=\infty$ for almost all $t\in X$. Then there exists a left-pure, left-full, left-invariant subspace $\mathfrak{M}_\infty$ of $\boldsymbol{L}^2$ such that $m[\mathfrak{M}_\infty](t)=\infty$ for almost all $t\in X$.*

PROOF. Since (X,μ) is σ-finite, there exists a family $\{E_n'\}_{n=1}^{\infty}$ of measurable subsets of X such that $X=\bigcup_{n=1}^{\infty}E_n'$, $E_1'\subset E_2'\subset\cdots\subset E_n'\subset\cdots$ and $\mu(E_n')<\infty$, $n\geqq 1$. Then we have $m(t)=\sum_{n=1}^{\infty}\chi_{E_n'}(t)=\infty$ a.e. Let $\{F_n\}_{n=1}^{\infty}$ be the family of mutually disjoint measurable subsets of X as in the proof of Theorem 3.5. Thus there exists a left-pure, left-invariant subspace $\mathfrak{M}$ of $\boldsymbol{L}^2$ such that $m[\mathfrak{M}](t)=\infty$, for almost all t in X and $\sum_{n\in Z}L_\delta^nP_{\mathfrak{F}}L_\delta^{*n}=R_{\chi_{\bigcup_{n=1}^{\infty}F_n}}$, where $\mathfrak{F}=\mathfrak{M}\ominus L_\delta\mathfrak{M}$. Put $F_0=X\setminus\bigcup_{n=1}^{\infty}F_n$. Define $\mathfrak{M}_\infty=\mathfrak{M}(F_0)\oplus\mathfrak{M}$. It is clear that $\mathfrak{M}_\infty$ is a left-full, left-pure, left-invariant subspace of $\boldsymbol{L}^2$ such that $m[\mathfrak{M}_\infty](t)=\infty$. This completes the proof.

By Corollary 3.6, we can construct a left-pure, left-full, left-invariant subspace of $\boldsymbol{L}^2$ such that $m(t)=\infty$ for almost all $t\in X$. We denote this space by $\mathfrak{M}_\infty$. Then we have the following theorem.

THEOREM 3.7. *Let $\mathfrak{M}$ be a left-pure, left-invariant subspace of $\boldsymbol{L}^2$. Then there exists a partial isometry V in $\mathfrak{R}$ such that $P_{\mathfrak{M}}=VP_{\mathfrak{M}_\infty}V^*$, so that $\mathfrak{M}=V\mathfrak{M}_\infty$.*

PROOF. Since $m[\mathfrak{M}](t)\leqq\infty=m[\mathfrak{M}_\infty](t)$, Proposition 2.2 implies the conclusion.

4. Remarks.

In this section, we shall remark the structure of two-sided invariant subspaces of $\boldsymbol{L}^2$. Keep the notations and the assumptions as in §2 and §3.

At first, we suppose that (X, μ) is non-atomic and $\mu(X)=\infty$. As in the proof of [**6**, Theorem 4.1], we have the following theorem.

THEOREM 4.1. *Let $m(t)$ be a non-zero measurable function with values in $\boldsymbol{Z}_+\cup\{\infty\}$. Then there is a two-sided invariant subspace $\mathfrak{M}$ with multiplicity function $m(t)$ if and only if there is a measurable function d on X with values in $\boldsymbol{Z}$ such that $d(t)-d(\tau^{-1}(t))=1-m(t)$ a.e. and $|d(t)|<\infty$ a.e.*

By Theorem 4.1, if $m(t)$ is a multiplicity function of a two-sided invariant subspace $\mathfrak{M}$ of $\boldsymbol{L}^2$, then $\mu(m^{-1}(\{\infty\}))=0$. However, by Corollary 3.6, we can construct a left-pure, left-full, left-invariant subspace $\mathfrak{M}_\infty$ such that $\{t\in X: m[\mathfrak{M}_\infty](t)=\infty\}=X$. Thus, $\mathfrak{M}_\infty$ is not two-sided invariant. Therefore, it is impossible to find a complete set of canonical models among the two-sided invariant subspaces.

Finally, we suppose that (X, μ) is atomic and $\mu(X)=\infty$. Thus the space X is countably discrete. Let $X=\{x_n\}_{n=-\infty}^{\infty}$ and the map τ will be the translation $\tau(x_i)=x_{i+1}$ of X. In this case, McAsey studied the structure of invariant subspaces in [**2**, Chapter IV]. He considered the four classes of all non-negative $\boldsymbol{Z}_+\cup\{\infty\}$-valued functions on X. A function m from X to $\boldsymbol{Z}_+\cup\{\infty\}$ is of type 0 (resp. 1, 2) in case the cardinality of the set $m^{-1}(\{\infty\})$ is 0 (resp. 1, 2). Such a function is of type 3 in case the cardinality is greater than or equal to 3. Further, he defined the notion of admissible functions. That is, the function m from X to $\boldsymbol{Z}_+\cup\{\infty\}$ is an admissible function in case m is either of

i) type 0, or

ii) type 1 (suppose $m(x_k)=\infty$) and one of the following conditions holds:

a) $\operatorname{supp} m=\{x_k\}$,

b) $\operatorname{supp} m\subset\{x_k\}\cup C$ and $\operatorname{supp} m\neq\{x_k\}$,

c) $\operatorname{supp} m\subset\{x_k\}\cup D$ and $\operatorname{supp} m\neq\{x_k\}$,

where $C=\{x_{k-1}, x_{k-2}, x_{k-3}, \cdots\}$ and $D=\{x_{k+1}, x_{k+2}, x_{k+3}, \cdots\}$,

iii) type 2 (suppose that $m(x_k)=m(x_j)=\infty$, $j>k$) and $\operatorname{supp} m\cap(C\cup E)=\emptyset$, where $C=\{x_{k-1}, x_{k-2}, x_{k-3}, \cdots\}$ and $E=\{x_{j+1}, x_{j+2}, x_{j+3}, \cdots\}$. By [**1**, Theorem 4.13], a function m from X to $\boldsymbol{Z}_+\cup\{\infty\}$ is an admissible multiplicity function if and only if it is the multiplicity function of a two-sided invariant subspace. However, in §3, we constructed a left-pure, left-full, left-invariant subspace $\mathfrak{M}_\infty$ such that $m[\mathfrak{M}_\infty](x_k)=\infty$ for all $k\in\boldsymbol{Z}$. Of course, $\mathfrak{M}_\infty$ is not two-sided invariant.

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