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Manifolds without conjugate points and with integral curvature zero

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0. Introduction.

A complete Riemannian manifold M is said to be *without conjugate points* if no geodesic contains a pair of mutually conjugate points. E. Hopf ([**9**]) and L. W. Green ([**7**]) have proved that the integral of the scalar curvature of a compact Riemannian manifold without conjugate points is nonpositive, and it vanishes only if the metric is flat. The non-conjugacy hypothesis was discussed in [**10**] and [**11**]. Namely, it follows that a compact Riemannian manifold is without focal points if there is a point which cannot be a focal point to any geodesic, although a pole and a point which is not a pole can exist simultaneously in a torus $\boldsymbol{T}^2$ of revolution. Recently, N. Innami ([**12**]) has proved that the integral of the scalar curvature of a complete simply connected Riemannian manifold $\boldsymbol{R}^n$ without conjugate points is nonpositive if the Ricci curvature is summable on the unit tangent bundle, and it vanishes only if the metric is flat. Here a function is called *summable* if its absolute integral exists. The purpose of the present paper is to improve the topological hypothesis more.

Let M be a complete Riemannian manifold and let SM be the unit tangent bundle of M. Let $f^t: SM\to SM$ be the geodesic flow, i.e., $f^t v=\dot\gamma_v(t)$ for any $v\in SM$ where $\gamma_v: (-\infty, \infty)\to M$ is the geodesic with $\dot\gamma_v(0)=v$. We say that a $v\in SM$ is *non-wandering* if there exist sequences $\{v_n\}\subset SM$ and $\{t_n\}\subset \boldsymbol{R}$ such that $t_n\to\infty$, $v_n\to v$ and $f^{t_n}v_n\to v$ as $n\to\infty$. We denote by Ω the set of all non-wandering points in SM under the geodesic flow.

THEOREM. *Let M be a complete Riemannian manifold without conjugate points. Suppose Ω decomposes into at most countably many f^t-invariant sets each of which has finite volume and the Ricci curvature is summable on SM. Then, the integral of the scalar curvature of M is nonpositive, and it vanishes only if M is flat.*

If the manifold M is flat outside a compact set, then the assumption of summability for the Ricci curvature is automatically satisfied. Furthermore, the theorem is true without assumption put on the set Ω of all non-wandering points (see Corollary 3). The proof of Theorem divides into two parts: One is for $SM-\Omega$ and the other is for Ω. The typical cases are the following.

COROLLARY 1 ([**12**]). *Let M be a complete simply connected Riemannian manifold without conjugate points. If the Ricci curvature of M is summable on SM, then the integral of the scalar curvature of M is nonpositive, and it vanishes only if M is Euclidean.*

S. Cohn-Vossen ([**4**]) has proved that a plane without conjugate points has the nonpositive integral curvature if it exists ([**2**]). Corollary 1 is the answer of the question when it vanishes. L. W. Green and R. Gulliver ([8]) give a partial answer as an application of the theorem of E. Hopf also, proving that a plane whose metric differs from the canonical flat metric at most on a compact set is Euclidean if there is no conjugate point.

COROLLARY 2. *Let M be a complete Riemannian manifold without conjugate points and with finite volume. If the Ricci curvature of M is summable on SM, then the integral of the scalar curvature of M is nonpositive, and it vanishes only if M is flat.*

It is the difficulty of the proof that the summability of $\operatorname{tr} A$ on SM is not established where $A(v)$ is the limit of the second fundamental forms at $\pi(v)$ of the geodesic spheres $S(\pi(v), \gamma_v(t))$ with center $\gamma_v(t)$ and through $\pi(v)$ in M as $t\to\infty$, where π is the projection of SM to M. In fact, Corollary 2 is a direct consequence of the method of E. Hopf and L. W. Green if we assume in addition any condition which ensure the summability of $\operatorname{tr} A$ on SM, for example, that the sectional curvature of M is bounded below ([**7**]). To escape from the summability argument we use the Fubini theorem for $SM-\Omega$ and the Birkhoff ergodic theorem for Ω. This is why we assume that Ω decomposes into at most countably many f^t-invariant sets each of which has finite volume.

There is a special case that we can calculate the integral of the Ricci curvature over Ω without assumption of decomposition.

COROLLARY 3. *Let M be a complete Riemannian manifold without conjugate points which is flat outside some compact set. Then, the integral of the scalar curvature of M is nonpositive, and it vanishes only if M is flat.*

The author would like to express his hearty thanks to the referee who suggests Corollary 3 without proof.

1. Preliminaries.

Let M be a complete Riemannian manifold and let SM be the unit tangent bundle. Let $f^t: SM \to SM$ be the geodesic flow, i.e., $f^t v = \dot{\gamma}_v(t)$ for any $t \in (-\infty, \infty)$ where $\gamma_v: (-\infty, \infty) \to M$ is the geodesic with $\dot{\gamma}_v(0) = v$. Let $d\sigma$ be the volume form induced from the Riemannian metric of M and let $d\theta$ be the canonical volume form on the unit sphere in the Euclidean space $\boldsymbol{E}^n$, $n = \dim M$. Then, $d\omega = d\sigma \wedge d\theta$ is a volume form on SM and f^t-invariant.

We define a Riemannian metric g_1 on SM as follows: Let $\xi \in T_v SM$, $v \in SM$ and let $c: (-\varepsilon, \varepsilon) \to SM$ be a curve with $\dot{c}(0) = \xi$. If $c(t) = (c_1(t), c_2(t))$ for any $t \in (-\varepsilon, \varepsilon)$ by the local trivialization, then

$$g_1(\xi, \xi) = g(\dot{c}_1(0), \dot{c}_1(0)) + g(\nabla_{c_1} c_2(0), \nabla_{c_1} c_2(0))$$

where g is the Riemannian metric of M and $\nabla_{c_1} c_2$ is the covariant derivative along c_1. The orbits of the geodesic flow are geodesics in SM with the Riemannian metric g_1. If $\gamma: [a, b] \to M$ is a minimizing geodesic ($a = -\infty$, $b = \infty$ admitted), then the lift $\dot{\gamma}$ of γ to SM is a minimizing geodesic in SM also.

1.1. The trajectories of the geodesic flow. We say that a $v \in SM$ is *non-wandering* if there exist sequences $\{v_n\} \subset SM$ and $\{t_n\} \subset \boldsymbol{R}$ such that $t_n \to \infty$, $v_n \to v$ and $f^{t_n} v_n \to v$. We denote by Ω the set of all non-wandering points in SM under the geodesic flow. It follows that Ω is closed and f^t-invariant. We introduce an equivalence relation $\sim$ in $SM - \Omega$ in such a way that $v \sim w$ if $v = f^t w$ for some $t \in (-\infty, \infty)$, where $v, w \in SM - \Omega$. Let N be the set of all equivalence classes $[v]$, $v \in SM - \Omega$. Since $SM - \Omega$ is open and f^t-invariant, there exists locally a hypersurface H in $SM - \Omega$ containing v and diffeomorphic to an open subset in $\boldsymbol{E}^{2n-2}$ such that $[w] \cap H = \{w\}$ and H intersects $[w]$ transversely for any $w \in H$. The collection of such hypersurfaces H yields a differentiable structure of N with dimension $2n-2$. We define the volume form $d\eta$ on N such that $d\eta_{[v]} \wedge dt = d\omega_v$ for any $[v] \in N$. Then we have, for any summable function F on $SM - \Omega$,

$$\int_{SM-\Omega} F\, d\omega = \int_{[v]\in N} d\eta \int_{-\infty}^{\infty} F(f^t v) dt\,, \tag{1.1}$$

where $F_{[v]}: [v] \to \boldsymbol{R}$ is given by $F_{[v]}(w) = F(w)$ for any $w \in [v]$.

1.2. The Birkhoff ergodic theorem. Let D be an f^t-invariant subset of SM with finite volume. The Birkhoff ergodic theorem says that for any summable function F on D

1) $$F^*(v) = \lim_{T\to\infty} \frac{1}{T} \int_0^T F(f^t v)\, dt$$

exist and are f^t-invariant for almost all $v \in D$,

2) for any f^t-invariant measurable subset $B \subset D$,

$$\int_B F^* d\omega = \int_B F\, d\omega .$$

We say that a $v \in D$ is *uniformly recurrent* if for any neighborhood U of v, we have

$$\liminf_{T \to \infty} \frac{1}{T} \int_0^T \chi_U(f^t v) dt > 0 ,$$

where $\chi_U : D \to \boldsymbol{R}$ is the characteristic function of U. We denote by $W(D)$ the set of all uniformly recurrent vectors in D. It follows from the Birkhoff ergodic theorem that $W(D)$ has full measure in D ([**1**]).

1.3. The limit of the second fundamental forms of geodesic spheres. Let R be the curvature tensor of M. For any $v \in SM$ let $R(v) : v^\perp \to v^\perp$ be a symmetric linear map given by $R(v)(x) = R(x, v)v$ for any $x \in v^\perp$, where $v^\perp = \{w \in T_{\pi(v)}M ; \langle v, w \rangle = 0\}$.

We assume hereafter that M is without conjugate points unless otherwise stated. Let $\tilde{M}$ be the universal covering space of M. Then, $\tilde{M}$ is diffeomorphic to $\boldsymbol{R}^n$ and all geodesics are minimizing in $\tilde{M}$. For any $v \in S\tilde{M}$ let $\tilde{A}_s(v)$ be the second fundamental form at $\pi(v)$ of the geodesic sphere $S(\pi(v), \dot{\gamma}_v(s))$ with center $\gamma_v(s)$ through $\pi(v)$ relative to $-v$. It follows from [**5**], [**6**], [**7**], [**9**], [**13**] that

$$\lim_{s \to \infty} \tilde{A}_s(v) = \tilde{A}(v)$$

exists and

$$|\langle \tilde{A}(v)x, x \rangle| \leqq \max\{|\langle \tilde{A}_{-1}(v)y, y \rangle|, |\langle \tilde{A}_1(v)y, y \rangle| ; y \in v^\perp, |y| = 1\}$$

for any $v \in S\tilde{M}$ and any $x \in v^\perp$, $|x| = 1$. The map

$$\tilde{A} : S\tilde{M} \longrightarrow \bigcup_{v \in S\tilde{M}} L(v^\perp)$$

satisfies the following, where $L(v^\perp) = \{h ; h$ is a linear map of $v^\perp$ into itself$\}$.

1) $\operatorname{tr} \tilde{A}$ is measurable.

2) $\tilde{A}(v)$ is symmetric for any $v \in S\tilde{M}$.

3) $\tilde{A}(f^t v)$ is of class C^∞ for $t \in (-\infty, \infty)$.

4) $\tilde{A}'(f^t v) + \tilde{A}(f^t v)^2 + R(f^t v) = 0$

for any $t \in (-\infty, \infty)$, where $\tilde{A}'(f^t v)$ is the covariant derivative of $\tilde{A}(f^t v)$ along γ_v at $\gamma_v(t)$.

5) For any compact set $K \subset \tilde{M}$ there is a constant $C(K) > 0$ such that $\|\tilde{A}(v)\| < C(K)$ for any $v \in SK$, where $\|\tilde{A}(v)\|$ is the norm of $A(v)$.

By the construction of the map $\tilde{A}$ we can induce the map A on SM which satisfies the same properties above.

1.4. The solution of a matrix equation of Riccati type. We consider the following $(n-1)\times(n-1)$ matrix differential equation of Riccati type.

$$\text{(J)} \qquad X'(t)+X(t)^2+R(t)=0$$

on $t\in(-\infty, \infty)$, where $R(t)$ is a symmetric matrix and $\operatorname{tr} R$ is summable on $(-\infty, \infty)$. The following lemma will be used in the case that $R(t)=R(f^t v)$ and $\operatorname{tr} R(t)=\operatorname{Ric}(f^t v)$ for almost all $v\in SM$ such that the Ricci curvature $\operatorname{Ric}(f^t v)$ is summable over $(-\infty, \infty)$.

LEMMA 1. *Suppose there exists a symmetric solution $A(t)$ of* (J) *on $t\in(-\infty, \infty)$. Then, the integral of* $\operatorname{tr} R(t)$ *on $(-\infty, \infty)$ is nonpositive. If it vanishes, then both $A(t)$ and $R(t)$ must be identically zero on $(-\infty, \infty)$.*

PROOF. The proof is the same as in [**12**]. We first prove that there exist sequences $\{a_n\}$ and $\{b_n\}\subset \boldsymbol{R}$ such that $a_n\to\infty$, $b_n\to-\infty$, $\operatorname{tr} A(a_n)\to 0$ and $\operatorname{tr} A(b_n)\to 0$ as $n\to\infty$. Suppose for indirect proof that an $\varepsilon>0$ and an s exist such that $|\operatorname{tr} A(t)|>\varepsilon$ for any $t>s$. Since

$$(\operatorname{tr} A(t))^2 \leqq n \operatorname{tr} A(t)^2$$

for any $t\in(-\infty, \infty)$, and, hence,

$$\int_s^t \operatorname{tr} A(t)^2 dt \geqq (\varepsilon^2/n)(t-s)$$

for any $t>s$, and since

$$\operatorname{tr} A(t)-\operatorname{tr} A(s)+\int_s^t \operatorname{tr} A(t)^2 dt+\int_s^t \operatorname{tr} R(t)\,dt=0$$

for any $t>s$, we see that $\operatorname{tr} A(t)\to-\infty$ as $t\to\infty$, since $\operatorname{tr} R(t)$ is summable over $(-\infty, \infty)$. If we take a $u>s$ such that $|\operatorname{tr} A(t)|>1$ for any $t\geqq u$, then

$$\begin{aligned}\frac{t-u}{n} \leqq \int_u^t \frac{\operatorname{tr} A(t)^2}{(\operatorname{tr} A(t))^2}dt &\leqq \left|\int_u^t \frac{\operatorname{tr} A'(t)}{(\operatorname{tr} A(t))^2}dt\right|+\left|\int_u^t \frac{\operatorname{tr} R(t)}{(\operatorname{tr} A(t))^2}dt\right| \\ &\leqq \left|-\frac{1}{\operatorname{tr} A(t)}+\frac{1}{\operatorname{tr} A(u)}\right|+\int_u^t |\operatorname{tr} R(t)|\,dt\,,\end{aligned}$$

a contradiction, because the right hand side is bounded above. The existence of a sequence $\{b_n\}\subset \boldsymbol{R}$ we want is proved similarly.

Integrating (J) after taking the trace on $[b_n, a_n]$ and taking $n\to\infty$, we obtain

$$\int_{-\infty}^{\infty} \operatorname{tr} R(t)\,dt=-\int_{-\infty}^{\infty} \operatorname{tr} A(t)^2 dt \leqq 0\,.$$

If the equality holds, then

$$\operatorname{tr} A(t)^2 = 0 \longrightarrow A(t) = 0 \longrightarrow A'(t) = 0 \longrightarrow R(t) = 0$$

for any $t \in (-\infty, \infty)$. Lemma 1 is proved.

2. The integral of the Ricci curvature on $SM-\Omega$.

Let M be a manifold as in Theorem. We will prove the following.

LEMMA 2. *The integral of the Ricci curvature of M on $SM-\Omega$ is nonpositive, and it vanishes only if $R(v)=R(\cdot, v)v=0$ for any $v \in SM-\Omega$.*

PROOF. Since the Ricci curvature is summable and by the formula (1.1), the integral of the absolute Ricci curvature is finite along the geodesic $\gamma_v : (-\infty, \infty) \to M$ with $\dot\gamma_v(0)=v$ for almost all $v \in SM-\Omega$. It follows from (1.3.4) and Lemma 1 that

$$\int_{-\infty}^{\infty} \operatorname{Ric}(f^t v)\, dt \leqq 0$$

for almost all $v \in SM-\Omega$. Integrating it on N as in 1.1, we obtain

$$\int_{SM-\Omega} \operatorname{Ric} d\omega = \int_{[v] \in N} d\eta \int_{-\infty}^{\infty} \operatorname{Ric}(f^t v) dt \leqq 0 .$$

The equality means from Lemma 1 that $R(v)=R(\cdot, v)v=0$ for almost all $v \in SM-\Omega$. Since $R(v)$ depends continuously on the points $v \in SM$, we see that R is identically zero on $SM-\Omega$. Lemma 2 is proved.

3. The integral of the Ricci curvature on Ω.

Let M be a manifold as in Theorem and let $\Omega_1 \subset \Omega$ be an f^t-invariant set which has finite volume. We will prove the following.

LEMMA 3. *The integral of the Ricci curvature of M over Ω_1 is nonpositive, and it vanishes only if $R(v)=R(\cdot, v)v=0$ for any $v \in \Omega_1$.*

PROOF. Let $X(\Omega_1)$ be the set of all vectors v such that $\operatorname{Ric}^*(v)$ exists as in (1.2.1). Then, $X(\Omega_1) \cap W(\Omega_1)$ has full measure in Ω_1. Let a $v \in X(\Omega_1) \cap W(\Omega_1)$ and let K be a compact neighborhood of v in Ω_1. It follows from (1.3.5) that there exists a constant $C(K)>0$ such that $\|A(w)\| < C(K)$ for any $w \in K$. Since v is uniformly recurrent, there exists a sequence $\{T_n\} \subset \boldsymbol{R}$ such that $T_n \to \infty$, $f^{T_n} v \to v$ as $n \to \infty$ and $f^{T_n} v \in K$ for all n. By (1.3.4), we have

$$\frac{1}{T_n}(\operatorname{tr} A(f^{T_n} n) - \operatorname{tr} A(v)) + \frac{1}{T_n}\int_0^{T_n} \operatorname{tr} A(f^t v)^2 dt + \frac{1}{T_n}\int_0^{T_n} \operatorname{Ric}(f^t v) dt = 0 .$$

Taking $n \to \infty$ we obtain

$$\mathrm{Ric}^*(v)=-\lim_{n\to\infty}\frac{1}{T_n}\int_0^{T_n}\mathrm{tr}\,A(f^tv)^2dt\leqq 0\,.$$

Hence, by the Birkhoff ergodic theorem (1.2.2), we get

$$\int_{\Omega_1}\mathrm{Ric}\,d\omega=\int_{\Omega_1}\mathrm{Ric}^*d\omega\leqq 0\,.$$

Suppose the equality holds. Then, $X_0(\Omega_1)=\{v\in\Omega_1;\mathrm{Ric}^*(v)=0\}$ has full measure in Ω_1, and, hence, $X_0(\Omega_1)\cap W(\Omega_1)$ has full measure in Ω_1. We will prove that $\mathrm{Ric}(v)=0$ for any $v\in X_0(\Omega_1)\cap W(\Omega_1)$. The idea of the proof is seen in [**14**]. Let a $v\in X_0(\Omega_1)\cap W(\Omega_1)$ and let $\gamma:[0,\infty)\to SM$ be a geodesic with $\gamma(t)=f^tv$ for any $t\in(-\infty,\infty)$. We put $A(t)=A(f^tv)$ and $\mathrm{Ric}(t)=\mathrm{Ric}(f^tv)$ for all $t\in(-\infty,\infty)$. Choose a positive l such that the geodesic open ball $B(l)$ in SM with center v and radius l is strongly convex. The convex ball $B(l)$ has a property that for any points $p,q\in\overline{B(l)}$ there is the unique minimizing geodesic joining p and q which is contained in $B(l)$ possibly except for p and q, where $\overline{B(l)}$ is the closure of $B(l)$ in SM. Since $\mathrm{Ric}^*(v)=0$ and $v\in W(\Omega_1)$, it follows from the argument above that

$$\lim_{n\to\infty}\frac{1}{T_n}\int_0^{T_n}\mathrm{tr}\,A(t)^2dt=0\,,$$

if a sequence $\{T_n\}\subset\boldsymbol{R}$ is such that $T_n\to\infty$ as $n\to\infty$ and $\gamma(T_n)$ lie in the boundary of $B(l)$ for all n.

ASSERTION. *There exists a sequence* $\{t_n\}\subset[0,\infty)$ *such that*

1) $t_n\to\infty$ *as* $n\to\infty$,

2) *if* $A_n(t)$ *is the matrix given by* $A_n(t)=A(t_n+t)$ *for any* $t\in[0,l]$, *then*

$$\int_0^l\mathrm{tr}\,A_n(t)^2dt\longrightarrow 0\quad as\quad n\to\infty\,,$$

and $\mathrm{tr}\,A_n(t)\to 0$ *for almost all* $t\in[0,l]$ *as* $n\to\infty$,

3) *if* $\gamma_n:[0,l]\to SM$ *is the geodesic given by* $\gamma_n(t)=f^{t_n+t}v$ *for any* $t\in[0,l]$, *then* γ_n *converges to the geodesic* $\gamma_0:[0,l]\to SM$ *with* $\gamma_0(t)=f^{t-l/2}v$ *for any* $t\in[0,l]$ *as* $n\to\infty$.

PROOF OF ASSERTION. Let $k\geqq 4$ be an integer. Since $B(l/k)$ is a convex ball and γ is a geodesic, $\gamma^{-1}(B(l/k))$ is the union of intervals whose lengths are less than or equal to $2l/k$, say

$$(a_1',b_1'),(a_2',b_2'),\cdots,(a_i',b_i'),\cdots;$$
$$a_1'<b_1'<a_2'<b_2'<\cdots<a_i'<b_i'<\cdots\longrightarrow\infty\,.$$

Put

$$a_i=\frac{a_i'+b_i'}{2}-\frac{l}{2}\,;\ b_i=\frac{a_i'+b_i'}{2}+\frac{l}{2}$$

for each $i=1, 2, \cdots$. Then, $\gamma([a_i, b_i])\subset B(l)$ and $\gamma(a_i), \gamma(b_i)\notin B(l/k)$, since

$$d_1(\gamma(t), v) \leqq d_1\Big(\gamma(t), \gamma\Big(a_i+\frac{l}{2}\Big)\Big)+d_1\Big(\gamma\Big(a_i+\frac{l}{2}\Big), v\Big) < \frac{l}{2}+\frac{l}{k} < l$$

for any $t\in[a_i, b_i]$, and since

$$d_1(\gamma(a_i), v) \geqq d_1\Big(\gamma(a_i), \gamma\Big(a_i+\frac{l}{2}\Big)\Big)-d_1\Big(\gamma\Big(a_i+\frac{l}{2}\Big), v\Big) > \frac{l}{2}-\frac{l}{k} \geqq \frac{l}{k},$$

from the choice of k, where $d_1(\cdot, \cdot)$ is the distance induced from the Riemannian metric defined on SM in Section 1. It follows similarly that $d_1(\gamma(b_i), v)> l/k$. Suppose

$$\liminf_{i\to\infty} \int_{a_i}^{b_i} \operatorname{tr} A(t)^2 dt > \alpha > 0 .$$

For any n, we have

$$\begin{aligned}
&\frac{1}{T_n}\int_0^{T_n} \operatorname{tr} A(t)^2 dt \geqq \frac{1}{T_n}\Big[\sum_{i=1}^{m_n}\int_{a_i}^{b_i} \operatorname{tr} A(t)^2 dt\Big] \\
&\geqq \frac{1}{T_n}\Big[\sum_{i=1}^{m}\int_{a_i}^{b_i} \operatorname{tr} A(t)^2 dt\Big]+\frac{\alpha}{lT_n}\sum_{i=m+1}^{m_n}(b_i-a_i) \\
&\geqq \frac{\alpha}{lT_n}\sum_{i=m+1}^{m_n}(b_i'-a_i') = \frac{\alpha}{lT_n}\int_0^{T_n}\chi_{B(l/k)}(\gamma(t))dt-\frac{\alpha}{lT_n}\sum_{i=1}^{m}(b_i'-a_i'),
\end{aligned}$$

where m_n and m are chosen so that

$$b_{m_n} < T_n < a_{m_n+1} \quad \text{and} \quad \inf_{i\geqq m}\int_{a_i}^{b_i} \operatorname{tr} A(t)^2 dt > \alpha .$$

This implies that

$$0 = \lim_{n\to\infty}\frac{1}{T_n}\int_0^{T_n} \operatorname{tr} A(t)^2 dt \geqq \frac{\alpha}{l}\liminf_{T\to\infty}\frac{1}{T}\int_0^{T}\chi_{B(l/k)}(f^t v)\,dt > 0 ,$$

a contradiction. Thus we can find an integer $i(k)\geqq k$ such that

$$\gamma\Big(\frac{a_{i(k)}+b_{i(k)}}{2}\Big) \in B(l/k) \quad \text{and} \quad \int_{a_{i(k)}}^{b_{i(k)}} \operatorname{tr} A(t)^2 dt \leqq \frac{1}{k} .$$

If $t_k=a_{i(k)}$ for all $k\geqq 4$, the sequence $\{t_k\}$ satisfies the condition 1) and the first part of 2). For the second part of 2) and 3) we have only to choose a suitable subsequence $\{t_n\}$ of $\{t_k\}$ if necessary.

We return to the proof of $\operatorname{Ric}(v)=0$. Rewritting (1.3.4) in terms of 2), we get for each n

$$\operatorname{tr} A_n'(t)+\operatorname{tr} A_n(t)^2+\operatorname{Ric}_n(t) = 0 \tag{3.4}$$

for any $t\in[0, l]$, where $\operatorname{Ric}_n(t)=\operatorname{Ric}(t_n+t)$. It should be noted that $\operatorname{Ric}_n(t)$ converges to $\operatorname{Ric}(t-l/2)$ uniformly in $t\in[0, l]$ as $n\to\infty$. Suppose $\operatorname{Ric}(0)=\operatorname{Ric}(v)\neq 0$, say $\mathrm{Ric}(v)>0$. Then, there exist a and $b\in[0, l]$, $a<l/2<b$, such that $\mathrm{Ric}(t-l/2)>0$ for any $t\in[a, b]$ and $\mathrm{tr}\,A_n(a)$, $\mathrm{tr}\,A_n(b)\to 0$ as $n\to\infty$. On the other hand, by integrating (3.4) on the interval $[a, b]$ and taking n to infinity, we have

$$\int_a^b \mathrm{Ric}\Big(t-\frac{l}{2}\Big)dt=0\,,$$

a contradiction. Therefore, $\mathrm{Ric}(v)=0$ for any $v\in X_0(\Omega_1)\cap W(\Omega_1)$. It follows from Lemma 1 that $R(v)=R(\cdot, v)v=0$ for any $v\in X_0(\Omega_1)\cap W(\Omega_1)$. Since $R(v)$ depends continuously on the points $v\in SM$, we see that R is identically zero on Ω_1. Lemma 3 is proved.

4. Proof of Theorem.

By Lemmas 2 and 3, we have

$$\frac{\theta_{n-1}}{n}\int_M S\,d\sigma=\int_{SM}\mathrm{Ric}\,d\omega=\int_{SM-\Omega}\mathrm{Ric}\,d\omega+\sum_{i=1}^{\infty}\int_{\Omega_i}\mathrm{Ric}\,d\omega\leqq 0\,,$$

where θ_{n-1} is the volume of the unit sphere in $\boldsymbol{E}^n$, S is the scalar curvature of M and $\Omega=\sum_{i=1}^{\infty}\Omega_i$ is the decomposition of f^t-invariant sets each of which has finite volume. If the equality holds, then

$$\int_{SM-\Omega}\mathrm{Ric}\,d\omega=\int_{\Omega_i}\mathrm{Ric}\,d\omega=0\,,$$

for all $i=1, 2, \cdots$. Lemmas 2 and 3 state that the curvature tensor $R(\cdot, v)v$ is zero for any $v\in SM$. Therefore, M is flat. This completes the proof of Theorem.

5. Proof of Corollaries.

If a complete simply connected Riemannian manifold M is without conjugate points, then all geodesics are minimizing in M. This implies that Ω is a empty set. Hence, Corollary 1 follows from Theorem. For Corollary 2 we have nothing to prove.

For the proof of Corollary 3 we need the notion of totally convex sets. We say that a set C in a complete Riemannian manifold M is *totally convex* if for any points $p, q\in C$ all geodesic curves joining p and q are entirely contained in C. It follows that any totally convex closed set C is an imbedded submanifold in M (possibly with not differentible boundary), and if $\gamma:[0, \infty)\to M$ is a geodesic such that $\gamma(0)$ is in the interior of C and $\gamma(s)$ is in the boundary of C for some s, then $\gamma(t)$ is outside C for any $t\in(s, \infty)$. G. Thorbergsson ([**15**]) proved by a slight modification of the Cheeger and Gromoll basic construction ([**3**]) that if M is a complete Riemannian manifold with nonnegative sectional curvature outside some compact set, then there is a family $\{K_t;\, t>0\}$ of compact totally convex sets with $M=\cup K_t$ and $K_t \subset K_s$ for $t \leqq s$.

5.1. Proof of Corollary 3. Let M be as in Corollary 3 and let K be a compact set in M such that the sectional curvature is zero outside K. By Thorbergsson's result we can find a compact set C such that the interior C^0 of C contains K. We want to prove that $SC^0 \cap \Omega$ is f^t-invariant, where $SC^0=\{v \in SM;\ \pi(v) \in C^0\}$. If this were not true, then there is a $v \in SC^0 \cap \Omega$ such that $\pi(f^s v)$ is in $M-C$ for some $s>0$, since Ω is f^t-invariant and C is a totally convex set. We can choose sequences $\{v_n\} \subset SC^0$ and $\{t_n\} \subset \boldsymbol{R}$ such that $t_n \to \infty$, $v_n \to v$ and $f^{t_n} v_n \to v$ as $n \to \infty$, since v is a non-wandering point under the geodesic flow. Then it follows that $f^s v_n \to f^s v$ as $n \to \infty$. Hence, we can find a sufficiently large m such that $\pi(v_m) \in C^0$, $\pi(f^{t_m} v_m) \in C^0$ and $\pi(f^s v_m) \notin C$. This contradicts that C is a totally convex set in M, since $\gamma: [0, \infty) \to M$ given by $\gamma(t)=\pi(f^t v_m)$ for any t is a geodesic with $\gamma(0) \in C^0$, $\gamma(t_m) \in C^0$ and $\gamma(s) \notin C$.

Thus, we can use Lemma 3 to integrate the Ricci curvature over $SC^0 \cap \Omega$, since $SC^0 \cap \Omega$ has finite volume. Now we have in the same notation in Section 4

$$\frac{\theta_{n-1}}{n}\int_M S d\sigma = \int_{SM} \mathrm{Ric}\, d\omega = \int_{SM-\Omega} \mathrm{Ric}\, d\omega + \int_{SC^0 \cap \Omega} \mathrm{Ric}\, d\omega + \int_{(SM-SC^0)\cap\Omega} \mathrm{Ric}\, d\omega \leqq 0\,,$$

because the third term in the right is zero, since the sectional curvature is zero on $M-C^0$. If the equality holds, then

$$\int_{SM-\Omega} \mathrm{Ric}\, d\omega = \int_{SC^0 \cap \Omega} \mathrm{Ric}\, d\omega = 0\,.$$

Lemmas 2 and 3 state that the curvature tensor $R(\cdot, v)v$ is zero for any $v \in (SM-\Omega) \cup (SC^0 \cap \Omega)$. Therefore, M is flat. This completes the proof of Corollary 3.

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