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# Interpolating sequences in the maximal ideal space of $H^{\infty}$

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## 1. Introduction.

Let $H^{\infty}$ be the space of bounded analytic functions on the open unit disc $D$. $H^{\infty}$ becomes a Banach algebra with the supremum norm. We denote by $M(H^{\infty})$ the maximal ideal space of $H^{\infty}$ with the weak*-topology. We identify a function in $H^{\infty}$ with its Gelfand transform. For points $x$ and $y$ in $M(H^{\infty})$, the pseudo-hyperbolic distance is defined by

$$\rho(x, y)=\sup\{|h(x)|;\ h\in \text{ball }(H^{\infty}),\ h(y)=0\},$$

where ball $(H^{\infty})$ stands for the unit closed ball of $H^{\infty}$. For $z$ and $w$ in $D$, we have $\rho(z, w)=|z-w|/|1-\bar{z}w|$. A sequence $\{x_j\}_j$ in $M(H^{\infty})$ is called interpolating if for every bounded sequence $\{a_j\}_j$ there is a function $f$ in $H^{\infty}$ such that $f(x_j)=a_j$ for every $j$. It is well known (see [2, p. 283]) that for a sequence $\{z_j\}_j$ in $D$, $\{z_j\}_j$ is interpolating if and only if

$$\inf_k \prod_{j\neq k} \rho(z_j, z_k)>0.$$

For a sequence $\{z_j\}_j$ in $D$ with $\sum_{j=1}^{\infty} 1-|z_j|<\infty$, a function

$$b(z)=\prod_{j=1}^{\infty} \frac{\bar{z}_j}{|z_j|} \frac{z_j-z}{1-\bar{z}_j z} \quad (z\in D)$$

is called a Blaschke product with zeros $\{z_j\}_j$, and $\{z_j\}_j$ is called the zero sequence of $b$. If $\{z_j\}_j$ is interpolating, we call $b$ interpolating. For a function $f$ in $H^{\infty}$, put $Z(f)=\{x\in M(H^{\infty});\ f(x)=0\}$. For a subset $E$ of $M(H^{\infty})$, we denote by $\mathrm{cl}\, E$ the weak*-closure of $E$ in $M(H^{\infty})$.

For a point $x$ in $M(H^{\infty})$, the set $P(x)=\{y\in M(H^{\infty});\ \rho(y, x)<1\}$ is called a Gleason part of $x$. If $P(x)\neq\{x\}$, $P(x)$ is called nontrivial. $D$ is a typical nontrivial part. We set

$$G=\{x\in M(H^{\infty});\ x \text{ is nontrivial}\}.$$

Hoffman [5] proved that for a point $x$ in $G$, there is an interpolating sequence $\{z_j\}_j$ such that $x$ is contained in $\mathrm{cl}\,\{z_j\}_j$, and there is a continuous map $L_x$ from $D$ onto $P(x)$ such that $f\circ L_x\in H^{\infty}$ for every $f\in H^{\infty}$, where $L_x$ is given by $L_x(z)=\lim_\alpha(z_{j_\alpha}-z)/(1-\bar{z}_{j_\alpha}z)$ for a net $\{z_{j_\alpha}\}_\alpha$ in $\{z_j\}_j$ with $z_{j_\alpha}\to x$. When $L_x$ is a homeomorphism, $P(x)$ is called a homeomorphic part.

Our problem is; if $\{x_j\}_j$ is an interpolating sequence in $G$, is there an interpolating Blaschke product $b$ such that $Z(b)\supset\{x_j\}_j$? Generally the converse is not true. For, let $b$ be an interpolating Blaschke product with zeros $\{z_n\}_n$ in $D$ and let $x$ be a cluster point of $\{z_n\}_n$. Put $\{x_j\}_j=\{z_n\}_n\cup\{x\}$. Then it is not difficult to see that $\{x_j\}_j$ is not interpolating and $Z(b)\supset\{x_j\}_j$. In [**3**] and [**6**], they independently proved that if $P$ is a homeomorphic part and $\{x_j\}_j\subset P$, then $\{x_j\}_j$ is interpolating if and only if $\{x_j\}_j=Z(b)\cap P$ for an interpolating Blaschke product $b$. In this paper, we study an interpolating sequence whose elements are contained in distinct parts in $G$. Our theorem is the following.

THEOREM. *Let $\{x_j\}_j$ be a sequence in $G$ such that $P(x_k)\cap\mathrm{cl}\{x_j\}_{j\neq k}=\phi$ for every $k$. Then the following conditions are equivalent.*

(i) *There is an interpolating Blaschke product $b$ such that $Z(b)\supset\{x_j\}_j$.*

(ii) *$\{x_j\}_j$ is an interpolating sequence.*

The idea to prove our theorem is basically the same as in [**6**]. The difference between them is; let $h$ be a function in $H^\infty$ with $h(x_1)\neq 0$ and $h(x_j)=0$ for $j\geqq 2$ and let $B$ be a Blaschke factor of $h$. If $\{x_j\}_j$ is contained in the same part, then $B(x_1)\neq 0$ and $B(x_j)=0$ for $j\geqq 2$, but under the assumption of our theorem we can not say anything about $B$. Previous paper's problem is how to construct an interpolating subproduct $b$ of $B$ such that $b(x_j)=0$ for $j\geqq 2$, but this paper's problem is how to construct an interpolating Blaschke product $b$ such that $b(x_j)=0$ for $j\geqq 2$ using the function $h$. Therefore this paper is a little bit complicated more than the previous one. The main part of this paper is to prove (ii)$\Rightarrow$(i). In Section 2, we give eight lemmas. Using them, we prove our theorem in Section 3.

## 2. Blaschke subproducts.

For an interpolating Blaschke product $b$ with zeros $\{z_j\}_j$, put

$$\delta(b)=\inf_k\prod_{j\neq k}\rho(z_j, z_k).$$

By Hoffman [**5**, p. 82], we have the following lemma.

LEMMA 1. *Let $x\in M(H^\infty)$ and let $b$ be an interpolating Blaschke product with $b(x)=0$. If $0<\delta<1$, then there is a subproduct $b_1$ of $b$ such that $b_1(x)=0$ and $\delta(b_1)>\delta$.*

We use the same idea to prove the following lemmas 2, 4 and 5, but these situations are different, so we shall give these detail proofs. Lemma 6 is a summary of these results. Let $\{x_j\}_j$ be an interpolating sequence. Then by the open mapping theorem, there is a universal constant $M$ such that for every sequence $\{a_j\}_j$ with $|a_j|\leqq 1$ for every $j$, there is a function $f$ in $H^\infty$ with $\|f\|\leqq M$ and $f(x_j)=a_j$ for every $j$. The constant $M$ is called an interpolation constant for $\{x_j\}_j$.

LEMMA 2. *Let $\{x_j\}_j$ be a sequence in $G$ and let $\{b_j\}_j$ be a sequence of interpolating Blaschke products with $b_j(x_j)=0$. Let $h$ be a function in ball $(H^\infty)$ with $Z(h)\cap D=\phi$ and $Z(h)\supset\{x_j\}_j$. If $x$ is a point in $M(H^\infty)$ with $h(x)\neq 0$, then for each $r$ with $0<r<1$ there is a Blaschke product $\prod_{j=1}^\infty \psi_j$ such that*

(i) *$\psi_j$ is a subproduct of $b_j$ with $\psi_j(x_j)=0$; and*

(ii) $\left|\left(\prod_{j=1}^\infty \psi_j\right)(x)\right| > r$.

PROOF. Let $\{z_{j,k}\}_k$ be the zero sequence of $b_j$. Since $b_j(x_j)=0$, by [4, p. 205], $x_j\in \mathrm{cl}\{z_{j,k}\}_k$. Let $M_j$ be an interpolation constant for $\{z_{j,k}\}_k$. Take a sequence $\{r_j\}_j$ such that

$$0<r_j<1 \quad \text{and} \quad \prod_{j=1}^\infty r_j > r .$$

Then take a sequence $\{\varepsilon_j\}_j$ such that

$$(1) \qquad 0<\varepsilon_j<1 \quad \text{and} \quad \prod_{j=1}^\infty \frac{r_j-M_j\varepsilon_j}{1+M_j\varepsilon_j} > r .$$

Put

$$(2) \qquad E=\{\zeta\in D;\ |h(\zeta)|>|h(x)|/2\} .$$

By the corona theorem (see [2, p. 318]), $x$ is contained in cl $E$.

Fix $j$ arbitrary. Then there is a positive integer $n$, depending on $j$, such that

$$(3) \qquad r_j^n < |h(x)|/2 .$$

Let $\psi_j$ be a subproduct of $b_j$ with zeros $F_j=\{z_{j,k};\ |h(z_{j,k})|<\varepsilon_j^n\}$. Since $h(x_j)=0$ and $x_j\in\mathrm{cl}\{z_{j,k}\}_k$, we have $x_j\in \mathrm{cl}\, F_j$, so that $\psi_j(x_j)=0$. Since $Z(h)\cap D=\phi$, we may consider that $h^{1/n}$ is a function in ball$(H^\infty)$. Since $|h^{1/n}|<\varepsilon_j$ on $F_j$ and the interpolating sequence $F_j$ has $M_j$ as an interpolation constant, there is a function $f$ in $H^\infty$ such that

$$\|f\|\leqq M_j\varepsilon_j \quad \text{and} \quad f(z_{j,k})=h^{1/n}(z_{j,k}) \quad \text{for every } z_{j,k}\in F_j .$$

Then there is a function $g$ in $H^\infty$ such that

$$f-h^{1/n}=\psi_j g .$$

Here we have $\|g\|\leqq 1+M_j\varepsilon_j$. Consequently we get

$$|h^{1/n}(z)|-M_j\varepsilon_j\leqq|(f-h^{1/n})(z)|\leqq(1+M_j\varepsilon_j)|\phi_j(z)| \tag{4}$$

for every $z\in D$. Therefore for $\zeta\in E$ we get

$$\begin{aligned} r_j &< (|h(x)|/2)^{1/n} && \text{by (3)} \\ &< |h^{1/n}(\zeta)| && \text{by (2)} \\ &\leqq (1+M_j\varepsilon_j)|\phi_j(\zeta)|+M_j\varepsilon_j && \text{by (4)}. \end{aligned}$$

Hence

$$\frac{r_j-M_j\varepsilon_j}{1+M_j\varepsilon_j}<|\phi_j(\zeta)| \qquad \text{for every} \quad \zeta\in E.$$

Consequently we have

$$\begin{aligned} r &< \prod_{j=1}^{\infty}\frac{r_j-M_j\varepsilon_j}{1+M_j\varepsilon_j} && \text{by (1)} \\ &< \left|\left(\prod_{j=1}^{\infty}\phi_j\right)(\zeta)\right| && \text{for every} \quad \zeta\in E. \end{aligned}$$

Since $x\in\operatorname{cl} E$, we get $r<\left|\left(\prod_{j=1}^{\infty}\phi_j\right)(x)\right|$.

The following lemma comes from [**5**, Theorem 5.2].

LEMMA 3. *Let $B$ be a Blaschke product with zeros $\{w_j\}_j$. Then there are subfactors $B_1$ and $B_2$ of $B$ such that $B=B_1B_2$ and $B_1=B_2=0$ on $Z(B)\setminus\operatorname{cl}\{w_j\}_j$.*

LEMMA 4. *Let $\{x_j\}_j$ be a sequence in $G$ and $\{b_j\}_j$ be a sequence of interpolating Blaschke products with $b_j(x_j)=0$. Let $B$ be a Blaschke product with zeros $\{w_k\}_k$ such that $Z(B)\supset\{x_j\}_j$ and $x_j\notin\operatorname{cl}\{w_k\}_k$ for every $j$. If $x$ is a point in $M(H^\infty)$ with $B(x)\neq 0$, then for each $r$ with $0<r<1$ there is a Blaschke product $\prod_{j=1}^{\infty}\phi_j$ such that*

(i) *$\phi_j$ is a subproduct of $b_j$ with $\phi_j(x_j)=0$; and*

(ii) $\left|\left(\prod_{j=1}^{\infty}\phi_j\right)(x)\right|>r.$

PROOF. Take $\{z_{j,k}\}_k$, $\{M_j\}_j$, $\{r_j\}_j$, and $\{\varepsilon_j\}_j$ as in the proof of Lemma 2. Put

$$E=\{\zeta\in D;\ |B(\zeta)|>|B(x)|/2\}.$$

Then $x\in\operatorname{cl} E$. Fix $j$ arbitrary. There is a positive integer $n$ such that $r_j{}^n<|B(x)|/2$. Applying Lemma 3 succeedingly $n$-times for $B$ and its subfactors, we get

$$B=B_1B_2\cdots B_n \quad \text{and} \quad B_i=0 \quad \text{on} \quad Z(B)\setminus\operatorname{cl}\{w_k\}_k$$

for every $1\leqq i\leqq n$. For each $i$, $1\leqq i\leqq n$, let $\phi_{j,i}$ be a subproduct of $b_j$ with zeros

$$F_{j,i}=\{z_{j,k};\ |B_i(z_{j,k})|<\varepsilon_j\}.$$

Since $B_i(x_j)=0$ and $x_j\in\mathrm{cl}\,\{z_{j,k}\}_k$, we have

$$\bigcap_{i=1}^{n} F_{j,i}\neq\phi \quad\text{and}\quad x_j\in\mathrm{cl}\bigcap_{i=1}^{n} F_{j,i}.$$

Let $\psi_j$ be a subproduct of $b_j$ with zeros $\bigcap_{i=1}^{n}F_{j,i}$. Then $\psi_j(x_j)=0$, $|\psi_{j,i}|\leqq|\psi_j|$ on $D$ for every $i$, and $|B_i|<\varepsilon_j$ on $F_{j,i}$. Since the interpolating sequence $F_{j,i}$ has $M_j$ as an interpolation constant, there is a function $f_i$ in $H^{\infty}$ such that

$$\|f_i\|\leqq M_j\varepsilon_j \quad\text{and}\quad f_i(z_{j,k})=B_i(z_{j,k}) \qquad\text{for}\quad z_{j,k}\in F_{j,i}.$$

Then there is a function $g_i$ in $H^{\infty}$ such that

$$f_i-B_i=\psi_{j,i}g_i.$$

Since $\|g_i\|\leqq 1+M_j\varepsilon_j$, we have

$$\begin{aligned}\frac{|B_i(z)|-M_j\varepsilon_j}{1+M_j\varepsilon_j}&\leqq|\psi_{j,i}(z)| \quad\text{for}\quad z\in D \quad\text{and}\quad 1\leqq i\leqq n\\ &\leqq|\psi_j(z)|.\end{aligned}$$

Let $\zeta\in E$. Since

$$\prod_{i=1}^{n}|B_i(\zeta)|=|B(\zeta)|>|B(x)|/2>r_j^{\,n},$$

we have $|B_i(\zeta)|>r_j$ for some $i$, where $i$ depends on $\zeta$. Hence

$$\frac{r_j-M_j\varepsilon_j}{1+M_j\varepsilon_j}\leqq|\psi_j(\zeta)| \qquad\text{for every}\quad \zeta\in E.$$

Consequently for every $\zeta\in E$ we have

$$r<\prod_{j=1}^{\infty}\frac{r_j-M_j\varepsilon_j}{1+M_j\varepsilon_j}\leqq\Big|\Big(\prod_{j=1}^{\infty}\psi_j\Big)(\zeta)\Big|.$$

Since $x\in\mathrm{cl}\,E$, we get $r<|(\prod_{j=1}^{\infty}\psi_j)(x)|$.

LEMMA 5. *Let $\{x_j\}_j$ be a sequence in $G$ such that $x_k$ is not contained in $\mathrm{cl}\,\{x_j\}_{j\neq k}$ for every $k$. Let $\{b_j\}_j$ be a sequence of interpolating Blaschke products with $b_j(x_j)=0$. Let $B$ be a Blaschke product with zeros $\{w_k\}_k$ such that $\{x_j\}_j\subset\mathrm{cl}\,\{w_k\}_k$. If $x$ is a point in $M(H^{\infty})$ such that $|B(x)|>\delta$, then there is a Blaschke product $\prod_{j=1}^{\infty}\psi_j$ such that*

(i) *$\psi_j$ is a subproduct of $b_j$ with $\psi_j(x_j)=0$; and*

(ii) $\Big|\Big(\prod_{j=1}^{\infty}\psi_j\Big)(x)\Big|>\delta$.

PROOF. Take $\{z_{j,k}\}_k$ and $\{M_j\}_j$ as in Lemma 2. Take $\sigma$ as $\delta<\sigma<|B(x)|$. Take a sequence $\{\varepsilon_j\}_j$ such that $\varepsilon_j>0$ and

$$\sigma \prod_{j=1}^{\infty} \frac{1-M_j\varepsilon_j\sigma^{-1}}{1+M_j\varepsilon_j} > \delta . \tag{5}$$

Put

$$E = \{\zeta \in D;\ |B(\zeta)| > \sigma\}. \tag{6}$$

Then $x \in \mathrm{cl}\, E$. By our assumption on $\{x_j\}_j$, there is a sequence of disjoint open subsets $\{U_j\}_j$ of $M(H^\infty)$ such that $x_j \in U_j$ for every $j$. Let $B_j$ be the Blaschke product with zeros $\{w_k\}_k \cap U_j$. Then $\prod_{j=1}^{\infty} B_j$ is a subproduct of $B$ and

$$|B_j| > \sigma \quad \text{on} \quad E . \tag{7}$$

Since $x_j \in \mathrm{cl}\, \{w_k\}_k$, $B_j(x_j)=0$.

Fix $j$ arbitrary. Let $\phi_j$ be the subproduct of $b_j$ with zeros $F_j=\{z_{j,k};\ |B_j(z_{j,k})|<\varepsilon_j\}$. Since $x_j \in \mathrm{cl}\, \{z_{j,k}\}_k$ and $B_j(x_j)=0$, we have $\phi_j(x_j)=0$. By the same way as Lemma 2 (replace $h^{1/n}$ by $B_j$), we have

$$\frac{|B_j(z)|-M_j\varepsilon_j}{1+M_j\varepsilon_j} \leqq |\phi_j(z)| \qquad \text{for every} \quad z \in D. \tag{8}$$

Therefore for $\zeta \in E$ we have

$$\begin{aligned}
\left|\left(\prod_{j=1}^{\infty}\phi_j\right)(\zeta)\right| &= \prod_{j=1}^{\infty}|\phi_j(\zeta)| \\
&\geqq \prod_{j=1}^{\infty}|B_j(\zeta)| \prod_{j=1}^{\infty}\frac{1-M_j\varepsilon_j|B_j(\zeta)|^{-1}}{1+M_j\varepsilon_j} && \text{by (8)} \\
&\geqq |B(\zeta)| \prod_{j=1}^{\infty}\frac{1-M_j\varepsilon_j\sigma^{-1}}{1+M_j\varepsilon_j} && \text{by (7)} \\
&\geqq \sigma \prod_{j=1}^{\infty}\frac{1-M_j\varepsilon_j\sigma^{-1}}{1+M_j\varepsilon_j} && \text{by (6)} \\
&> \delta . && \text{by (5)}
\end{aligned}$$

Since $x \in \mathrm{cl}\, E$, we get $|(\prod_{j=1}^{\infty}\phi_j)(x)|>\delta$.

The following lemma is a summary of Lemmas 2, 4 and 5.

LEMMA 6. *Let $\{x_j\}_j$ be a sequence in $G$ such that $x_k$ is not contained in* $\mathrm{cl}\,\{x_j\}_{j\neq k}$ *for every $k$. Let $\{b_j\}_j$ be a sequence of interpolating Blaschke products with $b_j(x_j)=0$. Let $x \in M(H^\infty)$. If $|f(x)|>\delta$ for some function $f$ in* ball$(H^\infty)$ *with $Z(f) \supset \{x_j\}_j$, then there is a Blaschke product $\prod_{j=1}^{\infty}\phi_j$ such that*

(i) *$\phi_j$ is a subproduct of $b_j$ with $\phi_j(x_j)=0$; and*

(ii) $\left|\left(\prod_{j=1}^{\infty}\phi_j\right)(x)\right| > \delta .$

PROOF. Let $f=Bh$, where $B$ is a Blaschke factor of $f$ and $Z(h)\cap D=\phi$. Let $\{w_k\}_k$ be a zero sequence of $B$. Put

$$\{x_{1,j}\}_j = \{x_i\,;\ B(x_i)=0 \quad \text{and} \quad x_i \in \mathrm{cl}\,\{w_k\}_k\}\,;$$
$$\{x_{2,j}\}_j = \{x_i\,;\ B(x_i)=0 \quad \text{and} \quad x_i \notin \mathrm{cl}\,\{w_k\}_k\}\,; \quad \text{and}$$
$$\{x_{3,j}\}_j = \{x_i\}_i \setminus (E_1 \cup E_2) = \{x_i\,;\ B(x_i) \neq 0\}\,.$$

Note that $|B(x)|>\delta$ and $h(x)\neq 0$. We devide $\{b_j\}_j$ into three parts $\{b_{1,j}\}_j$, $\{b_{2,j}\}_j$ and $\{b_{3,j}\}_j$ such that

$$b_{k,j}(x_{k,j})=0 \quad \text{for} \quad k=1, 2, 3 \quad \text{and} \quad j=1, 2, \cdots.$$

Take $\delta_1$ such that $\delta<\delta_1<|f(x)|$, and take $r$ such that

$$0<r<1 \quad \text{and} \quad \delta<\delta_1 r^2.$$

We apply Lemma 5 for $\{x_{1,j}\}_j$ and $\{b_{1,j}\}_j$. Then there is a subproduct $\psi_{1,j}$ of $b_{1,j}$ such that $\psi_{1,j}(x_{1,j})=0$ and $|(\prod_{j=1}^{\infty}\psi_{1,j})(x)|>\delta_1$. We apply Lemma 4 for $\{x_{2,j}\}_j$ and $\{b_{2,j}\}_j$. Then there is a subproduct $\psi_{2,j}$ of $b_{2,j}$ such that $\psi_{2,j}(x_{2,j})=0$ and $|(\prod_{j=1}^{\infty}\psi_{2,j})(x)|>r$. Since $Z(h)\supset\{x_{3,j}\}_j$, we can apply Lemma 2 for $\{x_{3,j}\}_j$ and $\{b_{3,j}\}_j$. Then there is a subproduct $\psi_{3,j}$ of $b_{3,j}$ such that $\psi_{3,j}(x_{3,j})=0$ and $|(\prod_{j=1}^{\infty}\psi_{3,j})(x)|>r$. Consequently, we have a desired Blaschke product $\prod_{i=1}^{3}\prod_{j=1}^{\infty}\psi_{i,j}$.

LEMMA 7. *Let $x\in G$ and let $b$ be an interpolating Blaschke product with $b(x)=0$. If $b_1$ and $b_2$ are subproducts of $b$ with $b_1(x)=b_2(x)=0$, then $x$ is contained in the closure of the intersection of zero sequences of $b_1$ and $b_2$.*

PROOF. Suppose not. Let $\{z_j\}_j$ and $\{w_j\}_j$ be the zero sequences of $b_1$ and $b_2$ respectively. Put $W=\{z_j\}_j\cap\{w_j\}_j$. Then $x\notin \mathrm{cl}\,W$, so that $x\in\mathrm{cl}\,(\{z_j\}_j\setminus W)$ and $x\in\mathrm{cl}\,(\{w_j\}_j\setminus W)$. Since disjoint subsets in an interpolating sequence have disjoint closures, we get a contradiction.

LEMMA 8. *Let $x\in G$ and let $E$ be a closed subset of $M(H^\infty)$ with $P(x)\cap E=\phi$. If $b$ is an interpolating Blaschke product with $b(x)=0$ and $0<r<1$, then there is a subproduct $\psi$ of $b$ such that $\psi(x)=0$ and $|\psi|>r$ on $E$.*

PROOF. For each $y\in E$, since $\rho(x, y)=1$ there is a function $h_y$ in ball$(H^\infty)$ such that $h_y(x)=0$ and $|h_y(y)|>r$. As a special case of Lemma 6, there is a subproduct $b_y$ of $b$ such that $b_y(x)=0$ and $|b_y(y)|>r$. Put

$$U_y=\{\zeta\in M(H^\infty)\,;\ |b_y(\zeta)|>r\}\,.$$

Then $\cup\{U_y\,;\ y\in E\}\supset E$. Hence there is a finite set $\{y_1, y_2, \cdots, y_n\}$ in $E$ such that $\cup\{U_{y_i}\,;\ 1\leqq i\leqq n\}\supset E$. Let $\psi$ be an interpolating Blaschke product with zeros $\bigcap_{i=1}^{n}Z(b_{y_i})\cap D$. By Lemma 7, we have $\psi(x)=0$. Since $|\psi|\geqq|b_{y_i}|$ on $D$, we have

$$|\psi(y)| \geqq \max\{|b_{y_i}(y)|\,;\ 1\leqq i\leqq n\}>r$$

for every $y\in E$.

### 3. Proof of Theorem.

PROOF. (i)$\Rightarrow$(ii) Let $b$ be an interpolating Blaschke product with zeros $\{z_k\}_k$ such that $Z(b)\supset\{x_j\}_j$. Since $x_j\notin \mathrm{cl}\{x_k\}_{k\neq j}$ for every $j$, there is a sequence of disjoint open subsets $\{U_j\}_j$ of $M(H^\infty)$ such that $x_j\in U_j$. Since $x_j$ is a cluster point of $\{z_k\}_k$, $\{z_k\}_k\cap U_j$ is an infinite set for each $j$. For a bounded sequence $\{a_j\}_j$, there is a function $h$ in $H^\infty$ such that $h(z_i)=a_j$ for every $z_i\in\{z_k\}_k\cap U_j$. Since $x_j\in \mathrm{cl}\{z_k\}_k\cap U_j$, we have $h(x_j)=a_j$ for every $j$. Therefore $\{x_j\}_j$ is an interpolating sequence.

(ii)$\Rightarrow$(i) Suppose that $\{x_j\}_j$ is an interpolating sequence. Since $x_j\in G$, there is an interpolating Blaschke product $b_j$ such that $b_j(x_j)=0$. By the open mapping theorem, there is a positive number $\delta$ such that

(#) $\inf_k \sup\{|h(x_k)|\,;\ h\in \mathrm{ball}(H^\infty),\ h(x_j)=0 \text{ for } j\neq k\}>\delta$.

Let $h_1$ be a function in ball$(H^\infty)$ such that $|h_1(x_1)|>\delta$ and $h_1(x_j)=0$ for $j\neq 1$. By Lemma 6 (consider as $x=x_1$ and $f=h_1$), there is a Blaschke product $B_1=\prod_{j=2}^\infty b_{1,j}$ such that $|B_1(x_1)|>\delta$ and $b_{1,j}$ is a subproduct of $b_j$ with $b_{1,j}(x_j)=0$ for $j\geqq 2$.

Let $\{r_j\}_j$ be a sequence of numbers such that

$$0<r_j<1 \quad\text{and}\quad \prod_{j=1}^\infty r_j>\delta .$$

By Lemma 8 (consider as $x=x_1$, $b=b_1$ and $E=\mathrm{cl}\{x_i\}_{i\neq 1}$), there is an interpolating Blaschke subproduct $\psi_1$ of $b_1$ such that $\psi_1(x_1)=0$ and $|\psi_1(x_i)|>r_1$ for $i\neq 1$. By Lemma 1, we may assume that $\delta(\psi_1)>\delta$. Since $|B_1(x_1)|>\delta$, there is a subsequence $\{z_{1,i}\}_i$ of the zero sequence of $\psi_1$ such that $|B_1(z_{1,i})|>\delta$ for every $i$. Then $x_1\in \mathrm{cl}\{z_{1,i}\}_i$. Let $\phi_1$ be the interpolating Blaschke product with zeros $\{z_{1,i}\}_i$. Then $\phi_1$ is a subproduct of $b_1$, $\delta(\phi_1)>\delta$, $\phi_1(x_1)=0$, and $|\phi_1(x_i)|>r_1$ for $i\neq 1$.

By induction, we shall construct a sequence of Blaschke products $\{B_j\}_{j\geq 2}$ and sequences of interpolating Blaschke products $\{\phi_j\}_{j\geq 2}$ and $\{b_{j,t}\}_{t>j}$ such that:

(a) $B_j=\prod_{t=j+1}^\infty b_{j,t}$ is a subproduct of $B_{j-1}=\prod_{t=j}^\infty b_{j-1,t}$ such that $|B_j(x_j)|>\delta$;

(b) $b_{j,t}$ is an interpolating Blaschke subproduct of $b_{j-1,t}$ such that $b_{j,t}(x_t)=0$ for $t\geqq j+1$;

(c) $\phi_j$ is a subproduct of $b_{j-1,j}$ with zeros $\{z_{j,i}\}_i$ and $\delta(\phi_j)>\delta$;

(d) $|B_j(z_{j,i})|>\delta$ for every $i$;

(e) $\phi_j(x_j)=0$ and $|\phi_j(x_i)|>r_j$ for $i\neq j$; and

(f) $|\phi_s(z_{j,i})|>r_s$ for every $s<j$ and $i$.

Our induction works on $k$. If we put $b_{0,t}=b_t$, then $B_1$, $\phi_1$ and $\{b_{1,t}\}_{t>1}$ satisfy all conditions $(a-f)$ for $k=1$.

Suppose that $\{B_j\}_{j<k}$, $\{\phi_j\}_{j<k}$ and $\{b_{j,t}\}_{t>j}(j<k)$ are already chosen. By

(#) and Lemma 6 (consider as $x=x_k$ and $\{b_j\}_j=\{b_{k-1,t}\}_{t\geqq k+1}$), there is a subproduct $B_k=\prod_{t=k+1}^{\infty}b_{k,t}$ of $B_{k-1}$ such that $|B_k(x_k)|>\delta$ and $b_{k,t}$ is an interpolating Blaschke subproduct of $b_{k-1,t}$ such that $b_{k,t}(x_t)=0$ for $t\geqq k+1$. Thus we get (a) and (b).

By Lemma 8 (consider as $x=x_k$, $b=b_{k-1,k}$ and $E=\mathrm{cl}\,\{x_j\}_{j\neq k}$), there is an interpolating Blaschke subproduct $\psi_k$ of $b_{k-1,k}$ such that $\psi_k(x_k)=0$ and $|\psi_k(x_i)|>r_k$ for $i\neq k$. By Lemma 1, we may assume that $\delta(\psi_k)>\delta$. Since $|B_k(x_k)|>\delta$, there is a subsequence $\{z_{k,i}\}_i$ of the zero sequence of $\psi_k$ such that $|B_k(z_{k,i})|>\delta$ for every $i$. Then we get (d) and $x_k\in\mathrm{cl}\,\{z_{k,i}\}_i$.

Let $\phi_k$ be the interpolating Blaschke product with zeros $\{z_{k,i}\}_i$. Then $\phi_k(x_k)=0$. Since $\phi_k$ is a subproduct of $\psi_k$, we get (c) and (e).

Since $|\phi_s(x_k)|>r_s$ for $s<k$ by (e), moreover we may assume that $\{z_{k,i}\}_i$ satisfies $|\phi_s(z_{k,i})|>r_s$ for every $s<k$ and $i$. Thus we get (f). This completes the induction.

Put $b=\prod_{k=1}^{\infty}\phi_k$. By (e), we have $Z(b)\supset\{x_j\}_j$. We shall prove that $b$ is an interpolating Blaschke product. We note that $\{z_{k,j}\}_{k,j}$ is the zero sequence of $b$. We have

$$\begin{aligned}
&\inf_{(k,i)}\prod_{(t,s)\neq(k,i)}\rho(z_{t,s},z_{k,i})\\
&\quad=\inf_{(k,i)}\Big[\prod_{t\neq k}\prod_{s=1}^{\infty}\rho(z_{t,s},z_{k,i})\Big]\Big[\prod_{s\neq i}\rho(z_{k,s},z_{k,i})\Big]\\
&\quad\geqq\inf_{(k,i)}\Big[\prod_{t\neq k}|\phi_t(z_{k,i})|\Big]\delta(\phi_k) && \text{by (c)}\\
&\quad\geqq\delta\inf_{(k,i)}\Big[\prod_{t<k}|\phi_t(z_{k,i})|\Big]\Big[\prod_{t>k}|\phi_t(z_{k,i})|\Big] && \text{by (c)}\\
&\quad\geqq\delta\inf_{(k,i)}\Big[\prod_{t<k}r_t\Big]|B_k(z_{k,i})| && \text{by (a), (b), (c), (f)}\\
&\quad\geqq\delta^2\prod_{t=1}^{\infty}r_t && \text{by (d)}\\
&\quad\geqq\delta^3.
\end{aligned}$$

Hence $b$ is an interpolating Blaschke product. This completes the proof.

REMARK. By the proof of (i)$\Rightarrow$(ii), for a sequence $\{x_k\}_k$ such that $Z(b)\supset\{x_k\}_k$ for some interpolating Blaschke product $b$, $\{x_k\}_k$ is interpolating if and only if $x_j\notin\mathrm{cl}\,\{x_k\}_{k\neq j}$ for every $j$.

## 4. Comments.

A closed subset $E$ of $M(H^{\infty})$ is called an interpolation set for $H^{\infty}$ if for every continuous function $f$ on $E$ there is a function $g$ in $H^{\infty}$ such that $g|_E=f$. In [7], Lingenberg proved that if $E$ is an interpolation set such that $E\subset G$ then there is an interpolating Blaschke product $b$ such that $Z(b)\supset E$. If $E$ is an interpolation set, then $E$ is $\rho$-separating, that is,

$$\inf\{\rho(x, y);\ x, y\in E, x\neq y\} > 0.$$

Recently Lingenberg and the author showed that if $E$ is a closed $\rho$-separating subset of $M(H^\infty)$ with $E\subset G$, $E$ is an interpolation set. Since every closed subset of $Z(b)$, where $b$ is an interpolating Blaschke product, is $\rho$-separating, the following conditions for closed subsets $E$ of $M(H^\infty)$ are equivalent:

(i) $E$ is an interpolation set and $E\subset G$;

(ii) $E$ is $\rho$-separating and $E\subset G$; and

(iii) there is an interpolating Blaschke product $b$ such that $Z(b)\supset E$.

The closedness of $E$ is an unremovable condition in the above assertion.

Now let $\{x_n\}_n$ be an interpolating sequence in $M(H^\infty)$. If $\{x_n\}_n$ is contained in $D$, then $\mathrm{cl}\{x_n\}_n\subset G$ by [**5**]. We have a following conjecture.

CONJECTURE. *If $\{x_n\}_n$ is an interpolating sequence in $G$, then* $\mathrm{cl}\{x_n\}_n\subset G$.

If this conjecture is affirmative, we may discuss as follows. Let $\{y_n\}_n$ be a sequence in $M(H^\infty)$. We put

$$\{y_{1,n}\}_n = \{y_n\}_n\cap M(L^\infty);$$
$$\{y_{2,n}\}_n = \{y_n\}_n\cap[M(H^\infty)\setminus(M(L^\infty)\cup G)];\quad \text{and}$$
$$\{y_{3,n}\}_n = \{y_n\}_n\cap G.$$

If $\{y_n\}_n$ is interpolating, then each $\{y_{k,n}\}_n$ is interpolating. We see the converse assertion is also true. Since $M(L^\infty)$ is closed, $\mathrm{cl}\{y_{1,n}\}_n\subset M(L^\infty)$. Since $\{y_{2,n}, y_{3,n}\}_n$ is a countable subset of $M(H^\infty)\setminus M(L^\infty)$, by [8] we have $\mathrm{cl}\{y_{2,n}, y_{3,n}\}_n\cap M(L^\infty)=\phi$. Since $G$ is an open subset of $M(H^\infty)$ [**5**], $\mathrm{cl}\{y_{2,n}\}_n\subset M(H^\infty)\setminus G$. Suppose that each $\{y_{k,n}\}_n$ is interpolating. Then $\mathrm{cl}\{y_{3,n}\}_n\subset G$ (if our conjecture is true), and $\mathrm{cl}\{y_{k,n}\}_n$, $k=1, 2, 3$, become mutually disjoint interpolation sets. Moreover

$$\rho(\mathrm{cl}\{y_{k,n}\}_n, \mathrm{cl}\{y_{j,n}\}_n) = 1 \qquad \text{for} \quad k\neq j.$$

Hence by [**9**], $\bigcup_{k=1}^3 \mathrm{cl}\{y_{k,n}\}_n$ is an interpolation set. Then $\{y_n\}_n=\bigcup_{k=1}^3\{y_{k,n}\}_n$ becomes an interpolating sequence.

Hence to determine whether $\{y_n\}_n$ is interpolating or not it is sufficient to study three sequences independently. Hoffman (unpublished note) proved that $\{y_{1,n}\}_n$ is interpolating if and only if $y_j\notin \mathrm{cl}\{y_{1,n}\}_{n\neq j}$ for every $j$. If $\{y_{3,n}\}_n$ is interpolating, then $\mathrm{cl}\{y_{3,n}\}_n$ is an interpolation set with $\mathrm{cl}\{y_{3,n}\}_n\subset G$ (if our conjecture is true) and $y_j\notin \mathrm{cl}\{y_{3,n}\}_{n\neq j}$ for every $j$. The converse is also true. For, by the first paragraph, there is an interpolating Blaschke product $b$ such that $Z(b)\supset\{y_{3,n}\}_n$. By the remark in Section 3, $\{y_{3,n}\}_n$ is interpolating.

But we do not know anything when $\{y_{2,n}\}_n$ is interpolating.

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