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Integral formulas for polyhedral and spherical billiards

Dedicated to Professor Yoshihiro Tashiro on his 70th birthday

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1. Introduction.

Let M^{n+1} be a complete Riemannian manifold with boundary $\partial M =: B \neq \phi$ which is a union of smooth hypersurfaces. We can see the precise definition of manifolds with boundary as billiard tables in [**17**]. Let $q \in B$ be an arbitrary point at which B is smooth and Q_q the symmetry with respect to T_qB, i.e.,

$$Q_q(w) = w - 2\langle w, N(q)\rangle N(q)$$

for any $w \in T_qM$, where $\langle \cdot, \cdot \rangle$ is the Riemannian metric in M and N is the unit normal vector field to B pointing inward. We say that $\gamma : [a, b] \to M$ is a *reflecting geodesic* or briefly a *geodesic* if there exists the partition $a = a_0 < a_1 < \cdots < a_m = b$ such that

(1.1) $\gamma(a_i) \in B$, B is smooth at $\gamma(a_i)$ and $\dot{\gamma}(a_i - 0) \notin T_{\gamma(a_i)}B$ for $i = 1, 2, \ldots, m-1$.

(1.2) $\gamma_i = \gamma | [a_{i-1}, a_i]$ is a geodesic in M in the usual sense for $i = 1, 2, \ldots, m$.

(1.3) $Q(\dot{\gamma}(a_i - 0)) = \dot{\gamma}(a_i + 0)$ for $i = 1, 2, \ldots, m-1$.

Throughout the paper the term "*geodesic*" means both usual one and reflecting one. We assume that geodesics are parametrized by arclength. As usual a variation of a geodesic γ through geodesics yields a Jacobi vector field Y along γ which satisfies the following properties at the boundary (see Section 2):

$$Q(Y(a_i - 0)) = Y(a_i + 0) \tag{1.4}$$

$$Q(\nabla_{\dot{\gamma}(a_i-0)} Y) - \nabla_{\dot{\gamma}(a_i+0)} Y = A(\dot{\gamma}(a_i + 0))(Y^{\perp}(a_i + 0)) \tag{1.5}$$

where $A(\dot{\gamma}(a_i + 0))$ is a symmetric endomorphism of n-dimensional subspace $\dot{\gamma}(a_i + 0)^{\perp}$ of $T_{\gamma(a_i)}M$ which is perpendicular to $\dot{\gamma}(a_i + 0)$ and ∇ is the Levi-Civita connection. We say that $\gamma(t_1)$, $t_0 \neq t_1 \in [a, b]$, is a *conjugate point* to $\gamma(t_0)$, $t_0 \in [a, b]$, if there exists a nontrivial Jacobi vector field Y along γ with $Y(t_0) = Y(t_1) = 0$.

Let T_1M be the unit tangent bundle of M. For a $v \in T_1M$ let γ_v be the geodesic with $\dot{\gamma}_v(0) = v$. If $\pi(v) \in B$ where $\pi : T_1M \to M$ is the natural projection, then $\dot{\gamma}_v(0)$ is considered either $\dot{\gamma}_v(+0)$ or $\dot{\gamma}_v(-0)$. The geodesics γ_v are defined on the whole real line $(-\infty, \infty)$ for almost all $v \in T_1M$. We denote the set of all such vectors by SM. Let $f^t : SM \to SM$ be a flow given by $f^t v = \dot{\gamma}_v(t)$ for any $v \in SM$. We denote the set of all vectors $v \in SM$ with $q = \pi(v) \in B$ and $\langle v, N(q) \rangle > 0$ by ∂SM. Let T be the ceiling function on ∂SM, i.e., $T(v)$ is the first parameter such that $\gamma_v(T(v)) \in B$, $T(v) > 0$ (possibly $+\infty$). Let $F : \partial SM \to \partial SM$ be a map given by $F(v) = \dot{\gamma}_v(T(v) + 0)$ for any $v \in \partial SM$.

The purpose of the present paper is to study the properties defined below and give necessary conditions.

(P_1) We say that M is *without conjugate points* if all geodesics have no conjugate points.

(P_2) We say that M is *with boundary isolated by conjugate points* if there exist positive measurable functions α and β on ∂SM such that $\gamma_v(\alpha(v))$ is the first conjugate point to $\gamma_v(-\beta(v))$ along γ_v and $T(v) \geq \alpha(v) + \beta(F(v))$ for any $v \in \partial SM$.

In order to state our results we need a few terminologies more. Let dM and dB be the volume formes on M and B (resp.) induced from the Riemannian metric and let S be the second fundamental form of B at differentiable points with respect to N.

THEOREM A. *Let M^{n+1} be a compact manifold with boundary $B \neq \phi$. If M is without conjugate points, then*

$$\int_B \operatorname{tr} S \, dB + \frac{1}{n+1} \int_M \mathrm{Sc} \, dM \leq 0$$

and the equality sign is true only if M is flat and B consists of totally geodesic hypersurfaces, where Sc *is the scalar curvature of M.*

It should be noted that $S \leq 0$ is a conclusion of nonconjugacy hypothesis (see Corollary 5.2). Moreover, by using the Gauss-Bonnet theorem (cf. [**16**]), Theorem A is altered to the following.

COROLLARY B. *Let M be a 2-dimensional compact manifold with boundary $B \neq \phi$ and without conjugate points. Then, $2\pi\chi(M) - \sum_{i=1}^{n} \alpha_i \leq 0$ and the equality sign is true only if M is flat and B consists of geodesic arcs, where $\chi(M)$ is the Euler characteristic of M and α_i are the outer angles at vertices $p_1, \ldots, p_n \in B$.*

Let $E = \{q \in B \mid S_q \neq 0\}$ and $I_b = \inf\{T(v) \mid v \in \partial SM \text{ with } \pi(v) \in E\}$. Let $d\omega$ be the Liouville volume on SM.

COROLLARY C. *Let M^{n+1} be a compact nonpositively curved manifold with boundary whose second fundamental form S is nonpositive. If $I_b > 0$, then*

$$-\int_{SM} \sum_{i=1}^{k} \chi_i(v) \, d\omega \leq \operatorname{vol} S^n(1) \sqrt{n \operatorname{vol}(M)} \sqrt{-\left(\int_B \operatorname{tr} S \, dB + \frac{1}{n+1} \int_M \mathrm{Sc} \, dM \right)}$$

*where $\chi_i(v) = \chi(v, \xi_i)$ are the Lyapunov characteristic numbers for a normal basis $\xi_1, \ldots, \xi_{2n+1}$ in the sense of Oceledec ([**18**]) with $\chi_i(v) \leq \chi_{i+1}(v)$ and $\chi_k(v) \leq 0$ and $\chi_{k+1}(v) > 0$, and the equality sign is true only if M is a space of constant curvature and the boundary is totally geodesic. If the dimension of M is two, then the inequality is* *altered to*

$$-\int_{SM}\chi(v)\,d\omega \le 2\pi\sqrt{\mathrm{vol}(M)}\sqrt{\sum_{i=1}^{n}\alpha_i - 2\pi\chi(M)}$$

where α_i are the outer angles at vertices $p_1,\dots,p_n \in B$.

Let λ_S denote the maximal eigenvalue function of S, i.e., $\lambda_S(q)$ is the maximal eigenvalue of S at $q \in B$. Concerning the condition (P_2) we prove the following theorem.

THEOREM D. *If M^{n+1} is compact, of nonpositive curvature and with boundary isolated by conjugate points, then*

$$\int_B \lambda_S\,dB \ge \frac{\mathrm{vol}(B)^2}{(n+1)\mathrm{vol}(M)}$$

and the equality sign is true only if M is a spherical domain with flat metric of radius ${\lambda_S}^{-1}$, where λ_S is constant.

If M is a simply connected domain in the 2-dimensional Euclidean space E^2, then the right hand side is greater than or equal to 2π because of the isoperimetric inequality (cf. [**22**]). Therefore, we get the following.

COROLLARY E. *Suppose $M \subset E^2$ is simply connected and the sum of outer angles are nonnegative. Then, M is with boundary isolated by conjugate points only if M is a circular domain in E^2.*

M. Bialy ([**3**]) proved Corollary E in the case that M is a domain surrounded by a smooth convex curve in E^2 under a certain condition which is equivalent to ours (see Section 10), and he partially answers the Birkhoff conjecture concerning the integrability of billiard ball maps. In Subsections 6.4 and 6.5 we give examples of domains with boundary isolated by conjugate points other than circular domains. Corollary E states that smooth simple plane curves are convex scattering if and only if they are circles. Taking Example 6.4 into consideration we see in [**14**] and [**15**] some remarks on the inequality in Theorem D as a pinching sphere theorem of new type.

In Section 2 we provide some facts concerning Jacobi vector fields along reflecting geodesics. In Section 3 we quote some comparison theorems for Jacobi vector fields along reflecting geodesics and the existence of limit solutions without proof, because they are proved in the same way as seen in [**4**]. In Section 4 we give the mirror equation which is a key lemma for the proof of Theorem D and make the relation between the second fundamental form and the reflection endomorphism clear. In Section 5 we show a local condition on the boundary which is necessary for nonconjugacy property. In Section 6 some examples of systems without conjugate points and related to our theorems will be given. In Section 7 we give the proof of Theorem A. In the proof we are free from the discussion of summability of the traces of minimal symmetric solutions

U of Riccati equation along reflecting geodesics which correspond, roughly speaking, to the second fundamental forms of horospheres. In Section 8 we prove Corollary C under the assumption $I_b > 0$ with which we establish the summability of $\operatorname{tr} U$. The significance of Theorem C is due to Pesin ([**20**], [**21**]), namely, he proved the negative integral of the sum of negative Lyapunov characteristic numbers of the geodesic flows on Riemannian manifolds without boundary and without conjugate points to be the measure theoretic entropy with respect to the normalized Liouville measure. In Section 9 Theorem D will be proved and we show in Section 10 that the property (P_2) on the boundary here is equivalent to one as discrete dynamical systems which is seen in [**3**].

The integral inequalities due to L. Green ([**6**]) and Ossermann and Sarnak ([**19**]), etc., can be proved even if the solutions of the differential equation of Jacobi type does not arise in Riemannian manifold without boundary, but is defined over flows preserving volume. From the point of view it is important to find such differential equations because they are candidates on which we establish those integral inequalities. In [**9**], [**10**], [**11**], [**12**] and [**13**] we see such examples. In the present paper we deal with billiard ball systems as an application of the differential equation of Jacobi type.

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2. Variation vector fields.

Let M be a Riemannian manifold with boundary $B \neq \phi$ and let $q \in B$ be a point at which B is differentiable. Let $\gamma : [a, b] \to M$ be a geodesic with $\gamma(t_0) = q$ and making the angle θ with the tangent space T_qB to B at q. We write $X_1(t) = \dot{\gamma}(t)$ for $a \leq t \leq t_0 - 0$ and $X_2(t) = \dot{\gamma}(t)$ for $t_0 + 0 \leq t \leq b$. We define a map $P_i : X_i(t_0)^{\perp} \to T_qB$ by

$$P_i(v) = v - \frac{\langle v, N \rangle}{\langle X_i(t_0), N \rangle} X_i(t_0)$$

for each $i = 1, 2$, where $X_i{}^{\perp} = \{w \in T_qM \mid \langle w, X_i \rangle = 0\}$. Let S be the second fundamental form with respect to the unit normal vector field N of B pointing inward which satisfies by definition

$$\nabla_Z N = -S_q(Z)$$

for any $Z \in T_qB$, where $S_q(Z) \in T_qB$. We define a map

$$A(X_2(t_0))(v) = 2(\langle X_2(t_0), N \rangle (S \circ P_2)(v) - \langle X_2(t_0), S \circ P_2(v) \rangle N)$$

for any $v \in X_2(t_0)^{\perp}$. Then, $A(X_2(t_0))$ is symmetric. We shall see the reason why A is defined in such a way as above.

Consider a variation $\varphi : [a, b] \times (-\varepsilon, \varepsilon) \to M$ such that $\varphi(t, 0) = \gamma(t)$ and $\varphi_s = \varphi(\cdot, s)$ is a geodesic for each s and the parameters $t_0(s)$ at which the geodesics reflect is smooth for s. Let $Y_1(t)$ be the variation vector field for $a \leq t \leq t_0 - 0$ and $Y_2(t)$ for $t_0 + 0 \leq t \leq b$. Then, we prove the following.

LEMMA 2.1.

$$\nabla_{X_i}\nabla_{X_i} Y_i + R(Y_i, X_i)X_i = 0, \tag{2.1}$$

$$Q(Y_1(t_0)) = Y_2(t_0), \tag{2.2}$$

$$Q(\nabla_{X_1} Y_1(t_0)) - \nabla_{X_2} Y_2(t_0) = A(X_2(t_0))(Y_2{}^{\perp}(t_0)), \tag{2.3}$$

where R is the Riemannian curvature tensor and $Y_2{}^{\perp}$ is the component perpendicular to X_2. Further, if $\langle Y_1(a), X_1(a)\rangle = 0$, then

$$X_i \perp Y_i \quad \textit{for } i = 1, 2.$$

PROOF. (1): Since φ is a variation through geodesics, Y_i is a Jacobi vector field along γ, and, hence, satisfies (1).

(2): Differentiating both sides of $\varphi(t_0(s) - 0, s) = \varphi(t_0(s) + 0, s)$ at $s = 0$, we have

$$t_0'(0)X_1(t_0) + Y_1(t_0) = t_0'(0)X_2(t_0) + Y_2(t_0),$$

and, hence,

$$\begin{aligned} Y_2(t_0) &= Y_1(t_0) + t_0'(0)(X_1(t_0) - X_2(t_0)) \\ &= Y_1(t_0) + 2t_0'(0)\langle X_1(t_0), N\rangle N, \end{aligned}$$

since φ_s are reflecting geodesics for all s. We also have

$$t_0'(0) = -\frac{\langle Y_1(t_0), N\rangle}{\langle X_1(t_0), N\rangle},$$

since $t_0'(0)X_1(t_0) + Y_1(t_0) \in T_{\gamma(t_0)}B$. Therefore, we get

$$Y_2(t_0) = Y_1(t_0) - 2\langle Y_1(t_0), N\rangle N = Q(Y_1(t_0)).$$

(3): Let $\psi : [a, b] \times (-\varepsilon, \varepsilon) \to M$ be a reparametrization of φ such that $\psi(\bar{t}(t, s), s) = \varphi(t, s), \bar{t}(t_0(s), s) = t_0$ and $\|\bar{X}_1(t_0, s)\| = \|\bar{X}_2(t_0, s)\|$ where $\bar{X}_i(\bar{t}, s) = (\partial\psi/\partial\bar{t})(\bar{t}, s)$ for $a \leq \bar{t} \leq t_0 - 0$ if $i = 1$ and $t_0 + 0 \leq \bar{t} \leq b$ if $i = 2$ as before. Let $\bar{Y}_i(\bar{t}, s) = (\partial\psi/\partial s)(\bar{t}, s)$ be the variation vector field for $i = 1, 2$ as before. Then, $\bar{Y}_1(t_0, s) = \bar{Y}_2(t_0, s)$ for all s. Since $X_1 - X_2 = 2\langle X_1, N\rangle N$ at $\bar{t} = t_0$, we see that

$$\begin{aligned} (\nabla_{\bar{X}_2} \bar{Y}_2)(t_0) &= (\nabla_{\bar{Y}_2} \bar{X}_2)(t_0) = \nabla_{\bar{Y}_1}(\bar{X}_1 - 2\langle \bar{X}_1, N\rangle N) \\ &= Q(\nabla_{\bar{X}_1} \bar{Y}_1(t_0)) + 2(\langle \bar{X}_2, S(\bar{Y}_2)\rangle N - \langle \bar{X}_2, N\rangle S(\bar{Y}_2)). \end{aligned}$$

It should be noted that

$$2(\langle \bar{X}_2, S(\bar{Y}_2)\rangle N - \langle \bar{X}_2, N\rangle S(\bar{Y}_2)) \in X_2(t_0)^{\perp}$$

and $\bar{X}_i$ can change to X_i because of the linear property of ∇. The image of φ has dimension 2, so that

$$Y_i^{\perp} := Y_i - \langle Y_i, X_i \rangle X_i = \overline{Y}_i - \langle \overline{Y}_i, X_i \rangle X_i$$

for $i = 1, 2$. Since φ is the variation through unit speed geodesics, we can see that $\langle Y_i, X_i \rangle = const.$, and, hence, $\nabla_{X_i} Y_i = \nabla_{X_i} Y_i^{\perp}$. Moreover, we have

$$\begin{aligned} \nabla_{X_2} Y_2^{\perp} &= \nabla_{X_2}(\overline{Y}_2 - \langle \overline{Y}_2, X_2 \rangle X_2) \\ &= Q(\nabla_{X_1} \overline{Y}_1) + 2(\langle X_2, S(\overline{Y}_2) \rangle N - \langle X_2, N \rangle S(\overline{Y}_2)) - \langle Q(\nabla_{X_1} \overline{Y}_1), X_2 \rangle X_2. \end{aligned}$$

Since $P_2(Y_2^{\perp}(t_0)) = \overline{Y}_2(t_0)$ and $Q(X_1) = X_2$, we see that

$$\nabla_{X_2} Y_2^{\perp} = Q(\nabla_{X_1} Y_1^{\perp}) - A(X_2(t_0))(Y_2^{\perp}(t_0))$$

and, therefore, (2.3) is proved.

(4): Since φ is a variation through unit speed geodesics, the length of each geodesic is

$$\begin{aligned} t - a &= \int_a^t \|X(t,s)\| \, dt \\ &= \int_a^{t_0(s)} \|X_1(t,s)\| \, dt + \int_{t_0(s)}^t \|X_2(t,s)\| \, dt \end{aligned}$$

for any $t \in [a, b]$. Differentiating at $s = 0$, we have

$$0 = \langle Y_1(t), X_1(t) \rangle - \langle Y_1(a), X_1(a) \rangle$$

if $a \le t \le t_0 - 0$, and

$$\begin{aligned} 0 =& \langle Y_1(t_0), X_1(t_0) \rangle - \langle Y_1(a), X_1(a) \rangle \\ &+ \langle Y_2(t), X_2(t) \rangle - \langle Y_2(t_0), X_2(t_0) \rangle \\ &+ t_0'(0)(\|X_1(t_0)\| - \|X_2(t_0)\|) \end{aligned}$$

if $t_0 + 0 \le t \le b$. It holds from the first equation and the assumption $\langle Y_1(a), X_1(a) \rangle = 0$ that $\langle Y_1(t), X_1(t) \rangle = 0$ for $a \le t \le t_0 - 0$. Since $Y_2(t_0) = Q(Y_1(t_0))$, $X_2(t_0) = Q(X_1(t_0))$ and $\|X_1(t_0)\| = \|X_2(t_0)\| = 1$, we also have that $\langle Y_2(t), X_2(t) \rangle = 0$ for $t_0 + 0 \le t \le b$. This completes the proof of the lemma.

The property (2.3) is written with different expression in [**24**]. We can show many properties for perpendicular Jacobi vector fields along a reflecting geodesic in the same way as proved for ordinary ones.

3. Comparison theorems and limit solutions.

Let $\gamma : (-\infty, \infty) \to M$ be a geodesic with $\gamma(a_i) \in B$ for $\cdots < a_{-1} < a_0 < a_1 < \cdots$. Let $e_1, \dots, e_{n+1} = \dot{\gamma}(0)$ be an orthonormal basis at $\gamma(0)$ and $e_1(t), \dots, e_{n+1}(t)$ the parallel vector fields along γ such that $e_j(0) = e_j$, $Q(e_j(a_i - 0)) = e_j(a_i + 0)$ for $j = 1, \dots, n+1$. Let $R(t)$ be the matrix representation of $R(\cdot, \dot{\gamma}(t))\dot{\gamma}(t)$ with respect to the basis $\{e_j(t)\}$ and A_i the one of $A(\dot{\gamma}(a_i+0))$. If $Y(t)=\sum_{j=1}^n Y_j(t)e_j(t)$ is a perpendicular variation vector field through geodesics, then we see from Lemma 2.1 that

$$Y''(t)+R(t)Y(t)=0 \quad \text{for } t\neq a_i \quad (J_R), \tag{3.1}$$

$$Y(t) \text{ is continuous}, \tag{3.2}$$

$$Y'(a_i-0)-Y'(a_i+0)=A_i(Y(a_i)). \tag{3.3}$$

These properties motivate us to study the solutions of the differential equation (3.1) of Jacobi type (called (J_R)) satisfying the condition (3.2) and (3.3). In this paper we need the following comparison theorems and the existence theorem of limit solutions. The lemmas can be proved in the same way as the well-known theorems corresponding to them, so we omit the proof. However, we note that in the course of the proof given in [**4**] we should use the index form replaced for our case by

$$I(Y,W)=-\sum_{i=1}^{k-1}\langle A_i(Y),W\rangle(a_i)+\int_a^b(\langle Y',W'\rangle-\langle RY,W\rangle)\,dt$$

for piecewise smooth vector-valued functions Y,W on $[a,b]$, where we assume that Y,Z are not differentiable possibly at $a<a_1<\cdots<a_{k-1}<b$.

LEMMA 3.1. *Let $R_1(t)$ and $R_2(t)$ be symmetric (n,n)-matrix valued functions on $[a,b]$ which are possibly discontinuous at $a<a_1<\cdots<a_{k-1}<b$ and let A_i and B_i be symmetric (n,n)-matrices for $i=1,\dots,k-1$. Suppose $R_1(t)\leq R_2(t)$ and $A_i\leq B_i$ for any $t\in[a,b]$ and $i=1,\dots,k-1$. If (J_{R_2}) is without conjugate points on $[a,b]$, then so is (J_{R_1}) in $[a,b]$.*

LEMMA 3.2. *Let $R_1(t)$ and $R_2(t)$ be symmetric (n,n)-matrix valued functions on $[a,b]$ which are possibly discontinuous at $a<a_1<\cdots<a_{k-1}<b$ and let A_i and B_i be symmetric (n,n)-matrices for $i=1,\dots,k-1$. Suppose $\langle R_1(t)x,x\rangle\leq\langle R_2(t)y,y\rangle$ and $\langle A_ix,x\rangle\leq\langle B_iy,y\rangle$ for any $t\in[a,b], i=1,\dots,k-1$ and any $x,y\in\mathbf{R}^n$ with $\|x\|=\|y\|$. Suppose (J_{R_2}) is without conjugate points on $[a,b]$. If $Y_1(t)$ and $Y_2(t)$ are solutions of (J_{R_1}) and (J_{R_2}) satisfying the above conditions* (3.2) *and* (3.3) *with $Y_1(a)=Y_2(a)=0$, $\|Y_1'(a)\|=\|Y_2'(a)\|$, respectively, then $\|Y_1(t)\|\geq\|Y_2(t)\|$ for any $t\in[a,b]$.*

We consider (J_R) the (n,n)-matrix valued equation. Let $D_s(t)$ be the solution of (J_R) satisfying (3.2) and (3.3) with $D_s(s)=0$ and $D_s'(s)=I$, and set $U_s(t)=D_s'(t)D_s(t)^{-1}$ in the interval $t\in I$ with $\det D_s(t)\neq 0$ for each s. By using U_s as usual we can prove the following.

LEMMA 3.3 (The conjugate points separation property). *Suppose (J_R) is considered in the interval $[a,b]$ and t_1 is conjugate to t_0. Then, any $t\notin[t_0,t_1]$ has an s in (t_0,t_1) which is conjugate to t.*

LEMMA 3.4. *If (J_R) is without conjugate points on $(-\infty,\infty)$,* then *$U(t)=\lim_{s\to\infty}U_s(t)$ exists and satisfies that*

$$U'(t) + U(t)^2 + R(t) = 0 \quad \textit{for } t \neq a_i, \tag{3.4}$$

$$U(a_i - 0) - U(a_i + 0) = A_i, \tag{3.5}$$

$U(t)$ is the minimal symmetric matrix-valued function satisfying (3.4) *and* (3.5) *defined on* $(-\infty, \infty)$, *i.e., if a symmetric matrix-valued function $V(t)$ defined on $(-\infty, \infty)$ satisfies* (3.4) *and* (3.5), *then $U(t) \leq V(t)$ for any $t \in (-\infty, \infty)$.*

4. The mirror equation.

In this section we prove the mirror equation and make the relation of S and A clear. As before, let λ_S and $\lambda_{A(X)}$ be functions given by the maximal eigenvalues of S and $A(X)$, respectively, at any point.

LEMMA 4.1 (The mirror equation). *Let B be a hypersurface in Euclidean space E^{n+1} and $q \in B$. Let $\gamma : [0, t_0] \to E^{n+1}$ be a geodesic reflecting at only one point $\gamma(a) = q$ with $X := \dot{\gamma}(a+0)$ such that the angle between $\dot{\gamma}(a)$ and T_qB is θ and $\gamma(t_0)$ is the first conjugate point to $\gamma(0)$ along γ. Let $b = t_0 - a$. Then,*

$$\frac{2\lambda_S}{\sin\theta} \geq \lambda_{A(X)} = \frac{1}{a} + \frac{1}{b}.$$

The equality sign is true in the first inequality if and only if there are the eigenvectors of $A(X)$ and S with eigenvalues $\lambda_{A(X)}$ and λ_S in the subspace spanned by $\{N, X\}$. In particular, the equality sign is always true if $n = 1$.

PROOF. Let $P(w) = w - (\langle w, N\rangle / \langle X, N\rangle)X$ for any $w \in X^{\perp}$. Then,

$$\begin{aligned}\langle A(X)(w), w\rangle &= 2\langle X, N\rangle\langle S \circ P(w), P(w)\rangle \\ &\leq \frac{2}{\sin\theta}\|w\|^2\left\langle S\left(\frac{P(w)}{\|P(w)\|}\right), \frac{P(w)}{\|P(w)\|}\right\rangle\end{aligned}$$

for any $w \in X^{\perp}$, since

$$\|P(w)\| \leq \frac{1}{\sin\theta}\|w\|.$$

This proves the first inequality. In E^{n+1} the matrix Jacobi field D along γ with $D(0) = 0$ and $D'(0) = I$ is written

$$D(t) = (t - a)(I - aA(X)) + aI$$

for $t \in [a, t_0]$. Hence, $D(t)$ is symmetric, and $D(t_0) \geq 0$ since $\gamma(t_0)$ is the first conjugate point to $\gamma(0)$. We see that

$$A(X) \leq \left(\frac{1}{a} + \frac{1}{b}\right)I$$

and

$$\lambda_{A(X)} = \frac{1}{a} + \frac{1}{b}.$$

This completes the proof.

We use the following lemma in the next section which is a straightforward modification of Lemma 4.1.

LEMMA 4.2. *If the Euclidean space is replaced by the hyperbolic space* $H(-k^2)$ *of constant curvature* $-k^2$ $(k > 0)$ *in Lemma* 4.1, *then the a and b in Lemma* 4.1 *change to* $(1/k)\tanh ka$ *and* $(1/k)\tanh kb$, *respectively.*

We conclude this section by showing the relation between S_q and $A(X)$.

LEMMA 4.3. *The following are true.*

(4.1) *If the dimension of* M *is two, then* $A(X) = 2\kappa/\sin\theta$, *where* κ *is the geodesic curvature of* B *at* q.

(4.2) $S_q = 0$ *if and only if* $A(X) = 0$.

(4.3) *If* $S_q \leq 0$, *then* $A(X) \leq 0$ *and* $\operatorname{tr} A(X) \geq (2/\sin\theta)\operatorname{tr} S_q$.

(4.4) *If* $S_q \geq 0$, *then* $A(X) \geq 0$ *and* $\operatorname{tr} A(X) \leq (2/\sin\theta)\operatorname{tr} S_q$.

(4.5) *If* $S_q = \lambda I$, *then* $\operatorname{tr} A(X) = (2\lambda/\sin\theta)(1 + (n-1)\sin^2\theta)$.

Here $\operatorname{tr} S$ *is by definition the trace of* S.

PROOF. Let $w_1, w_2 \in X^\perp$. We have that

$$\langle A(X)(w_1), w_2\rangle = 2\langle X, N\rangle\langle S \circ P(w_1), P(w_2)\rangle$$

Since P is surjective, the statement (4.2) and the first parts of (4.3) and (4.4) are clear.

In order to prove others we extend $S, A(X)$ and P linearly on T_qM by putting $S(N) = 0$, $A(X)(X) = 0$ and $P(X) = 0$. The traces of S and $A(X)$ do not change. Take an orthonormal basis $\{e_k\}$ such that $e_1, \ldots, e_n \in T_qB$ are eigenvectors of S with eigenvalues $\lambda_1, \ldots, \lambda_n$, respectively, and $e_{n+1} = N$. Then, we get

$$\begin{aligned}
\operatorname{tr} A(X) &= \sum_{k=1}^{n+1} \langle A(X)(e_k), e_k\rangle = 2\langle X, N\rangle \sum_{k=1}^{n+1} \langle S \circ P(e_k), P(e_k)\rangle \\
&= 2\langle X, N\rangle \left\{ \sum_{k=1}^{n} \langle S(e_k), e_k\rangle - \frac{1}{\langle X, N\rangle}\left\langle S(X), N - \frac{X}{\langle X, N\rangle}\right\rangle \right\} \\
&= 2\sin\theta \left\{ \sum_{k=1}^{n} \lambda_k + \frac{1}{\sin^2\theta}\langle S(X), X\rangle \right\}.
\end{aligned}$$

Since $S(X) = \sum_{k=1}^n \lambda_k \langle X, e_k\rangle e_k$, we have that

$$\operatorname{tr} A(X) = \frac{2}{\sin\theta} \sum_{k=1}^{n} \lambda_k(\sin^2\theta + \langle X, e_k\rangle^2).$$

Since $\sin\theta = \langle X, e_{n+1}\rangle$, we see that $\sin^2\theta + \langle X, e_k\rangle^2 \le 1$ for each k, and equality is true if $n = 1$. This completes the proof of (4.1), and

$$\operatorname{tr} A(X) \ge \frac{2}{\sin\theta}\operatorname{tr} S \quad \text{if } S \le 0,$$

$$\operatorname{tr} A(X) \le \frac{2}{\sin\theta}\operatorname{tr} S \quad \text{if } S \ge 0,$$

and

$$\operatorname{tr} A(X) = \frac{2\lambda}{\sin\theta}(1 + (n-1)\sin^2\theta) \quad \text{if } S = \lambda I.$$

5. Conjugacy property near the boundary.

This section is devoted to the proof of the following theorem which gives locally a necessary condition for nonconjugacy property.

THEOREM 5.1. *If $\lambda_S > 0$ at some point q of the boundary, then there is a geodesic reflecting at q with conjugate points.*

PROOF. Let $w \in T_qB$ be a unit eigenvector of S with eigenvalue $\lambda_S(q) > 0$ and $W \subset T_qB$ the subspace spanned by $N(q)$ and w. Let $Z = \exp_q W$, namely Z is a surface given by geodesics emanating from q and tangent to W at q, and let s be the distance from q to the point $q(s)$ in $Z \cap B$. We first estimate the length of geodesics reflecting at q as $s \to 0$, and secondly show the existence of conjugate points on those geodesic arcs.

Let $\theta(s)$ be the angle between the geodesic from $q(s)$ to q and w. Then, the geodesic curvature of the curve $Z \cap B$ is $\lambda_S(q)$ at q and

$$\lim_{s\to 0}\frac{\theta(s)}{s} = \frac{\lambda_S(q)}{2}.$$

From this we see that any geodesic reflecting at q from $q(s)$ is defined on $[0, 3s/2]$ if s is sufficiently small positive. In order to prove this we define $s_1 = s_1(s)$ so that $\theta(s_1) = \theta(s)$. If $s_1 \to 0$ (as $s \to 0$) is not true, we have nothing to prove. Otherwise,

$$\lim_{s\to 0}\frac{s_1(s)}{s} = \lim_{s\to 0}\frac{\theta(s)}{s}\frac{s_1}{\theta(s_1)} = 1.$$

Hence, $s_1(s) = s + o(s)$. Thus, the upper bound of the interval of reflecting geodesics is $2s$ as $s \to 0$.

Suppose the sectional curvature of M around $q \in B$ is greater than $-k^2$ $(k > 0)$. Take a hypersurface $\bar{B}$ in the space form $H(-k^2)$ of sectional curvature $-k^2$ such that $\bar{q} \in \bar{B}$ and $S_{\bar{q}} = S_q$. Let $\bar{w}$ be the unit eigenvector of $S_{\bar{q}}$ with eigenvalue $\lambda_S(q)$. For the geodesic γ from $q(s)$ to q we take the geodesic $\bar{\gamma}$ in $H(-k^2)$ such that $\bar{\gamma}(s) = \bar{q}, \dot{\bar{\gamma}}(s)$ belongs to the linear span $\{\bar{N}(\bar{q}), \bar{w}\}$ and $\langle\dot{\bar{\gamma}}(s), \bar{w}\rangle = \langle\dot{\gamma}(s), w\rangle =$

$\cos\theta(s)$. The time comparison theorem (Lemma 3.1) states that the time of the first conjugate point t along $\bar{\gamma}$ is not less than the one along γ. Hence we have only to show that the first conjugate point to $\bar{\gamma}(0)$ along $\bar{\gamma}$ appears within $[0, 3s/2]$. The equation with unknown variable t is from Lemma 4.2

$$\frac{2\lambda_S(q)}{\sin\theta(s)} = k(\coth ks + \coth k(t-s)).$$

There is a solution of this equation in $[0, 3s/2]$. This completes the proof.

A simple application of the theorem is the following.

COROLLARY 5.2. *Let M be a complete manifold with boundary $B \neq \phi$. If M is without conjugate points, then the second fundamental form S of M is nonpositive.*

6. Examples.

In this section we give some examples of geodesics without conjugate points, domains without conjugate points and with boundary isolated by conjugate points.

6.1. Let M be a compact simply connected domain with boundary $B \neq \phi$ in the 2-dimensional hyperbolic space form $H(-1)$ and let $\gamma : [0, \ell] \to M$ be a geodesic segment whose endpoints are on B and which are perpendicular to B. Extend γ as a reflecting geodesic defined on $(-\infty, \infty)$. If the geodesic curvatures κ of B at the intersection points satisfy that $0 < \kappa < (1/2)\tanh \ell$, then the geodesic γ defined on $(-\infty, \infty)$ is without conjugate points.

6.2. Let M be a domain in the unit 2-sphere satisfying the following conditions: (1) the diameter d is less than $\pi/2$. (2) B is the union of arcs of circles with center points outside of M and radii r with $\cot r > (1/2)\tan d$. The condition (1) implies that all geodesic segments in the interior are minimizing. And, (2) implies that the geodesic curvature of boundary curves satisfies $\kappa = -\cot r$, i.e., $-\kappa > (1/2)\tan d$. It follows from these properties that all geodesics in M are without conjugate points.

6.3. Let M be a complete Riemannian manifold of nonpositive curvature with boundary $B \neq \phi$ whose second fundamental form is nonpositive. Then, M is without conjugate points. This follows from Lemma 3.1 or 3.2.

The following examples are for "boundary isolated by conjugate points".

6.4. Let M be a complete flat manifold which is the union of balls. Then, M is with boundary isolated by conjugate points. If v is a point in SM with $\pi(v)$ belonging to a sphere of radius r, then $\alpha(v) = \beta(v) = r\sin\theta(v)$ where $\theta(v)$ is the angle between v and the tangent space to the boundary at $\pi(v)$. It should be noted that $\alpha(v)$ is half the length of chord. The mirror equation (Lemma 4.1) implies that $\gamma_v(\alpha(v))$ is the conjugate point to $\gamma_v(-\beta(v))$. We have to prove that $T(v) \geq \alpha(v) + \beta(F(v))$. Suppose for indirect proof that $T(v) < \alpha(v) + \beta(F(v))$. Then, $\alpha(v) > (T(v)/2)$ or $\beta(F(v)) >$

$(T(v)/2)$. If the former is true, then $\gamma_v(T(v))$ is not in the boundary of M, a contradiction. By the same reason the latter does not happen.

6.5. In [**23**] we see the definition of "*convex scattering*". We say that a smooth curve Γ in the Euclidean plane E^2 is *convex scattering* if for every segment $[\Gamma(s_0), \Gamma(s_1)], \Gamma(s_0) \neq \Gamma(s_1)$, such that the arc of Γ between $\Gamma(s_0)$ and $\Gamma(s_1)$ lies entirely on one side of the straight line passing through $\Gamma(s_0)$ and $\Gamma(s_1)$, we have that

$$D_2(s_0) \cap D_2(s_1) \cap [\Gamma(s_0), \Gamma(s_1)]$$

contains at most one point, where $D_2(s)$ is the closed disk tangent to Γ at $\Gamma(s)$ with radius half the radius of curvature and lying on the same side of Γ as the circle of curvature. M. Wojtkowski ([**23**]) gives some examples of curves isolated by conjugate points as the union of convex scattering curves. He designs those examples to show the existence of curves in which almost all trajectories of geodesic flow have nonvanishing Lyapunov characteristic numbers.

7. Proof of Theorem A.

In this section we give the proof of Theorem A. Put $SM_0 = \{v \in SM \mid \gamma_v(t) \notin B$ for any $t \in (-\infty, \infty)\}$ and $SM_1 = SM - SM_0$. Let Ω be the set of all uniformly recurrent vectors (see [**2**] for the definition). It is noticed that SM_0, SM_1, Ω are f^t-invariant. By the same argument as in [**6**], we get

$$\int_{SM_0} \mathrm{Ric}\, d\omega \leq 0$$

and the equality sign is true only if $R(\cdot, v)v = 0$ for any $v \in SM_0$, where Ric denotes the Ricci curvature.

Let $v \in \Omega \cap SM_1$. Then, there exists the sequence $\{t_j\}$ such that $t_j \to \pm\infty$ and $\gamma_v(t_j) \in B$. The special flow g^t of $\bigcup_{w \in \partial SM} \{w\} \times [0, T(w))$ is measure theoretically isometric to the geodesic flow of $\Omega \cap SM_1$ with the Liouville volume $d\omega = \sin\theta\, dB\, d\Theta dt$ at $v \in \Omega \cap SM_1$, where θ is the angle between γ_v and B at the last time $t_0 \leq 0$ with $\gamma(t_0) \in B$, $d\Theta$ is the natural volume form of the unit sphere with dimension n and dt is the dual to the vector field tangential to the flow. Put

$$G(w) = \int_0^{T(w)} \mathrm{tr}\, R(\cdot, f^t w) f^t w\, dt$$

for any $w \in \partial SM \cap \Omega$. By Lemma 3.4, we get

$$\mathrm{tr}\, U(f^{T(w)-0} w) - \mathrm{tr}\, U(w) + \int_0^{T(w)} \mathrm{tr}\, U(f^t w)^2\, dt + G(w) = 0$$

for any $w \in \partial SM \cap \Omega$, and, hence,

$$\mathrm{tr}\, A(F(w)) + \mathrm{tr}\, U(F(w)) - \mathrm{tr}\, U(w) + \int_0^{T(w)} \mathrm{tr}\, U(f^t w)^2\, dt + G(w) = 0.$$

By the same way as above for $F(w), \dots, F^n(w), n = 1, 2, \dots,$ we have

$$\sum_{i=1}^{n} \operatorname{tr} A(F^i(w)) + \operatorname{tr} U(F^n(w)) - \operatorname{tr} U(w) + \int_0^{T^n(w)} \operatorname{tr} U(f^t w)^2 \, dt + \sum_{i=1}^{n-1} G(F^i(w)) = 0$$

for every $n = 1, 2, \dots,$ where $T^n(w) = T(w) + \cdots + T(F^{n-1}(w))$. By Corollary 5.2 and Lemma 4.3, it follows that

$$\sum_{i=1}^{n} \frac{2}{\sin\theta(F^i(w))} \operatorname{tr} S(\gamma_w(T^i(w))) + \operatorname{tr} U(F^n(w)) - \operatorname{tr} U(w) + \int_0^{T^n(w)} \operatorname{tr} U(f^t w)^2 \, dt + \sum_{i=1}^{n-1} G(F^i(w)) \leq 0.$$

We know that $(2/\sin\theta) \operatorname{tr} S$ and G are summable on the set ∂SM with respect to the measure $d\omega_1 = \sin\theta \, dB \, d\Theta$ which is F-invariant (cf. [5]). Since w is a recurrent vector, there exists a subsequence $\{m\}$ of $\{n\}$ such that $F^m(w) \to w$ as $m \to \infty$. We can see that U is locally bounded because of the construction, and, hence,

$$\frac{1}{m} \operatorname{tr} U(F^m(w)) \to 0 \quad \text{and} \quad \frac{1}{m} \operatorname{tr} U(w) \to 0$$

as $m \to 0$. Consequently, by the Birkhoff ergodic theorem (cf. [1], [5]), we get

$$\left(\frac{2}{\sin\theta} \operatorname{tr} S\right)^*(w) + G^*(w) \leq 0$$

for almost all $w \in \partial SM$, where $H^*(w) = \lim_{n\to\infty} (1/n) \sum_{i=1}^{n-1} H(F^i(w))$ for any integrable function H on ∂SM. Integrate the inequality on ∂SM, and we have

$$\int_{\partial SM} \frac{2}{\sin\theta} \operatorname{tr} S \sin\theta \, dB \, d\Theta + \int_{\partial SM} G(w) \sin\theta dB \, d\Theta \leq 0,$$

and, hence, combining it with the integral over SM_0,

$$\operatorname{vol}(S^n(1)) \int_B \operatorname{tr} S \, dB + \frac{\operatorname{vol}(S^n(1))}{n+1} \int_M \operatorname{Sc} dM \leq 0,$$

where $\operatorname{vol}(S^n(1))$ is the volume of the unit sphere with dimension n. Then, we complete the proof of the inequality.

Suppose the equality sign is true. Then, for almost all $w \in \partial SM$ there exists a sequence $\{m\}$ such that

$$\lim_{m\to\infty} \frac{1}{m} \int_0^{T^m(w)} \operatorname{tr} U(f^t w)^2 \, dt = 0,$$

and $F^m(w) \to w$ as $m \to \infty$. From this we can take a sequence $\{k\}$ of $\{m\}$ such that

$$\int_{T^k(w)}^{T^{k+1}(w)} \operatorname{tr} U(f^t w)^2 \, dt \to 0 \quad \text{as} \quad k \to \infty \quad \text{(cf. [8])}$$

and, hence, $\operatorname{tr} U(f^t F^k(w)) \to 0$ as $k \to \infty$ for almost all $t, 0 \le t \le T^{k+1}(w) - T^k(w) = T(F^{k-1}(w))$. Since

$$\operatorname{tr} U(f^a w) - \operatorname{tr} U(f^b w) + \int_b^a \operatorname{tr} U(f^t w)^2 \, dt + \int_b^a \operatorname{Ric}(f^t w) \, dt = 0$$

for any a, b with $T^k(w) \le b < a \le T^{k+1}(w)$, we can see that $\operatorname{Ric}(f^t w) = 0$ for any t, $0 \le t \le T(w)$, because Ric is continuous although we do not know whether so is $\operatorname{tr} U$ or not. Hence, the integral of the scalar curvature over M is zero, and the boundary consists of totally geodesic hypersurfaces because $S \le 0$. Again, integrating the Riccati equation, we can show that $U(v) = 0$ for all $v \in SM$. This implies that the sectional curvature is zero identically. This completes the proof.

8. Proof of Corollary C.

As before, M denotes a manifold with boundary $B \ne \phi$. We need three lemmas for the proof of Corollary C.

LEMMA 8.1. *Let M^{n+1} be a compact manifold with $B \ne \phi$. Suppose M is without conjugate points. If $I_b > 0$, then* $\operatorname{tr} U$ *and* $\operatorname{tr} U^2$ *are summable on SM, where U is the minimal symmetric solution of the Riccati equation along γ_v, $v \in SM$, as seen in Lemma* 3.3.

PROOF. First of all we notice that $S \le 0$ because of Corollary 5.2. We use the notation SM_0 and SM_1 as in Section 7. Since $I_b > 0$, we can have a positive ε such that $T(v) > 2\varepsilon$ for any $v \in \partial SM$ with $\pi(v) \in E$. It follows from this that there exists a positive $a > 0$ depending on ε and the sectional curvature of M such that

$$U(f^{T(v)+0} v) \ge -aI \quad \text{and} \quad U(f^{T(v)-0} v) \le aI.$$

By Lemmas 3.4 and 4.3, we see

$$\begin{aligned} -na \le \operatorname{tr} U(f^{T(v)+0} v) &\le na + |\operatorname{tr} A(f^{T(v)+0} v)| \\ &\le na - \frac{2}{\sin\theta} \operatorname{tr} S. \end{aligned}$$

This implies that $\operatorname{tr} U$ is summable on ∂SM with respect to $d\omega_1 = \sin\theta \, dB \, d\Theta$. Since U satisfies the Riccati equation, we get

$$\operatorname{tr} U(f^{T(v)-0} v) - \operatorname{tr} U(v) + \int_0^{T(v)} \operatorname{tr} U(f^t v)^2 \, dt + \int_0^{T(v)} \operatorname{Ric}(f^t v) \, dt = 0$$

for any $v \in \partial SM$, and, hence,

$$\operatorname{tr} A(F(v)) + \operatorname{tr} U(F(v)) - \operatorname{tr} U(v) + \int_0^{T(v)} \operatorname{Ric}(f^t v) \, dt = -\int_0^{T(v)} \operatorname{tr} U(f^t v)^2 \, dt$$

Since $0 \ge \operatorname{tr} A \ge (2/\sin\theta) \operatorname{tr} S$, the left hand side is summable on ∂SM. Hence, $\operatorname{tr} U^2$ is summable on SM_1. The summability of $\operatorname{tr} U^2$ on SM_0 is proved similarly.

Therefore, $\operatorname{tr} U^2$ is summable on SM. Since the measure of SM is finite, $\operatorname{tr} U$ is summable on SM also. This completes the proof.

From the lemma the solutions of the differential equation $D' = UD$ with initial condition $D(0, v) = I$ define a multiplicative cocycle satisfying the condition given in [**18**]. Therefore, we can use the Oceledec multiplicative ergodic theorem for the multiplicative cocycle $D(t, v)$.

LEMMA 8.2. $(1/t)\operatorname{tr} U(f^t v)^2 \to 0$ *as* $t \to \infty$ *for almost all* $v \in \partial SM$.

PROOF. This follows from the facts that for any $\varepsilon > 0$,

$$\omega(\{v \in SM \,|\, \operatorname{tr} U(f^t v)^2 > t\varepsilon\}) = \omega(\{v \in SM \,|\, \operatorname{tr} U(v)^2 > t\varepsilon\})$$

is monotone nonincreasing for t and

$$\int_0^\infty \omega(\{v \in SM \,|\, \operatorname{tr} U(v)^2 > t\varepsilon\})\, dt = \frac{1}{\varepsilon} \int_{SM} \operatorname{tr} U^2 \, d\omega.$$

This completes the proof.

Let $Y(t)$ be a perpendicular Jacobi vector field along γ_v satisfying (2.1)–(2.3) such that $Y'(t) = U(f^t v) Y(t)$. Then, we can find a unique $\xi \in T_v T_1 M$ such that $Y(t) = \pi_* f^t{}_* \xi$. We write it $Y_\xi(t)$.

LEMMA 8.3. *For almost* $v \in SM$ *the following are true. If* $Y_\xi(t) = \pi_* f^t{}_* \xi$ *is given as above, then we see that*

$$\limsup_{t\to\infty} \frac{1}{t} \log\|f^t{}_*\xi\| = \limsup_{t\to\infty} \frac{1}{t} \log\|Y_\xi(t)\|$$

and

$$\liminf_{t\to\infty} \frac{1}{t} \log\|f^t{}_*\xi\| = \liminf_{t\to\infty} \frac{1}{t} \log\|Y_\xi(t)\|.$$

PROOF. Notice that $\langle UY_\xi, UY_\xi\rangle \le \operatorname{tr} U^2 \langle Y_\xi, Y_\xi\rangle$. Since we use the natural metric on $T_1 M$ for the estimates and

$$\|Y_\xi(t)\|^2 \le \|f^t{}_*\xi\|^2 = \|Y_\xi(t)\|^2 + \|Y'_\xi(t)\|^2.$$

we have

$$\frac{1}{t} \log\|Y_\xi(t)\| \le \frac{1}{t} \log\|f^t{}_*\xi\| \le \frac{1}{2t}(\operatorname{tr} U(f^t v)^2 + 1) + \frac{1}{t} \log\|Y_\xi(t)\|.$$

The statement follows from Lemma 8.2.

PROOF OF COROLLARY C. From Lemma 8.3 and Lemma 3.4 we see that $Y_\xi(t)$ is constructed as above if $\chi(v, \xi) < 0$. It follows from the Oceledec multiplicative ergodic theorem ([**18**]) that $\chi(v, \xi) := \limsup_{t\to\infty}(1/t)\log\|f^t{}_*\xi\| = \lim_{t\to\infty}(1/t)\log\|f^t{}_*\xi\|$ for almost all $v \in SM$, and

$$-\sum_{i=1}^{k} \chi(v,\xi_i) = -\sum_{i=1}^{n} \chi(v,\xi_i) = -\lim_{t\to\infty} \frac{1}{t} \log|\det D(t,v)|$$
$$= -\lim_{t\to\infty} \frac{1}{t} \int_0^t \operatorname{tr} U(f^t v)\, dt$$

for a subset $\xi_1,\dots,\xi_k$ of a normal basis. By estimating the integral of $\operatorname{tr} U^2$ as in the proof of Theorem A, we have

$$-\int_{SM} \sum_{i=1}^{k} \chi_i(v)\, d\omega = -\int_{SM} \operatorname{tr} U\, d\omega \le (n\operatorname{vol}(SM))^{1/2}\left(\int_{SM} \operatorname{tr} U^2\, d\omega\right)^{1/2}$$
$$\le (n\operatorname{vol}(SM))^{1/2}(\operatorname{vol} S^n(1))^{1/2}\left(-\left(\int_B \operatorname{tr} S\, dB + \frac{1}{n+1}\int_M \operatorname{Sc} dM\right)\right)^{1/2}$$

If the equality sign is true, then $U = aI$ for some constant a. Thus, $A = 0$ and $R(\cdot, f^t v)f^t v = -a^2 I$. This implies that M is a space of constant curvature and the boundary is totally geodesic. This completes the proof.

9. Proof of Theorem D.

In this section we prove Theorem D. We first of all notice that $\lambda_S > 0$ because of Lemma 4.1. Let Ω be the set of all recurrent vectors in ∂SM and let $v \in \Omega$ with $\pi(v) = p$. We take a small hypersurface H in E^{n+1} such that $\bar{p} \in H$ and the second fundamental form $\bar{S}$ of H at $\bar{p}$ is S_p under some identification by isometry Φ from T_pM to $T_{\bar{p}}E^{n+1}$. Let $\gamma : (-\infty,\infty) \to E^{n+1}$ be a geodesic reflecting at only one point $\bar{p} = \gamma(0)$ such that $\dot\gamma(+0) = \Phi(v)$. If $\gamma(a)$ is the first conjugate point to $\gamma(-\beta(v))$, then it follows from Lemma 4.1 that

$$\frac{2\lambda_S(\pi(v))}{\sin\theta(v)} \ge \frac{1}{a} + \frac{1}{\beta(v)} \ge \frac{4}{a+\beta(v)}$$

and, hence, by Lemma 3.1, we have that

$$\frac{\sin\theta(v)}{\lambda_S(\pi(v))} \le \frac{a+\beta(v)}{2} \le \frac{\alpha(v)+\beta(v)}{2}$$

from any $v \in \Omega$. Integrate it as follows:

$$\int_{0\le\theta\le\pi/2} \sin^2\theta\, d\Theta \int \frac{1}{\lambda_S}\, dB = \int_{\partial SM} \frac{\sin\theta}{\lambda_S}\, d\omega_1$$
$$\le \int_{\partial SM} \frac{\alpha(v)+\beta(v)}{2}\, d\omega_1 = \int_{\partial SM} \frac{\alpha(v)+\beta(F(v))}{2}\, d\omega_1$$
$$\le \int_{\partial SM} \frac{T}{2}\, d\omega_1 \le \frac{1}{2}\int_{SM} d\omega = \frac{1}{2}\operatorname{vol}(S^n(1))\operatorname{vol}(M).$$

On one hand, we have

$$\operatorname{vol}(B)^2 = \left(\int_B dB\right)^2 \le \int_B \frac{1}{\lambda_S} dB \int_B \lambda_S \, dB$$

and

$$\int_{S^n(1)} \sin^2\theta \, d\Theta = \frac{1}{n+1} \operatorname{vol} S^n(1).$$

Therefore, we get

$$\int_B \lambda_S \, dB \ge \frac{\operatorname{vol}(B)^2}{(n+1)\operatorname{vol}(M)}.$$

Assume that the equality sign is true. It follows that λ_S is constant, say $1/r$. By Lemma 4.1, one of eigenvectors of S with eigenvalue $1/r$ is in the subspace spanned by $\{N, v\}$ for $\pi(v) = p$ at which B is smooth. Since v moves in T_pM, the boundary is totally umbilical. Furthermore, $\alpha(v) = \beta(v)$, and, hence, $\alpha(v) = r\sin\theta(v)$ if v and T_pB make the angle $\theta(v)$. In addition, $T(v) = \alpha(v) + \beta(F(v))$ for almost all $v \in \partial SM$. From these facts we have to conclude that M is a spherical domain with flat metric. Let $p \in B$ be a point at which B is smooth and W a neighborhood of p where B is smooth. Let $\varphi(q, t) = \exp_q tN(q)$ for $q \in W$. Then, $\varphi(W, [0, r]) \subset M$ since $r = \alpha(N(q)) < T(N(q))$. Since conjugate points are continuous at $N(q)$, we see that $\varphi(W, r)$ is a single point p_0 which is the focal point of W. The comparison theorem of geodesic spheres implies that $\varphi(W, [0, t])$ is a flat cone sector. Suppose for indirect proof that B is not a sphere, namely the union of pieces of spheres $K_1, K_2, \ldots, K_m, \ldots, \operatorname{Int} K_i \cap \operatorname{Int} K_j = \phi$ where $\operatorname{Int} K_i$ is the interior of K_i as a submanifold. First we observe that $\varphi(\operatorname{Int} K_i, [0, r]) \cap \varphi(\operatorname{Int} K_j, [0, r]) = \phi$ if $K_i \cap K_j \ne \phi$. In fact, otherwise, we get a contradiction to $T(v) = \alpha(v) + \beta(F(v))$. Hence, we can find out a $v \in \partial SM$ such that $\pi(v) \in \operatorname{Int} K_1$, $\pi(F(v)) \in \operatorname{Int} K_j$, $j \ne 1$ and γ_v passes through ∂K_1 (the boundary of K_1 as a submanifold). If θ is the angle between v and K_1, then $\alpha(v) = r\sin\theta$ and $\beta(F(v)) < T(v) - 2r\sin\theta$. Therefore, we can find a $w \in \partial SM$ near v such that $T(w) > \alpha(v) + \beta(F(v))$, a contradiction. This completes the proof.

10. Nonconjugacy hypothesis as discrete dynamical systems.

The definition of nonconjugacy property in the study of billiards is given as discrete dynamical systems ([3]). In this section we show its equivalence to ours. The tangent space T_vT_1M is identified with $T_pM \oplus v^\perp$ by the map $\xi \to (\pi_*\xi, K(\xi))$ where $p \in \pi(v), v^\perp$ is the subspace of T_pM orthogonal to v and K is the connection map. Let $c(s)$ be a curve in ∂SM with $c(0) = v, \dot{c}(0) = \xi = (\xi_1, \xi_2)$. Then,

$$F^n(c(s)) = f^{T^n(c(s))-0}c(s) - 2\langle f^{T^n(c(s))-0}c(s), N\rangle N.$$

Differentiate it at $s = 0$, and we get

$$F^n{}_*\xi = (Y_\xi^\perp(T^n(v) - 0) + \langle \pi_*F^n{}_*\xi, f^{T^n(v)-0}v\rangle f^{T^n(v)-0}v, \nabla_{F^n(v)} Y_\xi^\perp),$$

where $Y_\xi(t) = \pi_* f^t{}_*\xi = Y_\xi^\perp(t) + \langle \xi_1, v\rangle f^t v$ is a Jacobi vector field along γ_v. Since $\dot\gamma_v(T^n(v) - 0) = f^{T^n(v)-0}v \notin T_{\pi F^n(v)}B$, and $\pi_* F^n{}_*\xi \in T_{\pi F^n(v)}B$, we have the following.

LEMMA 10.1. *There is a* $0 \neq \xi \in T_{\pi(v)}\partial SM$ *with* $\pi_*\xi = 0$ *such that* $\pi_* F^n{}_*\xi = 0$ *if and only if* $\pi F^n(v)$ *is a conjugate point to* $\pi(v)$ *along* γ_v.

The former condition is adopted as the definition of conjugate points in dealing with billiards as discrete dynamical systems ([**3**]).

We consider the relation further for the case of dim $M = 2$. Let $\Gamma(u)$, $0 \le u \le \ell$, be a simple C^∞ convex closed curve in E^2 which is parametrized by arclength and M the domain surrounded by Γ. We use the coordinates (u, θ) for ∂SM, namely $v \in \partial SM$ has the coordinates (u, θ) if $\pi(v) = \Gamma(u)$ and the angle between v and Γ is θ. Assume that any geodesic γ in M with $\gamma(0) \in \Gamma$ has no conjugate points on Γ, namely any conjugate point to $\gamma(0)$ along γ is in the interior of M if exists. Let a geodesic $\gamma_v : (-\infty, \infty) \to M$ be such that $\gamma(a_i) \in \Gamma$ for $\cdots < a_{-1} < a_0 = 0 < a_1 < \cdots$. Let $\xi \in T_v\partial SM$ with $\xi = (0, 1)$. It follows from Lemma 10.1 that $\pi_* F^n{}_*\xi \neq 0$ for all $n \neq 0$. Since $\pi_* F^n{}_*\xi$ is continuous for the variable θ, we see that $\pi_* F^n{}_*\xi > 0$. In fact, $u'(0) \ge \bar u'(0) = (1/\kappa(u))$ if $u(\theta) = \pi F^n(u, \theta)$ and $\bar u(\theta) = \pi F(u, \theta)$. This fact, $\pi_* F^n{}_*\xi > 0$, implies that any perpendicular Jacobi vector field $Y^\perp$ along γ_v with $Y^\perp(0) = 0$ has exactly one zero in each chord connecting $\pi F^n(v)$ and $\pi F^{n+1}(v)$ for $n \neq 0, -1$.

Conversely, this property implies that $\pi_* F^n{}_*\xi > 0$ for $n \neq 0$ if $\xi = (0, 1)$ because of the separation property of conjugate points (Lemma 3.3). From these arguments we get a straightforward modification of a theorem due to Bialy ([**3**]).

THEOREM 10.2. *Let* $\Gamma(s), 0 \le s \le \ell$, *be a simple* C^∞ *convex closed curve in* E^2 *with positive geodesic curvature surrounding* M *and let* $F : \Gamma \times (0, \pi) \to \Gamma \times (0, \pi)$ *be a billiard ball map. If* $\Gamma \times (0, \pi)$ *is foliated by F-invariant curves which are locally graphs over pieces of* Γ, *then* M *is with boundary isolated by conjugate points, and, therefore,* Γ *is a circle.*

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