J. Math. Soc. Japan
Vol. 52, No. 4, 2000

Complements of plane curves with logarithmic Kodaira dimension zero

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(Received Apr. 14, 1999)

Abstract. We prove that logarithmic geometric genus of a complement of plane curve with logarithmic Kodaira dimension zero is equal to one.

1. Introduction.

Let $B \subset \boldsymbol{P}^2$ be a reduced projective plane curve defined over the complex number field $\boldsymbol{C}$. To study the curve B, the logarithmic Kodaira dimension $\bar{\kappa}(\boldsymbol{P}^2 - B)$ of $\boldsymbol{P}^2 - B$ plays an important role. There are some results for the calculation of $\bar{\kappa}(\boldsymbol{P}^2 - B)$ (see [**18**] and [**20**], etc.). In [**12**], Miyanishi and Sugie studied the structure of $\boldsymbol{P}^2 - B$ when $\bar{\kappa}(\boldsymbol{P}^2 - B) = -\infty$ by using the $\boldsymbol{A}^1$-ruling theorem (cf. [**11**, Chapter I]). In [**16**] (or [**15**]), Tsunoda classified rational cuspidal curves (i.e., rational curves with only cusps as singularities) B of $\bar{\kappa}(\boldsymbol{P}^2 - B) = 1$ with unique singular points by using the structure theorem of non-complete algebraic surfaces with $\bar{\kappa} = 1$ due to Kawamata [**6**] (see also [**11**, Chapter II]). Recently, in [**8**], Kishimoto studied rational cuspidal curves B of $\bar{\kappa}(\boldsymbol{P}^2 - B) = 1$ with two singular points.

In the present article, we shall study the case $\bar{\kappa}(\boldsymbol{P}^2 - B) = 0$, mainly using the classification theory of affine surfaces with $\bar{\kappa} = 0$ in [**9**]. The main result is the following theorem.

THEOREM 1.1. *Let $B \subset \boldsymbol{P}^2$ be a reduced projective plane curve whose complement has logarithmic Kodaira dimension zero. Then the following assertions hold true*:

(1) $\bar{p}_g(\boldsymbol{P}^2 - B) = 1$, *where $\bar{p}_g(\boldsymbol{P}^2 - B)$ denotes the logarithmic geometric genus of $\boldsymbol{P}^2 - B$.*

(2) *If B is not an irreducible nonsingular cubic curve then each irreducible component of B is a rational curve.*

(3) $\sharp(B)$ (= *the number of irreducible components of B*) ≤ 3 *and the equality holds if and only if $\boldsymbol{P}^2 - B \cong \boldsymbol{C}^* \times \boldsymbol{C}^*$, where $\boldsymbol{C}^* = \boldsymbol{C} - \{0\}$.*

(4) *If B is an irreducible rational curve then B has unique singular point and the number of analytic branches of B at the singular point is equal to two.*

In [**15**], Tsunoda obtained the same result as Theorem 1.1 when B is irreducible.

As applications of Theorem 1.1, we study the fundamental groups and the topological Euler characteristics of the surfaces $\boldsymbol{P}^2 - B$ with $\bar{\kappa}(\boldsymbol{P}^2 - B) = 0$ in §5.

2000 *Mathematical Subject Classification*. Primary 14J26, Secondary 14H20.
Key Words and Phrases. Plane curves, logarithmic Kodaira dimension, logarithmic geometric genus.
The author is partially supported by JSPS Research Fellowships for Young Scientists and Grant-in-Aid for Scientific Research, the Ministry of Education, Science and Culture.

By a $(-n)$-curve $(n \geq 1)$ we mean a nonsingular complete rational curve with self-intersection number $(-n)$. A reduced effective divisor D is called an SNC-divisor (resp. an NC-divisor) if D has only simple normal crossings (resp. normal crossings). Let $f : X_1 \to X_2$ be a birational morphism between smooth surfaces X_1 and X_2 and let D_i $(i = 1, 2)$ be a divisor on X_i. We denote the direct image of D_1 on X_2 (resp. the total transform of D_2 on X_1, the proper transform of D_2 on X_1) by $f_*(D_1)$ (resp. $f^*(D_2)$, $f'(D_2)$). We refer to [**5**] for the definitions of the logarithmic Kodaira dimension $\bar{\kappa}$, the logarithmic geometric genus $\bar{p}_g$, the logarithmic n-genus $\bar{P}_n$ $(n \geq 1)$ and the logarithmic irregularity $\bar{q}$, etc.

The author would like to express his gratitude to Professor Masayoshi Miyanishi who gave the author valuable advice and encouragement during the preparation of the present article.

2. Preliminaries.

We recall some basic notions in the theory of peeling (cf. [**13**] and [**1**]). Let (X, B) be a pair of a nonsingular projective surface X and an SNC-divisor B on X. We call such a pair (X, B) an *SNC-pair*. A connected curve T consisting of irreducible components of B (a connected curve in B, for short) is a *twig* if the dual graph of T is a linear chain and T meets $B - T$ in a single point at one of the end components of T, the other end of T is called the *tip* of T. A connected curve R (resp. F) in B is a *rod* (resp. *fork*) if R (resp. F) is a connected component of B and the dual graph of R (resp. F) is a linear chain (resp. the dual graph of the exceptional curves of a minimal resolution of a non-cyclic quotient singularity). A connected curve E in B is *rational* (resp. *admissible*) if each irreducible component of E is rational (resp. if there are no (-1)-curves in $\mathrm{Supp}(E)$ and the intersection matrix of E is negative definite). An admissible rational twig T in B is *maximal* if T is not extended to an admissible rational twig with more irreducible components of B.

Let $\{T_\lambda\}$ (resp. $\{R_\mu\}$, $\{F_\nu\}$) be the set of all admissible rational maximal twigs (resp. all admissible rational rods, all admissible rational forks), where no irreducible components of T_λ's belong to R_μ's or F_ν's. Then there exists a unique decomposition of B as a sum of effective $\boldsymbol{Q}$-divisors $B = B^\sharp + \mathrm{Bk}(B)$ such that

i) $\mathrm{Supp}(\mathrm{Bk}(B)) = (\bigcup_\lambda T_\lambda) \cup (\bigcup_\mu R_\mu) \cup (\bigcup_\nu F_\nu)$,

ii) $(B^\sharp + K_X \cdot Z) = 0$ for every irreducible component Z of $\mathrm{Supp}(\mathrm{Bk}(B))$.

We call the divisor $\mathrm{Bk}(B)$ the *bark* of B and say that $B^\sharp + K_X$ is produced by the *peeling* of B.

Definition 2.1 (cf. [**13**, 1.11]). An SNC-pair (X, B) is *almost minimal* if, for every irreducible curve C on X, either $(B^\sharp + K_X \cdot C) \geq 0$ or the intersection matrix of $C + \mathrm{Bk}(B)$ is not negative definite.

We have the following result due to Miyanishi and Tsunoda [**13**].

Lemma 2.2 (cf. [**13**, Theorem 1.11]). *Let (X, B) be an SNC-pair. Then there exists a birational morphism $\mu : X \to W$ onto a nonsingular projective surface W such that the following four conditions* (i) $\sim$ (iv) *are satisfied*:

(i) *$C := \mu_*(B)$ is an SNC-divisor.*

(ii) *$\mu_* \operatorname{Bk}(B) \le \operatorname{Bk}(C)$ and $\mu_*(B^\sharp + K_X) \ge C^\sharp + K_W$.*

(iii) *$\bar{P}_n(X - B) = \bar{P}_n(W - C)$ for every integer $n \ge 1$. In particular, $\bar{\kappa}(X - B) = \bar{\kappa}(W - C)$.*

(iv) *The pair (W, C) is almost minimal.*

We call the pair (W, C) as in Lemma 2.2 an *almost minimal model* of (X, B).

The following result follows from [**1**, Lemma 6.20] and [**13**, Theorem 2.11 (1)]. Note that a rod (resp. a fork) is called a *club* (resp. an *abnormal club*) in [**1**].

Lemma 2.3. *Let (X, B) be an SNC-pair with $\bar{\kappa}(X - B) \ge 0$. Assume that any rational twig of B is admissible. If (X, B) is not almost minimal then there exists a (-1)-curve E, not contained in B, such that one of the following holds*:

(i) *$E \cap B = \varnothing$.*

(ii) *$(E \cdot B) = 1$ and E meets an irreducible component of* $\operatorname{Supp}(\operatorname{Bk}(B))$.

(iii) *$(E \cdot B) = 2$ and E meets two different connected components of B such that one of the connected components is a rational rod R_ν of B and E meets a tip of R_ν.*

Further, $\bar{P}_n(X - (B + E)) = \bar{P}_n(X - B)$ for any $n \ge 1$ and hence $\bar{\kappa}(X - (B + E)) = \bar{\kappa}(X - B)$.

Lemma 2.4. *Let (X, B) be an almost minimal SNC-pair with $\bar{\kappa}(X - B) = 0$ and $\bar{p}_g(X - B) = 1$. Assume that X is rational and B is connected. Then $B + K_X \sim 0$ and B is a nonsingular elliptic curve or a loop of nonsingular rational curves.*

Proof. See [**9**, Proposition 1.5 (1)] or [**21**, Reduction theorem]. □

Now we recall the construction of a strongly minimal model of a nonsingular affine surface with $\bar{\kappa} = 0$ (cf. [**9**, §**2**]). Let $S = \operatorname{Spec}(A)$ be a nonsingular affine surface with $\bar{\kappa}(S) = 0$ and let (X, B) be an SNC-pair with $X - B = S$. We call such a pair (X, B) an *SNC-completion of S*. Note that S is rational by [**9**, Theorem 1.6]. Let (W, C) be an almost minimal model of (X, B). By contracting (-1)-curves E with $(E \cdot C) \le 1$ successively, we obtain a birational morphism $\nu : W \to V$ such that $(F \cdot \nu_*(C)) > 1$ for any (-1)-curve F on V. Put $D := \nu_*(C)$ and $S' := V - D$. We call the surface S' a *strongly minimal model* of S. By [**9**, Lemmas 2.3 and 2.4 and Corollary 2.5], we have the following result.

Lemma 2.5. *With the same notation and the assumptions as above, the following assertions hold*:

(1) *S' is an affine open subset of S and $S - S'$ is an empty set or a disjoint union of the affine lines $\boldsymbol{A}^1$.*

(2) *D is an NC-divisor. Furthermore, if $\bar{p}_g(S) = 0$ then D becomes an SNC-divisor and the pair (V, D) is almost minimal.*

(3) *$\bar{P}_n(S') = \bar{P}_n(S)$ for any $n \ge 1$. In particular, $\bar{\kappa}(S') = \bar{\kappa}(S) = 0$.*

Definition 2.6. Let $S = \operatorname{Spec}(A)$ be a nonsingular affine surface with $\bar{\kappa}(S) = 0$ and let (X, B) be an SNC-completion of S. We call the pair (X, B) (resp. the surface S) to be *strongly minimal* if (X, B) is almost minimal and $(E \cdot B) > 1$ for any (-1)-curve E on X (resp. if there exists a strongly minimal model S' of S such that $S = S'$). Note that if S is strongly minimal and $\bar{p}_g(S) = 0$ then S has a strongly minimal SNC-completion by Lemma 2.5 (2).

Lemma 2.7. *Let* $S = \mathrm{Spec}(A)$ *be a nonsingular affine surface with* $\bar{\kappa}(S) = 0$ *and let* (X, B) *be an SNC-completion of S such that* $(B_i \cdot B - B_i) \geq 3$ *for any* (-1)*-curve* $B_i \subset B$*. If* (X, B) *is not strongly minimal then there exists a* (-1)*-curve E, not contained in B, such that* $(E \cdot B) = 1$ *and* $\bar{P}_n(X - (B + E)) = \bar{P}_n(X - B)$ *for any* $n \geq 1$.

Proof. If (X, B) is almost minimal then the assertion is clear by the definition of strongly minimality and Lemma 2.5 (3). Suppose that (X, B) is not almost minimal. Since $\bar{\kappa}(X - B) = 0$ and $(B_i \cdot B - B_i) \geq 3$ for any (-1)-curve $B_i \subset B$, we know that any rational twig of B is admissible by virtue of [**17**, Step (3) in the proof of Theorem 1.3]. Further, B is connected and S contains no complete curves since S is affine. Hence the assertion follows from Lemma 2.3. □

We state the classification of strongly minimal affine surfaces with $\bar{\kappa} = 0$. For more details, see [**9**].

Lemma 2.8 (cf. [**9**, Theorems 0.1, 4.5 and 5.4]). *Let S be a strongly minimal nonsingular affine surface with* $\bar{\kappa}(S) = 0$. *Then we have*:

(1) *S is one of the surfaces in Table* 1, *where* $m(S)$, $e(S)$ *and* $\pi_1(S)$ *are respectively*

Table 1

Type	$m(S)$	$\bar{q}(S)$	$e(S)$	$\pi_1(S)$
$*(9)$	1	0	3	$\boldsymbol{Z}/(3)$
$*(8)$	1	0	4	$\boldsymbol{Z}/(2)$
$O(8)$	1	0	3	$\boldsymbol{Z}/(2)$
$O(k+4, -k)$ $(k \geq 0)$	1	0	2	$\boldsymbol{Z}/(k+2)$
$O(4, 1)$	1	1	1	$\boldsymbol{Z}$
$O(2, 2)$	1	1	2	$\boldsymbol{Z}$
$O(1, 1, 1)$ $(\cong \boldsymbol{C}^* \times \boldsymbol{C}^*)$	1	2	0	$\boldsymbol{Z}^2$
$X[2]$	2	0	2	$\boldsymbol{Z}/(4)$
$H[-1, 0, -1]$	2	1	0	$\langle y, t \rangle / (yty^{-1}t)$
$H[0, 0]$	2	1	1	$\boldsymbol{Z}$
$H[k, -k]$ $(k \geq 1)$	2	0	1	$\boldsymbol{Z}/(4k)$
$Y\{3, 3, 3\}$	3	0	1	$\boldsymbol{Z}/(9)$
$Y\{2, 4, 4\}$	4	0	1	$\boldsymbol{Z}/(8)$
$Y\{2, 3, 6\}$	6	0	1	$\boldsymbol{Z}/(6)$

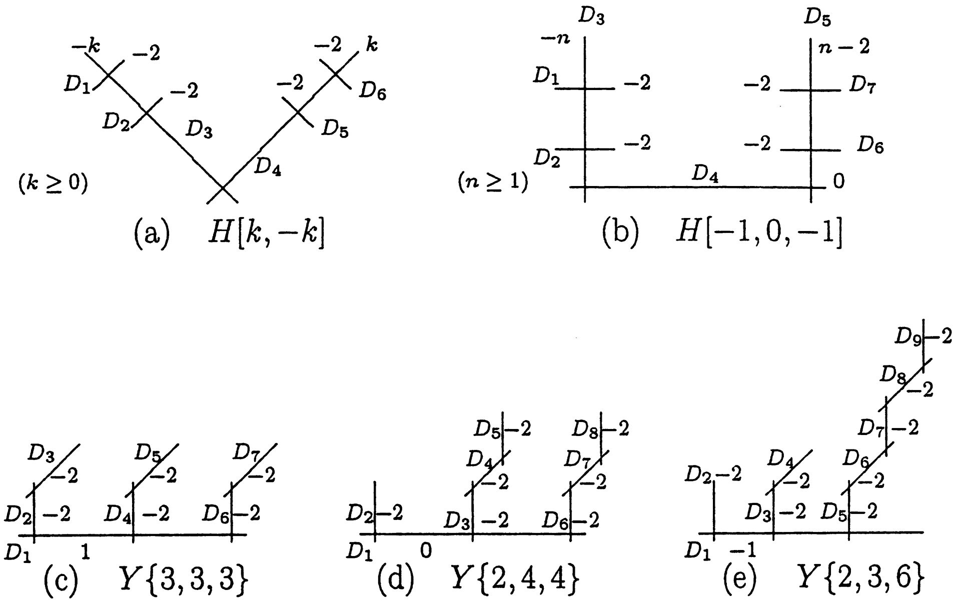

Figure 1

the least positive integer such that $\bar{P}_{m(S)}(S) > 0$, *the topological Euler characteristic of* S *and the fundamental group of* S.

(2) *Assume further that* $\bar{p}_g(S) = 0$ *and* $e(S) \leq 1$. *Let* (V, D) *be a strongly minimal SNC-completion of* S. *Then the configuration of* D *is one of* (a) $\sim$ (e) *in Figure* 1, *where each line represents a nonsingular rational curve and each number indicates the self-intersection number of the corresponding curve.*

Corollary 2.9. *Let* S *be a nonsingular affine surface with* $\bar{\kappa}(S) = 0$. *Then the following assertions hold*:

(1) $e(S) \geq 0$ *and the equality holds if and only if* S *is strongly minimal and of type* $O(1,1,1)$ *or* $H[-1,0,-1]$.

(2) *Assume that* $e(S) = \bar{p}_g(S) = 0$, *i.e.,* S *is of type* $H[-1,0,-1]$. *Let* (V, D) *be an SNC-completion of* S *such that* $(D_i \cdot D - D_i) \geq 3$ *for any* (-1)*-curve* $D_i \subset D$. *Then* (V, D) *is strongly minimal and the configuration of* D *is given as* (b) *in Figure* 1.

Proof. By Lemmas 2.5 (1), 2.7 and 2.8, the assertions are clear. $\square$

3. Proof of Theorem 1.1, part I.

In this section, we prove Theorem 1.1 when the curve B is reducible. We prove some lemmas to be used later.

Lemma 3.1. *Let* V *be a nonsingular projective surface with* $q(V) := h^1(V, \mathcal{O}_V) = 0$ *and let* D *be a non-zero reduced effective divisor on* V. *Then*

$$\bar{q}(V - D) \geq \sharp(D) - \rho(V),$$

where $\rho(V)$ *is the Picard number of* V. *Furthermore, the equality holds provided* $\rho(V) = 1$.

Proof. Let $D = \sum_i D_i$ be the irreducible decomposition of D. Since $q(V) = 0$, we get

$$\bar{q}(V - D) = \dim_{\boldsymbol{Q}} \operatorname{Ker}\left(\bigoplus_i \boldsymbol{Q}[D_i] \to \operatorname{Pic}(V) \otimes \boldsymbol{Q}\right)$$

by [**2**, Lemma 2]. Hence

$$\bar{q}(V - D) \geq \sharp(D) - \rho(V).$$

If $\rho(V) = 1$ then the natural map $\bigoplus_i \boldsymbol{Q}[D_i] \to \operatorname{Pic}(V) \otimes \boldsymbol{Q}$ is surjective. So $\bar{q}(V - D) = \sharp(D) - 1$. □

Lemma 3.2. *Let* S *be a nonsingular affine surface with* $\bar{\kappa}(S) = 0$. *Then* $\bar{q}(S) \leq 2$. *Moreover,* $\bar{q}(S) = 2$ *if and only if* $S \cong \boldsymbol{C}^* \times \boldsymbol{C}^*$.

Proof. By [**3**, Theorem II], the assertions are clear. See also [**7**, Theorem 2.8 and Corollary 2.9]. □

Now we shall prove Theorem 1.1 when B is reducible.

Lemma 3.3. *With the same notation as in Theorem* 1.1, $\sharp(B) \leq 3$ *and the equality holds if and only if* $\boldsymbol{P}^2 - B \cong \boldsymbol{C}^* \times \boldsymbol{C}^*$. *In particular, if* $\sharp(B) = 3$ *then* $\bar{p}_g(\boldsymbol{P}^2 - B) = 1$.

Proof. We note that $\bar{p}_g(\boldsymbol{C}^* \times \boldsymbol{C}^*) = 1$. Lemma 3.1 implies that $\bar{q}(\boldsymbol{P}^2 - B) = \sharp(B) - 1$. So, by Lemma 3.2, we know that $\sharp(B) \leq 3$ and the equality holds if and only if $\boldsymbol{P}^2 - B \cong \boldsymbol{C}^* \times \boldsymbol{C}^*$. □

Among the assertions of Theorem 1.1, (3) and (1) in the case $\sharp(B) \geq 3$ are verified. Next we consider the case $\sharp(B) = 2$.

Lemma 3.4. *With the same notation as in Theorem* 1.1, *assume that* $\sharp(B) = 2$. *Then* $\bar{p}_g(\boldsymbol{P}^2 - B) = 1$.

Proof. Put $S := \boldsymbol{P}^2 - B$. Note that $\bar{p}_g(S) \leq 1$ because $\bar{\kappa}(S) = 0$. Suppose to the contrary that $\bar{p}_g(S) = 0$. Let $B = B_1 + B_2$ be the irreducible decomposition of B. Let $\mu : W \to \boldsymbol{P}^2$ be a composite of blowing-ups such that $C := \mu^{-1}(B)$ becomes an SNC-divisor and that μ is the shortest among such birational morphisms. From now on, we call such a morphism μ a *minimal SNC-map for the pair* $(\boldsymbol{P}^2, B)$. Note that $W - C = S$. Since $\bar{p}_g(W - C) = \bar{p}_g(S) = 0$, each irreducible component of C is a nonsingular rational curve and the dual graph of C is a tree by [**11**, Lemma I.2.1.3]. So B_1 and B_2 are rational cuspidal curves and meet in only one point P. Hence $e(S) = e(\boldsymbol{P}^2) - e(B_1) - e(B_2 - \{P\}) = 3 - 2 - 1 = 0$. By Corollary 2.9 (1), S is of type $H[-1, 0, -1]$.

Let C_i $(i = 1, 2)$ be the proper transform of B_i on W. Assume that $(C_j \cdot C - C_j) \geq 3$ for any (-1)-curve $C_j \subset C$. Then, by Corollary 2.9 (2), the configuration of C is given as (b) in Figure 1. Since each component of $C - (C_1 + C_2)$ has negative self-intersection number, D_4 is one of $\{C_1, C_2\}$ and either $\{C_1, C_2\} \cap \{D_1, D_2, D_3\} = \emptyset$ or $\{C_1, C_2\} \cap \{D_5, D_6, D_7\} = \emptyset$. Then there exists $P_1 \in \boldsymbol{P}^2$ such that $D_i + D_{i+1} + D_{i+2} =$

$\mu^{-1}(P_1)$, where $i = 1$ or 5. This is a contradiction. So there exists a (-1)-curve H in $\mathrm{Supp}(C)$ such that $(H \cdot C - H) \leq 2$. By the minimality of μ, we know that $H = C_1$ or C_2. Assume that $H = C_1$. We claim that:

Claim. $(C_1 \cdot C - C_1) = 2$.

Proof. If $(C_1 \cdot C - C_1) = 1$ then $\bar{\kappa}(W - C) = \bar{\kappa}(W - (C - C_1)) = 0$. Since $W - (C - C_1) = \boldsymbol{P}^2 - B_2$, we have $\bar{\kappa}(\boldsymbol{P}^2 - B_2) = 0$. In the next section, we prove that if $D \subset \boldsymbol{P}^2$ be an irreducible rational cuspidal curve then $\bar{\kappa}(\boldsymbol{P}^2 - D) \neq 0$ (cf. Lemmas 4.1 and 4.2). So we have a contradiction.

The above claim implies that there exists a unique singular point $Q \in B_1$ other than P. Then, since Q is a cusp of B_1, there exists a unique decomposition of $\mu^{-1}(Q)$ as a sum of non-zero reduced effective divisors $\mu^{-1}(Q) = E + F + G$ such that the following three conditions are satisfied:

(i) F and G are connected.

(ii) E is a unique (-1)-curve in $\mu^{-1}(Q)$ and hence each component of $F + G$ has self-intersection number ≤ -2.

(iii) $(E \cdot F) = (E \cdot G) = (E \cdot C_1) = 1$.

The dual graph of C is given as in Figure 2, where we put $\tilde{C} := C - (C_1 + E + F + G)$. We have $(C_1 \cdot \tilde{C}) = 1$.

Let $\nu : W \to W'$ be a sequence of contractions of (-1)-curves and subsequently contractible curves in C, starting with the contraction of C_1, such that $C' := \nu_*(C)$ is an SNC-divisor and that the contraction of any (-1)-curve in C' makes the image of D' lose the simple normal crossing property (the SNC-property, for short). Then $(\nu_*(E)^2) \geq 0$ and the weighted dual graphs of $\nu_*(F)$ and $\nu_*(G)$ are the same as those of F and G. Further, $(C_i' \cdot C' - C_i') \geq 3$ for any (-1)-curve $C_i' \subset C'$ because the dual graph of C is a tree. Since $W' - C' = S$ is of type $H[-1, 0, -1]$, the configuration of C' is given as (b) in Figure 1 by Corollary 2.9 (2). Since $(\nu_*(E)^2) \geq 0$, $\nu_*(E) = D_4$ or D_5. If $\nu_*(E) = D_4$ then $\nu_*(C_1 + \tilde{C}) = 0$ and $D_5 + D_6 + D_7 = \nu_*(F)$ or $\nu_*(G)$. This is a contradiction because $\nu_*(F)$ and $\nu_*(G)$ contain no irreducible curves with self-intersection number ≥ -1. If $\nu_*(E) = D_5$ then $\nu_*(\tilde{C}) = D_1 + \cdots + D_4$ and F and G are irreducible (-2)-curves. This is also a contradiction because the intersection matrix of $E + F + G$ is then not negative definite. □

The assertion (2) of Theorem 1.1 follows from Lemma 3.5 below.

Lemma 3.5 (cf. [**10**, Lemma 4]). *Let $B \subset \boldsymbol{P}^2$ be a reduced curve. Assume that $\bar{\kappa}(\boldsymbol{P}^2 - B) \leq 1$ and B contains a non-rational curve. Then B is an irreducible nonsingular cubic curve.*

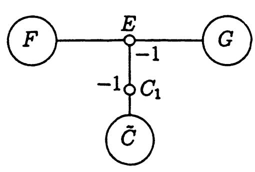

Figure 2

Proof. Since B contains a non-rational curve, $\deg B \geq 3$. By virtue of [**4**, Theorem 4], we have $\bar{\kappa}(\boldsymbol{P}^2 - B) = \kappa(B + K_{\boldsymbol{P}^2}, \boldsymbol{P}^2) = \kappa((\deg B - 3)\ell, \boldsymbol{P}^2)$, where ℓ is a line on $\boldsymbol{P}^2$ and $\kappa(B + K_{\boldsymbol{P}^2}, \boldsymbol{P}^2)$ denotes the $(B + K_{\boldsymbol{P}^2})$-dimension of $\boldsymbol{P}^2$ (cf. [**5**]). If $\deg B \geq 4$ then $\kappa((\deg B - 3)\ell, \boldsymbol{P}^2) = 2$. So $\deg B = 3$ and hence B is an irreducible nonsingular cubic curve. □

4. Proof of Theorem 1.1, part II.

In this section, we treat the case where B is irreducible. All results in this section except for Lemma 4.3 are stated in [**15**], where their proofs however are not given. For the sake of completeness, we give the proofs which use the classification theory of the affine surfaces with $\bar{\kappa} = 0$ (cf. §2). In [**14**], Orevkov independently gave the proofs of Lemmas 4.1 and 4.2. Our proofs are almost the same as Orevkov's.

Assume that $\bar{p}_g(\boldsymbol{P}^2 - B) = 0$ and B is irreducible. Then, by using the same argument as in the proof of Lemma 3.4, we know that B is a rational cuspidal curve. If B is nonsingular, B is a line or a conic and $\bar{\kappa}(\boldsymbol{P}^2 - B) = -\infty$. So $\sharp\mathrm{Sing}(B) \geq 1$. By [**18**, Theorem (II)], $\sharp\mathrm{Sing}(B) \leq 2$. Here we note that $e(\boldsymbol{P}^2 - B) = e(\boldsymbol{P}^2) - e(B) = 3 - 2 = 1$.

We shall consider the cases $\sharp\mathrm{Sing}(B) = 1$ and $\sharp\mathrm{Sing}(B) = 2$ separately.

Lemma 4.1. *If $B \subset \boldsymbol{P}^2$ is a rational cuspidal curve with $\sharp\mathrm{Sing}(B) = 1$. Then $\bar{\kappa}(\boldsymbol{P}^2 - B) \neq 0$.*

Proof. Suppose that $\bar{\kappa}(\boldsymbol{P}^2 - B) = 0$. Let $\mu : W \to \boldsymbol{P}^2$ be a minimal SNC-map for $(\boldsymbol{P}^2, B)$ (cf. the proof of Lemma 3.4) and let C_1 be the proper transform of B on W. Let P be the unique singular point of B. Then there exists a unique decomposition of $\mu^{-1}(P)$ as a sum of nonzero reduced effective divisors $\mu^{-1}(P) = E + F + G$ such that the conditions (i) $\sim$ (iii) for $\mu^{-1}(Q)$ as in the proof of Lemma 3.4 hold. The dual graph of $C := \mu^{-1}(B) = C_1 + E + F + G$ is given as in Figure 3.

Since $(C_1 \cdot C + K_W) = -1 < 0$ and $\bar{\kappa}(W - C) = 0$, we know that $(C_1)^2 < 0$ by the theory of Zariski decomposition (cf. [**17**, the proof of Theorem 1.3]). If $(C_1)^2 = -1$ then $\bar{\kappa}(W - C) = \bar{\kappa}(W - (C - C_1)) = 0$ because $(C_1 \cdot C - C_1) = 1$. Since $C - C_1 = \mu^{-1}(P)$ can be contracted to a smooth point, we have $\bar{\kappa}(W - (C - C_1)) = -\infty$, which is a contradiction. So $(C_1)^2 \leq -2$.

Suppose that (W, C) is strongly minimal (cf. Definition 2.6). Then, in view of $e(W - C) = 1$, we know that the configuration of C is given as one of (a), (c), (d) and (e) in Figure 1. Since C contains a unique (-1)-curve E, the configuration of C is either (a) or (e). If the case (a) occurs then C contains a curve with non-negative self-intersection number, which is a contradiction. If the case (e) occurs then $E + F + G = D_1 + D_3 + D_4 + \cdots + D_9$ since C_1 is irreducible. This is also a contradiction because

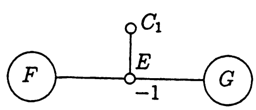

Figure 3

the intersection matrix of $D_1 + D_3 + D_4 + \cdots + D_9$ is then not negative definite. Hence there exists a (-1)-curve H, not contained in C, such that $(H \cdot C) = 1$ by Lemma 2.7.

Let $\nu : W \to W'$ be a sequence of contractions of (-1)-curves and subsequently contractible curves in $C + H$, starting with the contraction of H, such that $C' := \nu_*(C)$ is an SNC-divisor and that the contraction of any (-1)-curve in $\mathrm{Supp}(C')$ makes the image of C' lose the SNC-property. Then $(C_i' \cdot C' - C_i') \geq 3$ for any (-1)-curve $C_i' \subset C'$ because the dual graph of $C + H$ is a tree. Since $e(W' - C') = e(W - C) - 1 = 0$ and $\bar{p}_g(W' - C') = 0$ by Lemma 2.7, $W' - C'$ is of type $H[-1, 0, -1]$ and (W', C') is strongly minimal by Corollary 2.9 (2). The configuration of C' is then given as (b) in Figure 1. We note that $Q := \nu(H)$ is a unique fundamental point of ν. Since each component of C has negative self-intersection number, $Q \in D_4$. Then either $\nu'(D_1 + D_2 + D_3)$ or $\nu'(D_5 + D_6 + D_7)$ is contained in F or G. We consider the case $\nu'(D_1 + D_2 + D_3) \subset F$ or G. The case $\nu'(D_5 + D_6 + D_7) \subset F$ or G can be treated similarly. Since $Q \in D_4$, $\nu'(D_i)$ $(i = 1, 2)$ is a (-2)-curve and a terminal component of C. We can factor the map $\mu = \mu_1 \circ \mu_2 : W \to \boldsymbol{P}^2$ so that $\mu_{2*}(\nu'(D_3))$ is a unique (-1)-curve in $\mathrm{Supp}\, \mu_{2*}(E + F + G)$. Then, since $\nu'(D_i)$ $(i = 1, 2)$ is a (-2)-curve and a terminal component of C, $\mu_{2*}(\nu'(D_i))$ $(i = 1, 2)$ remains as a (-2)-curve. This is a contradiction because the intersection matrix of $\mu_{2*}(\nu'(D_1 + D_2 + D_3)) \subset \mu_1^{-1}(P)$ is then not negative definite. $\square$

Lemma 4.2. *If* $B \subset \boldsymbol{P}^2$ *be a rational cuspidal curve with* $\sharp \mathrm{Sing}(B) = 2$ *then* $\bar{\kappa}(\boldsymbol{P}^2 - B) \geq 1$.

Proof. By [**18**, Theorem (IV)], $\bar{\kappa}(\boldsymbol{P}^2 - B) \geq 0$. Suppose that $\bar{\kappa}(\boldsymbol{P}^2 - B) = 0$. Let P_1 and P_2 be two singular points of B. Let $\mu : W \to \boldsymbol{P}^2$ be a minimal SNC-map for $(\boldsymbol{P}^2, B)$. Then there exists a unique decomposition of $\mu^{-1}(P_i)$ $(i = 1, 2)$ as a sum of non-zero reduced effective divisors $\mu^{-1}(P_i) = E_i + F_i + G_i$ such that the conditions (i) $\sim$ (iii) for $\mu^{-1}(Q)$ as in the proof of Lemma 3.4 hold, where we consider respectively E, F and G as E_i, F_i and G_i. Let C_1 be the proper transform of B on W and $C := \mu^{-1}(B) = C_1 + \sum_{i=1}^{2}(E_i + F_i + G_i)$. The dual graph of C is given as in Figure 4.

We consider the following two cases separately.

Case 1: $(C_1)^2 \neq -1$. Then all (-1)-curves in C are exhausted by E_1 and E_2 and $(E_i \cdot C - E_i) = 3$ $(i = 1, 2)$. If (W, C) is strongly minimal then it follows from $e(W - C) = 1$ that the configuration of C is given as one of (a), (c), (d) and (e) in

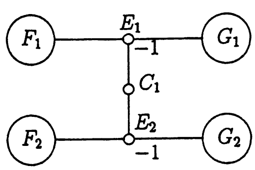

Figure 4

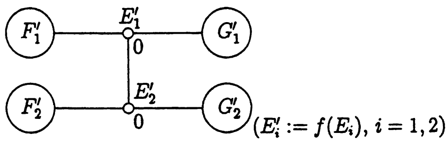

Figure 5

Figure 1. This is, however, a contradiction. So there exists a (-1)-curve H, not contained in C, such that $(H \cdot C) = 1$ by Lemma 2.7. Let $\nu : W \to W'$ be a sequence of contractions of (-1)-curves and subsequently contractible curves in $C + H$, starting with the contraction of H, such that $C' := \nu_*(C)$ is an SNC-divisor and that the contraction of any (-1)-curve in $\mathrm{Supp}(C')$ makes the image of C' lose the SNC-property. We know that $Q := \nu(H)$ is a unique fundamental point of ν and the configuration of C' is given as (b) in Figure 1 by the same argument as in the proof of Lemma 4.1.

Since $(H \cdot C) = 1$, we may assume that $(H \cdot E_1 + F_1 + G_1) = 0$. Then $\nu_*(E_1)^2 \geq -1$ and the dual graphs of $\nu_*(F_1)$ and $\nu_*(G_1)$ are the same as those of F_1 and G_1. So $\nu_*(E_1) = D_4$ or D_5. Since each component of $\nu_*(F_1)$ and $\nu_*(G_1)$ has self-intersection number ≤ -2, $\nu_*(E_1) = D_5$. Hence F_1 and G_1 are (-2)-curves. This contradicts that the intersection matrix of $E_1 + F_1 + G_1$ is negative definite.

Case 2: $(C_1)^2 = -1$. Let $f : W \to W'$ be the contraction of C_1 and put $C' := f_*(C)$. The dual graph of C' is given as in Figure 5, where the dual graphs of $F_i' := f_*(F_i)$ and $G_i' := f_*(G_i)$ $(i = 1, 2)$ are the same as those of F_i and G_i.

The divisor C' contains no (-1)-curves. If (W', C') is strongly minimal then the configuration of C' must be (a) in Figure 1. Then $k = 0$ in Figure 1 (a) and F_1, F_2, G_1 and G_2 are (-2)-curves. This is a contradiction because the intersection matrix of $E_i + F_i + G_i$ $(i = 1, 2)$ is then not negative definite. So there exists a (-1)-curve H', not contained in C', such that $(H' \cdot C') = 1$ by Lemma 2.7. Let $\nu : W' \to W''$ be a sequence of contractions of (-1)-curves and subsequently contractible curves in $C' + H'$, starting with the contraction of H', such that $C'' := \nu_*(C')$ is an SNC-divisor and that the contraction of any (-1)-curve in $\mathrm{Supp}(C'')$ makes the image of C'' lose the SNC-property. By the same argument as in the proof of Lemma 4.1, we know that $Q := \nu(H')$ is a unique fundamental point of ν and the configuration of C'' is given as (b) in Figure 1.

We may assume that $(H' \cdot C') = (H' \cdot E_2' + G_2') = 1$. Then $\nu_*(E_2')^2 \geq 0$, $(\nu_*(E_1') \cdot C'' - \nu_*(E_1')) = 3$ and the dual graphs of $\nu_*(F_1')$ and $\nu_*(G_1')$ are the same as those of F_1 and G_1. So $\nu_*(E_1') = D_5$ and F_1 and G_1 are (-2)-curves. This is a contradiction. □

The proof of (1) of Theorem 1.1 is thus completed by Lemmas 3.3, 3.4, 4.1 and 4.2.

Proof of (4) of Theorem 1.1. Let $B \subset \boldsymbol{P}^2$ be an irreducible rational curve with $\bar{\kappa}(\boldsymbol{P}^2 - B) = 0$. Then, by Lemmas 4.1 and 4.2 and [**18**, Theorems (II) and (III)], B has a unique singular point, say P, and P is not a cusp. We denote the number of analytic branches of B at P by $r_P(B)$. Then $r_P(B) = 2$ follows from Lemma 4.3 below.

LEMMA 4.3. *Let D be an irreducible rational curve on a nonsingular projective rational surface V with $\bar{\kappa}(V - D) = 0$. Let $s(D)$ be the number of singular points on D which are not cusps. Then $s(D) \le 1$ and if $s(D) = 1$ then $r_P(D) = 2$, where P is the singular point on D which is not a cusp.*

PROOF. Assume that $s(D) \ge 1$. Let $f : \tilde{V} \to V$ be a minimal SNC-map for (V, D) and let $\tilde{D} := f^{-1}(D)$. Then $\tilde{D}$ contains loops of nonsingular rational curves. So $\bar{p}_g(\tilde{V} - \tilde{D}) = \bar{p}_g(V - D) = 1$. Let (W, C) be an almost minimal model of $(\tilde{V}, \tilde{D})$. Lemma 2.4 implies that C is a loop of nonsingular rational curves. The dual graph of $\tilde{D}$ then contains only one loop by the construction of almost minimal models (cf. [**13**], etc.). Hence the assertions hold. $\square$

The proof of Theorem 1.1 is thus completed.

5. $\pi_1(\boldsymbol{P}^2 - B)$ and $e(\boldsymbol{P}^2 - B)$.

In this section, we study the fundamental groups $\pi_1(\boldsymbol{P}^2 - B)$ and the topological Euler characteristics $e(\boldsymbol{P}^2 - B)$ of the surfaces $\boldsymbol{P}^2 - B$ with $\bar{\kappa}(\boldsymbol{P}^2 - B) = 0$ by using Theorem 1.1.

PROPOSITION 5.1. *Let $B \subset \boldsymbol{P}^2$ be a reduced curve with $\bar{\kappa}(\boldsymbol{P}^2 - B) = 0$. Then $\pi_1(\boldsymbol{P}^2 - B)$ is abelian. In particular, if B is irreducible then $\pi_1(\boldsymbol{P}^2 - B) = \boldsymbol{Z}/(\deg B)\boldsymbol{Z}$.*

PROOF. Put $S := \boldsymbol{P}^2 - B$. Let S' be a strongly minimal model of S. Then, by Theorem 1.1 (1) and Lemma 2.8 (1), $\pi_1(S')$ is an abelian group. So $\pi_1(S)$ is abelian since S' is a Zariski open subset of S. If B is irreducible then $H_1(S; \boldsymbol{Z}) \cong \boldsymbol{Z}/(\deg B)\boldsymbol{Z}$ by the duality. $\square$

PROPOSITION 5.2. *Let $B \subset \boldsymbol{P}^2$ be a reduced curve with $\bar{\kappa}(\boldsymbol{P}^2 - B) = 0$. Then*

$$e(\boldsymbol{P}^2 - B) = \begin{cases} 3, & \text{if } B \text{ is a nonsingular cubic curve} \\ 3 - \sharp(B), & \text{otherwise.} \end{cases}$$

PROOF. By Theorem 1.1, the assertion holds unless $\sharp(B) = 2$. So we consider the case $\sharp(B) = 2$. Put $S := \boldsymbol{P}^2 - B$.

Assume that S is strongly minimal. Since $\bar{q}(S) = \sharp(B) - 1 = 1$ by Lemma 3.1, S is of type $O(4, 1)$ or $O(2, 2)$ (cf. Table 1). If the latter case occurs then $S \cong \boldsymbol{P}^1 \times \boldsymbol{P}^1 - (C_1 + C_2)$, where C_i $(i = 1, 2)$ is a curve of bidegree $(1, 1)$ and $C_1 + C_2$ is an SNC-divisor (cf. [**9**, Theorem 3.1]). This is a contradiction because $\mathrm{Pic}(S)$ is then not a finite group. Hence we know that $e(S) = 1$. Assume that S is not strongly minimal. Let S' be a strongly minimal model of S. Since $S - S'$ consists of disjoint r affine lines $\boldsymbol{A}^1$ $(r \ge 1)$ by Lemma 2.5 (1), we have $e(S) = e(S') + r$. Put $B' := \boldsymbol{P}^2 - S'$. Then B' is purely of codimension one. Since $\sharp(B') = \sharp(B) + r = 2 + r$, we have $S' \cong \boldsymbol{C}^* \times \boldsymbol{C}^*$ and $r = 1$ by Theorem 1.1 (3). Hence $e(S) = e(S') + r = 1$. $\square$

6. The case B is irreducible.

Let $B \subset \boldsymbol{P}^2$ be an irreducible curve with $\bar{\kappa}(\boldsymbol{P}^2 - B) = 0$. Throughout this section, we assume that B is not a nonsingular cubic curve. Theorem 1.1 (4) implies that B is a rational curve with unique singular point P and $r_P(B) = 2$. Let B_1 and B_2 be two analytic branches of B at P. Then we have the following three cases:

Case (I): P is a smooth point of B_1 and B_2.

Case (II): P is a smooth point of either B_1 or B_2, but not for both.

Case (III): P is a singular point of B_1 and B_2.

We call the curve B to be of type (I) (resp. (II), (III)) if the case (I) (resp. (II), (III)) occurs.

We consider the case (I).

Proposition 6.1. *Suppose that B is of type* (I). *Then B is projectively equivalent to one of the curves defined by the following polynomials, where (X, Y, Z) denotes the system of homogeneous coordinates in $\boldsymbol{P}^2$ and $d = \deg B$.*

d	defining equation
3	$XYZ - X^3 - Y^3$
4	$(YZ - X^2)^2 + tX^2Y^2 + XY^3,\ t \in \boldsymbol{C} - \{0\}$
5	$(YZ - X^2)(YZ^2 - X^2Z + tY^2Z - tX^2Y + 2XY^2) + Y^5,\ t \in \boldsymbol{C} - \{0\}$

Conversely, if C_t is a curve whose defining equation is one of the above list with $\deg C_t = 4$ *or* 5 *then* $\bar{\kappa}(\boldsymbol{P}^2 - C_t) = 0$. *Moreover, C_t and C_s are projectively equivalent if and only if $t^3 = s^3$, i.e., t^3 is the projective invariant.*

Proof. Since B_1 and B_2 are smooth at P, the multiplicity of B at P is equal to two. So the assertions follow from [**20**, Propositions 1 and 3] (or [**19**]). $\square$

We give examples of the cases (II) and (III). We denote by $\boldsymbol{F}_a$, M_a and ℓ a Hirzebruch surface of degree a, the minimal section of $\boldsymbol{F}_a$ and a general fiber of the ruling on $\boldsymbol{F}_a$, respectively.

Example 1. Let C_0, C_1 and C_2 be three irreducible curves on $\boldsymbol{F}_a$ $(a \geq 3)$ such that $C_0 \sim M_a + (a+1)\ell$ (the relation $\sim$ represents the linear equivalence of divisors), $C_1 = M_a$, $C_2 \sim \ell$ and $C_0 + C_1 + C_2$ is an SNC-divisor. See Figure 6-(i). Let $\mu : V \to \boldsymbol{F}_a$ be the composite of $(a-1)$-times blowing-ups such that the configuration of $C' := \mu^{-1}(C_0 + C_1 + C_2)$ is shown as in Figure 6-(ii), where C_i' $(i = 0, 1, 2)$ is the proper

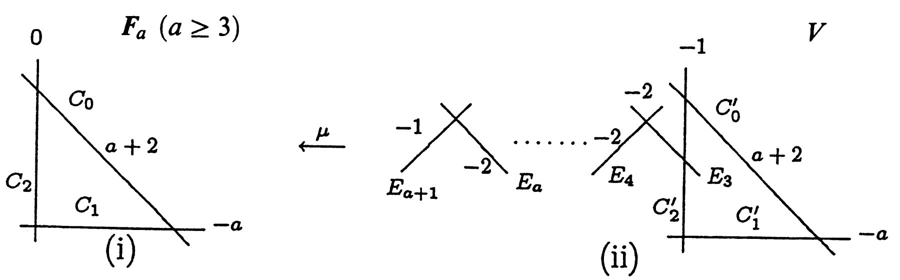

Figure 6

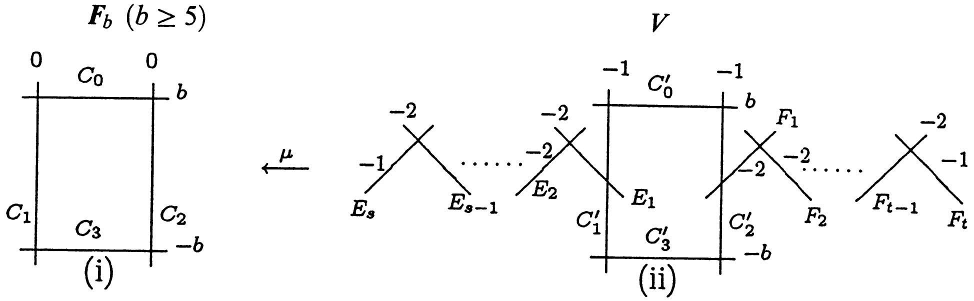

Figure 7

transform of C_i. Then we obtain the birational morphism $\nu : V \to \boldsymbol{P}^2$ which is the contraction of the curve $C' - (C_0' + E_{a+1})$ in the order $C_2', E_3, \dots, E_a, C_1'$. Put $B := \nu(C_0')$. We know that $\deg B = a + 1 \geq 4$. We have $\bar{\kappa}(V - C') = \bar{\kappa}(\boldsymbol{F}_a - (C_0 + C_1 + C_2)) = 0$ because $C_0 + C_1 + C_2 + K_{\boldsymbol{F}_a} \sim 0$. Since E_{a+1} is a (-1)-curve and $(E_{a+1} \cdot C' - E_{a+1}) = 1$, $\bar{\kappa}(V - (C' - E_{a+1})) = \bar{\kappa}(V - C')$. So $\bar{\kappa}(\boldsymbol{P}^2 - B) = \bar{\kappa}(V' - (C' - E_{a+1})) = 0$. The curve B is of type (II).

Example 2. Let b, s and t be three integers such that $b \geq 5$, $s, t \geq 2$ and $s + t = b - 1$. Let $C_0, \dots, C_3$ be four irreducible curves on $\boldsymbol{F}_b$ such that $C_0 \sim M_b + b\ell$, $C_1 \sim C_2 \sim \ell$, $C_3 = M_b$ and $C_0 + \cdots + C_3$ is an SNC-divisor. See Figure 7-(i). Let $\mu : V \to \boldsymbol{F}_b$ be the composite of $(s + t)$-times blowing-ups such that the configuration of $C' := \mu^{-1}(C_0 + \cdots + C_3)$ is shown as in Figure 7-(ii), where C_i' $(i = 0, \dots, 3)$ is the proper transform of C_i. Then we obtain the birational morphism $\nu : V \to \boldsymbol{P}^2$ which is the contraction of the curve $C' - (C_0' + E_s + F_t)$ in the order $C_1', E_1, \dots, E_{s-1}, C_2', F_1, \dots, F_{t-1}, C_3'$. Put $B := \nu(C_0')$. We know that $\deg B = b \geq 5$. Since $C_0 + \cdots + C_3 + K_{\boldsymbol{F}_b} \sim 0$, $\bar{\kappa}(\boldsymbol{P}^2 - B) = 0$ (cf. Example 1). The curve B is of type (III).

By the above examples, we have the following result.

Proposition 6.2. *For any integer $n \geq 4$ (resp. ≥ 5), there exists an irreducible rational curve $B \subset \boldsymbol{P}^2$ of degree n such that $\bar{\kappa}(\boldsymbol{P}^2 - B) = 0$ and B is of type* (II) *(resp.* (III)).

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