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Open surfaces of logarithmic Kodaira dimension zero in arbitrary characteristic

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Abstract. In this article we study nonsingular rational open surfaces of logarithmic Kodaira dimension zero with connected boundaries at infinity defined over an algebraically closed field of arbitrary characteristic. We establish a classification theory of nonsingular affine surfaces of logarithmic Kodaira dimension zero and give a characterization of $\boldsymbol{A}_*^1 \times \boldsymbol{A}_*^1$ in arbitrary characteristic.

0. Introduction.

Let k be an algebraically closed field. Let S be a nonsingular (not necessarily complete) algebraic surface defined over k. Let (X, B) be a pair of a nonsingular projective surface X and a reduced simple normal crossing divisor B on X such that $S = X - B$. We call such a pair an *SNC-completion* of S. Since $\dim S = 2$, SNC-completions of S exist. We say that S has a connected boundary at infinity if B is connected. Note that if S is a nonsingular affine surface then S has a connected boundary at infinity.

In [**13**], when $\mathrm{char}(k) = 0$, the author established a classification theory of nonsingular rational open surfaces of logarithmic Kodaira dimension zero with connected boundaries at infinity, which gives generalizations of Fujita's results concerning the classification of nonsingular affine surfaces of $\bar{\kappa} = 0$ with finite Picard groups (cf. [**2**, §8]). We can expect that the classification theory of such surfaces works also in the case $\mathrm{char}(k) > 0$ because the minimal model theory of open algebraic surfaces (cf. [**15**]) and the two-dimensional log abundance theorem (cf. [**10**], [**12**] and [**3**]) are valid in the case $\mathrm{char}(k) > 0$.

In the present article, we attempt to establish a classification theory of nonsingular rational open surfaces of logarithmic Kodaira dimension zero with connected boundaries at infinity defined over an algebraically closed field of arbitrary characteristic. In §1, following [**13**, §§1 and 2], we construct an almost minimal

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model (W, C) and a strongly minimal model (V, D) of a nonsingular rational open surface S of $\bar{\kappa}(S) = 0$ with a connected boundary at infinity. Further, we prove that a normal affine surface of logarithmic Kodaira dimension zero is rational (Theorem 1.10), which is a slight generalization of [**13**, Theorem 1.6]. In §2, by improving the arguments as in [**13**, §4], we classify the strongly minimal nonsingular rational open surfaces of $\bar{\kappa} = 0$ with connected boundaries at infinity (cf. Theorems 2.8, 2.10, 2.16 and 2.17). By using the results in §§1 and 2, we prove the following characterization of $\boldsymbol{A}_*^1 \times \boldsymbol{A}_*^1$ in §3.

Theorem 3.1. *Let* $S = \mathrm{Spec}(A)$ *be a normal affine surface defined over an algebraically closed field* k *of arbitrary characteristic. Assume that* $\bar{\kappa}(S) = 0$. *Then* $\mathrm{rank}_{\boldsymbol{Z}} A^*/k^* \leq 2$ *and the equality holds if and only if* $S \cong \boldsymbol{A}_*^1 \times \boldsymbol{A}_*^1$, *where* $\boldsymbol{A}_*^1 := \boldsymbol{A}_k^1 - \{0\}$.

Remark 0.1. (1) For a ring R we denote its multiplicative group consisting of invertible elements by R^*.

(2) For any algebraic variety X defined over k, the group $\Gamma(X, \mathcal{O}_X)^*/k^*$ is a finitely generated, free abelian group by Nagata's imbedding theorem (cf. [**17**]).

When $\mathrm{char}(k) = 0$, Iitaka [**6**, Theorem II] obtained a characterization of $\boldsymbol{A}_*^1 \times \boldsymbol{A}_*^1$ as follows: a normal affine surface S is isomorphic to $\boldsymbol{A}_*^1 \times \boldsymbol{A}_*^1$ if and only if $\bar{\kappa}(S) = 0$ and $\bar{q}(S) = 2$ (for definitions of $\bar{\kappa}(S)$ and $\bar{q}(S)$, see Definition 1.9). The result [**6**, Theorem II] was generalized by Kawamata [**11**] in any dimension. Iitaka's proof used Kawamata's addition formula on logarithmic Kodaira dimension for a fibered variety (cf. [**9**]), which is not yet proved in the case $\mathrm{char}(k) > 0$.

The terminology is the same as the one in [**15**] and [**13**]. By a $(-n)$-curve $(n \geq 1)$ we mean a nonsingular complete rational curve with self-intersection number $-n$. A reduced effective divisor D is called an NC-divisor (an SNC-divisor, resp.) if D has only normal crossings (simple normal crossings, resp.).

Let V be a nonsingular projective surface and let D, D_1 and D_2 be divisors on V. We employ the following notation. For the definitions of $\bar{\kappa}$, $\bar{p}_g$, $\bar{P}_m$ and $\bar{q}$, see [**7**] (see also [**8**] for the definitions in the case $\mathrm{char}(k) > 0$).

K_V: the canonical divisor on V.
$\bar{\kappa}(S)$: the logarithmic Kodaira dimension of a non-complete surface S.
$\bar{p}_g(S)$ (or $\bar{P}_1(S)$): the logarithmic geometric genus of S.
$\bar{P}_m(S)$ $(m \geq 2)$: the logarithmic m-genus of S.
$\bar{q}(S)$: the logarithmic irregularity of S.
$\rho(V)$: Picard number of V.
$\boldsymbol{F}_n$ $(n \geq 0)$: Hirzebruch surface of degree n.
M_n $(n \geq 0)$: the minimal section of $\boldsymbol{F}_n$.
$\overline{M}_n$ $(n \geq 0)$: a section of the ruling on $\boldsymbol{F}_n$ with $(\overline{M}_n \cdot M_n) = 0$.

$\sharp D$: the number of all irreducible components in $\mathrm{Supp}(D)$.
$f^*(D)$: total transform of D.
$f_*(D)$: direct image of D.
$f'(D)$: proper transform of D.
$[D^\sharp]$: the integral part of a $\boldsymbol{Q}$-divisor $D^\sharp$.
$D_1 \sim D_2$: D_1 and D_2 are linearly equivalent.
$D_1 \equiv D_2$: D_1 and D_2 are numerically equivalent.

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1. Preliminary results.

In this section we construct an almost minimal model and a strongly minimal model of a nonsingular rational open surface of $\bar{\kappa} = 0$ with a connected boundary at infinity. We will know that the results in [**13**, §§1 and 2] are valid in the case $\mathrm{char}(k) > 0$ by virtue of the two-dimensional log abundance theorem due to Fujita [**3**].

We recall some basic notions in the theory of peeling (cf. [**15**, Chapter 1]). Let (X, B) be a pair of a nonsingular projective surface X and an SNC-divisor B. We call such a pair (X, B) an *SNC-pair*. A connected curve T consisting of irreducible components of B (a connected curve in B, for short) is a *twig* if the dual graph of T is a linear chain and T meets $B - T$ in a single point at one of the end components of T, the other end of T is called the *tip* of T. A connected curve R (resp. F) in B is a *rod* (resp. *fork*) if R (resp. F) is a connected component of B and the dual graph of R (resp. F) is a linear chain (resp. the dual graph of the exceptional curves of a minimal resolution of a non-cyclic quotient singularity). A connected curve E in B is *rational* (resp. *admissible*) if each irreducible component of E is rational (resp. if there are no (-1)-curves in $\mathrm{Supp}(E)$ and the intersection matrix of E is negative definite). An admissible rational twig T in B is *maximal* if T is not extended to an admissible rational twig with more irreducible components of B.

Let $\{T_\lambda\}$ (resp. $\{R_\mu\}$, $\{F_\nu\}$) be the set of all admissible rational maximal twigs (resp. all admissible rational rods, all admissible rational forks), where no irreducible components of T_λ's belong to R_μ's or F_ν's. Then there exists a unique decomposition of B as a sum of effective $\boldsymbol{Q}$-divisors $B = B^\sharp + \mathrm{Bk}(B)$ such that the following two conditions i) and ii) are satisfied:

i) $\mathrm{Supp}(\mathrm{Bk}(B)) = \left(\bigcup_\lambda T_\lambda\right) \cup \left(\bigcup_\mu R_\mu\right) \cup \left(\bigcup_\nu F_\nu\right)$.

ii) $(B^\sharp + K_X \cdot Z) = 0$ for every irreducible component Z of $\mathrm{Supp}(\mathrm{Bk}(B))$.

We call the divisor $\mathrm{Bk}(B)$ the *bark* of B and say that $B^\sharp + K_X$ is produced by the *peeling* of B.

We define the almost minimality of an SNC-pair.

Definition 1.1 (cf. [**15**, §1.11]). An SNC-pair (X, B) is *almost minimal* if, for every irreducible curve C on X, either $(B^\sharp + K_X \cdot C) \geq 0$ or the intersection matrix of $C + \mathrm{Bk}(B)$ is not negative definite and $(B^\sharp + K_X \cdot C) < 0$.

Lemma 1.2 (cf. [**15**, Theorem 1.11]). *Let (X, B) be an SNC-pair. Then there exists a birational morphism $\mu : X \to W$ onto a nonsingular projective surface W such that the following four conditions* (i)$\sim$(iv) *are satisfied*:

(i) *$C := \mu_*(B)$ is an SNC-divisor.*

(ii) *$\mu_* \mathrm{Bk}(B) \leq \mathrm{Bk}(C)$ and $\mu_*(B^\sharp + K_X) \geq C^\sharp + K_W$.*

(iii) *$\bar{P}_n(X - B) = \bar{P}_n(W - C)$ for every integer $n \geq 1$. In particular, $\bar{\kappa}(X - B) = \bar{\kappa}(W - C)$.*

(iv) *The pair (W, C) is almost minimal.*

We call the pair (W, C) an almost minimal model *of (X, B).*

We note the following result.

Lemma 1.3. *Let (X, B) be an SNC-pair. Assume that (X, B) is almost minimal and $\bar{\kappa}(X - B) = 0$. Then $n(B^\sharp + K_X) \sim 0$ for some integer $n > 0$. In particular, $B^\sharp + K_X \equiv 0$.*

Proof. By the assumptions and [**15**, Theorem 2.11], we know that $B^\sharp + K_X$ is nef. Then $B + K_X \equiv (B^\sharp + K_X) + \mathrm{Bk}(B)$ gives rise to the Zariski-Fujita decomposition of $B + K_X$ (cf. [**14**, Chapter I]), where $B^\sharp + K_X$ is the nef part. Hence the assertion follows from the two-dimensional log abundance theorem [**3**, Theorem (5.12)]. □

Now, let S be a nonsingular open surface with a connected boundary at infinity and let (X, B) be an SNC-completion of S. Let (W, C) be an almost minimal model of (X, B). We call the surface $W - C$ an *almost minimal model* of $S = X - B$.

Lemma 1.4. *With the same notation as above, the following assertions hold*:
(1) *C is connected.*
(2) *If S is affine then $W - C$ is an affine open subset of S.*

Proof. By using the same argument as in the proof of [**13**, Lemma 1.4], we obtain the assertions. □

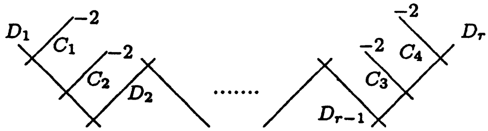

Figure 1.

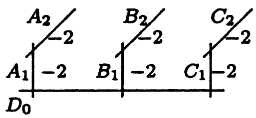

Figure 2.

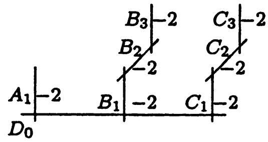

Figure 3.

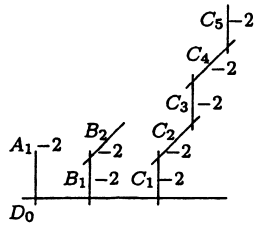

Figure 4.

Lemma 1.5 (cf. [**13**, Proposition 1.5]). *Let S be a nonsingular rational open surface of $\bar{\kappa}(S)=0$ with a connected boundary at infinity and let (W, C) be an almost minimal model of an SNC-completion of S. Then the following assertions hold*:

(1) *If $\bar{p}_g(S) \geq 1$ then $\bar{p}_g(S) = 1$, $C + K_W \sim 0$ and C is a nonsingular elliptic curve or a loop of nonsingular rational curves.*

(2) *If $\bar{p}_g(S) = 0$ and $\bar{P}_2(S) \geq 1$ then $\bar{P}_2(S) = 1$. Furthermore, if $[C^\sharp] = 0$ then C is either a single (-4)-curve or an admissible rational rod with (-3)-curves as tip components and n $(0 \leq n \leq 8)$ (-2)-curves as middle components, and if $[C^\sharp] \neq 0$ then the configuration of C is given as in Figure* 1.

(3) *If $\bar{p}_g(S) = \bar{P}_2(S) = 0$ and $\bar{P}_3(S) \geq 1$ then $\bar{P}_3(S) = 1$ and the configuration of C is given as in Figure* 2.

(4) *If $\bar{P}_n(S) = 0$ for $n = 1, 2, 3$ and $\bar{P}_4(S) \geq 1$ then $\bar{P}_4(S) = 1$ and the configuration of C is given as in Figure* 3.

(5) *If $\bar{P}_n(S) = 0$ for $1 \leq n \leq 4$ then $\bar{P}_5(S) = 0$, $\bar{P}_6(S) = 1$ and the configuration of C is given as in Figure* 4.

In Figures $1 \sim 4$, each line represents a nonsingular rational curve and each number indicates the self-intersection number of the corresponding curve.

Proof. Since $\bar{\kappa}(S) = 0$, $\bar{P}_n(S) \leq 1$ for any $n \geq 1$.

Suppose that $\bar{p}_g(S) = 1$. Then, since $h^0(W, [C^\sharp + K_W]) = h^0(W, [C^\sharp] + K_W) = \bar{p}_g(S) = 1$ by [**15**, Lemma 1.10], it follows from [**14**, Lemma I.2.1.3] that one of the following two cases takes place.

(A) There exists an irreducible curve $A \leq [C^\sharp]$ such that $p_a(A) \geq 1$, where $p_a(A)$ denotes the arithmetic genus of A.

(B) Every irreducible curve of $[C^\sharp]$ is rational and the dual graph of $[C^\sharp]$ contains a loop, say A, of nonsingular rational curves.

Then $|A + K_W| \neq \emptyset$ (cf. [**14**, Lemma I.2.1.3]). Since $C^\sharp + K_W \equiv 0$ by Lemma 1.3, we know that $C^\sharp = [C^\sharp] = A$ and $A + K_W \sim 0$. Then A is a connected component of C and hence $C = A$. The assertion (1) thus follows.

Suppose that $\bar{p}_g(S) = 0$. Then, by virtue of Lemma 1.3, we know that [**18**, Proposition 2.2] is valid in any characteristic. So the assertions (2) $\sim$ (5) follow from [**18**, Proposition 2.2]. □

Let S be a nonsingular rational open surface of $\bar{\kappa}(S) = 0$ with a connected boundary at infinity and let (W, C) be an almost minimal model of an SNC-completion (X, B) of S. Then, by contracting (-1)-curves E with $(E \cdot C) \leq 1$ successively, we obtain a birational morphism $\mu : W \to V$ such that $(F \cdot \mu_*(C)) > 1$ for any (-1)-curve F on W. Put $D = \mu_*(C)$ and $S' := V - D$. The pair (V, D) is called a *strongly minimal model* of (X, B).

Note that our definition of a strongly minimal model differs slightly from the definition in [**4**]. In fact, let E be an irreducible curve such that $(E \cdot C^\sharp + K_W) = 0$, $E \subsetneq \mathrm{Supp}(C)$ and the intersection matrix of $E + \mathrm{Bk}(C)$ is negative definite. Then E is a (-1)-curve (if $(E \cdot C^\sharp) > 0$) or a (-2)-curve (if $(E \cdot C^\sharp) = 0$). In our case, E is contracted if $(E \cdot C^\sharp) > 0$ and not contracted if $(E \cdot C^\sharp) = 0$. Furthermore, we also contract (-1)-curves E contained in $\mathrm{Supp}(C)$ provided $(C \cdot E) = 1$.

Lemma 1.6. *With the same notation as above, the following assertions hold*:

(1) *D is connected.*

(2) *If S is affine then S' is an affine open subset of S.*

Proof. The assertions are verified by the same argument as in the proof of [**13**, Lemma 1.4]. □

Lemma 1.7. *With the same notation and assumptions as above, the following assertions hold true.*

(1) *The divisor D is an SNC-divisor or an irreducible rational curve with one node. In particular, D is an NC-divisor.*

(2) *$\bar{P}_n(S') = \bar{P}_n(S)$ for any $n \geq 1$. In particular, $\bar{\kappa}(S') = 0$.*

(3) *If $\bar{p}_g(S) = 0$ then D is an SNC-divisor and (V, D) is almost minimal.*

(4) *Assume that $\bar{p}_g(S) = 0$ and $\bar{P}_2(S) = 1$. If $[D^\sharp] = 0$ then D is either a single (-4)-curve or an admissible rational rod with (-3)-curves as tip components and n $(0 \leq n \leq 8)$ (-2)-curves as middle components, and if $[D^\sharp] \neq 0$ then the configuration of D is given as in Figure 1.*

(5) *If $\bar{p}_g(S) = \bar{P}_2(S) = 0$ and $\bar{P}_3(S) = 1$ then the configuration of D is given as in Figure 2.*

(6) *If $\bar{P}_n(S) = 0$ for $n = 1, 2, 3$ and $\bar{P}_4(S) = 1$ then the configuration of D is given as in Figure 3.*

(7) *If* $\bar{P}_n(S) = 0$ *for* $1 \leq n \leq 4$ *then the configuration of* D *is given as in Figure* 4.

Proof. By virtue of Lemmas 1.3 and 1.5, we know that [**13**, Lemma 2.4] is valid in any characteristic. So we obtain the assertions. □

We define the strongly minimality of a nonsingular rational open surface with $\bar{\kappa} = 0$ and with a connected boundary at infinity as follows.

Definition 1.8. Let S be a nonsingular rational open surface of $\bar{\kappa}(S) = 0$ with a connected boundary at infinity. We say that S is *strongly minimal* if there exists a strongly minimal model (V, D) of an SNC-completion of S such that $S = V - D$. We then call such a pair (V, D) an *SM-completion* of S.

We shall give a slight generalization of [**13**, Theorem 1.6]. We give some definitions needed later.

Definition 1.9. Let S be a normal (not necessarily complete) algebraic surface and let $\pi : \tilde{S} \to S$ be a resolution of singularities of S. We define the *logarithmic Kodaira dimension* $\bar{\kappa}(S)$ and the *logarithmic irregularity* $\bar{q}(S)$ of S by $\bar{\kappa}(S) := \bar{\kappa}(\tilde{S})$ and $\bar{q}(S) := \bar{q}(\tilde{S})$.

Our definition of $\bar{\kappa}$ for a normal surface differs from the one in [**4**]. Note that $\bar{\kappa}(S)$ and $\bar{q}(S)$ do not depend on the choice of π.

We prove the following result.

Theorem 1.10. *Let* S *be a normal affine surface of* $\bar{\kappa}(S) = 0$. *Then* S *is a rational surface.*

Proof. Let $\pi : \tilde{S} \to S$ be a resolution of singularities of S. Let (X, B) be an SNC-completion of $\tilde{S}$. Since S is a normal affine surface, B is connected and $\kappa(B, X) = 2$, where $\kappa(B, X)$ denotes the B-dimension of X (cf. [**7**]). Let (W, C) be an almost minimal model of (X, B). Then there exists a birational morphism $f : X \to W$ such that $f_*(B) = C$. We have $\kappa(C, W) = 2$ because $B \leq f^*(C)$.

Suppose to the contrary that S is not a rational surface. Then, by using Lemma 1.3 and the same argument as in the proof of [**13**, Theorem 1.6], we know that the pair (W, C) is one of the following (i) and (ii):

(i) W is a minimal surface of Kodaira dimension zero and either $C = 0$ or C is an admissible rational rod or fork consisting entirely of (-2)-curves.

(ii) W is a ruled surface with $h^1(W, \mathcal{O}_W) = 1$ (i.e., W is an elliptic ruled surface) and C ($\equiv -K_W$) is an irreducible nonsingular elliptic curve with $(C^2) \leq 0$.

Both in the cases (i) and (ii), we have $\kappa(C, W) \leq 1$. This is a contradiction. □

2. Classification.

In this section we classify the strongly minimal nonsingular rational open surfaces of $\bar{\kappa} = 0$ with connected boundaries at infinity defined over an algebraically closed field k of arbitrary characteristic. First of all, we give some examples (Examples 2.1 ∼ 2.7) of such surfaces for the case $\bar{p}_g = 0$.

Example 2.1 (cf. [**13**, Example 4.1]). Let $V_0 = \boldsymbol{P}^1 \times \boldsymbol{P}^1$. Let C_1 be an irreducible curve with $C_1 \sim 2M_0 + \ell$, where ℓ is a fiber of a fixed ruling π on V_0. Assume that a morphism $\pi|_{C_1} : C_1 \to \boldsymbol{P}^1$ is separable. Let ℓ_i $(i = 1, 2)$ be a fiber of π such that ℓ_i meets C_1 in two distinct points, say P_i and P_i'. Let $\mu_0 : V_1 \to V_0$ be the blowing-up with centers P_1 and P_2. Put $E_i := \mu_0^{-1}(P_i)$ $(i = 1, 2)$. Let $\mu_1 : V_2 \to V_1$ be the blowing-up with centers $Q_i := E_i \cap \mu_0'(\ell_i)$ $(i = 1, 2)$. Put $V := V_2$ and

$$D := \mu_1'(E_1 + E_2 + \mu_0'(C_1 + \ell_1 + \ell_2)).$$

Then $\bar{\kappa}(V - D) = \bar{p}_g(V - D) = 0$ and $\bar{P}_2(V - D) = 1$. Further, the configuration of D is given as in Figure 1, where $r = 1$ and $(D_1^2) = 2$. We denote the surface $S := V - D$ by $X[2]$. By the argument as in the proof of [**13**, Proposition 4.13], $\mathrm{Pic}(S) \cong \boldsymbol{Z} \oplus \boldsymbol{Z}/4\boldsymbol{Z}$.

Example 2.2 (cf. [**2**, §8.5], [**13**, Example 4.2]). Let $V_0 = \boldsymbol{F}_n$ $(n \geq 1)$ and let ℓ_0, ℓ_1 and ℓ_2 be three distinct fibers of the ruling on V_0. Put $P_i := \ell_i \cap \overline{M}_n$ $(i = 1, 2)$. Let $\mu_0 : V_1 \to V_0$ be the blowing-up with centers P_1 and P_2. Put $E_i := \mu_0^{-1}(P_i)$ $(i = 1, 2)$. Furthermore, let $\mu_1 : V_2 \to V_1$ be the blowing-up with centers $Q_i := E_i \cap \mu_0'(\ell_i)$ $(i = 1, 2)$. Put $V := V_2$ and

$$D := \mu_1'\left(E_1 + E_2 + \mu_0'\left(\sum_{i=0}^{2} \ell_i + M_n + \overline{M}_n\right)\right).$$

Then $\bar{\kappa}(V - D) = \bar{p}_g(V - D) = 0$ and $\bar{P}_2(V - D) = 1$. Further, the configuration of D is given as in Figure 1, where $r = 3$, $(D_1^2) = -n$, $(D_2^2) = 0$ and $(D_3^2) = n - 2$ $(n \geq 1)$. Put $S := V - D$. If $n = 1$, this is Fujita's $H[-1, 0, -1]$. If $n > 1$, the elementary transformations with centers at $P_0 := \ell_0 \cap \overline{M}_n$ and its infinitely near points will reduce the case $n > 1$ to the case $n = 1$. Hence, for any $n \geq 1$, the surface S is $H[-1, 0, -1]$. By the argument as in [**2**, §8.6], $\mathrm{Pic}(S) \cong \boldsymbol{Z}/2\boldsymbol{Z}$.

Example 2.3 (cf. [**2**, §8.26], [**13**, Example 4.3]). Let $V_0 = \boldsymbol{P}^1 \times \boldsymbol{P}^1$. Let ℓ_0, ℓ_1 and ℓ_2 be three distinct irreducible curves with $\ell_i \sim \ell$ $(i = 0, 1, 2)$, where ℓ is a fiber of a fixed ruling on V_0, and let $\bar{\ell}_0$, $\bar{\ell}_1$ and $\bar{\ell}_2$ be three distinct curves with

$\bar{\ell}_j \sim M_0$ $(j = 0, 1, 2)$. Put $P_{ij} := \ell_i \cap \bar{\ell}_j$ $(1 \le i, j \le 2)$. Let $\mu_0 : V_1 \to V_0$ be the blowing-up with centers P_{ij} $(1 \le i, j \le 2)$. Put $V := V_1$ and

$$D := \mu_0'\left(\sum_{i=0}^{2}(\ell_i + \bar{\ell}_i)\right).$$

Then $\bar{\kappa}(V - D) = \bar{p}_g(V - D) = 0$ and $\bar{P}_2(V - D) = 1$. Further, the configuration of D is given as in Figure 1, where $r = 2$ and $(D_1^2) = (D_2^2) = 0$. The surface $S := V - D$ is Fujita's $H[0, 0]$. By the argument as in [**2**, §8.26] (see also [**13**, Proposition 4.13]), $\mathrm{Pic}(S) \cong \boldsymbol{Z}$.

Example 2.4 (cf. [**2**, §8.26], [**13**, Example 4.4]). Let $V_0 = \boldsymbol{F}_n$ $(n \ge 1)$. Let $C_1 = M_n$ and C_2 a nonsingular irreducible curve with $C_2 \sim M_n + (n+1)\ell$, where ℓ is a fiber of the ruling on $\boldsymbol{F}_n$. Let ℓ_1 and ℓ_2 be fibers of the ruling with $P_i := \ell_i \cap C_2 \notin C_1 \cap C_2$ $(i = 1, 2)$.

Let $\mu_0 : V_1 \to V_0$ be the blowing-up with centers P_1 and P_2. Put $E_i := \mu_0^{-1}(P_i)$ $(i = 1, 2)$, $\ell_i' := \mu_0'(\ell_i)$ $(i = 1, 2)$ and $C_i' := \mu_0'(C_i)$ $(i = 1, 2)$. Let $\mu_1 : V_2 \to V_1$ be the blowing-up with centers $Q_i := E_i \cap \ell_i'$ $(i = 1, 2)$. Put $V := V_2$ and

$$D := \mu_1'\left(\sum_{i=1}^{2}(E_i + \ell_i' + C_i')\right).$$

Then $\bar{\kappa}(V - D) = \bar{p}_g(V - D) = 0$ and $\bar{P}_2(V - D) = 1$. Further, the configuration of D is given as in Figure 1, where $r = 2$, $(D_1^2) = -n$ and $(D_2^2) = n$ $(n \ge 1)$. The surface $S := V - D$ is Fujita's $H[n, -n]$. By the argument as in [**2**, §8.26], $\mathrm{Pic}(S) \cong \boldsymbol{Z}/4n\boldsymbol{Z}$.

Example 2.5 (cf. [**2**, §8.37], [**13**, Example 5.1]). Let H_i $(i = 1, 2, 3)$ be three non-concurrent lines on $V_0 := \boldsymbol{P}^2$. Put $P_1 := H_1 \cap H_2$, $P_2 := H_2 \cap H_3$ and $P_3 := H_3 \cap H_1$. Let H_4 be a fourth line not passing any of the P_i $(1 \le i \le 3)$. Let $\mu_0 : V_1 \to V_0$ be the blowing-up with centers P_i $(i = 1, 2, 3)$. Put $E_i := \mu_0^{-1}(P_i)$ $(i = 1, 2, 3)$. Let $\mu_1 : V_2 \to V_1$ be the blowing-up with centers $Q_i := \mu_0'(H_i) \cap E_i$ $(i = 1, 2, 3)$. Put $V := V_2$ and

$$D := \mu_1'\left(E_1 + E_2 + E_3 + \mu_0'\left(\sum_{i=1}^{4} H_i\right)\right).$$

The surface $S := V - D$ is Fujita's $Y\{3, 3, 3\}$. By the argument as in [**2**, §8.38], $\mathrm{Pic}(S) \cong \boldsymbol{Z}/9\boldsymbol{Z}$.

Example 2.6 (cf. [**2**, §8.53], [**13**, Example 5.2]). Let $V_0 = \boldsymbol{P}^1 \times \boldsymbol{P}^1$. Let ℓ_1, ℓ_2 and ℓ_3 be three distinct irreducible curves with $\ell_i \sim \ell$, where ℓ is a fiber of a fixed ruling on V_0, and let $\bar{\ell}_1$, $\bar{\ell}_2$, and $\bar{\ell}_3$ be three distinct curves with $\bar{\ell}_i \sim M_0$. Put $P_1 := \ell_1 \cap \bar{\ell}_1$, $P_2 := \ell_2 \cap \bar{\ell}_1$, $P_3 := \ell_2 \cap \bar{\ell}_3$ and $P_4 := \ell_3 \cap \bar{\ell}_3$. Let $\mu_0 : V_1 \to V_0$ be the blowing-up with centers $P_1, \dots, P_4$. Put $E_1 := \mu_0^{-1}(P_1)$ and $E_4 := \mu_0^{-1}(P_4)$. Let $\mu_1 : V_2 \to V_1$ be the blowing-up with centers $Q_1 := E_1 \cap \mu_0'(\ell_1)$ and $Q_2 := E_4 \cap \mu_0'(\ell_3)$. Put $V := V_2$ and

$$D := \mu_1'\left(E_1 + E_4 + \mu_0'\left(\sum_{i=1}^{3}(\ell_i + \bar{\ell}_i)\right)\right).$$

The surface $S := V - D$ is Fujita's $Y\{2,4,4\}$. By the argument as in [**2**, §8.55], $\mathrm{Pic}(S) \cong \boldsymbol{Z}/8\boldsymbol{Z}$.

Example 2.7 (cf. [**2**, §8.59], [**13**, Example 5.3]). Put $V_0 := \boldsymbol{P}^1 \times \boldsymbol{P}^1$. Let ℓ_i $(i = 1,2,3)$ and $\bar{\ell}_j$ $(j = 1,2,3)$ be the same as in Example 2.6. Put $P_1 := \ell_1 \cap \bar{\ell}_2$, $P_2 := \ell_2 \cap \bar{\ell}_3$, $P_3 := \ell_3 \cap \bar{\ell}_1$ and $P_4 := \ell_1 \cap \bar{\ell}_3$. Let $\mu_0 : V_1 \to V_0$ be the blowing-up with centers $P_1, \dots, P_4$. Put $E_i := \mu_0^{-1}(P_i)$, $1 \le i \le 3$. Let $\mu_1 : V_2 \to V_1$ be the blowing-up with centers $Q_1 := E_1 \cap \mu_0'(\overline{\ell_2})$, $Q_2 := E_2 \cap \mu_0'(\ell_2)$ and $Q_3 := E_3 \cap \mu_0'(\ell_3)$. Put $V := V_2$ and

$$D := \mu_1'\left(E_1 + E_2 + E_3 + \mu_0'\left(\sum_{i=1}^{3}(\ell_i + \bar{\ell}_i)\right)\right).$$

The surface $S := V - D$ is Fujita's $Y\{2,3,6\}$. By the argument as in [**2**, §8.62], $\mathrm{Pic}(S) \cong \boldsymbol{Z}/6\boldsymbol{Z}$.

Now, let S be a strongly minimal nonsingular rational open surface of $\bar{\kappa}(S) = 0$ with a connected boundary at infinity and let (V, D) be an SM-completion of S. We shall consider the following three cases I$\sim$III separately.

Case I: $\bar{p}_g(S) \ge 1$. In this case, the proof of [**13**, Theorem 3.1] works in positive characteristic case as well. Since $\bar{\kappa}(S) = 0$, we have $\bar{p}_g(S) = 1$. Lemmas 1.5 (1) and 1.7 (1) imply that D is an NC-divisor and $D + K_V \sim 0$. If there exists a (-1)-curve E on V then $(E \cdot D) = -(K_V \cdot E) = 1$. This contradicts the hypothesis that (V, D) is a strongly minimal model of an SNC-completion of S. Hence $V \cong \boldsymbol{P}^2$ or $\boldsymbol{F}_n$, $n \ne 1$. Then such pairs (V, D) can be classified easily. Namely, we have the following result.

Theorem 2.8. *Let S be a strongly minimal nonsingular rational open surface of $\bar{\kappa}(S) = 0$ with a connected boundary at infinity and let (V, D) be an SM-completion of S. Suppose that $\bar{p}_g(S) \ge 1$. Then $\bar{p}_g(S) = 1$, D is an NC-divisor and the pair (V, D) is one of the following* 1)$\sim$16), *where if $V = \boldsymbol{F}_n$ we denote by ℓ a general fiber of the ruling on $\boldsymbol{F}_n$. Furthermore, if S is affine then (V, D) is one of* 1)$\sim$13n).

1) $(*(9))$ $V = \boldsymbol{P}^2$, *D is a nonsingular cubic curve.*
2) $(O(9))$ $V = \boldsymbol{P}^2$, *D is a cubic curve with one node.*
3) $(*(8))$ $V = \boldsymbol{P}^1 \times \boldsymbol{P}^1$, $D \sim -K_{\boldsymbol{P}^1 \times \boldsymbol{P}^1}$ *is a nonsingular elliptic curve.*
4) $(O(8))$ $V = \boldsymbol{P}^1 \times \boldsymbol{P}^1$, $D \sim -K_{\boldsymbol{P}^1 \times \boldsymbol{P}^1}$ *is a rational curve with one node.*
5) (Fujita's $O(4,0)$) $V = \boldsymbol{P}^1 \times \boldsymbol{P}^1$, $D = G + C$, *where* $G \sim M_0$ *and* $C \sim M_0 + 2\ell$.
6n) (Fujita's $O(n+4,-n)$) $V = \boldsymbol{F}_n$ $(n \geq 2)$, $D = M_n + C$, *where* $C \sim M_n + (n+2)\ell$.
7) (Fujita's $O(4,1)$) $V = \boldsymbol{P}^2$, $D = H + C$, *where H is a line and C is a conic.*
8) $V = \boldsymbol{P}^1 \times \boldsymbol{P}^1$, $D = H + G + C$, *where* $H \sim M_0$, $G \sim \ell$ *and* $C \sim M_0 + \ell$.
9n) $V = \boldsymbol{F}_n$ $(n \geq 2)$, $D = F + M_n + C_3$, *where F is a fiber of the ruling and* $C_3 \sim M_n + (n+1)\ell$ *is a nonsingular rational curve.*
10) $(O(2,2))$ $V = \boldsymbol{P}^1 \times \boldsymbol{P}^1$, $D = C_1 + C_2$, *where* $C_i \sim M_0 + \ell$ $(i = 1,2)$.
11) (Fujita's $O(1,1,1)$) $V = \boldsymbol{P}^2$, $D = H_1 + H_2 + H_3$, *where* H_i *are lines in general position in* $\boldsymbol{P}^2$.
12) $V = \boldsymbol{P}^1 \times \boldsymbol{P}^1$, $D = H_1 + H_2 + G_1 + G_2$, *where* $H_i \sim M_0$ $(i = 1,2)$ *and* $G_j \sim \ell$ $(j = 1,2)$.
13n) $V = \boldsymbol{F}_n$ $(n \geq 2)$, $D = M_n + \overline{M}_n + F_1 + F_2$, *where* F_1 *and* F_2 *are fibers of the ruling.*
14) $(\overline{*}(8))$ $V = \boldsymbol{F}_2$, $D \sim -K_{\boldsymbol{F}_2}$ *is a nonsingular elliptic curve.*
15) $(\overline{O}(8))$ $V = \boldsymbol{F}_2$, $D \sim -K_{\boldsymbol{F}_2}$ *is a rational curve with one node.*
16) $(\overline{O}(2,2))$ $V = \boldsymbol{F}_2$, $D = C_1 + C_2$, *where* $C_i \sim \overline{M}_2$ $(i = 1,2)$.

REMARK 2.9. If (V,D) is one of 7), 8) and 9n) in Theorem 2.8 then $S = V - D$ is of type $O(4,1)$. If (V,D) is one of 11), 12) and 13n) then $S = V - D \cong \boldsymbol{A}^1_* \times \boldsymbol{A}^1_*$.

CASE II: $\bar{p}_g(S) = 0$ and $\bar{P}_2(S) \geq 1$. Then $\bar{P}_2(S) = 1$ and D is an SNC-divisor by Lemma 1.7 (3). If $[D^\sharp] = 0$ then D is either a single (-4)-curve or an admissible rational rod with (-3)-curves as tip components and n $(0 \leq n \leq 8)$ (-2)-curves as middle components by Lemma 1.7 (4). We shall prove the following result (Theorem 2.10). When $\mathrm{char}(k) = 0$, Theorem 2.10 is [**13**, Theorem 4.5]. The proof of [**13**, Theorem 4.5] used a covering method and the classification of Gorenstein log del Pezzo surfaces of rank one (cf. [**16**]), which do not hold in the case $\mathrm{char}(k) > 0$.

THEOREM 2.10. *Let S be a strongly minimal nonsingular rational open surface of* $\bar{\kappa}(S) = 0$ *with a connected boudary at infinity and let* (V,D) *be an SM-completion of S. Assume that* $\bar{p}_g(S) = 0$, $\bar{P}_2(S) \geq 1$ *and* $[D^\sharp] \neq 0$. *Then the pair* (V,D) *is one of the pairs enumerated in Examples* 2.1 ∼ 2.4.

In what follows ($\sim$Lemma 2.15), we prove Theorem 2.10.

Since $[D^\sharp] \neq 0$, the configuration of D is given as in Figure 1 by Lemma 1.7 (4). Then $D^\sharp + K_V \equiv 0$ and $D^\sharp$ is given as

$$D^\sharp = \sum_{i=1}^{r} D_i + \frac{1}{2}\sum_{j=1}^{4} C_j.$$

Put $D' := \sum_{j=1}^{4} C_j$.

Lemma 2.11. *The pair (V, D') is almost minimal and $\bar{\kappa}(V - D) = -\infty$.*

Proof (cf. [**13**, Lemma 4.6]). Since D' consists of disjoint four (-2)-curves, we have $D'^\sharp = 0$. Hence $\bar{\kappa}(V - D') = -\infty$ because V is a rational surface. Suppose that (V, D') is not almost minimal. Since $D'^\sharp = 0$, there exists a (-1)-curve E such that the intersection matrix of $E + \mathrm{Bk}(D') = E + D'$ is negative definite. Then $(E \cdot D') = 0$ or 1. Assume that $(E \cdot D') = 0$. Since $D^\sharp + K_V \equiv 0$ and $D^\sharp = \sum_{i=1}^{r} D_i + 1/2D'$, we have

$$(E \cdot D) = (E \cdot D^\sharp) = -(E \cdot K_V) = 1.$$

This contradicts that (V, D) is a strongly minimal model of an SNC-completion of S. If $(E \cdot D') = 1$ then $1 = -(E \cdot K_V) = (E \cdot D^\sharp) = (E \cdot \sum_{i=1}^{r} D_i) + 1/2$, which is also a contradiction. Hence (V, D') is almost minimal. □

Lemma 2.12. *With the above notation, we have $\rho(V) = 5$ or 6.*

Proof (cf. [**13**, Lemma 4.7]). By [**1**], there exists a birational morphism $\pi : V \to \overline{V}$ which is the contraction of $\mathrm{Supp}(D')$. Then $\overline{V}$ has four rational double points of type A_1 as singularities and hence $K_V \sim \pi^*(K_{\overline{V}})$. By Lemma 2.11 and [**15**, Theorem 2.11], $K_{\overline{V}}$ is not nef. Hence there exists an extremal rational curve $\bar{\ell}$ on $\overline{V}$ (cf. [**12**], [**15**]). Let ℓ be the proper transform of $\bar{\ell}$ on V. Since (V, D') is almost minimal by Lemma 2.11, $\overline{V}$ is relatively minimal, i.e., there are no irreducible curves $\overline{C}$ on $\overline{V}$ with $(\overline{C}^2) < 0$ and $(\overline{C} \cdot K_{\overline{V}}) < 0$ (cf. [**4**, p. 469]). Hence, by [**15**, Lemma 2.7], one of the following two cases takes place:

(A) The intersection matrix of $\ell + \mathrm{Bk}(D')$ is negative semi-definite, but not negative definite. Furthermore, $(\bar{\ell}^2) = 0$.

(B) $\rho(\overline{V}) = 1$ and $-K_{\overline{V}}$ is ample.

If the case (A) takes place then $\rho(\overline{V}) = 2$. A little explanation is desirable. □

Lemma 2.13. *There exists a (-1)-curve E such that E is not a component of D and*

$$(E \cdot C_i) = (E \cdot C_j) = 1,$$

where $i \neq j$ if $r = 1$ and $i \in \{1, 2\}$ and $j \in \{3, 4\}$ if $r \geq 2$.

Proof. The proof consists of two steps (I) and (II) below.

Step (I). We shall find a (-1)-curve which meets two of the C_i $(1 \leq i \leq 4)$. The argument in this step is similar to [**2**, §8.24].

Since $\rho(V) \geq 5$ by Lemma 2.12, there exist (-1)-curves. Let E be a (-1)-curve. If $E \subset D$ then $r \geq 2$. Indeed, if $r = 1$ then $D_1 = E$ is a (-1)-curve. Since $D^\sharp + K_V \equiv 0$ and $D^\sharp = D_1 + 1/2D'$, $(K_V^2) = (D^\sharp)^2 = 1$ and hence $\rho(V) = 10 - (K_V^2) = 9$, which contradicts Lemma 2.12. Since (V, D) is a strongly minimal model of an SNC-completion of S, we know that $E = D_1$ or D_r. Thus, we obtain a (-1)-curve meeting two of the C_i $(1 \leq i \leq 4)$.

Assume that D contains no (-1)-curves. For any (-1)-curve C, we have $(C \cdot D^\sharp) = 1/2(C \cdot D') + (C \cdot \sum_{i=1}^r D_i) = -(C \cdot K_V) = 1$. So $(C \cdot D) = (C \cdot D') = 2$ because (V, D) is a strongly minimal model of an SNC-completion of S. Suppose that, for any (-1)-curve C, we have $(C \cdot D') = (C \cdot C_j) = 2$ for some j, $1 \leq j \leq 4$. Let E_1 be a (-1)-curve on V. We may assume that $(E_1 \cdot C_1) = 2$. Let $\mu_1 : V \to V_1$ be the contraction of E_1. Since $\rho(V_1) = \rho(V) - 1 \geq 4$, there exists a (-1)-curve, say E_2. Then $(E_2 \cdot \mu_{1*}(D^\sharp)) = 1/2(E_2 \cdot \mu_{1*}(D')) + (E_2 \cdot \mu_{1*}(\sum_{i=1}^r D_i)) = -(K_{V_1} \cdot E_2) = 1$. Since $\mu_{1*}(D)$ contains no (-1)-curves and (V, D) is a strongly minimal model of an SNC-completion of S, we have $(E_2 \cdot \mu_{1*}(D')) = 2$. Then $(E_2 \cdot \mu_{1*}(D')) = (E_2 \cdot \mu_{1*}(C_\ell))$ for some ℓ, $1 \leq \ell \leq 4$. Indeed, if not, the proper transform of E_2 on V is a (-1)-curve meeting two of the C_i $(1 \leq i \leq 4)$, which is a contradiction.

Assume that $\ell \geq 2$. The proper transform $\mu_1'(E_2)$ of E_2 is then a (-1)-curve with $(E_1 \cdot \mu_1'(E_2)) = 0$. Then $(2E_1 + C_1)^2 = (2\mu_1'(E_2) + C_\ell)^2 = 2$ and $(2E_1 + C_1 \cdot 2\mu_1'(E_2) + C_\ell) = 0$. This contradicts the Hodge index theorem. So $\ell = 1$. In this way, we obtain a birational morphism $\mu : V \to \boldsymbol{F}_m$ onto a Hirzebruch surface $\boldsymbol{F}_m$ such that $\mu_*(C_2)$, $\mu_*(C_3)$ and $\mu_*(C_4)$ remain as (-2)-curves. This is a contradiction. Hence we obtain a (-1)-curve meeting two of the C_i $(1 \leq i \leq 4)$.

Step (II). Let E be a (-1)-curve meeting two of the C_i $(1 \leq i \leq 4)$. Since $D^\sharp + K_V \equiv 0$ and (V, D) is a strongly minimal model of an SNC-completion of S, we may assume that one of the following three cases (a)$\sim$(c) occurs:

(a) $E = D_1$ or D_r.

(b) $(E \cdot C_1) = (E \cdot C_2) = 1$ and E is not a component of D.

(c) $(E \cdot C_1) = (E \cdot C_3) = 1$ (and hence E is not a component of D).

It suffices to show the assertion of Lemma 2.13 if the case (a) or (b) takes place. We consider the cases (a) and (b) separately.

Case (a). We may assume that $E = D_1$. Note that $r \geq 2$. Then a divisor

$F := 2D_1 + C_1 + C_2$ moves to define a $\boldsymbol{P}^1$-fibration $\Phi := \Phi_{|F|} : V \to \boldsymbol{P}^1$ and D_2 becomes a 2-section of Φ. We consider the following two subcases (a)-1 and (a)-2 separately.

Subcase (a)-1: $r \geq 3$. Then $D_3 + \cdots + D_r + C_3 + C_4$ is contained in a fiber G of Φ. Since $\rho(V) \leq 6$ by Lemma 2.12 and $\sharp G \geq 3$, we know that $\rho(V) = 6$, $\sharp G = 3$ and Φ has no singular fibers other than F and G. Then $r = 3$ and D_r $(= D_3)$ is a (-1)-curve. Since $(K_V \cdot D + K_V) = 1/2(K_V \cdot D') = 0$, $(K_V^2) = 10 - \rho(V) = 4$ and $(D_1^2) = (D_3^2) = -1$, we have $(D_2^2) = 0$.

A complete linear system $|D_2|$ defines a $\boldsymbol{P}^1$-fibration $\Psi : V \to \boldsymbol{P}^1$ and D_1 and D_3 become sections of Ψ. Let F_i $(i = 1, 2)$ be a fiber of Ψ containing C_i. Since $\rho(V) = 6$ and $\sharp F_i \geq 3$, $i = 1, 2$, we may assume that there exist (-1)-curves E_1 and E_2 such that $F_i = C_i + E_i + C_{i+2}$ for $i = 1, 2$. Then $(E_i \cdot C_i) = (E_i \cdot C_{i+2}) = 1$, $i = 1, 2$. In this subcase, Lemma 2.13 is thus verified.

Subcase (a)-2: $r = 2$. Let G_1 and G_2 be fibers of Φ containing C_3 and C_4, respectively. Since $\rho(V) \leq 6$ by Lemma 2.12 and $\sharp G_i \geq 3$ for $i = 1, 2$, we know that $\rho(V) = 6$, $G_1 = G_2$ and there exists a (-1)-curve E_1 such that $G_1 = C_3 + C_4 + 2E_1$. Since $D^\sharp = D_1 + D_2 + 1/2D'$ and $D^\sharp + K_V \equiv 0$, we have $(D_1^2) + (D_2^2) = 0$. So $(D_2^2) = 1$. Let $f : V \to Y$ be the contraction of E_1, C_4, D_1 and C_2. Then $Y \cong \boldsymbol{F}_m$ and $f_*(D_2) \sim 2M_m + (m+1)\ell$, where ℓ is a fiber of a $\boldsymbol{P}^1$-fibration $\Phi \circ f^{-1} : Y \to \boldsymbol{P}^1$, because $(f_*(D_2)^2) = (D_2^2) + 3 = 4$, $f_*(D^\sharp) \equiv -K_Y$ and $f_*(C_1) \sim f_*(C_3) \sim \ell$. Since

$$0 \leq (M_m \cdot f_*(D_2)) = 1 - m,$$

we have $m \leq 1$.

If $m = 1$ then $(f_*(D_2) \cdot M_1) = 0$. Since the fundamental points of f lie on $f_*(D_2)$, the proper transform $f'(M_1)$ of M_1 on V is a (-1)-curve with $(f'(M_1) \cdot C_1) = (f'(M_1) \cdot C_3) = 1$. Suppose that $m = 0$. Then $f_*(D_2) \sim 2M_0 + \ell$. Let $L \sim M_0$ be an irreducible curve on $F_0 = \boldsymbol{P}^1 \times \boldsymbol{P}^1$ such that $Q := f(D_1 + C_2) \in L$. Since $(L \cdot f_*(D_2)) = 1$, $(L^2) = 0$ and $(L \cdot f_*(C_3)) = 1$, the proper transform $f'(L)$ of L on V is a (-1)-curve with $(f'(L) \cdot C_2) = (f'(L) \cdot C_3) = 1$. In this subcase, the assertion of Lemma 2.13 is thus verified.

Case (b). If $r = 1$ then Lemma 2.13 is clear. So we assume that $r \geq 2$. Put $F := 2E + C_1 + C_2$. Then a complete linear system $|F|$ defines a $\boldsymbol{P}^1$-fibration $\Phi := \Phi_{|F|} : V \to \boldsymbol{P}^1$ and D_1 becomes a 2-section of Φ. Let G be a fiber of Φ containing $D_2 + \cdots + D_r + C_3 + C_4$, where we note that $r \geq 2$. Since $\sharp G \geq 3$ and $\rho(V) \leq 6$ by Lemma 2.12, we know that $\sharp G = 3$ and $\rho(V) = 6$. Hence $r = 2$ and $(D_r^2) = -1$. In this case, by using the same argument as in Subcase (a)-2, Lemma 2.13 is verified. □

In Figure 1, we put $a_i := (D_i^2)$, $i = 1, \dots, r$.

LEMMA 2.14. *Assume that* $r = 1$. *Then* $a_1 = 2$ *and* (V, D) *is the pair considered in Example* 2.1.

PROOF. By Lemma 2.13, there exists a (-1)-curve E meeting two of the C_i $(1 \le i \le 4)$. We may assume that $(E \cdot C_1) = (E \cdot C_3) = 1$. Then a divisor $F := 2E + C_1 + C_3$ defines a $\boldsymbol{P}^1$-fibration $\Phi := \Phi_{|F|} : V \to \boldsymbol{P}^1$, D_1 is a 2-section of Φ and C_2 and C_4 are contained in fibers of Φ. Let G be a fiber of Φ containig C_2. Since $\sharp G \ge 3$ and $\rho(V) \le 6$ by Lemma 2.12, we know that $\rho(V) = 6$, $\sharp G = 3$ and G contains C_4. Then we obtain a (-1)-curve E' such that $G = 2E' + C_2 + C_4$ and $(E' \cdot C_2) = (E' \cdot C_4) = 1$. Since $\rho(V) = 6$ and $D^\sharp + K_V \equiv 0$, we have

$$a_1 + 2 = (D^\sharp)^2 = (K_V^2) = 10 - \rho(V) = 4.$$

So $a_1 = 2$.

Let $h : V \to Y$ be the contraction of E, E', C_3 and C_4. Then $Y \cong \boldsymbol{F}_n$ and $(h_*(D_1)^2) = a_1 + 2 = 4$. Since $h_*(D_1) \sim 2M_n + \alpha\ell$, where $\alpha \in \boldsymbol{Z}$ and ℓ is a fiber of a $\boldsymbol{P}^1$-fibration $\Phi \circ h^{-1} : Y \to \boldsymbol{P}^1$, we have

$$4 = (h_*(D_1)^2) = 4(\alpha - n).$$

So $\alpha = n + 1$. Further, since $0 \le (M_n \cdot h_*(D_1)) = 1 - n$, we have $n = 0$ or 1. If $n = 1$ then $(M_1 \cdot h_*(D_1)) = 0$. So, by contracting C_2 instead of C_4, we may assume that $n = 0$, i.e., $Y = \boldsymbol{P}^1 \times \boldsymbol{P}^1$. Then $\bar{D}_1 := h_*(D_1) \sim 2M_0 + \ell$. Note that the morphism $\Phi \circ h^{-1}|_{\bar{D}_1} : \bar{D}_1 \to \boldsymbol{P}^1$ is separable even if $\mathrm{char}(k) = 2$. Hence we obtain the case considered in Example 2.1. □

LEMMA 2.15. *Assume that* $r \ge 2$. *Then* (V, D) *is one of the pairs enumerated in Examples* 2.2 ∼ 2.4.

PROOF. Assume that $a_1 \ge a_r$. We consider the following two cases separately.

CASE 1: $r = 2$. By Lemma 2.13, there exists a (-1)-curve E such that $(E \cdot C_1) = (E \cdot C_3) = 1$. Put $F := 2E + C_1 + C_3$. Then F defines a $\boldsymbol{P}^1$-fibration $\Phi := \Phi_{|F|} : V \to \boldsymbol{P}^1$ and D_1 and D_2 become sections of Φ. Let G be a fiber of Φ containing C_2. Since $\rho(V) \le 6$ and $\sharp G \ge 3$, we know that $\rho(V) = 6$, $\sharp G = 3$ and Φ has no singular fibers other than F and G. Then there exists a (-1)-curve E' such that $G = 2E' + C_2 + C_4$. Since $D^\sharp = D_1 + D_2 + 1/2D'$, $D^\sharp + K_V \equiv 0$ and $\rho(V) = 6$, we have $a_1 + a_2 = 0$. Let $h : V \to Y$ be the contraction of E, E', C_3 and C_4. Then $Y \cong \boldsymbol{F}_m$, $m = -a_2$ and $h_*(D_2) = M_m$. Therefore, we obtain the case considered in Example 2.4 if $m \ge 1$.

We see that if $a_1 = a_2 = 0$ then the pair (V, D) is constructed in the fashion as in Example 2.3. A complete linear system $|D_2|$ then defines a $\boldsymbol{P}^1$-fibration

$\Phi := \Phi_{|D_2|} : V \to \boldsymbol{P}^1$ and D_1, C_3 and C_4 become sections of Φ. Let F_i $(i = 1, 2)$ be a fiber of Φ containing C_i. Then $F_1 \neq F_2$ and $\sharp F_i \geq 3$, $i = 1, 2$. Since $\rho(V) = 6$, we know that $\sharp F_1 = \sharp F_2 = 3$ and Φ has no singular fibers other than F_1 and F_2. In particular, $\mathrm{Supp}(F_i)$ $(i = 1, 2)$ consists entirely of (-1)-curves and (-2)-curves. If $\mathrm{Supp}(F_i)$ ($i = 1$ or 2) contains a (-2)-curve C_i' other than C_i then $C_i' \cap D = \emptyset$ because $D^\sharp + K_V \equiv 0$ and $D^\sharp = D_1 + D_2 + 1/2D'$. Then a unique (-1)-curve in $\mathrm{Supp}(F_i)$, which has the multiplicity 2 in F_i, must meet C_3 and C_4 which are sections of Φ. This is a contradiction. Hence there exist four (-1)-curves E_1, E_1', E_2 and E_2' such that $F_i = E_i + C_i + E_i'$, $i = 1, 2$. Since C_3 and C_4 are sections of Φ and $D^\sharp + K_V \equiv 0$, we may assume that $(E_i \cdot C_3) = (E_i' \cdot C_4) = 1$, $i = 1, 2$. Let $h : V \to Y$ be the contraction of E_1, E_1', E_2 and E_2'. Then $Y \cong \boldsymbol{F}_m$ and $(h_*(D_1)^2) = 0$. So $Y \cong \boldsymbol{P}^1 \times \boldsymbol{P}^1$ because $h_*(D_1)$ is a section of the $\boldsymbol{P}^1$-fibration $\Phi \circ h^{-1} : Y \to \boldsymbol{P}^1$. Therefore, we obtain the case considered in Example 2.3.

CASE 2: $r \geq 3$. By Lemma 2.13, there exists a (-1)-curve E such that $(E \cdot C_1) = (E \cdot C_3) = 1$. Put $F := 2E + C_1 + C_3$. Then $|F|$ defines a $\boldsymbol{P}^1$-fibration $\Phi := \Phi_{|F|} : V \to \boldsymbol{P}^1$ and D_1 and D_r become sections of Φ. Let G and H be a fiber of Φ containing $\{D_2, \ldots, D_{r-1}\}$ and C_2, respectively. Then $\mathrm{Supp}(G)$ contains none of the C_i $(1 \leq i \leq 4)$ and $\sharp H \geq 3$. Since $\rho(V) \leq 6$, we know that $\rho(V) = 6$ and $\sharp H = 3$. Then G is an irreducible fiber and there exists a (-1)-curve E' such that $H = 2E' + C_2 + C_4$. In particular, $r = 3$ and $a_2 = 0$.

Since $(K_V \cdot D + K_V) = (K_V \cdot D^\sharp + K_V) = 0$ and $\rho(V) = 6$, we have

$$(K_V \cdot D_1 + D_2 + D_3) = -(6 + a_1 + a_2 + a_3) = -(K_V^2) = -4.$$

So, $a_3 = -(a_1 + 2)$. Then $a_1 < 0$ or $a_3 < 0$. So we may assume that $a_3 < 0$. Let $h : V \to Y$ be the contraction of E, E', C_1 and C_2. Then $Y \cong \boldsymbol{F}_m$, $m = -a_3$, $h_*(D_3) = M_m$ and $h_*(D_1) \sim \overline{M}_m$. Therefore, we obtain the case considered in Example 2.2. □

The proof of Theorem 2.10 is thus completed.

CASE III: $\bar{P}_2(S) = 0$. The proof of [**13**, Theorem 5.4] works also in positive characteristic case by virtue of Lemmas 1.3 and 1.7. So we have the following result.

THEOREM 2.16. *Let S be a strongly minimal nonsingular rational open surface of $\bar{\kappa}(S) = \bar{P}_2(S) = 0$ with a connected boundary at infinity and let (V, D) be an SM-completion of S. Then $\bar{P}_i(S) = 1$ for $i = 3, 4$ or 6. Furthermore, the following assertions hold*:

(1) *If $\bar{P}_3(S) \geq 1$ then $\bar{P}_3(S) = 1$ and S is Fujita's $Y\{3, 3, 3\}$. The pair (V, D) can be constructed in the fashion as in Example 2.5.*

Table 1.

Type	m	r	Pic(S)	for details, see:
$*(9)$	1	0	$\boldsymbol{Z}/3\boldsymbol{Z}$	Theorem 2.8
$O(9)$	1	0	$\boldsymbol{Z}/3\boldsymbol{Z}$	Theorem 2.8
$*(8)$	1	0	$\boldsymbol{Z} \oplus \boldsymbol{Z}/2\boldsymbol{Z}$	Theorem 2.8
$O(8)$	1	0	$\boldsymbol{Z} \oplus \boldsymbol{Z}/2\boldsymbol{Z}$	Theorem 2.8
$O(n+4, -n)$ $(n \geq 0, n \neq 1)$	1	0	$\boldsymbol{Z}/(n+2)\boldsymbol{Z}$	Theorem 2.8
$O(4, 1)$	1	1	0	Theorem 2.8
$O(2, 2)$	1	1	$\boldsymbol{Z}$	Theorem 2.8
$O(1, 1, 1)$	1	2	0	Theorem 2.8
$X[2]$	2	0	$\boldsymbol{Z} \oplus \boldsymbol{Z}/4\boldsymbol{Z}$	Example 2.1
$H[-1, 0, -1]$	2	1	$\boldsymbol{Z}/2\boldsymbol{Z}$	Example 2.2
$H[0, 0]$	2	1	$\boldsymbol{Z}$	Example 2.3
$H[n, -n]$ $(n \geq 1)$	2	0	$\boldsymbol{Z}/4n\boldsymbol{Z}$	Example 2.4
$Y\{3, 3, 3\}$	3	0	$\boldsymbol{Z}/9\boldsymbol{Z}$	Example 2.5
$Y\{2, 4, 4\}$	4	0	$\boldsymbol{Z}/8\boldsymbol{Z}$	Example 2.6
$Y\{2, 3, 6\}$	6	0	$\boldsymbol{Z}/6\boldsymbol{Z}$	Example 2.7

(2) *If* $\bar{P}_3(S) = 0$ *and* $\bar{P}_4(S) \geq 1$ *then* $\bar{P}_4(S) = 1$ *and* S *is Fujita's* $Y\{2, 4, 4\}$. *The pair* (V, D) *can be constructed in the fashion as in Example* 2.6.

(3) *If* $\bar{P}_3(S) = \bar{P}_4(S) = 0$ *then* $\bar{P}_6(S) = 1$ *and* S *is Fujita's* $Y\{2, 3, 6\}$. *The pair* (V, D) *can be constructed in the fashion as in Example* 2.7.

By using Theorems 2.8, 2.10 and 2.16, we obtain the following result.

Theorem 2.17. *Let* R *be a nonsingular affine surface defined over an algebraically closed field* k. *Assume that* $\bar{\kappa}(R) = 0$. *Then there exists an affine open subset* $S = \operatorname{Spec}(A) \subset R$ *such that*

(i) $\bar{P}_n(S) = \bar{P}_n(R)$ *for any* $n \geq 1$, *in particular*, $\bar{\kappa}(S) = \bar{\kappa}(R) = 0$, *and*

(ii) S *is strongly minimal in the sense of Definition* 1.8.

Furthermore, the surface S *is one of the surfaces in Table* 1, *where* m *is the least positive integer such that* $\bar{P}_m(S) > 0$ *and* $r := \operatorname{rank}_{\boldsymbol{Z}} A^*/k^*$.

Proof. Let (V, D) be a strongly minimal model of an SNC-completion of R. Then the surface $S := V - D$ is an affine open subset of R by Lemma 1.6

TABLE 2.

Type	$\bar{q}$	e	π_1
$*(9)$	0	3	$\boldsymbol{Z}/3\boldsymbol{Z}$
$O(9)$	0	2	$\boldsymbol{Z}/3\boldsymbol{Z}$
$*(8)$	0	4	$\boldsymbol{Z}/2\boldsymbol{Z}$
$O(8)$	0	3	$\boldsymbol{Z}/2\boldsymbol{Z}$
$O(n+4,-n)\ \ (n \geq 0, n \neq 1)$	0	2	$\boldsymbol{Z}/(n+2)\boldsymbol{Z}$
$O(4,1)$	1	1	$\boldsymbol{Z}$
$O(2,2)$	1	2	$\boldsymbol{Z}$
$O(1,1,1)$	2	0	$\boldsymbol{Z}^2$
$X[2]$	0	2	$\boldsymbol{Z}/4\boldsymbol{Z}$
$H[-1,0,-1]$	1	0	$\langle y,t\rangle/(yty^{-1}t)$
$H[0,0]$	1	1	$\boldsymbol{Z}$
$H[k,-k]\ \ (k \geq 1)$	0	1	$\boldsymbol{Z}/4k\boldsymbol{Z}$
$Y\{3,3,3\}$	0	1	$\boldsymbol{Z}/9\boldsymbol{Z}$
$Y\{2,4,4\}$	0	1	$\boldsymbol{Z}/8\boldsymbol{Z}$
$Y\{2,3,6\}$	0	1	$\boldsymbol{Z}/6\boldsymbol{Z}$

(2). By Lemma 1.7, S satisfies the conditions (i) and (ii) in Theorem 2.17. Note that if $\bar{p}_g(S) = 0$ (then D is an SNC-divisor by Lemma 1.7 (3)), $\bar{P}_2(S) = 1$ and $[D^\sharp] = 0$ then $V - D$ cannot be affine because D is an admissible rational rod (cf. Lemma 1.7 (4)). Hence (V, D) is one of the pairs in Theorems 2.8, 2.10 and 2.16. When $\bar{p}_g(S) = 1$, the Picard group $\mathrm{Pic}(S)$ can be calculated easily. We put $S := \mathrm{Spec}(A)$. By using Lemma 3.4 in §3, we can calculate $\mathrm{rank}_{\boldsymbol{Z}} A^*/k^*$. Hence S is one of the surfaces in Table 1. □

REMARK 2.18. When k is the complex number field $\boldsymbol{C}$, we have the following list (Table 2) of the logarithmic irregularities $\bar{q}$, the Euler numbers e and the fundamental groups π_1 of the strongly minimal nonsingular affine surfaces of $\bar{\kappa} = 0$. For details, see [**13**].

3. Characterization of $\boldsymbol{A}^1_* \times \boldsymbol{A}^1_*$.

In this section we give a characterization of $\boldsymbol{A}^1_* \times \boldsymbol{A}^1_*$ in arbitrary characteristic. A main result of this section is the following theorem.

THEOREM 3.1. *Let* $S = \mathrm{Spec}(A)$ *be a normal affine surface defined over an algebraically closed field* k. *Assume that* $\bar{\kappa}(S) = 0$. *Then* $\mathrm{rank}_{\boldsymbol{Z}} A^*/k^* \le 2$ *and the equality holds if and only if* $S \cong \boldsymbol{A}^1_* \times \boldsymbol{A}^1_*$.

We give a definition which we need later.

DEFINITION 3.2. Let (X, B) be a pair of a nonsingular projective surface X and a reduced effective divisor B and let $B = \sum_{i=1}^r B_i$ be the decomposition of B into irreducible components. We define $\bar{r}(X, B)$ as follows:

$$\bar{r}(X, B) := \mathrm{rank}_{\boldsymbol{Z}} \mathrm{Ker}\left(\bigoplus_{i=1}^r \boldsymbol{Z}[B_i] \to \mathrm{Pic}(X)\right),$$

where $\bigoplus_{i=1}^r \boldsymbol{Z}[B_i] \to \mathrm{Pic}(X)$ is the natural mapping.

Now, let $S = \mathrm{Spec}(A)$ be a normal affine surface defined over an algebraically closed field k. Let $\pi : \tilde{S} \to S$ be a minimal resolution of S and let (X, B) be an SNC-completion of $\tilde{S}$. Then B is connected. We note the following result.

LEMMA 3.3. *With the above notation, we have* $\mathrm{rank}_{\boldsymbol{Z}} A^*/k^* = \bar{r}(X, B)$.

PROOF. An element $f \in A$ is invertible if and only if the divisor (f) on X is supported by B. Hence any element $f \in A^*$ gives rise to a relation among the irreducible components of B, and two such relations given by f, $g \in A^*$ are the same if and only if $g = cf$ with $c \in k^*$. □

When $\mathrm{char}(k) = 0$ and $\bar{\kappa}(S) = 0$, we have the following result.

LEMMA 3.4. *With the above notation, we assume further that* $\mathrm{char}(k) = 0$ *and* $\bar{\kappa}(S) = 0$. *Then* $\bar{q}(S) = \mathrm{rank}_{\boldsymbol{Z}} A^*/k^*$.

PROOF. Since $\bar{\kappa}(S) = 0$, S is a rational surface by Theorem 1.10. So $q(X) = h^1(X, \mathcal{O}_X) = 0$. By [**5**, Lemma 2], we have $\bar{q}(S) = \bar{r}(X, B)$. Hence the assertion follows from Lemma 3.3. □

REMARK 3.5. The author does not know whether Lemma 3.4 is true or not in the case $\mathrm{char}(k) > 0$.

We assume that $\bar{\kappa}(S) = 0$. By Theorem 1.10, S is a rational surface. Let (V, D) be a strongly minimal model of (X, B). Then there exists a birational morphism $f : X \to V$ such that $f_*(B) = D$. Let $f = f_\ell \circ \cdots \circ f_0$ $(\ell \ge -1)$ be a decomposition of f into blowing-ups at single points $f_i : X_i \to X_{i+1}$, where we define $f = \mathrm{id}$ if $\ell = -1$ and put $X_0 := X$ and $X_{\ell+1} := V$. Put $B_0 := B$ and $B_{i+1} := f_{i*}(B_i)$, $i = 0, \ldots, \ell$. It is then clear that

$$\bar{r}(X_i, B_i) \le \bar{r}(X_{i+1}, B_{i+1})$$

for $i = 0, \ldots, \ell$.

Lemma 3.6. *Let (V, D) be an SM-completion of a strongly minimal nonsingular rational open surface of $\bar{\kappa} = 0$ with a connected boundary at infinity. Then $\bar{r}(V, D) \leq 2$ and the equality holds if and only if (V, D) is one of the pairs* 11), 12), *and* 13n) *in Theorem* 2.8. *In particular*, $V - D \cong \boldsymbol{A}^1_* \times \boldsymbol{A}^1_*$ *if* $\bar{r}(V, D) = 2$.

Proof. By using the classification of SM-completions of strongly minimal nonsingular rational open surfaces of $\bar{\kappa} = 0$ with connected boundaries at infinity in §2 (Theorems 2.8, 2.10 and 2.16), we obtain the assertion. □

Proof of Theorem 3.1. We use the same notation and assumptions as above. The first assertion easily follows from Lemma 3.6 and the above remark

$$\operatorname{rank}_{\boldsymbol{Z}} A^*/k^* = \bar{r}(X, B) \leq \bar{r}(V, D).$$

The "if" part of the second assertion is clear. We prove the "only if" part of the second assertion.

If $\operatorname{rank}_{\boldsymbol{Z}} A^*/k^* = 2$ then, by Lemma 3.6, (V, D) is one of the pairs 11), 12) and 13n) in Theorem 2.8. The fundamental points of the birational morphism $f : X \to V$ lie on D. Indeed, if there exists a fundamental point $Q \in V - D$ then $\tilde{S}$ contains a (-1)-curve, which is a contradiction because S is affine and the resolution of singularities $\pi : \tilde{S} \to S$ is minimal. Hence $X_i - B_i \supseteq X_i - \operatorname{Supp} f_i^*(B_{i+1})$ for $i = 0, \dots, \ell$.

If $X_i - B_i = X_i - \operatorname{Supp} f_i^*(B_{i+1})$ for all $i = 0, \dots, \ell$ then $\tilde{S} = X - B = V - D$. So $S = \tilde{S} = \boldsymbol{A}^1_* \times \boldsymbol{A}^1_*$ by Lemma 3.6. Suppose that there exists an integer j $(0 \leq j \leq \ell)$ such that $X_j - B_j \supsetneqq X_j - \operatorname{Supp} f_j^*(B_{j+1})$, i.e., $B_j < (f_j^*(B_{j+1}))_{\text{red}}$. We assume that j is maximal among the integers j' $(0 \leq j' \leq \ell)$ such that $B_{j'} < (f_{j'}^*(B_{j'+1}))_{\text{red}}$. Let P_{j+1} be the center of the blowing-up $f_j : X_j \to X_{j+1}$. Note that $P_{j+1} \in B_{j+1}$. Put $E_j := f_j^{-1}(P_{j+1})$. Then E_j is not a component of B_j.

Claim 1. *P_{j+1} is a nonsingular point of B_{j+1}, i.e., there exists a unique component of B_{j+1} passing through P_{j+1}.*

Proof. By Lemma 3.6, B_{j+1} is an SNC-divisor and each component of B_{j+1} is a rational curve. Further, B_{j+1} contains a unique loop of curves, say $\tilde{B}_{j+1}$.

Suppose that there exist two components $B_{j+1,1}$ and $B_{j+1,2}$ such that $P_{j+1} = B_{j+1,1} \cap B_{j+1,2}$. If one of $B_{j+1,1}$ and $B_{j+1,2}$ is not a component of $\tilde{B}_{j+1}$ then $B_j = f_j'(B_{j+1})$ is not connected. So $B = B_0$ is not connected, which is a contradiction. Assume that both $B_{j+1,1}$ and $B_{j+1,2}$ are components of $\tilde{B}_{j+1}$. Then the dual graph of B_j is a tree because there is a unique loop of curves in B_{j+1}. So $\bar{p}_g(X_j - B_j) = 0$ by [**14**, Lemma I.2.1.3]. On the other hand, $\bar{p}_g(X_j - B_j) =$

$\bar{p}_g(V - D) = 1$ by Lemmas 1.7 (2) and 3.6. This is also a contradiction. This proves Claim 1.

Let $B_{j+1} = \sum_{i=1}^{r} B_{j+1,i}$ be the decomposition of B_{j+1} into irreducible components. Assume that $P_{j+1} \in B_{j+1,1}$.

Claim 2. There exist integers $\alpha_1 > 0$, $\alpha_2, \ldots, \alpha_r$ such that

$$\sum_{i=1}^{r} \alpha_i B_{j+1,i} \sim 0.$$

Proof. Let D_1 be a component of D such that $f_\ell \circ \cdots \circ f_{j+1}(B_{j+1,1}) \subset D_1$. Since (V, D) is one of the pairs 11), 12) and 13n) in Theorem 2.8, for any irreducible curve $D_2 \subset D$ adjacent to D_1, we can find positive integers β_1 and β_2 and a divisor D' with $\mathrm{Supp}(D') \subseteq \mathrm{Supp}(D - (D_1 + D_2))$ such that $\beta_1 D_1 + \beta_2 D_2 + D' \sim 0$. Then $(0 \sim)$ $(f_{j+1} \circ \cdots \circ f_\ell)^*(\beta_1 D_1 + \beta_2 D_2 + D')$ can be expressed as in Claim 2, where we take D_2 so that $f_\ell \circ \cdots \circ f_{j+1}(B_{j+1,1}) = D_1 \cap D_2$ if $f_\ell \circ \cdots \circ f_{j+1}(B_{j+1,1})$ is a singular point of D. This proves Claim 2.

The natural map $f_{j_*} : \mathrm{Pic}(X_j) \to \mathrm{Pic}(X_{j+1})$ induces an injection

$$f_{j_*} : \mathrm{Ker}\left(\bigoplus_{i=1}^{r} \boldsymbol{Z}[B_{j,i}] \to \mathrm{Pic}(X_j)\right) \to \mathrm{Ker}\left(\bigoplus_{i=1}^{r} \boldsymbol{Z}[B_{j+1,i}] \to \mathrm{Pic}(X_{j+1})\right),$$

where $B_j = \sum_{i=1}^{r} B_{j,i}$ is the decomposition of B_j into irreducible components. Note that $\sharp B_j = \sharp B_{j+1} = r$. For $m = j, j+1$, we define $K_m := \mathrm{Ker}(\bigoplus_{i=1}^{r} \boldsymbol{Z}[B_{m,i}] \to \mathrm{Pic}(X_m))$. Let $\alpha_1 > 0$, $\alpha_2, \ldots, \alpha_r$ be integers specified as in Claim 2. Then $f_j'(\sum_{i=1}^{r} \alpha_i B_{j+1,i}) = \sum_{i=1}^{r} \alpha_i f_j'(B_{j+1,i}) \notin K_j$ because $0 \sim f_j^*(\sum_{i=1}^{r} \alpha_i B_{j+1,i}) = f_j'(\sum_{i=1}^{r} \alpha_i B_{j+1,i}) + \alpha_1 E_1$ by Claim 1. So, $\mathrm{rank}_{\boldsymbol{Q}}(K_{j+1}/f_{j_*}(K_j)) \otimes_{\boldsymbol{Z}} \boldsymbol{Q} \geq 1$. Then

$$2 = \bar{r}(V, D) = \bar{r}(V_{j+1}, D_{j+1}) > \bar{r}(V_j, D_j) \geq \mathrm{rank}_{\boldsymbol{Z}} A^*/k^* = 2,$$

which is a contradiction.

The proof of Theorem 3.1 is thus completed.

As a consequence of Theorem 3.1, we have the cancellation theorem for $\boldsymbol{A}_*^1 \times \boldsymbol{A}_*^1$ (Theorem 3.8).

Definition 3.7 (cf. [**8**, §1]). Let Y be an algebraic variety defined over an algebraically closed field k, not necessarily nonsingular or complete over k. We call Y a *resoluble* variety if there exists a resolution of singularities $\pi : X \to Y$ such that X has an NC-completion (V, D), where V is a nonsingular complete algebraic variety and D is an NC-divisor on V such that $X = V - D$.

Note that an algebraic variety Y defined over k is resoluble if $\mathrm{char}(k) = 0$ or if $\dim Y \leq 2$; also if $\dim Y = 3$ and $\mathrm{char}(k) \geq 7$.

Theorem 3.8. *Let* $S = \mathrm{Spec}(A)$ *be an affine algebraic variety and* T *a resoluble algebraic variety, both defined over an algebraically closed field* k. *Assume that there exists a* k*-isomorphism* $S \times_k T \cong (\boldsymbol{A}_*^1 \times \boldsymbol{A}_*^1) \times_k T$. *Then* $S \cong \boldsymbol{A}_*^1 \times \boldsymbol{A}_*^1$.

Proof. By the same argument as in [**8**, §3.2], we know that S is a non-singular affine surface. By [**8**, Theorem 1.6], $\bar{\kappa}(S) = \bar{\kappa}(\boldsymbol{A}_*^1 \times \boldsymbol{A}_*^1) = 0$.

For normal algebraic varieties V and W defined over k, we have the following natural isomorphism

$$\Gamma(V \times_k W, \mathcal{O}_{V \times_k W})^*/k^* \cong \Gamma(V, \mathcal{O}_V)^*/k^* \oplus \Gamma(W, \mathcal{O}_W)^*/k^*$$

by virtue of [**8**, §4.5]. Let N be the singular locus of T. Then, $S \times_k T \cong (\boldsymbol{A}_*^1 \times \boldsymbol{A}_*^1) \times_k T$ implies that $S \times_k (T - N) \cong (\boldsymbol{A}_*^1 \times \boldsymbol{A}_*^1) \times_k (T - N)$. By the above remark, we have $\mathrm{rank}_{\boldsymbol{Z}} A^*/k^* = 2$. Hence $S \cong \boldsymbol{A}_*^1 \times \boldsymbol{A}_*^1$ by Theorem 3.1. □

Remark 3.9. When $\mathrm{char}(k) = 0$, we can drop the assumption that S is affine in Theorem 3.8. See [**2**, §9].

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