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PARETO-EFFICIENT TARGET BY OBTAINING THE FACETS OF THE EFFICIENT FRONTIER IN DEA

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ABSTRACT. In this paper, we propose an algorithm to calculate an improvement target for each inefficient DMU in the CCR model by calculating all equations forming the facets of the efficient frontier. By introducing a parameter into the algorithm, we calculate a minimal distance point or a Pareto-efficient point on the efficient frontier as an improvement target. All improvement targets are obtained by solving quadratic mathematical problems.

KEYWORDS : DEA; Efficient frontier; Production possibility set; Pareto-efficient.

MSC : Primary 46N10; Secondary 93B15.

1. Introduction

DEA(Data Envelopment Analysis) is a non-parametric analytical methodology used for efficiency analysis of a DMU(Decision Making Unit) that consumes inputs to produce outputs. Each DMU is classified as either inefficient or efficient unit according to the optimal value of the CCR model defined in [3]. Moreover, the efficient DMUs are split between Pareto-efficient and Pareto-inefficient units depending on the positive optimal slackness of the CCR model. In the radial measure models, an improvement target can be obtained simply by using the optimal value. However, it is often diffcult to improve the values of inputs and outputs according to the improvement, because the improvements obtained by the radial measure models improve the only input (or output) values at the same rate. Therefore, Frei and Harker have proposed the minimal distance projection to the efficient frontier by using the Euclidean norm in [5]. Takeda and Nishino have proposed minimal norm problem to the efficient frontier from an inefficient DMU in [8]. Recently, improvement of efficiency for each inefficient DMU is one of the important subjects in DEA.

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Aparicio, Ruiz and Sirvent have formulated several mixed integer linear programs for typical norms to obtain a closest target on the efficient frontier in [1]. Further, Lozano and Villa have proposed a gradual efficiency improvement strategy in [7].

In this paper, we propose three kinds of improvement targets for each inefficient DMU in the CCR model. In order to calculate the targets, we use all equations forming the facets of the efficient frontier. The first and second targets are obtained by an algorithm with a parameter. By considering the convex combination of their targets and its projection to the efficient frontier, we suggest the third target as more flexible improvement.

The constitution of this paper is as follows. In Section 2, we introduce the CCR model and some definitions. In Section 3, we propose an algorithm to calculate a improvement target by introducing a parameter and a symmetric positive semi-definite matrix. In order to obtain improvements of DMUs, we use all equations forming the facets of the efficient frontiers. In Section 4, we show a numerical experiment.

Throughout this paper, we use the following notation: Let $\mathbb{R}^n$ be an n-dimensional Euclidean space. For a nutural number m, $\mathbb{R}^m_+ := \{x \in \mathbb{R}^m : x_i \geqslant 0,\, i = 1, \ldots, m\}$ and $\mathbb{R}^m_- := \{x \in \mathbb{R}^m : x_i \leqslant 0,\, i = 1, \ldots, m\}$. For a vector $a \in \mathbb{R}^n, a^\top$ denotes the transposed vector of a. Let I_n be the unit matrix on $\mathbb{R}^n$. For a subset $S \subset \mathbb{R}^n$, $\dim S$ denotes the dimension of S. For a subset $S \subset \mathbb{R}^n$, $\mathrm{int}\, S$ and $\mathrm{bd}\ S$ denote the interior and boundary of S, respectively. For subsets S_1 and $S_2 \subset \mathbb{R}^n$, $S_1 + S_2 := \{a + b : a \in S_1, b \in S_2\}$.

2. CCR model

In this section, we introduce the basic DEA model proposed by Charns, Cooper and Rhodes [3]. Through this paper, n denotes the number of DMUs. Each DMU consumes m different inputs to produce s different outputs. For each $j \in \{1, \ldots, n\}$, DMU(j) has an input vector $x(j) := (x(j)_1, \ldots, x(j)_m)^\top$ and an output vector $y(j) := (y(j)_1, \ldots, y(j)_s)^\top$. Moreover, we assume the following conditions.

(A1): $x(j) > 0$, $y(j) > 0$ for each $j \in \{1, \ldots, n\}$.
(A2): $(x(j_1)^\top, y(j_1)^\top)^\top \neq (x(j_2)^\top, y(j_2)^\top)^\top$ for each $j_1, j_2 \in \{1, \ldots, n\}$ $(j_1 \neq j_2)$.
(A3): $n > m + s$.
(A4): $\dim(\{x(1), \ldots, x(n)\} \times \{y(1), \ldots, y(n)\}) = m + s$.

Almost DEA models have Assumption (A1). Assumptions (A2), (A3) and (A4) are necessary to execute an algorithm to calculate all facets forming the efficient frontier. However, they are satisfied for almost practical problems. Assumption (A4) means that the convex hull of all DMUs has an interior point.

The CCR model formulated by Charnes, Cooper and Rhodes [3] evaluates the ratio between weighted sums of inputs and outputs. The CCR model provides for constant returns to scale(CRS). Therefore, some researchers call the CCR model the CRS model. In order to calculate an efficiency of DMU$(k)(1 \leq k \leq n)$, the CCR model is formulated as follows:

$$
(\mathrm{CCR}(k)) \begin{cases} \text{maximize} & \dfrac{u^\top y(k)}{v^\top x(k)} \\ \text{subject to} & \dfrac{u^\top y(j)}{v^\top x(j)} \leq 1,\ j = 1, ..., n, \\ & u_r \geq 0,\ r = 1, ..., s, \\ & v_i \geq 0,\ i = 1, ..., m. \end{cases}
$$

Since Problem $(\mathrm{CCR}(k))$ is a fractional programming problem, we can not solve it easily. Therefore, we transform Problem $(\mathrm{CCR}(k))$ into the linear programming problem by setting the denominator of the objective function equals to 1:

$$
(\mathrm{CCRLP}(k)) \begin{cases} \text{maximize} & u^\top y(k) \\ \text{subject to} & v^\top x(k) = 1, \\ & u^\top y(j) - v^\top x(j) \leq 0,\ j = 1, \ldots, n, \\ & u_r \geq 0,\ r = 1, ..., s, \\ & v_i \geq 0,\ i = 1, ..., m. \end{cases}
$$

Then, the dual problem of Problem $(\mathrm{CCR}(k))$ is defined as a linear programming problem as follows:

$$
(\mathrm{CCRD}(k)) \begin{cases} \text{minimize} & \theta \\ \text{subject to} & \theta x(k)_i - \displaystyle\sum_{j=1}^{n} \lambda_j x(j)_i \geq 0,\ i = 1, \ldots, m, \quad (1) \\ & \displaystyle\sum_{j=1}^{n} \lambda_j y(j)_r - y(k)_r \geq 0,\ r = 1, \ldots, s, \quad (2) \\ & \lambda_j \geq 0,\ j = 1, \ldots, n, \quad (3) \\ & \theta \in \mathbb{R}. \end{cases}
$$

Let $\theta^*_{\mathrm{CCR}}(k)$ denote the optimal value of (CCRD(k)). By conditions (2) and (3), we have that $(\lambda_1, \ldots, \lambda_n) \neq (0, \ldots, 0)$ and hence $\lambda_{\hat{j}} > 0$ for some $\hat{j} \in \{1, \ldots, n\}$. Then, it follows from (2) that $\theta^*_{\mathrm{CCR}} x(k)_i - \sum_{j=1}^{n} \lambda_j x(j)_i \geq \theta^*_{\mathrm{CCR}} x(k)_i - \lambda_{\hat{j}} x(\hat{j})_i \geq 0$. This implies that $\theta^*_{\mathrm{CCR}}(k) > 0$. Moreover, we note that $(\lambda^{'}, \theta^{'})$ is a feasible solution of (CCRD(k)) if $\theta^{'} = 1, \lambda^{'}_k = 1$ and $\lambda^{'}_j = 0$ for each $j \in \{1, \ldots, n\} \backslash \{k\}$. Therefore, $0 < \theta^*_{\mathrm{CCR}}(k) \leq 1$. By using the optimal value $\theta^*_{\mathrm{CCR}}(k)$ of (CCRD(k)), the efficiency of DMU(k) for the CCR model is defined as follows:

Definition 2.1. If $\theta^*_{\mathrm{CCR}}(k) = 1$ then DMU(k) is said to be CCR-efficient. Otherwise, DMU(k) is said to be CCR-inefficient.

Sometimes, there exists i (or r) such that $v_i = 0$ (or $u_r = 0$). This means the i (or r)th input(output) is not completely used to evaluate DMU(k). In order to resolve this shortage, Charns, Cooper and Rhodes have modified the CCR model by introducing a positive lower limit $(\varepsilon > 0)$ in [4]. Then the constraint conditions of Problems $(\mathrm{CCR}(k))$ and $(\mathrm{CCRLP}(k))$ are replaced as follows:

$$
\begin{matrix} v_i \geq 0,\ i = 1, \ldots, m, \\ u_r \geq 0,\ r = 1, \ldots, s. \end{matrix} \Rightarrow \begin{matrix} v_i \geq \varepsilon,\ i = 1, \ldots, m, \\ u_r \geq \varepsilon,\ r = 1, \ldots, s. \end{matrix}
$$

Then, Problem $(\mathrm{CCRD}(k))$ can be reformulated as follows:

$$
(\mathrm{CCRD}\varepsilon(k)) \begin{cases} \text{minimize} & \theta - \varepsilon \left(\displaystyle\sum_{i=1}^{m} s_{ix} + \sum_{r=1}^{s} s_{ry} \right) \\ \text{subject to} & \theta x(k)_i - \displaystyle\sum_{j=1}^{n} \lambda_j x(j)_i - s_{ix} = 0,\ i = 1, \ldots, m, \\ & \displaystyle\sum_{j=1}^{n} \lambda_j y(j)_r - y(k)_r - s_{ry} = 0,\ r = 1, \ldots, s, \\ & \lambda_j \geq 0,\ j = 1, \ldots, n, \\ & s_{ix} \geq 0,\ i = 1, \ldots, m, \\ & s_{ry} \geq 0,\ r = 1, \ldots, s, \\ & \theta \in \mathbb{R}. \end{cases}
$$

By using an optimal solution $(\theta^*_{\mathrm{CCR}}(k), s^*_x, s^*_y)$ of Problem (CCRD$\varepsilon(k)$), the efficiency of DMU(k) for the CCR model is more strictly evaluated.

Definition 2.2. If $\theta^*_{\mathrm{CCR}}(k) = 1$ and $(s^*_x, s^*_y) = (0,0)$ then DMU(k) is said to be CCR-Pareto-efficient. If $\theta^*_{\mathrm{CCR}}(k) = 1$ and $(s^*_x, s^*_y) \neq (0,0)$ then DMU(k) is said to be CCR-Pareto-inefficient. Otherwise, DMU(k) is said to be CCR-inefficient.

Let T_{CCR} be the production possibility set(PPS) of the CCR model defined in [3] as follows:

$$T_{\mathrm{CCR}} := \left\{ (x,y) : x \geq \sum_{j=1}^{n} \lambda_j x(j),\ 0 \leq y \leq \sum_{j=1}^{n} \lambda_j y(j) \text{ for some } \lambda \geq 0 \right\}.$$

Definition 2.3. (Conical hull) Let E be a nonempty subset in $\mathbb{R}^n$. Then, conic E is called the conical hull of E if it is defined as follows.

$$\text{conic}\, E := \left\{ x \in \mathbb{R}^n : x = \sum_{j=1}^{n} \lambda_j x(j),\ x(j) \in E,\ \lambda_j \geqslant 0,\ j = 1, \ldots, n \right\}.$$

By the definitions of T_{CCR} and conical hull, T_{CCR} is represented as follows:

$$T_{\mathrm{CCR}} = (\text{conic}\,\{(x(1), y(1)), \ldots, (x(n), y(n))\} + (\mathbb{R}^m_+ \times \mathbb{R}^s_-)) \cap (\mathbb{R}^m \times \mathbb{R}^s_+).$$

Hence, T_{CCR} is a closed convex set. We define the efficient frontier of the CCR model as follows:

$$F_{\mathrm{CCR}} = \mathrm{bd}(T_{\mathrm{CCR}} + (\mathbb{R}^m_+ \times \mathbb{R}^s_-)) \cap (\mathbb{R}^m \times \mathbb{R}^s_+).$$

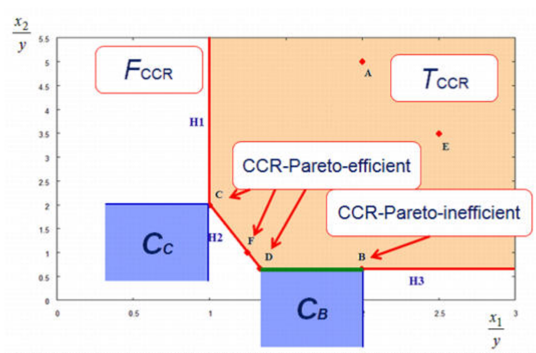

FIGURE 1. CCR-Pareto-efficiency

We explain the efficiency of the CCR model by using Figure 1. There are six DMUs and each DMU have two inputs (x_1, x_2) and one output (y). By Definition 2.1, B,C,D and F are evaluated as CCR-efficient DMUs. Next, we consider a cone $(\mathbb{R}^2_- \times \mathbb{R}^1_+)$+DMU($k$) for each CCR-efficient DMU(k). For example, for C, we consider $C_C := (\mathbb{R}^2_- \times \mathbb{R}^1_+) + (x(C), y(C))$. Then, $(C_C \cap T_{\mathrm{CCR}}) \backslash \mathrm{C} = \phi$, hence C is CCR-Pareto-efficient DMU. Similarly, D and F are evaluated as CCR-Pareto-efficient DMUs. In contrast, let $C_B := (\mathbb{R}^2_- \times \mathbb{R}^1_+) + (x(B), y(B))$ then $(C_B \cap T_{\mathrm{CCR}}) \backslash \mathrm{B} \neq \phi$, hence B is CCR-Pareto-inefficient DMU.

3. Improvements for inefficient DMUs

In this section, we propose three types of improvements for making inefficient DMUs efficient in the CCR model with the minimal change of input and output values. The first improvement is unrestricted, that is, we consider only the minimal change of input and output values. The inefficient DMUs can become efficient units by the smallest change under the condition which the improvement target is feasible. However, the improvement is sometimes Pareto-inefficient in the CCR model. Therefore, we propose the second improvement by forcing the Pareto-efficiency of the CCR model. Moreover, we calculate the third improvement intermediate between the first and second improvements by considering the convex combination and a projection.

First, we define the norm depending on a symmetric positive semidefinite matrix $A \in \mathbb{R}^{(m+s)\times(m+s)}$ as follows.

$$||Z||_A := \sqrt{Z^\top A Z},\ Z \in \mathbb{R}^{m+s}.$$

Under this norm, we consider the minimal change of input and output values for each inefficient DMUs.

Example 3.1. In the case of $A = I_{m+s}$, $||\cdot||_A$ corresponds to the Euclidean norm. If A is defined by

$$A = M_k := \begin{pmatrix} \left(\frac{1}{P(k)_1}\right)^2 & & \underset{\sim}{0} \\ & \ddots & \\ \underset{\sim}{0} & & \left(\frac{1}{P(k)_{m+s}}\right)^2 \end{pmatrix},$$

then $||\cdot||_A$ means the norm which considered the ratio of input and output values.

Let N_c be the number of facets forming the efficient frontier of the CCR model and let S_c be the index set of all facets. Then, we note that $N_c < \infty$ and we can calculate the coefficients of equations forming the facets(see [9]). Let $W_j := (-p_j^\top, q_j^\top)^\top$ for each $j \in S_c$, where, $p_j, q_j \geq 0$, $p_j \in \mathbb{R}^m$ and $q_j \in \mathbb{R}^s$. By using W_j, we represent T_{CCR} and F_{CCR} as follows.

Theorem 3.2. $T_{\mathrm{CCR}} = \displaystyle\bigcap_{j\in S_c} \{Z : W_j^\top Z \leq 0\}$.

Proof. Firstly, we shall show that $T_{\mathrm{CCR}} \subset \bigcap_{j\in S_c}\{Z : W_j^\top Z \leq 0\}$. For each $Z := (x^\top, y^\top)^\top \in T_{\mathrm{CCR}}$, there exists $\lambda^{'} \geq 0$ such that $x \geq \sum_{i=1}^n \lambda_i^{'} x(i),\ y \leq \sum_{i=1}^n \lambda_i^{'} y(i)$. Since $W_j = (-p_j^\top, q_j^\top)^\top$, then $W_j^\top Z = -p_j^\top x + q_j^\top y \leq -p_j^\top \sum_{i=1}^n \lambda_i^{'} x(i) + q_j^\top \sum_{i=1}^n \lambda_i^{'} y(i)$. By the definition of F_{CCR}, $-p_j^\top x(i) + q_j^\top y(i) \leq 0$ for each $i \in \{1, \ldots, n\}$. Hence, $W_j^\top Z \leq 0$ and $(x^\top, y^\top)^\top \in \bigcap_{j\in S_c}\{Z : W_j^\top Z \leq 0\}$. Therefore, $T_{\mathrm{CCR}} \subset \bigcap_{j\in S_c}\{Z : W_j^\top Z \leq 0\}$. Secondly, we shall show that $T_{\mathrm{CCR}} \supset \bigcap_{j\in S_c}\{Z : W_j^\top Z \leq 0\}$. For each $Z \in \bigcap_{j\in S_c}\{Z : W_j^\top Z \leq 0\}$, the following two cases occur.

(i): There exists $j \in S_c$ such that $W_j^\top Z = 0$.

(ii): There exist no $j \in S_c$ such that $W_j^\top Z = 0$.

In Case (i), by the definition of W_j, there exists $\lambda \geq 0$ such that $x = \sum_{i=1}^n \lambda_i x(i),\ y = \sum_{i=1}^n \lambda_i y(i)$. Hence, $Z \in T_{\mathrm{CCR}}$. In Case (ii), there exist $\delta > 0$ and $j \in S_c$ such that $W_j^\top (Z + \delta W_j) = 0$ and $W_k^\top (Z + \delta W_k) \leq 0$ for each $k \in S_c$. Let $Z^{'} := Z + \delta W_j$. Then, $x \geq x^{'}$ and $y \leq y^{'}$. By definition of W_j, there exists $\lambda \geq 0$ such that

$x^{'} = \sum_{i=1}^{n} \lambda_i x(i)$, $y^{'} = \sum_{i=1}^{n} \lambda_i y(i)$. Hence, $Z^{'} \in T_{\mathrm{CCR}}$ and $Z \in T_{\mathrm{CCR}}$. Therefore, $T_{\mathrm{CCR}} \supset \bigcap_{j\in S_c}\{Z : W_j^\top Z \leq 0\}$. Consequently, $T_{\mathrm{CCR}} = \bigcap_{j\in S_c}\{Z : W_j^\top Z \leq 0\}$. □

Theorem 3.3. $F_{\mathrm{CCR}} = \left(\bigcup_{j\in S_c}\{Z : W_j^\top Z = 0\}\right) \cap T_{\mathrm{CCR}}$.

Proof. Firstly, we shall show that $F_{\mathrm{CCR}} \subset \left(\bigcup_{j\in S_c}\{Z : W_j^\top Z = 0\}\right) \cap T_{\mathrm{CCR}}$. For each $Z^{'} := (x^{'\top}, y^{'\top})^\top \in F_{\mathrm{CCR}}$, $(x^{'\top}, y^{'\top})^\top \in T_{\mathrm{CCR}}$. Let $(\theta^*_{\mathrm{CCR}}(Z^{'}), \lambda_1^*, \ldots, \lambda_n^*)$ be an optimal solution of the CCR model for $Z^{'}$, that is $\theta^*_{\mathrm{CCR}}(Z^{'})$ solves the following problem.

$$(\mathrm{CCR(Z^{'})})\left\{\begin{array}{ll} \text{minimize} & \theta \\ \text{subject to} & \theta x_i^{'} - \displaystyle\sum_{j=1}^{n} \lambda_j x(j)_i \geq 0 \;\; i = 1, \ldots, m, \\ & \displaystyle\sum_{j=1}^{n} \lambda_j y(j)_r - y_r^{'} \geq 0 \;\; r = 1, \ldots, s, \\ & \lambda_j \geq 0 \;\; j = 1, \ldots, n, \\ & \theta \in \mathbb{R}. \end{array}\right.$$

Since $\theta^*_{\mathrm{CCR}}(Z^{'}) = 1$, there exists i such that $x_i^{'} = \sum_{j=1}^{n} \lambda_j^* x(j)_i$. Hence, $(x^{'\top}, y^{'\top})^\top \in \mathrm{bd}(T_{\mathrm{CCR}})$. By Theorem 3.2, there exists $j \in S_c$ such that $W_j^\top Z^{'} = 0$. Hence, $Z^{'} \in \bigcup_{j\in S_c}\{Z : W_j^\top Z = 0\}$. Therefore, $F_{\mathrm{CCR}} \subset \left(\bigcup_{j\in S_c}\{Z : W_j^\top Z = 0\}\right) \cap T_{\mathrm{CCR}}$. Secondly, we shall show that $F_{\mathrm{CCR}} \supset \left(\bigcup_{j\in S_c}\{Z : W_j^\top Z = 0\}\right) \cap T_{\mathrm{CCR}}$. For each $Z^{'} \in \left(\bigcup_{j\in S_c}\{Z : W_j^\top Z = 0\}\right) \cap T_{\mathrm{CCR}}$, by Theorem 3.2, $Z^{'} \in \mathrm{bd}(T_{\mathrm{CCR}})$. By definition of F_{CCR}, $Z^{'} \in F_{\mathrm{CCR}}$. Therefore, $F_{\mathrm{CCR}} \supset \left(\bigcup_{j\in S_c}\{Z : W_j^\top Z = 0\}\right) \cap T_{\mathrm{CCR}}$. Consequently, $F_{\mathrm{CCR}} = \left(\bigcup_{j\in S_c}\{Z : W_j^\top Z = 0\}\right) \cap T_{\mathrm{CCR}}$. □

We propose the following algorithm for obtaining the improvements $d^\alpha(k)$, where $\alpha \in \{0, 1\}$. Improvements for $\mathrm{DMU}(k)$ are obtained by the following algorithm:

Algorithm GIT:

Step 0: @

Select $\alpha \in \{0, 1\}$ (Choose the type of the improvment). Set $j := 1$ and go to Step 1.

Step 1: @

If $\alpha = 1$, then set

$$S_c^{'} := \{l \in S_c : W_{li} \neq 0 \;\; i \in \{1, \ldots, m+s\}\} \text{ and } S := S_c^{'}.$$

If $\alpha = 0$, then set

$$S := S_c.$$

Let N be the number of elements of S. Go to Step 2.

Step 2: @

Let $d_j^\alpha(k)$ be an optimal solution of Problem $(\mathrm{MIT}_j^\alpha(k))$ defined as follows:

$$(\mathrm{MIT}_j^\alpha(k))\left\{\begin{array}{ll} \text{minimize} & ||Z||_A \\ \text{subject to} & (Z + P(k))^\top W_j = 0, \\ & \alpha(Z + P(k))^\top W_o \leq 0 \text{ for each } o \in S, \end{array}\right.$$

where, j denote the jth element of S. If $j = N$, then go to Step 3. Otherwise, set $j \leftarrow j + 1$ and go to Step 2.

Step 3: @
Select $j^{'} \in \arg\min\{||d_j^{\alpha}(k)||_A : j \in S\}$ and set $d^{\alpha}(k) := d_{j^{'}}^{\alpha}(k)$. This algorithm terminates.

We can execute Algorithm GIT using the existing nonlinear optimization techniques (e.g. [2]). The existence and properties of an optimal solution are proved by the following theorems.

Theorem 3.4. *For each $\alpha \in \{0,1\}$, Problem $(\mathrm{MIT}_j^{\alpha}(k))$ has an optimal solution.*

Proof. Let $B_j^{\alpha}(k)$ be the feasible sets of Problem $(\mathrm{MIT}_j^{\alpha}(k))$ $(\alpha \in \{0,1\})$. We show the case of $\alpha = 0$. For the case of $\alpha = 1$, we can complete the proof in a way similar to the case of $\alpha = 0$. By the definition of T_{CCR}, $0 \in T_{\mathrm{CCR}}$. Since T_{CCR} is closed, by Theorem 3.3, F_{CCR} is closed and $0 \in F_{\mathrm{CCR}}$. Hence, $Z = -P(k)$ is a feasible solution and $\{Z : (Z + P(k))^{\top} W_j = 0\}$ is closed. Therefore, $B_j^0(k)$ is nonempty and closed. Since $B_j^0(k)$ is nonempty, for each $(x^{'}, y^{'}) \in B_j^0(k)$, $\bar{B}_j^0(k) := B_j^0(k) \cap \{(x^{\top}, y^{\top})^{\top} : ||(x^{\top}, y^{\top})^{\top}||_A \leq ||(x^{'\top}, y^{'\top})^{\top}||_A\}$ is compact. Therefore, we note that Problem $(\mathrm{MIT}_j^0(k))$ is equivalent to the following problem.

$$(\overline{\mathrm{MIT}}_j^0(k)) \left\{ \begin{array}{ll} \text{minimize} & ||Z||_A \\ \text{subject to} & Z \in \bar{B}_j^0(k). \end{array} \right.$$

Since $\bar{B}_j^0(k)$ is compact, by the continuity of the objective function, Problem $(\overline{\mathrm{MIT}}_j^0(k))$ has an optimal solution. By the definition of $\bar{B}_j^0(k)$, an optimal solution of Problem $(\overline{\mathrm{MIT}}_j^0(k))$ is also an optimal solution of Problem $(\mathrm{MIT}_j^0(k))$. Therefore, Problem $(\mathrm{MIT}_j^0(k))$ has an optimal solution. □

Theorem 3.5. *For each CCR-inefficient $\mathrm{DMU}(k)$, let $d^{\alpha}(k)$ $(\alpha \in \{0,1\})$ be an optimal solution calculated by Algorithm GIT. Then, $P(k) + d^{\alpha}(k) \in F_{\mathrm{CCR}}$.*

Proof. We prove the case of $\alpha = 0$. In order to obtain a contradiction, we suppose that $P(k) + d^0(k) \notin F_{\mathrm{CCR}}$. By Theorem 3.3, $P(k) + d^0(k) \notin T_{\mathrm{CCR}}$, and by Theorem 3.2, there exists $j \in S_c$ such that $(P(k) + d^0(k))^{\top} W_j > 0$. Since $\mathrm{DMU}(k)$ is a CCR-inefficient DMU, $P(k) \in \mathrm{int}\, T_{\mathrm{CCR}}$. Hence, from Theorem 3.2, $P(k)^{\top} W_j < 0$ and $(\gamma(P(k) + d^0(k)) + (1-\gamma)P(k))^{\top} W_j = (P(k) + \gamma d^0(k))^{\top} W_j = 0$, where $\gamma := -\frac{P(k)^{\top} W_j}{d^0(k)^{\top} W_j}$. Since $(P(k) + d^0(k))^{\top} W_j > 0$, we obtain $0 < \gamma < 1$. Therefore, $\gamma d^0(k)$ is a feasible solution of Problem $(\mathrm{MIT}_j^0(k))$. By the definition of $d_j^0(k)$, we have the following inequality: $||d_j^0(k)||_A \leq ||\gamma d^0(k)||_A < ||d^0(k)||_A$. This contradicts the optimality of $d^0(k)$ for Algorithm GIT. Consequently, $P(k) + d^0(k) \in F_{\mathrm{CCR}}$. For the case of $\alpha = 1$, we replace S_c by $S_c^{'}$ and can complete the proof in a way similar to the case of $\alpha = 0$. □

By Theorem 3.5, we note that $P(k) + d^0(k)$ is a CCR-efficient point for each CCR-inefficient $\mathrm{DMU}(k)$. Moreover, we obtain a Pareto-efficient point based on parameter $\alpha = 1$ as indicated by the following theorem.

Theorem 3.6. *For each CCR-inefficient $\mathrm{DMU}(k)$, let $d^1(k)$ be an optimal solution calculated by modified Algorithm $\mathrm{GIT}(\alpha = 1)$. Then, $P(k) + d^1(k) \in F_{\mathrm{CCR}}$ is a CCR-Pareto-efficient point.*

Proof. By Theorems 3.4 and 3.5, the existence of an optimal solution and $P(k) + d^1(k) \in F_{\mathrm{CCR}}$ are proved. In order to obtain a contradiction, we suppose that $P(k) + d^1(k)$ has positive slack, that is, there exist slack vectors $s^x \geq 0 \in \mathbb{R}^m$ and $s^y \geq 0 \in \mathbb{R}^s$ satisfying $(s^{x\top}, s^{y\top}) \neq (0, 0)$, and $P(k) + d^1(k) + (-s^{x\top}, s^{y\top})^{\top} \in F_{\mathrm{CCR}}$.

Since $d^1(k)$ is an optimal solution of Problem $(\mathrm{MIT}^1_j(k))$ for some $j \in \{1, \ldots, N\}$, there exists $j \in S$ such that $(d^1(k) + P(k))^\top W_j = 0$. Then $(d^1(k) + P(k) + (-s^{x\top}, s^{y\top})^\top)^\top W_j = (-s^{x\top}, s^{y\top})W_j > 0$. By Theorems 3.2 and 3.3, this contradicts $P(k)+d^1(k)+(-s^{x\top}, s^{y\top})^\top \in F_{\mathrm{CCR}}$. Therefore, $P(k)+d^1(k)$ is a CCR-Pareto-efficient point. □

By Theorems 3.3 and 3.5, $P(k) + d^0(k)$ and $P(k) + d^1(k)$ are contained in T_{CCR}. Since T_{CCR} is a closed convex set, $d^\lambda(k) := \lambda(P(k)+d^0(k))+(1-\lambda)(P(k)+d^1(k)) \in T_{\mathrm{CCR}}$ for each $\lambda \in (0, 1)$, where $d^0(k)$ and $d^1(k)$ are optimal solutions calculated by modified Algorithm $\mathrm{GIT}(\alpha = 0)$ and $(\alpha = 1)$, respectively. However, we note that $d^\lambda(k)$ is not always contained in F_{CCR}, since F_{CCR} is not convex set. In order to calculate a point on F_{CCR} based on $d^\lambda(k)$, we consider a projection. Let $\bar{\beta} := \min\{\beta : (P(k) + \beta(d^\lambda(k) - P(k)))^\top W_j = 0 \text{ for some } j \in S_c\}$. Then, by Theorems 3.2 and 3.3, $P(k)+\bar{\beta}(d^\lambda(k)-P(k)) \in F_{\mathrm{CCR}}$. We propose this point $P(k)+\bar{\beta}(d^\lambda(k)-P(k))$ as improvement intermediate between the two improvements which are obtained based on $d^0(k)$ and $d^1(k)$.

We explain the improvements proposed in this paper by using Figure 2. Now, we consider the improvements for E which is CCR-inefficient DMU. By using the optimal value of Problem (CCRD(k)), we obtain a traditional improvement target d_{CCR}. This improvement improves the only input values at the same rate, that is the output value is fixed. In contrast, we calculate three kinds of improvements which improve the input and output values. If the decision makers of E want to calculate a minimal distance target, then they select $\alpha = 0$ and obtain $d^0(\mathrm{E})$. If they want to calculate a Pareto-efficien target, then they select $\alpha = 1$ and obtain $d^1(\mathrm{E})$. In order to calculate a intermediate target, we consider a convex combination and project the point on F_{CCR} as shown in Figure 2.

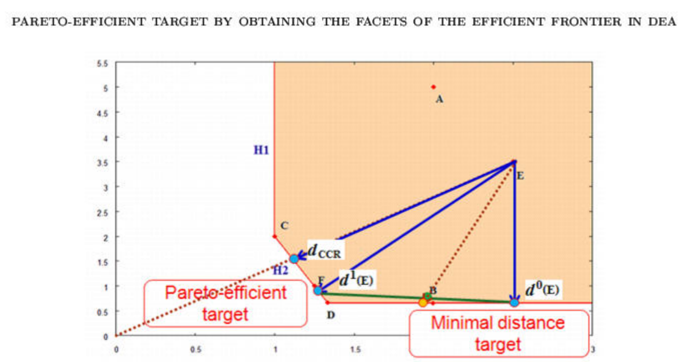

FIGURE 2. Improvement targets

4. EXAMPLE

In this section, we perform a numerical analysis for 10 Japanese banks by utilizing algorithms provided in this paper. As shown in Table 1, each bank has the ordinary profit as the single output. The number of employees and total assets are the two inputs used to generate the output.

TABLE 1. Inputs and Output values for 10 Japanese banks, 2008

Bank	Input1 (persons)	Input2 (one hundred million Japanese yen)	Output (one hundred million Japanese yen)
B1	3701	119895	3179
B2	3675	98359	2688
B3	3659	80955	2180
B4	3004	59600	1563
B5	2887	66373	1477
B6	2872	90984	2450
B7	2752	60770	1852
B8	2506	49008	1137
B9	2268	41151	1148
B10	2148	41158	1124

In this example, we obtain four hyperplanes forming F_{CCR} by using Algorithm FFC as follows.

$$\begin{aligned}
H_1 &:= \{(x_1, x_2, y) : -71.9x_1 - x_2 + 121.4y = 0\},\\
H_2 &:= \{(x_1, x_2, y) : -13.8x_1 - x_2 + 53.3y = 0\},\\
H_3 &:= \{(x_1, x_2, y) : -x_1 + 1.2y = 0\},\\
H_4 &:= \{(x_1, x_2, y) : -x_2 + 32.8y = 0\}.
\end{aligned}$$

By using the coefficients of the equations forming the hyperplanes, we can calculate CCR-efficiency scores without solving linear programming problem for each DMU(see [6, 9]). The CCR-efficiency scores are shown in the Table 2. Three banks (B1, B6 and B7) are evaluated as CCR-efficient DMUs and they do not have positive slack, hence, they do not have to think the improvement. The other bank's improvements are given by Tables 3 and 4. The minimal distance improvement target of each CCR-inefficient DMU is given in Table 3. The improvement shown in Table 4 is CCR-Pareto improvement. The two improvements for B2 coincide and the other DMUs have different two improvements. The improvements over the efficient frontier of the CCR model think decreasing inputs values and increasing output value.

TABLE 2. DEA analysis for 10 Japanese banks, 2008

Bank	CCR
B1	1.000000
B2	0.961359
B3	0.884268
B4	0.860520
B5	0.741447
B6	1.000000
B7	1.000000
B8	0.761275
B9	0.915398
B10	0.896108

TABLE 3. Minimal distance improvement ($A = M_k$)

Bank	Input1	Input2	Output
B1	-	-	-
B2	-32	-900	83
B3	0.00	-5290	126
B4	0.00	-4770	108
B5	0.00	-11665	190
B6	-	-	-
B7	-	-	-
B8	0.00	-7415	131
B9	0.00	-1896	48
B10	0.00	-2368	58

TABLE 4. Pareto-efficient improvement ($A = M_k$)

Bank	Input1	Input2	Output
B1	-	-	-
B2	-32	-900	83
B3	-215	-1529	201
B4	-324	-412	241
B5	-208	-7193	326
B6	-	-	-
B7	-	-	-
B8	-521	-3567	229
B9	-466	-1119	69
B10	-346	-1136	93

5. CONCLUSIONS

In this paper, we have proposed Algorithm GIT for calculating three kinds of improvements for CCR-inefficient DMUs. In order to calculate the improvements, all equations forming F_{CCR} have been used. By using a property of the coefficients of them, we have calculated three kinds of improvements.

The first improvement turns to the closest point over F_{CCR}. Sometimes, the improvement have a positive slack, that is, it may be a CCR-Pareto-inefficient point. In order to calculate CCR-Pareto-efficient point on F_{CCR}, we have proposed the second improvement. The decision makers can select either a minimal distance improvement or Pareto-efficient improvement by choosing a parameter in Algorithm GIT. Moreover, to calculate more flexible improvements, we have suggested a method using the convex combination and a projection. In the convex combination, the decision makers can adjust which they emphasize, feasibility or efficiency.

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