Evaluation methods for intuitionistic fuzzy sets based on set-relations

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ABSTRACT

Fuzzy sets are used to deal with data that are vague or imprecise. The intuitionistic fuzzy set, developed by Atanassov, is a generalization of a fuzzy set. It is characterized by two functions expressing degrees of belongingness and nonbelongingness, respectively.

Ordering fuzzy numbers and fuzzy sets are vital for fuzzy logic and decision-making. In classical fuzzy set theory, there are several ranking methods for fuzzy numbers and intuitionistic fuzzy numbers. In 1985, the fuzzy max order for fuzzy numbers was introduced and since then, many researchers have extended this order for fuzzy sets. In 2016, a ranking method for intuitionistic fuzzy numbers was introduced which used a total ordering.

On the other hand, Ike and Tanaka introduced a new evaluation method via level sets of two fuzzy sets in 2018. The method is based on set-relations proposed by Kuroiwa, Tanaka, and Ha using a vector ordering by a convex cone.

In this study, we present new evaluation methods for intuitionistic fuzzy sets derived through a partial ordering defined on their cut mappings. From the viewpoint of set optimization, eight types of intuitionistic fuzzy-set relations based on set relations in a vector space are proposed as a new comparison criteria of intuitionistic fuzzy sets. Moreover, an evaluation measure of the difference between two intuitionistic fuzzy sets are introduced and related properties are investigated.

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Though there is a possibility that these words may never reach their ears, may it be a comfort that somehow, in a single page of this dissertation, rests the proof of their good deeds, and know that they are remembered.

Contents

Chapter 1

Introduction

A fuzzy set is a mathematical concept introduced by Lotfi Zadeh [17] in 1965 as an extension of classical set theory. While classical sets allow an element to either fully belong or not belong to a set, fuzzy sets introduce the concept of partial membership. In other words, fuzzy sets allow for degrees of membership between 0 and 1, representing the extent to which an element belongs to a set.

When comparing fuzzy numbers, an inequality relation can be defined to determine the relative ordering between them. The most commonly used inequality relation for fuzzy numbers is the α-cut-based inequality relation, where the comparison is made based on the α-cut values of the fuzzy numbers. These inequality relations can be used to compare fuzzy numbers and establish their relative order in terms of their membership degrees.

In this research, we study evaluation methods for intuitionistic fuzzy sets which is a generalization of a fuzzy set. The concept of intuitionistic fuzzy set was introduced by Atanassov [1] and it is characterized by two functions expressing degree of belongingness and the degree of nonbelongingness, respectively. This kind of general fuzzy sets seems to be powerful in modeling mathematical problems by means of fuzzy set theory.

Intuitionistic fuzzy sets provide a more comprehensive representation of uncertainty compared to traditional fuzzy sets and find applications in various fields, including decision-making, pattern recognition, classification, and expert systems. They allow for a more nuanced handling of incomplete and imprecise information, making them a valuable tool in dealing with real-world uncertainties.

There are several ranking methods for fuzzy numbers and intuitionistic fuzzy numbers, which are specific concepts of fuzzy set; see Ramík and Řimánek [10] in 1985. In [10], the fuzzy max order for fuzzy numbers is introduced, and then several researchers have extended this order for fuzzy vectors. In [6], the fuzzy max order is extended for fuzzy sets by using orderings of level sets of fuzzy sets. In 2016, Nayagam, Jeevaraj, and Sivaraman [5] introduce a ranking method for intuitionistic fuzzy numbers which is a total ordering for intuitionistic fuzzy numbers.

Ike and Tanaka [4] in 2018 give a new evaluation method via level sets of two fuzzy sets which is similar to the approach of Kon [6] in 2014. The method is based on set-relations proposed by Kuroiwa, Tanaka, and Ha [13] using a vector ordering by a convex cone, and related studies are made in [3, 16, 11]. Convex cones play an essential role in convex optimization, as many optimization problems involve finding the minimum of a convex function over a convex cone. The convexity of the cone ensures that the optimization problem is well-structured, and efficient algorithms can be employed to find the optimal solution. Additionally, convex cones have applications in various fields, such as economics, engineering, and computer science.

Most of the results are derived in a similar way as in [4] but we do not require the norm structure in the topological vector space, and we use general concepts of upper and lower continuity along with Hausdorff upper and lower continuity. Also, Pareto minimal and maximal points in a partially ordered set are considered instead of singleton sets. Detailed descriptions with complete proofs are given as a self-contained paper for the convenience of readers.

The main results of this research are the correspondences between the eight types of intuitionistic fuzzy-set relations and their difference evaluation functions. The proofs for the eight theorems are narrowed down to four distinct proofs using established relationships and properties of set relations.

This thesis is organized as follows. We present a recollection of basic concepts and terminology in Chapter 2. The notions of intuitionistic fuzzy-set relations and their difference evaluation functions are introduced in Chapter 3. Related properties as well as the correspondences between them are discussed in the same chapter. Finally, summary and recommendations are provided in Chapter 4.

Chapter 2

Mathematical preliminaries

We begin by recalling some fundamental concepts and related properties in the areas of topological vector spaces, fuzzy set theory, and set optimization.

2.1 Topological vector spaces

Let Z be a real vector space and $\mathcal{P}(Z)$ denote the set of all nonempty subsets of Z. The *addition* and *scalar multiplication* are defined respectively by

$$A + B \coloneqq \{a + b \mid a \in A, b \in B\},\ \lambda A \coloneqq \{\lambda a \mid a \in A\}$$

for $A, B \in \mathcal{P}(Z)$ and $\lambda \in \mathbb{R}$. A real topological vector space Z is a vector space equipped with a topology such that the addition $Z \times Z \ni (z, z') \mapsto$

$z + z' \in Z$ and scalar multiplication $\mathbb{R} \times Z \ni (\lambda \times Z) \mapsto \lambda z \in Z$ are both continuous. The topological interior, topological closure, and complement of a set A are denoted by $\operatorname{int} A$, $\operatorname{cl} A$, and A^c, respectively.

For a topological space X and $x_0 \in X$, we denote by $\mathcal{N}_X(x_0)$ the set of all neighborhoods of x_0 in X.

Remark 2.1.1. The topological structure of a topological vector space Z about any point is determined by a base of neighborhoods of θ_Z. If $\mathcal{U}$ is a base of neighborhoods of θ_Z, then the sets $z + U$ constitute a base of neighborhoods of z for some $U \in \mathcal{U}$ and then $\mathcal{U}$ is called the *local base* in Z. Also, every $U \in \mathcal{U}$ is absorbing, and for $U \in \mathcal{U}$, there exists a balanced neighborhood $V \in \mathcal{U}$ such that $V + V \subset U$; V is called a balanced set when $tV \subset V$ for $|t| \leq 1$. V is balanced if and only if V is symmetric $(-V = V)$ and $tV \subset V$ for $0 \leq t < 1$. See [2] for more details.

Definition 2.1.1 (Continuity notions for set-valued map, [18])**.** Let Z be a topological vector space and X a topological space. A set-valued map $F : X \to \mathcal{P}(Z)$ is said to be
(i) *upper continuous (u.c.) at* $x_0 \in X$ if

$$\forall W \in \mathcal{P}(Z), W \text{ open}, F(x_0) \subset W, \exists U \in \mathcal{N}_X(x_0), \forall x \in U, F(x) \subset W;$$

(ii) *lower continuous (l.c.) at* $x_0 \in X$ if

$$\forall W \in \mathcal{P}(Z), W \text{ open}, F(x_0) \cap W \neq \emptyset, \exists U \in \mathcal{N}_X(x_0), \forall x \in U, F(x) \cap W \neq \emptyset;$$

(iii) *Hausdorff upper continuous (H-u.c.) at* $x_0 \in X$ if

$$\forall W \in \mathcal{U}, \exists U \in \mathcal{N}_X(x_0), \forall x \in U, F(x) \subset F(x_0) + W;$$

(iv) *Hausdorff lower continuous (H-l.c.) at* $x_0 \in X$ if

$$\forall W \in \mathcal{U}, \exists U \in \mathcal{N}_X(x_0), \forall x \in U, F(x_0) \subset F(x) + W;$$

(v) *upper continuous (resp., l.c, H-u.c, H-l.c.)* if F is so at every $x \in X$.

Remark 2.1.2. If a set-valued map $F : X \to \mathcal{P}(Z)$ is u.c. at $x_0 \in X$, then F is H-u.c. at x_0; the converse is true when $F(x_0)$ is compact. If F is H-l.c. at $x_0 \in X$, then F is l.c. at x_0; the converse is true when $F(x_0)$ is compact; see [18].

Lemma 2.1.1 ([4, Lemma 3.1])**.** *Let* $\emptyset \neq K \subset Z$ *be compact and* $\emptyset \neq O \subset Z$ *be open. Then* $\bigcap_{v \in K}(v + O)$ *is open.*

For a topological space X, a function $f : X \to \mathbb{R} \cup \{\pm\infty\}$ is said to be lower semicontinuous at $x_0 \in X$ if for any $s < f(x_0)$, there exists $U \in \mathcal{N}_X(x_0)$ such that $s \leq f(x)$ for all $x \in U$. f is lower semicontinuous, denoted by l.s.c., if it is so at every $x \in X$. As is well known, a lower semicontinuous function defined on a compact space always has a minimum.

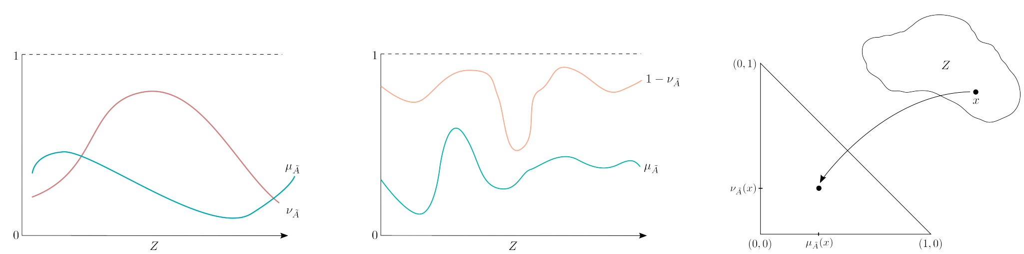
2.2 Intuitionistic fuzzy sets

The concept of a *fuzzy set* introduced by Zadeh [17] is a generalization of an ordinary set or *crisp set*. A fuzzy set $\widetilde{A}$ on Z is uniquely determined by its *membership function* $\mu_{\widetilde{A}} : Z \to [0,1]$ which represents the grade of membership of an element z in $\widetilde{A}$.

Definition 2.2.1 (intuitionistic fuzzy set, [1])**.** A pair $\widetilde{A} = (\mu_{\widetilde{A}}, \nu_{\widetilde{A}})$ is called an *intuitionistic fuzzy set* or *IFS* on Z, where

$$\mu_{\widetilde{A}} : Z \to [0,1] \text{ and } \nu_{\widetilde{A}} : Z \to [0,1]$$

are the membership and non-membership functions, respectively, if for all $z \in Z$, $0 \le \mu_{\widetilde{A}}(z) + \nu_{\widetilde{A}}(z) \le 1$. When $\mu_{\widetilde{A}}(z) + \nu_{\widetilde{A}}(z) = 1$, $\widetilde{A}$ is called a *fuzzy set*.



1st form 2nd form 3rd form

Geometrical interpretations of an IFS

Example 2.2.1. Let z be a natural number and τ be the function that determines the number of the natural numbers smaller than a fixed number z, which do not divide z. Hence for $z \ge 2$, define the set

$$F(z) := \{x \in \mathbb{N} : 1 \le x < z \text{ and } x \nmid z\},$$

and $\tau(z) \coloneqq n(F(z))$, where $n(X)$ is the cardinality of X. Let

$$\mu_1(z) \coloneqq n(\{x \in \mathbb{N} : 1 < x \leq z \text{ and } x \mid z\})$$

and $\nu_1(z) = \tau(z)$. Then we can define the function $\widetilde{A}$ that associates each natural number $z \geq 2$ with the pair

$$\widetilde{A}(z) = \left(\frac{\mu_1(z)}{z}, \frac{\nu_1(z)}{z}\right).$$

Thus, $\widetilde{A}$ is an IFS. ◀

Example 2.2.2. Let φ be the Euler's totient function that determines the number of the natural numbers smaller than a fixed number $z \in \mathbb{N}$, which do not have common divisors with z. For $z \geq 2$, let

$$G(z) \coloneqq \{y \in \mathbb{N} : 1 \leq y < z \text{ and } (y, z) = 1\},$$

and $\varphi(z) \coloneqq n(G(z))$, where (p, q) is the greatest common divisor of the natural numbers p and q. Define

$$\mu_2(z) \coloneqq n(\{y \in \mathbb{N} : 1 < y \leq z \text{ and } y \mid z\}),$$

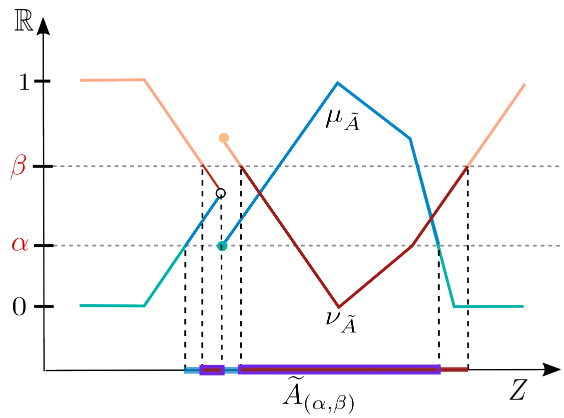
and $\nu_2(z) = \varphi(z)$. Then we can define the function $\widetilde{B}$ that associates each natural number $z \geq 2$ with the pair

$$\widetilde{B}(z) = \left(\frac{\mu_2(z)}{z}, \frac{\nu_2(z)}{z}\right).$$

Hence, $\widetilde{B}$ is an IFS. ◀

Let $\ell = \{(\alpha, \beta) \in [0,1]^2 \mid \alpha + \beta \leq 1\}$ be the set which is used to give values for α, β in the (α, β)-*cut* of $\widetilde{A}$ defined as

$$\widetilde{A}_{(\alpha,\beta)} := \begin{cases} \{z \in Z \mid \mu_{\widetilde{A}}(z) \geq \alpha \text{ and } \nu_{\widetilde{A}}(z) \leq \beta\} & \text{if } (\alpha,\beta) \in \ell \setminus \{(0,1)\}; \\ \operatorname{cl}\{z \in Z \mid \mu_{\widetilde{A}}(z) > 0 \text{ and } \nu_{\widetilde{A}}(z) < 1\} & \text{if } (\alpha,\beta) = (0,1). \end{cases}$$



$\widetilde{A}$ is said to be normal if $\widetilde{A}_{(1,0)} \neq \emptyset$ (or equivalently, $\widetilde{A}_{(\alpha,\beta)} \neq \emptyset$ for all $(\alpha, \beta) \in \ell$). We denote by $\mathcal{F}_N(Z)$ the set of all normal intuitionistic fuzzy sets on Z. For convenience, the set-valued mapping

$$\ell \ni (\alpha,\beta) \mapsto \widetilde{A}_{(\alpha,\beta)} \in \mathcal{P}(Z)$$

is referred to as the *cut mapping* of $\widetilde{A}$.

Example 2.2.3. Let $U = \{x, y, z\}$ and $\widetilde{A}$ be an IFS on U defined as follows:

$$\widetilde{A}(x) = (0.7, 0.1),\ \widetilde{A}(y) = (0.1, 0.1),\ \widetilde{A}(z) = (0.4, 0.5)$$

Then $\widetilde{A}_{(0.1,0.7)} = \{x, y, z\}$, $\widetilde{A}_{(0.2,0.8)} = \{x, z\}$, $\widetilde{A}_{(0.4,0.2)} = \{x\}$. ◀

For each IFS $\widetilde{A}$, we define the translation $\widetilde{A}+z$ for $z \in Z$ by

$$\mu_{\widetilde{A}+z}(z') \coloneqq \mu_{\widetilde{A}}(z'-z) \text{ and } \nu_{\widetilde{A}+z}(z') \coloneqq \nu_{\widetilde{A}}(z'-z)$$

and the scalar multiplication $\lambda\widetilde{A}$ for $\lambda \neq 0$ by

$$\mu_{\lambda\widetilde{A}}(z') \coloneqq \mu_{\widetilde{A}}\left(\tfrac{1}{\lambda}z'\right) \text{ and } \nu_{\lambda\widetilde{A}}(z') \coloneqq \nu_{\widetilde{A}}\left(\tfrac{1}{\lambda}z'\right).$$

By definition, it follows that

$$(\widetilde{A}+z)_{(\alpha,\beta)} = \widetilde{A}_{(\alpha,\beta)} + z \text{ and } (\lambda\widetilde{A})_{(\alpha,\beta)} = \lambda\widetilde{A}_{(\alpha,\beta)}$$

for any $(\alpha,\beta) \in \ell$.

For any intuitionistic fuzzy sets $\widetilde{A}$ and $\widetilde{B}$ defined on Z, their union and intersection are defined respectively as

$$\widetilde{A} \cup \widetilde{B} = \left(\mu_{\widetilde{A}\cup\widetilde{B}}, \nu_{\widetilde{A}\cup\widetilde{B}}\right) = \left(\max\{\mu_{\widetilde{A}}, \mu_{\widetilde{B}}\}, \min\{\nu_{\widetilde{A}}, \nu_{\widetilde{B}}\}\right)$$

and

$$\widetilde{A} \cap \widetilde{B} = \left(\mu_{\widetilde{A}\cap\widetilde{B}}, \nu_{\widetilde{A}\cap\widetilde{B}}\right) = \left(\min\{\mu_{\widetilde{A}}, \mu_{\widetilde{B}}\}, \max\{\nu_{\widetilde{A}}, \nu_{\widetilde{B}}\}\right).$$

By definition, it can be shown that

$$(\widetilde{A} \cup \widetilde{B})_{(\alpha,\beta)} \supset \widetilde{A}_{(\alpha,\beta)} \cup \widetilde{B}_{(\alpha,\beta)} \text{ and } (\widetilde{A} \cap \widetilde{B})_{(\alpha,\beta)} = \widetilde{A}_{(\alpha,\beta)} \cap \widetilde{B}_{(\alpha,\beta)}.$$

2.3 Set relations in set optimization

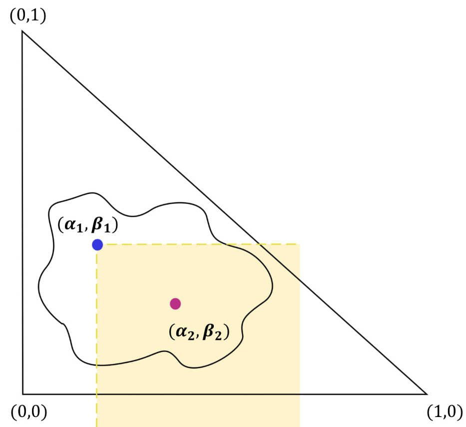
Let X be nonempty set and $\preccurlyeq$ a binary relation on X. The relation $\preccurlyeq$ is said to be *reflexive* if for all $x \in X, x \preccurlyeq x$; *irreflexive* if for all $x \in X, x \not\preccurlyeq x$; *transitive* if for all $x, y, z \in X$, $x \preccurlyeq y$ and $y \preccurlyeq z$ imply $x \preccurlyeq z$; *antisymmetric* if for all $x, y \in X$, $x \preccurlyeq y$ and $y \preccurlyeq x$ imply $x = y$; and *complete* if for all $x, y \in X, x \preccurlyeq y$ or $y \preccurlyeq x$. The relation $\preccurlyeq$ is called a *preorder* if it is reflexive and transitive; a *strict order* if it is irreflexive and transitive; a *partial order* if it is reflexive, transitive, and antisymmetric; a *linear* or *total order* if it is reflexive, transitive, antisymmetric, and complete.

A nonempty subset C of a linear space Z is called *convex* if for all $x, y \in C$, $[x, y] \coloneqq \{\lambda x + (1 - \lambda)y \mid \lambda \in [0, 1]\} \subset C$. A set $C \in \mathcal{P}(Z)$ is called a *cone* if $tz \in C$ for every $z \in C$ and $t > 0$. A cone C is convex if $z_1 + z_2 \in C$, for any $z_1, z_2 \in C$. The transitive relation $\leq_C$ is induced by a convex cone C as follows: for $z, z' \in Z$,

$$z \leq_C z' \iff z' - z \in C.$$

If C is a convex cone in Z, then $C + C = C$. Moreover, $\operatorname{int} C$ and $\operatorname{cl} C$ are also convex cones. In addition, assuming that C is *pointed* (i.e., $C \cap (-C) = \{\theta_Z\}$), we have $x \leq_C y, y \leq_C x \Longrightarrow y - x \in C \cap (-C) = \{\theta_Y\} \Longrightarrow y = x$. Hence, $\leq_C$ is antisymmetric and becomes a partial order.

We define $\preceq$ as the partial order where $(\alpha_1, \beta_1) \preceq (\alpha_2, \beta_2)$ means $\alpha_1 \leq \alpha_2$ and $\beta_1 \geq \beta_2$. Let $K \coloneqq \{(x, y) \in \mathbb{R}^2 \mid x \geq 0 \text{ and } y \leq 0\}$. Then K is pointed and $(\alpha_2, \beta_2) \in (\alpha_1, \beta_1) + K$ if and only if $(\alpha_1, \beta_1) \preceq (\alpha_2, \beta_2)$.



It can be seen that $\preceq$ is a partial order on ℓ and for any $(\alpha_1, \beta_1), (\alpha_2, \beta_2) \in \ell$,

$$\widetilde{A}_{(\alpha_2,\beta_2)} \subset \widetilde{A}_{(\alpha_1,\beta_1)} \text{ whenever } (\alpha_1, \beta_1) \preceq (\alpha_2, \beta_2). \tag{2.1}$$

For $\mathit{\Delta} \subset \ell$, the sets of *Pareto minimal points* and *Pareto maximal points* of $\mathit{\Delta}$ with respect to K as

$$\operatorname{Min} \mathit{\Delta} \coloneqq \{(\alpha, \beta) \in \mathit{\Delta} \mid \mathit{\Delta} \cap ((\alpha, \beta) - K) = \{(\alpha, \beta)\}\}$$

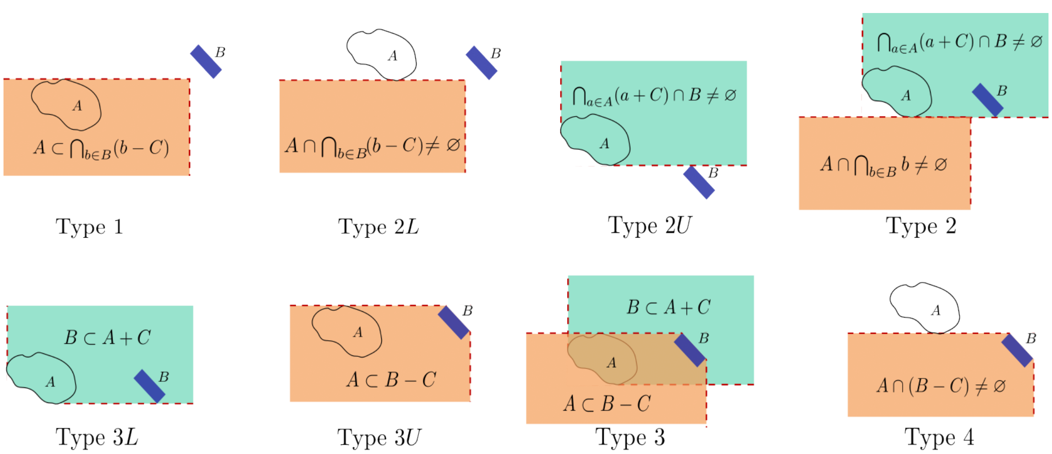
and

$$\operatorname{Max} \mathit{\Delta} \coloneqq \{(\alpha, \beta) \in \mathit{\Delta} \mid \mathit{\Delta} \cap ((\alpha, \beta) + K) = \{(\alpha, \beta)\}\},$$

respectively. Every point of $\mathit{\Delta}$ is said to be *dominated* by a minimal (resp., maximal) point of $\mathit{\Delta}$ if $\mathit{\Delta} \subset \operatorname{Min} \mathit{\Delta} + K$ (resp., $\mathit{\Delta} \subset \operatorname{Max} \mathit{\Delta} - K$), and this is called the *domination property* with respect to K (resp., $-K$).

Definition 2.3.1 (set relations, [13])**.** Let $C \subset Z$ be a convex cone. For $A, B \in \mathcal{P}(Z)$, the eight types of *set relations* are defined by

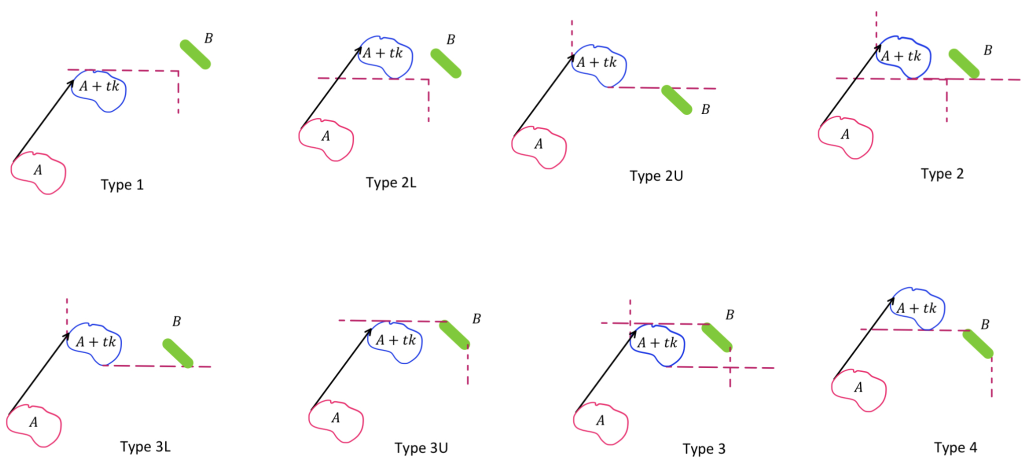
$$
\begin{aligned}
A \leq_C^{(1)} B &\overset{\text{def}}{\iff} \forall a \in A, \forall b \in B, a \leq_C b \iff A \subset \bigcap_{b \in B}(b - C);\\
A \leq_C^{(2L)} B &\overset{\text{def}}{\iff} \exists a \in A \text{ s.t. } \forall b \in B, a \leq_C b \iff A \cap \left(\bigcap_{b \in B}(b - C)\right) \neq \emptyset;\\
A \leq_C^{(2U)} B &\overset{\text{def}}{\iff} \exists b \in B \text{ s.t. } \forall a \in A, a \leq_C b \iff \left(\bigcap_{a \in A}(a + C)\right) \cap B \neq \emptyset;\\
A \leq_C^{(2)} B &\overset{\text{def}}{\iff} A \leq_C^{(2L)} B \text{ and } A \leq_C^{(2U)} B;\\
A \leq_C^{(3L)} B &\overset{\text{def}}{\iff} \forall b \in B, \exists a \in A, a \leq_C b \iff B \subset A + C;\\
A \leq_C^{(3U)} B &\overset{\text{def}}{\iff} \forall a \in A, \exists b \in B, a \leq_C b \iff A \subset B - C;\\
A \leq_C^{(3)} B &\overset{\text{def}}{\iff} A \leq_C^{(3L)} B \text{ and } A \leq_C^{(3U)} B;\\
A \leq_C^{(4)} B &\overset{\text{def}}{\iff} \exists a \in A, \exists b \in B, a \leq_C b \iff A \cap (B - C) \neq \emptyset.
\end{aligned}
$$



Optimization problems with set-valued functions become more convenient to deal with when we convert vectors or sets into a scalar value. Hence, scalarization plays a key role in vector and set optimization. Many researchers investigated scalarization functions for sets related to relations based on Tammer's sublinear scalarization function for vectors [14, 11, 16] given by

$$h_C(v; d) \coloneqq \inf\{t \in \mathbb{R} \mid v \in td - C\},$$

where C is a convex cone and $d \in C$.



Chapter 3

Evaluation methods for intuitionistic fuzzy sets

3.1 Intutionistic fuzzy-set relations

By considering the set relations between (α, β)-cuts of two intuitionistic fuzzy sets, the intuitionistic fuzzy-set relations are defined as follows.

Definition 3.1.1 (intuitionistic fuzzy-set relations)**.** Let $C \subset Z$ be a convex cone and $\emptyset \neq \Delta \subset \ell$. For each $j = 1, 2L, 2U, 2, 3L, 3U, 3, 4$, the *intuitionistic fuzzy-set relation* $\leq_C^{\Delta(j)}$, IFS relation shortly, is defined by

$$\widetilde{A} \leq_C^{\Delta(j)} \widetilde{B} \Longleftrightarrow \forall(\alpha, \beta) \in \Delta, \widetilde{A}_{(\alpha,\beta)} \leq_C^{(j)} \widetilde{B}_{(\alpha,\beta)}$$

for $\widetilde{A}, \widetilde{B} \in \mathcal{F}_N(Z)$.

The values α and β in the (α, β)-cut of $\widetilde{A}$ indicate the suitable or preferable degrees for membership and nonmembership functions of the fuzzy set $\widetilde{A}$. The set Δ can be defined as a collection of such values in comparing intuitionistic fuzzy sets. The IFS relations seem to be a multi-criteria comparison in intuitionistic fuzzy sets.

From the definition, we easily obtain the following implications:

$$\begin{aligned} \widetilde{A} \leq_C^{\Delta(1)} \widetilde{B} \Longrightarrow \widetilde{A} \leq_C^{\Delta(2L)} \widetilde{B} \Longrightarrow \widetilde{A} \leq_C^{\Delta(3L)} \widetilde{B} \Longrightarrow \widetilde{A} \leq_C^{\Delta(4)} \widetilde{B}; \\ \widetilde{A} \leq_C^{\Delta(1)} \widetilde{B} \Longrightarrow \widetilde{A} \leq_C^{\Delta(2U)} \widetilde{B} \Longrightarrow \widetilde{A} \leq_C^{\Delta(3U)} \widetilde{B} \Longrightarrow \widetilde{A} \leq_C^{\Delta(4)} \widetilde{B}; \\ \widetilde{A} \leq_C^{\Delta(1)} \widetilde{B} \Longrightarrow \widetilde{A} \leq_C^{\Delta(2)} \widetilde{B} \Longrightarrow \widetilde{A} \leq_C^{\Delta(3)} \widetilde{B} \Longrightarrow \widetilde{A} \leq_C^{\Delta(4)} \widetilde{B} \end{aligned} \tag{3.1}$$

for any intuitionistic fuzzy sets $\widetilde{A}, \widetilde{B}$.

The following proposition shows that the IFS relation property is inherited by the intersection of intuitionistic fuzzy sets.

Proposition 3.1.1. *Let $C \subset Z$ be a convex cone, Δ a nonempty subset of ℓ, and $\widetilde{A}, \widetilde{B}, \widetilde{S} \in \mathcal{F}_N(Z)$. If $\widetilde{A} \leq_C^{\Delta(1)} \widetilde{B}$, then $\widetilde{A} \cap \widetilde{S} \leq_C^{\Delta(j)} \widetilde{B} \cap \widetilde{S}$ for each $j = 1$, $2L$, $2U$, 2, $3L$, $3U$, 3, 4.*

Proof. Let $(\alpha, \beta) \in \Delta$. For all $x \in (\widetilde{A} \cap \widetilde{S})_{(\alpha,\beta)}$ and $y \in (\widetilde{B} \cap \widetilde{S})_{(\alpha,\beta)}$, we have $x \in \widetilde{A}_{(\alpha,\beta)}$ and $y \in \widetilde{B}_{(\alpha,\beta)}$. By assumption, $\widetilde{A}_{(\alpha,\beta)} \leq_C^{(1)} \widetilde{B}_{(\alpha,\beta)}$ for all $(\alpha, \beta) \in \Delta$, and hence $x \in y - C$ for $x \in \widetilde{A}_{(\alpha,\beta)}$ and $y \in \widetilde{B}_{(\alpha,\beta)}$. This implies that $(\widetilde{A} \cap \widetilde{S})_{(\alpha,\beta)} \leq_C^{(1)} (\widetilde{B} \cap \widetilde{S})_{(\alpha,\beta)}$. Hence, $\widetilde{A} \cap \widetilde{S} \leq_C^{\Delta(1)} \widetilde{B} \cap \widetilde{S}$. Using (3.1), the conclusion follows. □

The following proposition is crucial in proving some results in this thesis.

Proposition 3.1.2. *Let $C \subset Z$ be a convex cone, $k \in C$, and $\emptyset \neq \Delta \subset \ell$, and $\widetilde{A}, \widetilde{B} \in \mathcal{F}_N(Z)$. For each $j = 1, 2L, 2U, 2, 3L, 3U, 3, 4$,*
(i) if $\widetilde{A}+\bar{s}k \leq_C^{\Delta(j)} \widetilde{B}$ for some $\bar{s} \in \mathbb{R}$, then $\widetilde{A}+sk \leq_C^{\Delta(j)} \widetilde{B}$ for all $s \in (-\infty, \bar{s}]$;
(ii) if $\widetilde{A}+\bar{s}k \not\leq_C^{\Delta(j)} \widetilde{B}$ for some $\bar{s} \in \mathbb{R}$, then $\widetilde{A}+sk \not\leq_C^{\Delta(j)} \widetilde{B}$ for all $s \in [\bar{s}, +\infty)$.

Proof. Since C is a convex cone and $k \in C$, $C + tk \subset C$ for any $t > 0$.
(i) Let $s \in (-\infty, \bar{s}]$. Then $C + (\bar{s} - s)k \subset C$. Thus, $a + \bar{s}k \leq_C b$ implies that $a + sk \leq_C b$ for any $a, b \in Z$. Hence, for all $(\alpha, \beta) \in \Delta$, $\widetilde{A}_{(\alpha,\beta)} + \bar{s}k \leq_C^{(j)} \widetilde{B}_{(\alpha,\beta)}$ implies that $\widetilde{A}_{(\alpha,\beta)} + sk \leq_C^{(j)} \widetilde{B}_{(\alpha,\beta)}$. Therefore, $\widetilde{A} + sk \leq_C^{\Delta(j)} \widetilde{B}$.
(ii) The proof is similar to (i). □

Proposition 3.1.3. *Let $C \subset Z$ be a convex cone and $\emptyset \neq \Delta \subset \ell$. If Δ has the domination property with respect to K, then*

$$\widetilde{A} \leq_C^{\Delta(1)} \widetilde{B} \Longleftrightarrow \widetilde{A} \leq_C^{\operatorname{Min}\Delta(1)} \widetilde{B}. \tag{3.2}$$

Also, if Δ has the domination property with respect to $-K$, then

$$\widetilde{A} \leq_C^{\Delta(4)} \widetilde{B} \Longleftrightarrow \widetilde{A} \leq_C^{\operatorname{Max}\Delta(4)} \widetilde{B}. \tag{3.3}$$

Proof. Assume that $\widetilde{A} \leq_C^{\Delta(1)} \widetilde{B}$. Since $\operatorname{Min}\Delta \subset \Delta$, $\widetilde{A}_{(\alpha,\beta)} \leq_C^{(1)} \widetilde{B}_{(\alpha,\beta)}$ for all $(\alpha, \beta) \in \operatorname{Min}\Delta$. Hence, $\widetilde{A} \leq_C^{\operatorname{Min}\Delta(1)} \widetilde{B}$. Conversely, suppose $\widetilde{A} \leq_C^{\operatorname{Min}\Delta(1)} \widetilde{B}$ and let $(\alpha, \beta) \in \Delta$. Since Δ has the domination property with respect to K, there exists $(\alpha_0, \beta_0) \in \operatorname{Min}\Delta$ such that $(\alpha, \beta) \in (\alpha_0, \beta_0) + K$, that is, $(\alpha_0, \beta_0) \preceq (\alpha, \beta)$. By (2.1), $\widetilde{A}_{(\alpha,\beta)} \subset \widetilde{A}_{(\alpha_0,\beta_0)}$ and $\widetilde{B}_{(\alpha,\beta)} \subset \widetilde{B}_{(\alpha_0,\beta_0)}$. By assumption, $\widetilde{A}_{(\alpha_0,\beta_0)} \leq_C^{(1)} \widetilde{B}_{(\alpha_0,\beta_0)}$, which means $z_1 \leq_C z_2$ for $z_1 \in \widetilde{A}_{(\alpha_0,\beta_0)}$

and $z_2 \in \widetilde{B}_{(\alpha_0,\beta_0)}$. Therefore, $\widetilde{A}_{(\alpha,\beta)} \leq_C^{(1)} \widetilde{B}_{(\alpha,\beta)}$ and consequently, $\widetilde{A} \leq_C^{\Delta(1)} \widetilde{B}$. Similarly, (3.3) holds. □

When we compare two intuitionistic fuzzy sets with respect to IFS relation, we can consider evaluation measure of the difference between them in the same manner as in [4].

Definition 3.1.2 (difference evaluation function for IFS relations)**.** Let C be a convex cone in Z, $k \in \operatorname{int} C$, and $\emptyset \neq \Delta \subset \ell$. For each $j = 1$, $2L$, $2U$, 2, $3L$, $3U$, 3, 4, the *difference evaluation function* $D_{C,k}^{\Delta(j)} : \mathcal{F}_N(Z) \times \mathcal{F}_N(Z) \to \mathbb{R} \cup \{\pm\infty\}$ is defined by

$$D_{C,k}^{\Delta(j)}(\widetilde{A}, \widetilde{B}) \coloneqq \sup \left\{ t \in \mathbb{R} \,\middle|\, \widetilde{A} + tk \leq_C^{\Delta(j)} \widetilde{B} \right\}.$$

Proposition 3.1.4. *Let $C \subset Z$ be a convex cone, $k \in \operatorname{int} C$, $\emptyset \neq \Delta \subset \ell$, and $\widetilde{A}, \widetilde{B} \in \mathcal{F}_N(Z)$. Then for each $j = 1, 2L, 2U, 2, 3L, 3U, 3, 4$,*

$$D_{C,k}^{\Delta(j)}(\widetilde{A}, \widetilde{B}) = \inf_{(\alpha,\beta)\in\Delta} \sup \left\{ t \in \mathbb{R} \,\middle|\, \widetilde{A}_{(\alpha,\beta)} + tk \leq_C^{(j)} \widetilde{B}_{(\alpha,\beta)} \right\}.$$

Proof. For each $j = 1, 2L, 2U, 2, 3L, 3U, 3, 4$ and for any $(\alpha', \beta') \in \Delta$, we have

$$\left\{ t \,\middle|\, \forall(\alpha,\beta) \in \Delta, \widetilde{A}_{(\alpha,\beta)} + tk \leq_C^{(j)} \widetilde{B}_{(\alpha,\beta)} \right\} \subset \left\{ t \,\middle|\, \widetilde{A}_{(\alpha',\beta')} + tk \leq_C^{(j)} \widetilde{B}_{(\alpha',\beta')} \right\}$$

where it follows that

$$\sup \left\{ t \,\middle|\, \widetilde{A} + tk \leq_C^{\Delta(j)} \widetilde{B} \right\} \leq \sup \left\{ t \,\middle|\, \widetilde{A}_{(\alpha',\beta')} + tk \leq_C^{(j)} \widetilde{B}_{(\alpha',\beta')} \right\}.$$

By taking the infimum over (α', β') on both sides of this inequality, we obtain

$$D_{C,k}^{\Delta(j)}(\widetilde{A}, \widetilde{B}) \le \inf_{(\alpha,\beta)\in\Delta} \sup \left\{ t \in \mathbb{R} \,\middle|\, \widetilde{A}_{(\alpha,\beta)} + tk \le_C^{(j)} \widetilde{B}_{(\alpha,\beta)} \right\}.$$

Assume that there exists $\bar{s} \in \mathbb{R}$ such that

$$D_{C,k}^{\Delta(j)}(\widetilde{A}, \widetilde{B}) < \bar{s} < \inf_{(\alpha,\beta)\in\Delta} \sup \left\{ t \in \mathbb{R} \,\middle|\, \widetilde{A}_{(\alpha,\beta)} + tk \le_C^{(j)} \widetilde{B}_{(\alpha,\beta)} \right\}.$$

From the second inequality, there exists $s_{(\alpha',\beta')} > \bar{s}$ for any $(\alpha', \beta') \in \Delta$ such that $\widetilde{A}_{(\alpha',\beta')} + s_{(\alpha',\beta')}k \le_C^{(j)} \widetilde{B}_{(\alpha',\beta')}$. By Proposition 3.1.2 (i), we deduce that $\widetilde{A}_{(\alpha',\beta')} + \bar{s}k \le_C^{(j)} \widetilde{B}_{(\alpha',\beta')}$. This implies that $\bar{s} \le D_{C,k}^{\Delta(j)}(\widetilde{A}, \widetilde{B})$ and thus contradicts the assumption $D_{C,k}^{\Delta(j)}(\widetilde{A}, \widetilde{B}) < \bar{s}$. □

The next proposition provides alternative formula for the difference evaluation function when the domination property is satisfied.

Proposition 3.1.5. *Let $C \subset Z$ be a convex cone, $k \in \operatorname{int} C$, $\emptyset \ne \Delta \subset \ell$, and $\widetilde{A}, \widetilde{B} \in \mathcal{F}_N(Z)$. Then*
(i) $\Delta \subset \operatorname{Min} \Delta + K$ *implies* $D_{C,k}^{\Delta(1)}(\widetilde{A}, \widetilde{B}) = \sup \left\{ t \in \mathbb{R} \,\middle|\, \widetilde{A} + tk \le_C^{\operatorname{Min}\Delta(1)} \widetilde{B} \right\}$;
(ii) $\Delta \subset \operatorname{Max} \Delta - K$ *implies* $D_{C,k}^{\Delta(4)}(\widetilde{A}, \widetilde{B}) = \sup \left\{ t \in \mathbb{R} \,\middle|\, \widetilde{A} + tk \le_C^{\operatorname{Max}\Delta(4)} \widetilde{B} \right\}$;
(iii) $D_{C,k}^{\Delta(2)}(\widetilde{A}, \widetilde{B}) = \min \left\{ D_{C,k}^{\Delta(2L)}(\widetilde{A}, \widetilde{B}), D_{C,k}^{\Delta(2U)}(\widetilde{A}, \widetilde{B}) \right\}$;
(iv) $D_{C,k}^{\Delta(3)}(\widetilde{A}, \widetilde{B}) = \min \left\{ D_{C,k}^{\Delta(3L)}(\widetilde{A}, \widetilde{B}), D_{C,k}^{\Delta(3U)}(\widetilde{A}, \widetilde{B}) \right\}$.

Proof. By Definition 3.1.2, $D_{C,k}^{\Delta(j)}(\widetilde{A}, \widetilde{B}) = \sup \left\{ t \in \mathbb{R} \,\middle|\, \widetilde{A} + tk \le_C^{\Delta(j)} \widetilde{B} \right\}$ for each j.

(i) By (3.2), $\widetilde{A} \leq_C^{\Delta(1)} \widetilde{B} \Longleftrightarrow \widetilde{A} \leq_C^{\mathrm{Min}\,\Delta(1)} \widetilde{B}$. Hence,

$$D_{C,k}^{\Delta(1)}(\widetilde{A},\widetilde{B}) = \sup\left\{t \in \mathbb{R} \,\middle|\, \widetilde{A} + tk \leq_C^{\mathrm{Min}\,\Delta(1)} \widetilde{B}\right\}.$$

(ii) By (3.3), $\widetilde{A} \leq_C^{\Delta(4)} \widetilde{B} \Longleftrightarrow \widetilde{A} \leq_C^{\mathrm{Max}\,\Delta(4)} \widetilde{B}$ Hence,

$$D_{C,k}^{\Delta(4)}(\widetilde{A},\widetilde{B}) = \sup\left\{t \in \mathbb{R} \,\middle|\, \widetilde{A} + tk \leq_C^{\mathrm{Max}\,\Delta(4)} \widetilde{B}\right\}.$$

(iii) By Proposition 3.1.4,

$$D_{C,k}^{\Delta(2)}(\widetilde{A},\widetilde{B}) = \inf_{(\alpha,\beta)\in\Delta} \sup\left\{t \in \mathbb{R} \,\middle|\, \widetilde{A}_{(\alpha,\beta)} + tk \leq_C^{(2)} \widetilde{B}_{(\alpha,\beta)}\right\}.$$

We show that $\sup(E \cap F) = \min\{\sup E, \sup F\}$ where for each $(\alpha,\beta) \in \Delta$,

$$E := \left\{t \in \mathbb{R} \,\middle|\, \widetilde{A}_{(\alpha,\beta)} + tk \leq_C^{(2L)} \widetilde{B}_{(\alpha,\beta)}\right\}$$

and

$$F := \left\{t \in \mathbb{R} \,\middle|\, \widetilde{A}_{(\alpha,\beta)} + tk \leq_C^{(2U)} \widetilde{B}_{(\alpha,\beta)}\right\}.$$

Generally, $\sup(E \cap F) \leq \min\{\sup E, \sup F\}$. Assume that there exists $\bar{s} \in \mathbb{R}$ such that $\sup(E \cap F) < \bar{s} < \min\{\sup E, \sup F\}$. From the second inequality, there exist $s'_{(\alpha,\beta)} > \bar{s}$ and $s''_{(\alpha,\beta)} > \bar{s}$ such that $\widetilde{A}_{(\alpha,\beta)} + s'_{(\alpha,\beta)}k \leq_C^{(2L)} \widetilde{B}_{(\alpha,\beta)}$ and $\widetilde{A}_{(\alpha,\beta)} + s''_{(\alpha,\beta)}k \leq_C^{(2U)} \widetilde{B}_{(\alpha,\beta)}$. By Proposition 3.1.2 (i), we deduce that $\widetilde{A}_{(\alpha,\beta)} + \bar{s}k \leq_C^{(j)} \widetilde{B}_{(\alpha,\beta)}$ for $j = 2L, 2U$, which implies that $\bar{s} \leq \sup(E \cap F)$. This contradicts $\sup(E \cap F) < \bar{s}$. Hence, $\sup(E \cap F) = \min\{\sup E, \sup F\}$.

Therefore, by Definition 2.3.1 and Proposition 3.1.4,

$$\begin{aligned} D_{C,k}^{\Delta(2)}(\widetilde{A},\widetilde{B}) &= \inf_{(\alpha,\beta)\in\Delta} \min_{j=2L,2U} \sup\left\{ t\in\mathbb{R} \,\middle|\, \widetilde{A}_{(\alpha,\beta)} + tk \leq_C^{(j)} \widetilde{B}_{(\alpha,\beta)} \right\} \\ &= \min\{D_{C,k}^{\Delta(2L)}(\widetilde{A},\widetilde{B}), D_{C,k}^{\Delta(2U)}(\widetilde{A},\widetilde{B})\}, \end{aligned}$$

which implies the result.

(iv) The assertion is proved in a similar way to (iii). □

3.2 Correspondences between IFS relations and difference evaluation functions

The difference evaluation functions described in Definition 3.1.2 measure how different two intuitionistic fuzzy sets are. By using them, we can characterize evaluation of intuitionistic fuzzy sets. In this section, we prove several results on the characterizations of IFS relations in the same manner as in [4]. Specifically, we establish eight theorems which describe correspondences between the IFS relations and the difference evaluation functions for IFS defined in Section 2.

Definition 3.2.1 (compactness)**.** Let $\emptyset \neq \Delta \subset \ell$. An IFS $\widetilde{A}$ on Z is said to be *Δ-compact* if $\widetilde{A}_{(\alpha,\beta)}$ is compact for all $(\alpha,\beta) \in \Delta$. When $\widetilde{A}$ is $\{(\alpha,\beta)\}$-compact for some $(\alpha,\beta) \in \ell$, we simply say $\widetilde{A}$ is (α,β)-compact.

Using Lemma 2.1.1, the correspondence for the first IFS relation is obtained below.

Theorem 3.2.1 (Type 1)**.** *Let C be a convex cone in Z, $k \in \operatorname{int} C$, Δ a nonempty subset of ℓ, and $\widetilde{A}, \widetilde{B} \in \mathcal{F}_N(Z)$. Then the following statements hold:*

(i) if $\widetilde{A} \leq^{\Delta(1)}_{\operatorname{cl} C} \widetilde{B}$, then $D^{\Delta(1)}_{C,k}(\widetilde{A}, \widetilde{B}) \geq 0$;

(ii) if $D^{\Delta(1)}_{C,k}(\widetilde{A}, \widetilde{B}) \geq 0$, then $\widetilde{A} \leq^{\Delta(1)}_{\operatorname{cl} C} \widetilde{B}$;

(iii) if $\Delta \subset \operatorname{Min} \Delta + K$, $\operatorname{Min} \Delta$ is closed, the cut mappings of $\widetilde{A}$ and $\widetilde{B}$ are lower and upper continuous, respectively, and $\widetilde{A}$, $\widetilde{B}$ are $(\operatorname{Min} \Delta)$-compact, then $\widetilde{A} \leq^{\Delta(1)}_{\operatorname{int} C} \widetilde{B}$ implies $D^{\Delta(1)}_{C,k}(\widetilde{A}, \widetilde{B}) > 0$;

(iv) if $D^{\Delta(1)}_{C,k}(\widetilde{A}, \widetilde{B}) > 0$, then $\widetilde{A} \leq^{\Delta(1)}_{\operatorname{int} C} \widetilde{B}$.

Proof. (i) Assume that $\widetilde{A} \leq^{\Delta(1)}_{\operatorname{cl} C} \widetilde{B}$ and let $s_0 < 0$. Since $\operatorname{cl} C \subset C + s_0 k$, we have $\widetilde{A}_{(\alpha,\beta)} \subset \bigcap_{b \in \widetilde{B}_{(\alpha,\beta)}} (b - \operatorname{cl} C) \subset \bigcap_{b \in \widetilde{B}_{(\alpha,\beta)}} (b - C) - s_0 k$, $\forall (\alpha, \beta) \in \Delta$. This implies that $\widetilde{A} + s_0 k \leq^{\Delta(1)}_{C} \widetilde{B}$, and thus

$$D^{\Delta(1)}_{C,k}(\widetilde{A}, \widetilde{B}) = \sup \left\{ s \in \mathbb{R} \;\middle|\; \widetilde{A} + sk \leq^{\Delta(1)}_{C} \widetilde{B} \right\} \geq s_0.$$

By making s_0 sufficiently close to 0 from below, we obtain $D^{\Delta(1)}_{C,k}(\widetilde{A}, \widetilde{B}) \geq 0$.

(ii) Assume that $\widetilde{A} \not\leq^{\Delta(1)}_{\operatorname{cl} C} \widetilde{B}$. Then there exists $(\bar{\alpha}, \bar{\beta}) \in \Delta$ and $a \in \widetilde{A}_{(\bar{\alpha},\bar{\beta})}$ such that $a \notin \bigcap_{b \in \widetilde{B}_{(\bar{\alpha},\bar{\beta})}} (b - \operatorname{cl} C)$. Since $\displaystyle\bigcap_{b \in \widetilde{B}_{(\bar{\alpha},\bar{\beta})}} (b - \operatorname{cl} C)$ is closed, it follows that there exists $\bar{s} > 0$ such that $a - \bar{s}k \notin \bigcap_{b \in \widetilde{B}_{(\bar{\alpha},\bar{\beta})}} (b - \operatorname{cl} C)$. Thus, we have

$$\widetilde{A}_{(\bar{\alpha},\bar{\beta})} - \bar{s}k \not\subset \bigcap_{b \in \widetilde{B}_{(\bar{\alpha},\bar{\beta})}} (b - C),$$

which implies that $\widetilde{A} - \bar{s}k \not\leq_C^{\Delta(1)} \widetilde{B}$. Consequently,

$$D_{C,k}^{\Delta(1)}(\widetilde{A}, \widetilde{B}) = \sup\left\{ s \in \mathbb{R} \,\middle|\, \widetilde{A} + sk \leq_C^{\Delta(1)} \widetilde{B} \right\} \leq -\bar{s} < 0$$

holds by Proposition 3.1.2 (ii).

(iii) Assume that $\widetilde{A} \leq_{\operatorname{int} C}^{\Delta(1)} \widetilde{B}$ and let

$$O \coloneqq \bigcap_{b \in F} (b - \operatorname{int} C), \text{ where } F \coloneqq \bigcup_{(\alpha,\beta) \in \operatorname{Min} \Delta} \widetilde{B}_{(\alpha,\beta)}.$$

Since $\widetilde{A} \leq_C^{\Delta(1)} \widetilde{B}$ if and only if $\widetilde{A} \leq_C^{\operatorname{Min} \Delta(1)} \widetilde{B}$, we have

$$a \in O \text{ for all } a \in E \coloneqq \bigcup_{(\alpha,\beta) \in \operatorname{Min} \Delta} \widetilde{A}_{(\alpha,\beta)}.$$

Since the cut mapping of $\widetilde{B}$ is upper continuous and compact-valued defined on $\operatorname{Min} \Delta$ which is a closed set in ℓ and hence compact, F is compact; see [12]. Thus, O is open by Lemma 2.1.1. It follows that there exists $s_a > 0$ such that $a + s_a k \in O$. Thus, $E \subset \bigcup_{a \in E} (O - s_a k)$. This indicates that $\{O - s_a k\}_{a \in E}$ is an open cover of E. By the compactness of E, there exist m vectors $a_1, \ldots, a_m \in E$ such that $E \subset \bigcup_{i=1}^m (O - s_{a_i} k)$. Let $\bar{s} = \min\{s_{a_1}, \ldots, s_{a_m}\} > 0$. Since $\operatorname{int} C + s_{a_i} k \subset \operatorname{int} C + \bar{s}k$ for every $i = 1, \ldots, m$, we deduce that $\bigcup_{i=1}^m (O - s_{a_i} k) \subset O - \bar{s}k$. Hence, $E + \bar{s}k \subset O \subset \bigcap_{b \in F} (b - C)$, which implies that $\widetilde{A} + \bar{s}k \leq_C^{\Delta(1)} \widetilde{B}$. Therefore,

$$D_{C,k}^{\Delta(1)}(\widetilde{A}, \widetilde{B}) = \sup\left\{ s \in \mathbb{R} \,\middle|\, \widetilde{A} + sk \leq_C^{\Delta(1)} \widetilde{B} \right\} \geq \bar{s} > 0.$$

(iv) Assume that $D^{\Delta(1)}_{C,k}(\widetilde{A},\widetilde{B}) = \sup\left\{ s\in\mathbb{R} \,\middle|\, \widetilde{A}+sk \leq^{\Delta(1)}_{C} \widetilde{B}\right\} > 0$. Then for each $(\alpha,\beta)\in\Delta$, there exists $\bar{s}>0$ such that for all $(\alpha,\beta)\in\Delta$,

$$\widetilde{A}_{(\alpha,\beta)} + \bar{s}k \subset \bigcap_{b\in\widetilde{B}_{(\alpha,\beta)}} (b-C).$$

Since $C+\bar{s}k\subset \operatorname{int} C$, we have

$$\widetilde{A}_{(\alpha,\beta)} \subset \bigcap_{b\in\widetilde{B}_{(\alpha,\beta)}} (b-C) - \bar{s}k \subset \bigcap_{b\in\widetilde{B}_{(\alpha,\beta)}} (b-\operatorname{int} C)$$

for all $(\alpha,\beta)\in\Delta$. Therefore, $\widetilde{A} \leq^{\Delta(1)}_{\operatorname{int} C} \widetilde{B}$. □

Lemma 3.2.1. *Let $C\subset Z$ be a convex cone with $\operatorname{int} C\neq\emptyset$, $\emptyset\neq\Delta\subset\ell$, and $\widetilde{B}$ an IFS on Z. If the cut mapping of $\widetilde{B}$ is an upper continuous set-valued map from Δ to $\mathcal{P}(Z)$, then the set-valued map*

$$\Delta \ni (\alpha,\beta) \mapsto \bigcap_{b\in\widetilde{B}_{(\alpha,\beta)}} (b-C) \in \mathcal{P}(Z)$$

is Hausdorff lower continuous.

Proof. For each $(\bar{\alpha},\bar{\beta})\in\Delta$, we show that for all open sets $W\in\mathcal{U}$, there exists $U\in\mathcal{N}_{\mathbb{R}^2}((\bar{\alpha},\bar{\beta}))\cap\Delta$ such that $\bigcap_{b\in\widetilde{B}_{(\bar{\alpha},\bar{\beta})}}(b-C)\subset\bigcap_{b\in\widetilde{B}_{(\alpha,\beta)}}(b-C)+W$ for all $(\alpha,\beta)\in U$. Since $\operatorname{int} C\neq\emptyset$, there exist $k\in\operatorname{int} C\cap W$ and $V\in\mathcal{U}$ such that $k+V\subset W\cap\operatorname{int} C$. Thus, $k\in k+V\subset C\cap W$ which implies that $-V\subset k-C$. Since the cut mapping of $\widetilde{B}$ is upper continuous at $(\bar{\alpha},\bar{\beta})$, there exists $U\in\mathcal{N}_{\mathbb{R}^2}((\bar{\alpha},\bar{\beta}))\cap\Delta$ such that for all $(\alpha,\beta)\in U$, $\widetilde{B}_{(\alpha,\beta)}\subset\widetilde{B}_{(\bar{\alpha},\bar{\beta})}+V$. Now for all $(\alpha,\beta)\in U$ and $b\in\widetilde{B}_{(\alpha,\beta)}$, there exists $v_b\in\widetilde{B}_{(\bar{\alpha},\bar{\beta})}$ such that $b\in v_b+V$

or $v_b \in b - V$. Hence, $\bigcap_{v \in \widetilde{B}_{(\bar{\alpha},\bar{\beta})}} (v - C) \subset v_b - C \subset b - V - C \subset b + k - C$, which implies that

$$\bigcap_{v \in \widetilde{B}_{(\bar{\alpha},\bar{\beta})}} (v - C) \subset \bigcap_{b \in \widetilde{B}_{(\alpha,\beta)}} (b - C) + k \subset \bigcap_{b \in \widetilde{B}_{(\alpha,\beta)}} (b - C) + W.$$

Hence, the conclusion follows. □

Lemma 3.2.2. *Let $C \subset Z$ be a convex cone with $k \in \operatorname{int} C$, $\emptyset \neq \mathit{\Delta} \subset \ell$, and $\widetilde{A}, \widetilde{B} \in \mathcal{F}_N(Z)$. If the cut mappings of $\widetilde{A}$ and $\widetilde{B}$ are lower and upper continuous, respectively, then the function*

$$f((\alpha,\beta)) := \sup \left\{ t \in \mathbb{R} \;\middle|\; \widetilde{A}_{(\alpha,\beta)} + tk \leq_C^{(2L)} \widetilde{B}_{(\alpha,\beta)} \right\}$$

is lower semicontinuous.

Proof. By Definition 2.3.1,

$$f((\alpha,\beta)) = \sup \left\{ t \in \mathbb{R} \;\middle|\; (\widetilde{A}_{(\alpha,\beta)} + tk) \cap \left(\bigcap_{b \in \widetilde{B}_{(\alpha,\beta)}} (b - C) \right) \neq \emptyset \right\}$$

for all $(\alpha,\beta) \in \mathit{\Delta}$. For each $(\bar{\alpha},\bar{\beta}) \in \mathit{\Delta}$, we show that for all $s < f((\bar{\alpha},\bar{\beta}))$, there exists $U \in \mathcal{N}_{\mathbb{R}^2}((\bar{\alpha},\bar{\beta})) \cap \mathit{\Delta}$ such that $s \leq f((\alpha,\beta))$ for all $(\alpha,\beta) \in U$. Since $s < f((\bar{\alpha},\bar{\beta}))$, there exists $t > 0$ such that $s + t < f((\bar{\alpha},\bar{\beta}))$, which means that

$$\left(\widetilde{A}_{(\bar{\alpha},\bar{\beta})} + (s+t)k\right) \cap \left(\bigcap_{b \in \widetilde{B}_{(\bar{\alpha},\bar{\beta})}} (b - C) + V \right) \neq \emptyset.$$

Since $tk \in \operatorname{int} C$, there exist $W \in \mathcal{U}$ such that $tk + W \subset \operatorname{int} C \subset C$ and $V \in \mathcal{U}$ such that $V = -V$ and $V + V \subset W$. Since the cut mapping of $\widetilde{A}$ is lower continuous, $(\alpha, \beta) \mapsto \widetilde{A}_{(\alpha,\beta)} + (s+t)k$ is also lower continuous at $(\bar{\alpha}, \bar{\beta})$. Hence, there exists $U_1 \in \mathcal{N}_{\mathbb{R}^2}((\bar{\alpha}, \bar{\beta})) \cap \Delta$ such that for all $(\alpha, \beta) \in U_1$,

$$\left(\widetilde{A}_{(\alpha,\beta)} + (s+t)k\right) \cap \left(\bigcap_{b \in \widetilde{B}_{(\bar{\alpha},\bar{\beta})}} (b - C) + V \right) \neq \emptyset. \tag{5}$$

On the other hand, since the cut mapping of $\widetilde{B}$ is upper continuous,

$$\Delta \ni (\alpha, \beta) \mapsto \bigcap_{b \in \widetilde{B}_{(\alpha,\beta)}} (b - C)$$

is Hausdorff lower continuous by Lemma 3.2.1. It follows that there exists $U_2 \in \mathcal{N}_{\mathbb{R}^2}((\bar{\alpha}, \bar{\beta})) \cap \Delta$ such that for all $(\alpha, \beta) \in U_2$ and

$$\bigcap_{b \in \widetilde{B}_{(\bar{\alpha},\bar{\beta})}} (b - C) \subset \bigcap_{b \in \widetilde{B}_{(\alpha,\beta)}} (b - C) + V. \tag{6}$$

Let $U = U_1 \cap U_2$. Then by (5) and (6), for all $(\alpha, \beta) \in U$,

$$\left(\widetilde{A}_{(\alpha,\beta)} + (s+t)k\right) \cap \left(\bigcap_{b \in \widetilde{B}_{(\alpha,\beta)}} (b - C) + V + V \right) \neq \emptyset.$$

Hence, there exist $z \in \widetilde{A}_{(\alpha,\beta)} + sk$, $v_1, v_2 \in V$, and $y \in \bigcap_{b \in \widetilde{B}_{(\alpha,\beta)}} (b - C)$ such that $z + tk = y + v_1 + v_2$. Hence,

$$z = y - (tk - (v_1 + v_2)) \in y - C \subset \bigcap_{b \in \widetilde{B}_{(\alpha,\beta)}} (b - C) - C,$$

which implies that

$$\left(\widetilde{A}_{(\alpha,\beta)} + sk\right) \cap \left(\bigcap_{b \in \widetilde{B}_{(\alpha,\beta)}} (b - C) \right) \neq \emptyset.$$

Therefore, $s \leq f((\alpha, \beta))$ for any $(\alpha, \beta) \in U$. □

Theorem 3.2.2 (Type 2L). *Let $C \subset Z$ be a convex cone, $k \in \operatorname{int} C$, $\emptyset \neq \Delta \subset \ell$, and $\widetilde{A}, \widetilde{B} \in \mathcal{F}_N(Z)$. Then the following statements hold:*
(i) if $\widetilde{A} \leq^{\Delta(2L)}_{\operatorname{cl} C} \widetilde{B}$, then $D^{\Delta(2L)}_{C,k}(\widetilde{A}, \widetilde{B}) \geq 0$;
(ii) if $\widetilde{A}$ is Δ-compact, and $D^{\Delta(2L)}_{C,k}(\widetilde{A}, \widetilde{B}) \geq 0$, then $\widetilde{A} \leq^{\Delta(2L)}_{\operatorname{cl} C} \widetilde{B}$;
(iii) if Δ is closed, the cut mappings of $\widetilde{A}$ and $\widetilde{B}$ are lower and upper continuous, respectively, and $\widetilde{B}$ is Δ-compact, then $\widetilde{A} \leq^{\Delta(2L)}_{\operatorname{int} C} \widetilde{B}$ implies $D^{\Delta(2L)}_{C,k}(\widetilde{A}, \widetilde{B}) > 0$;
(iv) if $D^{\Delta(2L)}_{C,k}(\widetilde{A}, \widetilde{B}) > 0$, then $\widetilde{A} \leq^{\Delta(2L)}_{\operatorname{int} C} \widetilde{B}$.

Proof. (i) Assume that $\widetilde{A} \leq^{\Delta(2L)}_{\operatorname{cl} C} \widetilde{B}$. Then $\widetilde{A}_{(\alpha,\beta)} \cap \left(\bigcap_{b \in \widetilde{B}_{(\alpha,\beta)}} (b - \operatorname{cl} C)\right) \neq \emptyset$ for all $(\alpha, \beta) \in \Delta$. Let $s_0 < 0$. Since $\operatorname{cl} C \subset C + s_0 k$,

$$\left(\widetilde{A}_{(\alpha,\beta)} + s_0 k\right) \cap \left(\bigcap_{b \in \widetilde{B}_{(\alpha,\beta)}} (b - C) \right) \neq \emptyset.$$

for all $(\alpha, \beta) \in \Delta$ This implies that $\widetilde{A} + s_0 k \leq_C^{\Delta(2L)} \widetilde{B}$, and thus

$$D_{C,k}^{\Delta(2L)}(\widetilde{A}, \widetilde{B}) = \sup \left\{ s \in \mathbb{R} \,\middle|\, \widetilde{A} + sk \leq_C^{\Delta(2L)} \widetilde{B} \right\} \geq s_0.$$

By making s_0 sufficiently close to 0 from below, we obtain $D_{C,k}^{\Delta(2L)}(\widetilde{A}, \widetilde{B}) \geq 0$.
(ii) Assume that $\widetilde{A} \not\leq_{\mathrm{cl}\, C}^{\Delta(2L)} \widetilde{B}$. Then there exists $(\bar{\alpha}, \bar{\beta}) \in \Delta$ such that

$$\widetilde{A}_{(\bar{\alpha},\bar{\beta})} \cap \left(\bigcap_{b \in \widetilde{B}_{(\bar{\alpha},\bar{\beta})}} (b - \mathrm{cl}\, C) \right) = \emptyset.$$

Let $D = \bigcap_{b \in \widetilde{B}_{(\bar{\alpha},\bar{\beta})}} (b - \mathrm{cl}\, C)$. For any $a \in \widetilde{A}_{(\bar{\alpha},\bar{\beta})}$, we have $a \notin D$. Due to the closedness of D, it follows that there exists $s_a > 0$ such that $a - s_a k \notin D$ or $a \in D^c + s_a k$, and thus $\widetilde{A}_{(\bar{\alpha},\bar{\beta})} \subset \bigcup_{a \in \widetilde{A}_{(\bar{\alpha},\bar{\beta})}} (D^c + s_a k)$. This indicates that $\{D^c + s_a k\}_{a \in \widetilde{A}_{(\bar{\alpha},\bar{\beta})}}$ is an open cover of $\widetilde{A}_{(\bar{\alpha},\bar{\beta})}$. By the compactness of $\widetilde{A}_{(\bar{\alpha},\bar{\beta})}$, there exist $a_1, \ldots, a_m \in \widetilde{A}_{(\bar{\alpha},\bar{\beta})}$ such that $\widetilde{A}_{(\bar{\alpha},\bar{\beta})} \subset \bigcup_{i=1}^m (D^c + s_{a_i} k)$. Taking $\bar{s} = \min\{s_{a_1}, \ldots, s_{a_m}\} > 0$, we deduce $\bigcup_{i=1}^m (D^c + s_{a_i} k) \subset D^c + \bar{s} k$ because $-\mathrm{cl}\, C + \bar{s} k \subset -\mathrm{cl}\, C + s_{a_i} k$ for every $i = 1, \ldots, m$. We obtain

$$\left(\widetilde{A}_{(\bar{\alpha},\bar{\beta})} - \bar{s} k \right) \cap \left(\bigcap_{b \in \widetilde{B}_{(\bar{\alpha},\bar{\beta})}} (b - C) \right) = \emptyset,$$

which implies $\widetilde{A} - \bar{s} k \not\leq_C^{\Delta(2L)} \widetilde{B}$. Consequently, $D_{C,k}^{\Delta(2L)}(\widetilde{A}, \widetilde{B}) \leq -\bar{s} < 0$ holds by Proposition 3.1.2 (ii).

(iii) Assume that $\widetilde{A} \leq_{\mathrm{int}\, C}^{\Delta(2L)} \widetilde{B}$ and define $f : \Delta \to \mathbb{R} \cup \{\pm\infty\}$ by

$$f((\alpha,\beta)) = \sup \left\{ t \in \mathbb{R} \;\middle|\; \left(\widetilde{A}_{(\alpha,\beta)} + tk\right) \cap \left(\bigcap_{b \in \widetilde{B}_{(\alpha,\beta)}} (b - C) \right) \neq \emptyset \right\}.$$

We first prove $f((\alpha,\beta)) > 0$ for all $(\alpha,\beta) \in \Delta$. Since $\widetilde{B}_{(\alpha,\beta)} \subset \widetilde{A}_{(\alpha,\beta)} + \mathrm{int}\, C$ for any $(\alpha,\beta) \in \Delta$, there exists $z_{(\alpha,\beta)} \in \widetilde{A}_{(\alpha,\beta)} \cap \left(\bigcap_{b \in \widetilde{B}_{(\alpha,\beta)}} (b - \mathrm{int}\, C)\right)$ for each $(\alpha,\beta) \in \Delta$. Since $\bigcap_{b \in \widetilde{B}_{(\alpha,\beta)}} (b - \mathrm{int}\, C)$ is open by the compactness of $\widetilde{B}_{(\alpha,\beta)}$ via Lemma 2.1.1, it follows that there exists $\bar{s}_{(\alpha,\beta)} > 0$ such that

$$z_{(\alpha,\beta)} + \bar{s}_{(\alpha,\beta)} k \in \bigcap_{b \in \widetilde{B}_{(\alpha,\beta)}} (b - \mathrm{int}\, C).$$

Hence, $\left(\widetilde{A}_{(\alpha,\beta)} + \bar{s}_{(\alpha,\beta)} k\right) \cap \left(\bigcap_{b \in \widetilde{B}_{(\alpha,\beta)}} (b - C)\right) \neq \emptyset$ and $f((\alpha,\beta)) \geq \bar{s}_{(\alpha,\beta)} > 0$.

Next, we prove that $D_{C,k}^{\Delta(2L)}(\widetilde{A}, \widetilde{B}) > 0$. Since the cut mappings of $\widetilde{A}$ and $\widetilde{B}$ are lower and upper continuous, respectively, f is l.s.c. by Lemma 3.2.2. Since Δ is compact, we deduce that f has a minimum at some point on Δ, say $(\bar{\alpha}, \bar{\beta})$. Therefore, by Proposition 3.1.4,

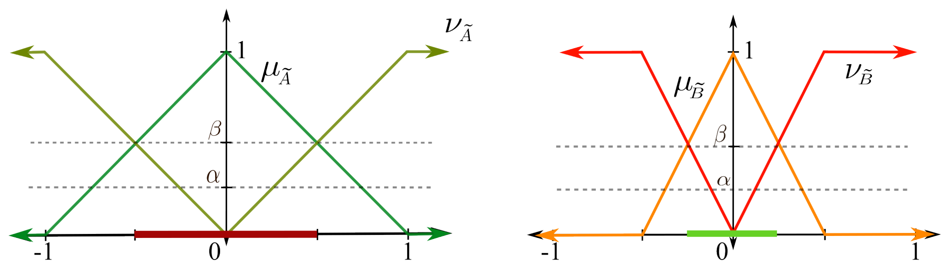
$$D_{C,k}^{\Delta(2L)}(\widetilde{A}, \widetilde{B}) = \inf_{(\alpha,\beta) \in \Delta} f((\alpha,\beta)) = f((\bar{\alpha},\bar{\beta})) \geq \bar{s}_{(\bar{\alpha},\bar{\beta})} > 0.$$

(iv) Assume that $D_{C,k}^{\Delta(2L)}(\widetilde{A}, \widetilde{B}) = \sup\left\{ s \in \mathbb{R} \;\middle|\; \widetilde{A} + sk \leq_C^{\Delta(2L)} \widetilde{B} \right\} > 0$. Then there exists $\bar{s} > 0$ such that $\left(\widetilde{A}_{(\alpha,\beta)} + \bar{s}k\right) \cap \left(\bigcap_{b \in \widetilde{B}_{(\alpha,\beta)}} (b - C)\right) \neq \emptyset$ for all $(\alpha,\beta) \in \Delta$. Since $C + \bar{s}k \subset \mathrm{int}\, C$, we have $\widetilde{A}_{(\alpha,\beta)} \cap \left(\bigcap_{b \in \widetilde{B}_{(\alpha,\beta)}} (b - \mathrm{int}\, C)\right) \neq \emptyset$ for all $(\alpha,\beta) \in \Delta$. Therefore, $\widetilde{A} \leq_{\mathrm{int}\, C}^{\Delta(2L)} \widetilde{B}$. $\square$

Example 3.2.1. Let $Z = \mathbb{R}$, $C = \mathbb{R}_+$ and define two IFS $\widetilde{A}, \widetilde{B}$ by

$$\mu_{\widetilde{A}}(x) := \begin{cases} 0, & x < -1 \\ x+1, & -1 \leq x < 0 \\ 1-x, & 0 \leq x < 1 \\ 0, & x > 1 \end{cases}, \quad \mu_{\widetilde{B}}(x) := \begin{cases} 0, & x < -0.5 \\ 2x+1, & -0.5 \leq x < 0.5 \\ 1-2x, & 0 \leq x < 0.5 \\ 0, & x > 1, \end{cases},$$

$\nu_{\widetilde{A}} := 1 - \mu_{\widetilde{A}}$ and $\nu_{\widetilde{B}} := 1 - \mu_{\widetilde{B}}$. A geometric interpretation is provided below.



In this example, we have $\widetilde{A}_{(\alpha,\beta)} = \left[-\frac{1}{2}, \frac{1}{2}\right]$ and $\widetilde{B}_{(\alpha,\beta)} = \left[-\frac{1}{4}, \frac{1}{4}\right]$ when $\Delta = \left\{\left(\frac{1}{4}, \frac{1}{2}\right)\right\} \subset \ell$. Hence, both are Δ-compact and $\widetilde{A} \leq_C^{\Delta(2L)} \widetilde{B}$. Thus, $D_{C,k}^{\Delta(2L)}(\widetilde{A}, \widetilde{B}) = \frac{1}{4} > 0$. ◀

The correspondence for Type 2U IFS relation is obtained similarly with Type 2L and the facts that in general, the following hold:

$$\widetilde{A} \leq_C^{\Delta(2U)} \widetilde{B} \Longleftrightarrow \widetilde{B} \leq_{-C}^{\Delta(2L)} \widetilde{A} \text{ and } D_{C,k}^{\Delta(2U)}(\widetilde{A}, \widetilde{B}) = D_{-C,-k}^{\Delta(2L)}(\widetilde{B}, \widetilde{A}).$$

Theorem 3.2.3 (Type 2U). *Let C be a convex cone in Z, $k \in \operatorname{int} C$, $\emptyset \neq \Delta \subset \ell$, and $\widetilde{A}, \widetilde{B} \in \mathcal{F}_N(Z)$. Then the following statements hold:*
(i) if $\widetilde{A} \leq_{\operatorname{cl} C}^{\Delta(2U)} \widetilde{B}$, then $D_{C,k}^{\Delta(2U)}(\widetilde{A}, \widetilde{B}) \geq 0$;
(ii) if $\widetilde{B}$ is Δ-compact and $D_{C,k}^{\Delta(2U)}(\widetilde{A}, \widetilde{B}) \geq 0$, then $\widetilde{A} \leq_{\operatorname{cl} C}^{\Delta(2U)} \widetilde{B}$;

(iii) if Δ is closed, the cut mappings of $\widetilde{A}$ and $\widetilde{B}$ are upper and lower continuous, respectively, and $\widetilde{A}$ is Δ-compact, then $\widetilde{A} \preceq^{\Delta(2U)}_{\operatorname{int} C} \widetilde{B}$ implies $D^{\Delta(2U)}_{C,k}(\widetilde{A}, \widetilde{B}) > 0$;

(iv) if $D^{\Delta(2U)}_{C,k}(\widetilde{A}, \widetilde{B}) > 0$, then $\widetilde{A} \preceq^{\Delta(2U)}_{\operatorname{int} C} \widetilde{B}$.

The next theorem is a natural consequence of Theorems 3.2.3, 3.2.2, and Proposition 3.1.5 (iii).

Theorem 3.2.4 (Type 2)**.** *Let $C \subset Z$ be a convex cone, $k \in \operatorname{int} C$, Δ a nonempty subset of ℓ, and $\widetilde{A}, \widetilde{B} \in \mathcal{F}_N(Z)$. Then the following statements hold:*

(i) if $\widetilde{A} \preceq^{\Delta(2)}_{\operatorname{cl} C} \widetilde{B}$, then $D^{\Delta(2)}_{C,k}(\widetilde{A}, \widetilde{B}) \geq 0$;

(ii) if $\widetilde{A}$ and $\widetilde{B}$ are Δ-compact and $D^{\Delta(2)}_{C,k}(\widetilde{A}, \widetilde{B}) \geq 0$, then $\widetilde{A} \preceq^{\Delta(2)}_{\operatorname{cl} C} \widetilde{B}$;

(iii) if Δ is closed, the cut mappings of $\widetilde{A}$ and $\widetilde{B}$ are both lower and upper continuous simultaneously, and $\widetilde{A}, \widetilde{B}$ are Δ-compact, then $\widetilde{A} \preceq^{\Delta(2)}_{\operatorname{int} C} \widetilde{B}$ implies $D^{\Delta(2)}_{C,k}(\widetilde{A}, \widetilde{B}) > 0$;

(iv) if $D^{\Delta(2)}_{C,k}(\widetilde{A}, \widetilde{B}) > 0$, then $\widetilde{A} \preceq^{\Delta(2)}_{\operatorname{int} C} \widetilde{B}$.

Lemma 3.2.3. *Let $C \subset Z$ be a convex cone with $k \in \operatorname{int} C$, $\emptyset \neq \Delta \subset \ell$, and $\widetilde{A}, \widetilde{B} \in \mathcal{F}_N(Z)$. If the cut mappings of $\widetilde{A}$ and $\widetilde{B}$ are lower and upper continuous, respectively, and $\widetilde{A}$ is Δ-compact, then the function*

$$f((\alpha, \beta)) \coloneqq \sup \left\{ t \in \mathbb{R} \;\middle|\; \widetilde{A}_{(\alpha,\beta)} + tk \preceq^{(3L)}_{C} \widetilde{B}_{(\alpha,\beta)} \right\}$$

is lower semicontinuous.

Proof. By Definition 2.3.1, $f((\alpha, \beta)) = \sup \left\{ t \in \mathbb{R} \;\middle|\; \widetilde{B}_{(\alpha,\beta)} \subset \widetilde{A}_{(\alpha,\beta)} + tk + C \right\}$ for all $(\alpha, \beta) \in \Delta$. Since the cut mapping of $\widetilde{A}$ is lower continuous and compact-valued by Δ-compactness of $\widetilde{A}$, $(\alpha,\beta) \mapsto \widetilde{A}_{(\alpha,\beta)} + tk$ is Hausdorff lower continuous for $t \in \mathbb{R}$; see Remark 2.1.2. The remainder of the proof is similar to the approach of that of Lemma 3.2.2. □

Theorem 3.2.5 (Type 3L). *Let $C \subset Z$ be a convex cone, $k \in \operatorname{int} C$, $\emptyset \neq \Delta \subset \ell$, and $\widetilde{A}, \widetilde{B} \in \mathcal{F}_N(Z)$. Then the following statements hold:*
(i) if $\widetilde{A} \leq_{\operatorname{cl} C}^{\Delta(3L)} \widetilde{B}$, then $D_{C,k}^{\Delta(3L)}(\widetilde{A}, \widetilde{B}) \geq 0$;
(ii) if $\widetilde{A}$ is Δ-compact and $D_{C,k}^{\Delta(3L)}(\widetilde{A}, \widetilde{B}) \geq 0$, then $\widetilde{A} \leq_{\operatorname{cl} C}^{\Delta(3L)} \widetilde{B}$;
(iii) if Δ is closed, the cut mappings of $\widetilde{A}$ and $\widetilde{B}$ are lower and upper continuous, respectively, and $\widetilde{A}, \widetilde{B}$ are Δ-compact, then $\widetilde{A} \leq_{\operatorname{int} C}^{\Delta(3L)} \widetilde{B}$ implies $D_{C,k}^{\Delta(3L)}(\widetilde{A}, \widetilde{B}) > 0$;
(iv) if $D_{C,k}^{\Delta(3L)}(\widetilde{A}, \widetilde{B}) > 0$, then $\widetilde{A} \leq_{\operatorname{int} C}^{\Delta(3L)} \widetilde{B}$.

Proof. To prove statement (iv), we can take the same approach as in the proof of Theorem 3.2.1.
(i) Assume that $\widetilde{A} \leq_{\operatorname{cl} C}^{\Delta(3L)} \widetilde{B}$. For all $(\alpha,\beta) \in \Delta$, $\widetilde{B}_{(\alpha,\beta)} \subset \widetilde{A}_{(\alpha,\beta)} + \operatorname{cl} C$. Let $s_0 < 0$. Since $\operatorname{cl} C \subset C + s_0 k$ for all $(\alpha,\beta) \in \Delta$, we have

$$\widetilde{B}_{(\alpha,\beta)} \subset \widetilde{A}_{(\alpha,\beta)} + C + s_0 k$$

implying that $\widetilde{A} + s_0 k \leq_{C}^{\Delta(3L)} \widetilde{B}$. Thus,

$$D_{C,k}^{\Delta(3L)}(\widetilde{A}, \widetilde{B}) = \sup \left\{ s \in \mathbb{R} \;\middle|\; \widetilde{A} + sk \leq_{C}^{\Delta(3L)} \widetilde{B} \right\} \geq s_0.$$

By making s_0 sufficiently close to 0 from below, we obtain $D_{C,k}^{\Delta(3L)}(\widetilde{A}, \widetilde{B}) \geq 0$.

(ii) Assume that $\widetilde{A} \not\leq^{\Delta(3L)}_{\operatorname{cl} C} \widetilde{B}$. Then there exists $(\bar{\alpha}, \bar{\beta}) \in \Delta$ such that

$$\widetilde{B}_{(\bar{\alpha},\bar{\beta})} \not\subset \widetilde{A}_{(\bar{\alpha},\bar{\beta})} + \operatorname{cl} C.$$

Thus, there exists $b \in \widetilde{B}_{(\bar{\alpha},\bar{\beta})}$ such that $b \notin \widetilde{A}_{(\bar{\alpha},\bar{\beta})} + \operatorname{cl} C$. Since $\widetilde{A}_{(\bar{\alpha},\bar{\beta})}$ is compact, it follows that $\widetilde{A}_{(\bar{\alpha},\bar{\beta})} + \operatorname{cl} C$ is closed. Hence, there exists $\bar{s} > 0$ such that $b + \bar{s}k \notin \widetilde{A}_{(\bar{\alpha},\bar{\beta})} + \operatorname{cl} C$. Since $b \notin \widetilde{A}_{(\bar{\alpha},\bar{\beta})} + \operatorname{cl} C - \bar{s}k$, we deduce $\widetilde{B}_{(\bar{\alpha},\bar{\beta})} \not\subset \widetilde{A}_{(\bar{\alpha},\bar{\beta})} - \bar{s}k + C$, and thus $\widetilde{A} - \bar{s}k \not\leq^{\Delta(3L)}_{C} \widetilde{B}$. By Proposition 3.1.2, $\widetilde{A} - sk \not\leq^{\Delta(3L)}_{C} \widetilde{B}$ for all $s \in (-\infty, \bar{s}]$. Consequently,

$$D^{\Delta(3L)}_{C,k}(\widetilde{A}, \widetilde{B}) = \sup \left\{ s \in \mathbb{R} \;\middle|\; \widetilde{A} + sk \leq^{\Delta(3L)}_{C} \widetilde{B} \right\} \leq -\bar{s} < 0.$$

(iii) Assume that $\widetilde{A} \leq^{\Delta(3L)}_{\operatorname{int} C} \widetilde{B}$ and define $f : \Delta \to \mathbb{R} \cup \{\pm\infty\}$ by

$$f((\alpha, \beta)) \coloneqq \sup \left\{ t \in \mathbb{R} \;\middle|\; \widetilde{B}_{(\alpha,\beta)} \subset \widetilde{A}_{(\alpha,\beta)} + tk + C \right\}.$$

We first prove that $f((\alpha, \beta)) > 0$ for all $(\alpha, \beta) \in \Delta$. Since for all $(\alpha, \beta) \in \Delta$, $\widetilde{B}_{(\alpha,\beta)} \subset \widetilde{A}_{(\alpha,\beta)} + \operatorname{int} C$, we have $b \in O \coloneqq \widetilde{A}_{(\alpha,\beta)} + \operatorname{int} C$ for any $b \in \widetilde{B}_{(\alpha,\beta)}$. It follows that there exists $s_b > 0$ such that $b - s_b k \in O$ because O is open, and thus

$$\widetilde{B}_{(\alpha,\beta)} \subset \bigcup_{b \in \widetilde{B}_{(\alpha,\beta)}} (O + s_b k).$$

This indicates that $\{O + s_b k\}_{b \in \widetilde{B}_{(\alpha,\beta)}}$ is an open cover of $\widetilde{B}_{(\alpha,\beta)}$. Since $\widetilde{B}_{(\alpha,\beta)}$ is compact, there exists m vectors $b_1, \ldots, b_m \in \widetilde{B}_{(\alpha,\beta)}$ such that

$$\widetilde{B}_{(\alpha,\beta)} \subset \bigcup_{i=1}^{m} (O + s_{b_i} k).$$

Taking $\bar{s}_{(\alpha,\beta)} \coloneqq \min\{s_{b_1}, \ldots, s_{b_m}\} > 0$, we deduce

$$\bigcup_{i=1}^{m}(O + s_{b_i}k) \subset O + \bar{s}_{(\alpha,\beta)}k$$

because $\operatorname{int} C + s_{b_i}k \subset \operatorname{int} C + \bar{s}_{(\alpha,\beta)}k$ for every $i = 1, \ldots, m$. Therefore, $\widetilde{B}_{(\alpha,\beta)} \subset \widetilde{A}_{(\alpha,\beta)} + \bar{s}_{(\alpha,\beta)}k + C$, and thus $f((\alpha, \beta)) \geq \bar{s}_{(\alpha,\beta)} > 0$.

Next, we show that $\widetilde{A} \leq_{\operatorname{int} C}^{\Delta(3L)} \widetilde{B}$ implies $D_{C,k}^{\Delta(3L)}(\widetilde{A}, \widetilde{B}) > 0$. By assumption, $\widetilde{A}$ is Δ-compact and the cut mappings of $\widetilde{A}, \widetilde{B}$ are lower and upper continuous, respectively, which implies that f is l.s.c. by Lemma 3.2.3. From the compactness of Δ, we deduce that f has a minimum at some point on Δ, say $(\bar{\alpha}, \bar{\beta})$. Therefore, by using Proposition 3.1.4,

$$D_{C,k}^{\Delta(3L)}(\widetilde{A}, \widetilde{B}) = \inf_{(\alpha,\beta)\in\Delta} f((\alpha, \beta)) = f((\bar{\alpha}, \bar{\beta})) \geq \bar{s}_{(\bar{\alpha},\bar{\beta})} > 0,$$

which completes the proof. □

The next result can be obtained by using the above theorem and the facts that $\widetilde{A} \leq_{C}^{\Delta(3U)} \widetilde{B} \Longleftrightarrow \widetilde{B} \leq_{-C}^{\Delta(3L)} \widetilde{A}$ and $D_{C,k}^{\Delta(3U)}(\widetilde{A}, \widetilde{B}) = D_{-C,-k}^{\Delta(3L)}(\widetilde{B}, \widetilde{A})$ hold in general.

Theorem 3.2.6 (Type 3U)**.** *Let $C \subset Z$ be a convex cone, $k \in \operatorname{int} C$, $\emptyset \neq \Delta \subset \ell$, and $\widetilde{A}, \widetilde{B} \in \mathcal{F}_N(Z)$. Then the following statements hold:*
(i) if $\widetilde{A} \leq_{\operatorname{cl} C}^{\Delta(3U)} \widetilde{B}$, then $D_{C,k}^{\Delta(3U)}(\widetilde{A}, \widetilde{B}) \geq 0$;
(ii) if $\widetilde{B}$ is Δ-compact and $D_{C,k}^{\Delta(3U)}(\widetilde{A}, \widetilde{B}) \geq 0$, then $\widetilde{A} \leq_{\operatorname{cl} C}^{\Delta(3U)} \widetilde{B}$;
(iii) if Δ is closed, the cut mappings of $\widetilde{A}$ and $\widetilde{B}$ are upper and lower continuous, respectively, and $\widetilde{A}, \widetilde{B}$ are Δ-compact, then $\widetilde{A} \leq_{\operatorname{int} C}^{\Delta(3U)} \widetilde{B}$ implies

$D^{\Delta(3U)}_{C,k}(\widetilde{A},\widetilde{B}) > 0$;

(iv) if $D^{\Delta(3U)}_{C,k}(\widetilde{A},\widetilde{B}) > 0$*, then* $\widetilde{A} \leq^{\Delta(3U)}_{\operatorname{int} C} \widetilde{B}$*.*

The next theorem is a consequence of Theorems 3.2.5, 3.2.6, and Proposition 3.1.5 (iv).

Theorem 3.2.7 (Type 3)**.** *Let* $C \subset Z$ *be a convex cone,* $k \in \operatorname{int} C$*,* Δ *a nonempty subset of* ℓ*, and* $\widetilde{A},\widetilde{B} \in \mathcal{F}_N(Z)$*. Then the following statements hold:*

(i) if $\widetilde{A} \leq^{\Delta(3)}_{\operatorname{cl} C} \widetilde{B}$*, then* $D^{\Delta(3)}_{C,k}(\widetilde{A},\widetilde{B}) \geq 0$*;*

(ii) if $\widetilde{A}$ *and* $\widetilde{B}$ *are* Δ*-compact and* $D^{\Delta(3)}_{C,k}(\widetilde{A},\widetilde{B}) \geq 0$ *implies* $\widetilde{A} \leq^{\Delta(3)}_{\operatorname{cl} C} \widetilde{B}$*;*

(iii) if Δ *is closed, the cut mappings of* $\widetilde{A}$ *and* $\widetilde{B}$ *are both lower and upper continuous simultaneously, and* $\widetilde{A},\widetilde{B}$ *are* Δ*-compact, then* $\widetilde{A} \leq^{\Delta(3)}_{\operatorname{int} C} \widetilde{B}$ *implies* $D^{\Delta(3)}_{C,k}(\widetilde{A},\widetilde{B}) > 0$*;*

(iv) if $D^{\Delta(3)}_{C,k}(\widetilde{A},\widetilde{B}) > 0$*, then* $\widetilde{A} \leq^{\Delta(3)}_{\operatorname{int} C} \widetilde{B}$*.*

Lemma 3.2.4. *Let* $C \subset Z$ *be a convex cone with* $k \in \operatorname{int} C$*,* $\emptyset \neq \Delta \subset \ell$*, and* $\widetilde{A},\widetilde{B} \in \mathcal{F}_N(Z)$*. If the cut mappings of* $\widetilde{A}$ *and* $\widetilde{B}$ *are both lower continuous, and* $\widetilde{A},\widetilde{B}$ *are* Δ*-compact, then the function*

$$f((\alpha,\beta)) := \sup\left\{t \in \mathbb{R} \;\middle|\; \widetilde{A}_{(\alpha,\beta)} + tk \leq^{(4)}_{C} \widetilde{B}_{(\alpha,\beta)}\right\}$$

is lower semicontinuous.

Proof. By Definition 2.3.1,

$$f((\alpha,\beta)) = \sup\left\{t \in \mathbb{R} \;\middle|\; \left(\widetilde{A}_{(\alpha,\beta)} + tk\right) \cap \left(\widetilde{B}_{(\alpha,\beta)} - C\right) \neq \emptyset\right\}$$

for all $(\alpha, \beta) \in \Delta$. Since the cut mappings of $\widetilde{A}$ and $\widetilde{B}$ are lower continuous, and $\widetilde{A}$ and $\widetilde{B}$ are compact-valued by assumption, the mappings

$$(\alpha, \beta) \mapsto \widetilde{A}_{(\alpha,\beta)} + tk \text{ and } (\alpha, \beta) \mapsto \widetilde{B}_{(\alpha,\beta)}$$

are Hausdorff lower continuous simultaneously. The remainder of the proof is similar to the approach of that of Lemma 3.2.2. □

Theorem 3.2.8 (Type 4). *Let $C \subset Z$ be a convex cone, $k \in \operatorname{int} C$, Δ a nonempty subset of ℓ, and $\widetilde{A}, \widetilde{B} \in \mathcal{F}_N(Z)$. Then the following statements hold:*
(i) if $\widetilde{A} \leq_{\operatorname{cl} C}^{\Delta(4)} \widetilde{B}$, then $D_{C,k}^{\Delta(4)}(\widetilde{A}, \widetilde{B}) \geq 0$;
(ii) if $\widetilde{A}$, $\widetilde{B}$ are Δ-compact and $D_{C,k}^{\Delta(4)}(\widetilde{A}, \widetilde{B}) \geq 0$, then $\widetilde{A} \leq_{\operatorname{cl} C}^{\Delta(4)} \widetilde{B}$;
(iii) if Δ is closed, $\widetilde{A}, \widetilde{B}$ are Δ-compact and have lower continuous cut mappings, then $\widetilde{A} \leq_{\operatorname{int} C}^{\Delta(4)} \widetilde{B}$ implies $D_{C,k}^{\Delta(4)}(\widetilde{A}, \widetilde{B}) > 0$;
(iv) if $D_{C,k}^{\Delta(4)}(\widetilde{A}, \widetilde{B}) > 0$, then $\widetilde{A} \leq_{\operatorname{int} C}^{\Delta(4)} \widetilde{B}$.

Proof. The proof for statements (i) and (iv) are similar to that of Theorem 3.2.1 while the proof for statement (iii) is similar to that of Theorem 3.2.5 via Lemma 3.2.4.
(ii) We assume that $\widetilde{A} \not\leq_{\operatorname{cl} C}^{\Delta(4)} \widetilde{B}$. Then there exists $(\bar{\alpha}, \bar{\beta}) \in \Delta$ such that $\widetilde{A}_{(\bar{\alpha},\bar{\beta})} \not\leq_{\operatorname{cl} C}^{(4)} \widetilde{B}_{(\bar{\alpha},\bar{\beta})}$, which implies $\widetilde{A}_{(\bar{\alpha},\bar{\beta})} \cap \left(\widetilde{B}_{(\bar{\alpha},\bar{\beta})} - \operatorname{cl} C\right) = \emptyset$. Since $\widetilde{B}_{(\bar{\alpha},\bar{\beta})}$ is compact and $\operatorname{cl} C$ is closed, $G := \widetilde{B}_{(\bar{\alpha},\bar{\beta})} - \operatorname{cl} C$ is closed. For each $a \in \widetilde{A}_{(\bar{\alpha},\bar{\beta})}$, there exists $s_a > 0$ such that $a - s_a k \in G^c$ which means that

$$a \in G^c + s_a k \subset \bigcup_{a \in \widetilde{A}_{(\bar{\alpha},\bar{\beta})}} (G^c + s_a k).$$

This implies that $\bigcup_{a\in\widetilde{A}_{(\bar{\alpha},\bar{\beta})}}(G^c+s_a k)$ is an open cover for $\widetilde{A}_{(\bar{\alpha},\bar{\beta})}$. Since $\widetilde{A}_{(\bar{\alpha},\bar{\beta})}$ is compact, there exists $a_1,\dots,a_m\in\widetilde{A}_{(\bar{\alpha},\bar{\beta})}$ such that $\widetilde{A}_{(\bar{\alpha},\bar{\beta})}\subset\bigcup_{i=1}^{m}(G^c+s_{a_i}k)$. By taking $\bar{s}:=\min\{s_{a_1},\dots,s_{a_m}\}>0$, we deduce that $\widetilde{A}_{(\bar{\alpha},\bar{\beta})}\subset G^c+\bar{s}k$. Hence, $\left(\widetilde{A}_{(\bar{\alpha},\bar{\beta})}-\bar{s}k\right)\cap G=\emptyset$. Consequently,

$$D_{C,k}^{\Delta(4)}(\widetilde{A},\widetilde{B})=\sup\left\{s\in\mathbb{R}\,\middle|\,\widetilde{A}+sk\preceq_C^{\Delta(4)}\widetilde{B}\right\}\leq-\bar{s}<0$$

and this completes the proof. □

Chapter 4

Conclusion

In this study, various types of intuitionistic fuzzy-set relations based on a convex cone have been introduced as a new comparison criteria of intuitionistic fuzzy sets. Several results related to the correspondences between IFS and their difference evaluation functions have been obtained under some assumptions of certain compactness and continuity. In particular, comparing intuitionistic fuzzy sets yields several results related to the sign of the difference evaluation function value between them with respect to the type of set relations. They are useful when we evaluate intuitionistic fuzzy sets with respect to multi-criteria comparison methods.

The following are left to be explored.

Choice of membership and non-membership functions In this research, several intuitionistic fuzzy sets are assumed to exist for convenience.

It would be useful to other fields if some results are obtained related to defining particular functions.

Relationship between different nonempty subsets Δ of ℓ. In Chapter 3, the nonempty subset Δ of ℓ is taken by preference to satisfy different results in this research. In the future, it would an interesting theme to investigate relationships between various types of Δ and their implications.

Bibliography

[1] K.T. Atanassov. Intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, 20(1):87–96, 1986.

[2] R.B. Holmes. *Geometric Functional Analysis and its Applications.* Springer-Verlag, New York, 1975.

[3] K. Ike. Further characterizations of possibility-theoretical indices in fuzzy optimization. *Fuzzy Sets and Systems*, 422(1):149–160, 2021.

[4] K. Ike and T. Tanaka. Convex-cone-based comparisons of and difference evaluations for fuzzy sets. *Optimization*, 67(1):1051–1066, 2018.

[5] V. Nayagam, S. Jeevaraj, and G. Sivaraman. Complete ranking of intuitionistic fuzzy numbers. *Fuzzy Information and Engineering*, 8(2):237–254, 2016.

[6] M. Kon. Operation and ordering of fuzzy sets, and fuzzy set-valued convex mappings. *J. Fuzzy Set Valued Analysis*, 2014(2014):1–17, 2014.

[7] P. Huidobro, P. Alonso, V. Janiš, S. Montes. Convexity and level sets for interval-valued fuzzy sets. *Fuzzy Optimization and Decision Making*, 21(1):553–580, 2022.

[8] L. Platil and T. Tanaka. A generalization of fuzzy-set relations for intuitionistic fuzzy sets. In *RIMS Kokyuroku 2240*, pages 78–83, Kyoto, JP, 2023. Kyoto University.

[9] L. Platil and T. Tanaka. Multi-criteria evaluation for intuitionistic fuzzy sets based on set-relations. *Nihonkai Math. J.*, 34(1):1–18, 2023.

[10] J. Ramík and J. Řimánek. Inequality relation between fuzzy numbers and its use in fuzzy optimization. *Fuzzy Sets and Systems*, 16(2):123–138, 1985.

[11] H. Yu, K. Ike, Y. Ogata, Y. Saito, and T. Tanaka. Computational methods for set-relation-based scalarizing functions. *Nihonkai Math. J.*, 28(1):139–149, 2017.

[12] A.A. Khan, C. Riahi, C. Tammer, and C. Zălinescu. *Set-valued Optimization–An Introduction with Applications*. Springer, Berlin, 2015.

[13] D. Kuroiwa, T. Tanaka, and T.X.D. Ha. On cone convexity of set-valued maps. *Nonlinear Analysis*, 30(1):1487–1496, 1997.

[14] K. Ike, M. Liu, Y. Ogata, T. Tanaka. Semicontinuity of the composition of set-valued map and scalarization function for sets. *J. Appl. Numer. Optim.*, 1(3):267–276, 2019.

[15] K. Ike, Y. Ogata, T. Tanaka, and H. Yu. Sublinear-like scalarization scheme for sets and its applications to set-valued inequalities. In *Variational Analysis and Set Optimization*, chapter 4, pages 72–91. CRC Press, 2019.

[16] Y. Ogata, Y. Saito, T. Tanaka, and S. Yamada. Sublinear scalarization methods for sets with respect to set-relations. *Linear and Nonlinear Analysis*, 3(1):121–132, 2017.

[17] L.A. Zadeh. Fuzzy sets. *Information and control*, 8(1):338–353, 1965.

[18] A. Göpfert, H. Riahi, C. Tammer, C. Zălinescu. *Variational methods in partially ordered spaces*. Springer-Verlag, New York, 2003.